AN OPERATOR EMBEDDING THEOREM
FOR COMPLEXITY CLASSES OF
RECURSIVE FUNCTIONS

Robert Moll

May 1973

MAC TECHNICAL MEMORANDUM 32

AN OPERATOR EMBEDDING THEOREM FOR COMPLEXITY CLASSES

OF RECURSIVE FUNCTIONS

Robert Moll

May 1973

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

PROJECT MAC

CAMBRIDGE   MASSACHUSETTS 02139

*This empty page was substituted for a
blank page in the original document.*

AN OPERATOR EMBEDDING THEOREM FOR COMPLEXITY CLASSES
OF RECURSIVE FUNCTIONS

Robert Moll[†]

Massachusetts Institute of Technology

Spring 1973

1. INTRODUCTION

Let $\mathcal{F}(t)$ be the set of functions computable by some machine using no more than $t(x)$ machine steps on all but finitely many arguments x. If we order the $\mathcal{F}$-classes under set inclusion as t varies over the recursive functions, then it is natural to ask how rich a structure is obtained. We show that this structure is very rich indeed. If R is any countable partial order and $\underset{\sim}{F}$ is any total effective operator, then we show that there is a recursively enumerable sequence of recursive machine running times $\{\Phi_{s(k)}\}_{k \in N}$ such that if jRk, then $\mathcal{F}(\underset{\sim}{F}(\Phi_{s(j)})) \subsetneq \mathcal{F}(\Phi_{s(k)})$, and if j and k are incomparable, then $\underset{\sim}{F}(\Phi_{s(j)}) < \Phi_{s(k)}$ on infinitely many arguments, and $\underset{\sim}{F}(\Phi_{s(k)}) < \Phi_{s(j)}$ on infinitely many arguments.

An interesting feature of our proof is that we avoid appealing explicitly to the continuity of total effective operators; indeed our proof follows directly from a single appeal to the recursion theorem.

Several investigators have considered this and related problems, and in Section 4 we briefly summarize these investigations and compare them to our own.

2. PRELIMINARIES

For notation from recursive function theory we follow Rogers [2].

For each $n \in N$, P_n stands for the partial recursive functions of n-variables, and R_n stands for the total recursive functions of n variables.

We use (a.e.) to denote "almost everywhere", which for our purposes stands for "all but finitely many". Similarly (i.o.) stands for "infinitely often".

Suppose $\{\varphi_0, \varphi_1, \ldots\}$ is a Godel numbering of P_1. A <u>measure on Computation</u> [1] $\Phi = \{\Phi_0, \Phi_1, \ldots\}$ is a sequence of functions in P_1 satisfying

1. $\forall i \in N \; [\text{dom}(\varphi_i) = \text{dom}(\Phi_i)]$

2. $\lambda ixy[\Phi_i(x) = y]$ is a recursive predicate.

If we think of our Gödel numbering in the usual one-tape Turing machine formalism, then

$\Phi_i(x) =$ "the number of steps in the computation of the ith Turing machine on argument x" is a measure on computation.

Henceforth let Φ be some fixed measure on computation. Then we define for any total function t

$$F(t) = \{i \in N \mid \varphi_i \in R, \text{ and } \Phi_i \leq t \; (a.e.)\},$$

and

$$\mathscr{F}(t) = \{\varphi_i \mid i \in F(t)\}.$$

That is, $F(t)$ is the set of (indices of) total machines which run in time t, and $\mathscr{F}(t)$ is the set of total functions computable within time t. $\mathscr{F}(t)$ is called a <u>complexity class</u>.

A sequence of partial functions $\Psi = \{\psi_0, \psi_1, \ldots\}$ is said to be an r.e. sequence of partial functions if $\lambda ix[\psi_i(x)] \in \rho_2$.

The following theorem of Blum [1] shows that we can uniformly enlarge complexity classes $\mathscr{F}(t)$ if t is a sufficiently well-behaved function.

<u>Theorem</u>. (Compression Theorem) There is a $g \in \aleph_2$ such that for every $\Phi_i \in \aleph_1$, $\mathscr{F}(\Phi_i) \subsetneq \mathscr{F}(\lambda xg(x, \Phi_i(x)))$. g is called a compression function for Φ.

An operator is a map which takes functions to functions; we write $\underset{\sim}{F}(f)(x)$ to mean the value of the operator $\underset{\sim}{F}$ applied to the function f, evaluated at x. An operator $\underset{\sim}{F}: D \subseteq \rho_1 \rightarrow \rho_1$ is called an <u>effective operator</u> if there is an $s \in \aleph_1$ such that $\underset{\sim}{F}(\wp_e)(x) = \wp_{s(e)}(x)$.

An effective operator $\underset{\sim}{F}$ is <u>total effective</u> if for every $f \in \aleph_1$, $\underset{\sim}{F}(f)$ is defined and $\underset{\sim}{F}(f) \in \aleph_1$.

3. THE EMBEDDING THEOREM

<u>Theorem</u>. Let $\underset{\sim}{F}$ be any total effective operator, and let R be any recursive countable partial order on N. Then there exists an r.e. sequence of recursive functions $p_0, p_1, \ldots p_n \ldots$ such that if jRk, then $\underset{\sim}{F}(p_j) < p_k$ (a.e.), and if j and k are incomparable, then $\underset{\sim}{F}(p_j) < p_k$ (i.o.), and $p_k < \underset{\sim}{F}(p_j)$ (i.o.).

-4-

Proof. We assume without loss of generality that R orders N-{0}

rather than N, and in addition that R contains kR0 for each k > 0.

Let $a_0 = \langle i_0, k_0 \rangle$, $a_1 = \langle i_1, k_1 \rangle$, ... $a_n = \langle i_n, k_n \rangle$, ... be a

recursive listing of all incomparable pairs in R such that if x and

y are incomparable, then $\langle x, y \rangle$ and $\langle y, x \rangle$ both appear infinitely

often in the list. As a technical convience we define $\underline{\max}[\phi] = 0$.

Let $s \in \aleph_2$ be the s_1^1 function of the s-m-n theorem defined by the

equation

$$\varphi_e(\langle x, y \rangle) = \varphi_{s(e,x)}(y).$$

Define $\psi \in P_2$ as follows:

$$\psi(e, \langle k, x \rangle) = \begin{cases}
0 \quad \text{if } x < k \text{ or } \exists n < k \text{ such that } \Phi_e(\langle 0, n \rangle) > x, \quad (1) \\[2em]
\underline{\max}_{\substack{j \leq x \\ jRk}} [\varphi_{s(e,j)}(x) + \underset{\sim}{F}(\varphi_{s(e,j)})(x)]) + \quad\quad (2)(i) \\[1.5em]
\quad [\varphi_{s(e,i_n)}(x) + \underset{\sim}{F}(\varphi_{s(e,i_n)})(x)], \quad\quad (2)(ii) \\[1em]
\quad \text{where } n = \mu m \leq x[((m = 0) \underline{\text{ and }} (x = k_0)) \underline{\text{ or }} \\[0.5em]
\quad [(m > 0) \underline{\text{ and }} (k = k_m) \underline{\text{ and }} [(\forall i \ (0 \leq i \leq m)) \\[0.5em]
\quad (\exists z_i \leq x) \text{ such that } (z_0 = k_0) \underline{\text{ and }} \\[0.5em]
\quad (z_{i+1} = z_i + \Phi_{s(e,k_i)}(z_i)) \underline{\text{ and }} (z_m = x)]]], \text{ if} \\[0.5em]
\quad \text{such an n exists and (1) is not true, and} \\[2em]
\underline{\max}_{\substack{j \leq x \\ jRk}} [\varphi_{s(e,j)}(x) + \underset{\sim}{F}(\varphi_{s(e,j)})(x)] \text{ otherwise.} \quad (3)
\end{cases}$$

$\psi \in P_2$ since all the test computations in clauses (1) and (2) are recursive by the second measure on computation axiom. By the recursion theorem there is an e such that $\psi(e, <k,x>) = \varphi_e(<k,x>)$; we apply the s-1-1 version of the s-m-n theorem to obtain $\psi(e, <k,x>) = \varphi_{s(e,k)}(x)$. To simplify our notation we now suppress mention of e and write $p_k(x) = \varphi_{s(e,k)}(x)$. Similarly we write $\Phi_{p_k}(x)$ for $\Phi_{s(e,k)}(x)$. Our definition now becomes

$$p_k(x) = \begin{cases} 0 \quad \text{if } x < k \text{ } \underline{\text{or}} \text{ } \exists n < k \text{ such that } \Phi_{p_0}(n) < x, \quad (1) \\[2em] \underset{\substack{j \leq x \\ jRk}}{\max}[p_j(x) + \underset{\sim}{F}(p_j)(x)]) \quad + \quad (2)(i) \\ \qquad\qquad [p_{i'_n}(x) + \underset{\sim}{F}(p_{i_n})(x)], \quad (2)(ii) \\ \text{where } n = \mu m \leq x[((m = 0) \text{ } \underline{\text{and}} \text{ } (x = k_0)) \text{ } \underline{\text{or}} \\ [(m > 0) \text{ } \underline{\text{and}} \text{ } (k = k_m) \text{ and } [(\forall i(0 \leq i \leq m))(\exists z_i \leq x) \\ \text{such that } (z_0 = k_0) \text{ } \underline{\text{and}} \text{ } (z_{i+1} = z_i + \Phi_{p_{k_i}}(z_i)) \text{ } \underline{\text{and}} \\ (z_m = x)]]], \text{ if such an } n \text{ exists and (1) is not} \\ \text{true, and} \\[3em] \underset{\substack{j \leq x \\ jRk}}{\max}[p_j(x) + \underset{\sim}{F}(p_j)(x)] \text{ otherwise.} \quad (3) \end{cases}$$

We first establish that at most finitely many of the functions $\{p_k\}_{k \in N}$ can be non-total. Suppose $p_k(x)$ diverges. Since p_0 is defined by (3) at all arguments, $p_0(x)$ must diverge, and so by (1) $p_j(x) \equiv 0$ for all $j > x$.

We now prove that for all k p_k is total.

Say that a_n is <u>serviced</u> <u>at</u> <u>x</u> if $p_{k_n}(x)$ is defined by (2), and if n is the least $m \le x$ satisfying the body of (2) in the definition of $p_{k_n}(x)$. We allow the possibility that $p_{k_n}(x)$ may diverge. If a_n is serviced at x, (2) guarantees that $x = z_n = \sum_{i=1}^{n-1} z_i + \Phi_{p_{k_i}}(z_i)$, and so a_n is serviced at no other argument. Moreover, if a_n is serviced at x and $p_{k_n}(x)$ diverges, then for $n' > n$ $a_{n'}$ will never be serviced, since $a_{n'}$ is serviced at y only when y bounds the computation of $\Phi_{p_{k_n}}(x)$.

Let k be an R-minimal element in the finite set $\{k' \mid p_{k'}$ non-total$\}$. Then if $p_k(x)$ diverges, it must do so because of (2)(ii). That is, a_n is serviced at x for some n, and p_{i_n} must be non-total.

But suppose $p_{i_n}(y)$ diverges by an instance of (2)(ii) for some y. This means that $i_n = k_j$ for some j and a_j is serviced at y. If $j < n$, then y must equal z_j, but since a_n is serviced x, $\Phi_{p_{k_j}}(z_j) < x$ and hence $p_{k_j}(z_j)$ must converge. If $j > n$, then since a_n is serviced at x and $p_k(x)$ is assumed to diverge, a_j is never serviced. Moreover j cannot equal n, for then i_n would equal k_n. Hence p_{i_n} must be non-total because of (2)(i) or (3), and so some function $p_{i'}$, such that $i'Ri_n$ is non-total.

Let i be R minimal among $\{i' \mid i'R\ i_n$ and i' non-total$\}$. Then p_i must be non-total by an instance of (2)(ii), say at argument y. Hence $i = k_j$ for some j, and a_j must be serviced at $y = \sum_{m=0}^{j-1} z_m + \Phi_{p_{k_j}}(z_m)$. If $j < n$, $p_{k_j}(y)$ must converge since a_n is serviced at x by assumption; and if $j = n$, then i_n and k_n are comparable, a contradiction. Furthermore if $j > n$, then a_j will never be serviced. Hence p_i is total, and we conclude that for every k $\dot{p}_k \in \aleph_1$.

If jRk, then $\underset{\sim}{F}(p_j)(z) \le p_k(z)$ for all $z \ge m_0 = \underline{\max}[k, j, \Phi_{p_0}(0), \Phi_{p_0}(1), \ldots \Phi_{p_0}(k-1)]$.

If j and k are incomparable, then $< j, k > = a_{n_0}, a_{n_1}, \cdots a_{n_q}, \cdots$ for some infinite sequence $n_0 < n_1 < n_2 \cdots n_q \cdots$.

For arguments $z \ge m_0$ $p_k(z)$ is defined by (2) or (3). Since the sequence of z_i's is strictly increasing, there is an i_0 such that for $i > i_0, z_i \ge m_0$. At those arguments z_i for $i > i_0$, $i = n_q$, $p_k(z_i)$ will be defined by clause (2) and $p_k(z_i) > F(\dot{p}_j)(z_i)$. A symmetric argument shows that $p_j > F(\dot{p}_k)$(i.o.), and the theorem is proved.

Corollary. Let $\underset{\sim}{F}$ be any total effective operator, and let R be any countable partial order on N. Then there exists an r.e. sequence of recursive measure functions $\Phi_{r(0)}, \Phi_{r(1)}, \ldots$ such that if jRk, then $\underset{\sim}{F}(\Phi_{r(j)}) < \Phi_{r(k)}$ (a.e.) and $\mathcal{F}(\underset{\sim}{F}(\Phi_{r(j)})) \subsetneq \mathcal{F}(\Phi_{r(k)})$, and if j and k are incomparable, then $F(\Phi_{r(j)}) < \Phi_{r(k)}$(i.o.), and $\underset{\sim}{F}(\Phi_{r(k)}) < \Phi_{r(j)}$(i.o.).

Proof. Mostowski [3] has shown that there is a countable partial order R^* into which any countable partial order may be embedded. Moreover, Sacks [4] has shown that R^* is recursive.

We assume without loss of generality that $\underset{\sim}{F}$ is at least as large as the identity operator, and that the compression function for Φ, g, is strictly increasing in its second argument. Blum [1] has shown that there is an $h \in \aleph_2$ such that for all i $\varphi_i(x) \leq h(x, \Phi_i(x))$(a.e.). We assume that h is strictly increasing in its second argument. To prove the corollary, apply the theorem to R^*, rewrite clause (2) as

$$\underset{\substack{j \leq x \\ jRk}}{\max} [p_j(x) + h(x, g(x, \underset{\sim}{F}(\Phi_{p_j})(x)))] + [p_{i_n}(x) + h(x, g(x, \underset{\sim}{F}(\Phi_{p_{i_n}})(x)))],$$

and we rewrite clause (3) as

$$\underset{\substack{j \leq x \\ jRk}}{\max} [p_j(x) + h(x, g(x, \underset{\sim}{F}(\Phi_{p_j})(x)))].$$

It is easy to see that the theorem goes through as before, and the monotonicity restrictions on g and h guarantee that the functions $\{\Phi_{p_k}\}_{k \in N}$ satisfy the corollary.

4. RELATION TO OTHER WORK, AND OPEN PROBLEMS

McCreight [5] is the first investigator to prove an embedding
theorem for subrecursive classes. He shows that any countable partial
order can be embedded in the complexity classes ordered under set
inclusion. However, his theorem is weaker than our results in that
the functions of his partial order are "separated" by composition
with a fixed recursive function, whereas our functions are separated
by a total effective operator. In [6] Enderton also proves a universal
embedding theorem for subrecursive classes. His notion of a sub-
recursive class is quite weak, however, and his result is an immediate
corollary of McCreight's theorem.

Early work on the structure of subrecursive classes was done by
Feferman [12], Meyer and Ritchie [7], and Basu [8]. Feferman shows
that dense chains exist for various notions of subrecursive classes.
Meyer and Ritchie define what they call elementary honest classes, and
they show the existence of dense chains and infinite anti-chains for
such classes. Moreover, they are able to exhibit certain functions f
such that dense chains of classes will exist between f and the iterate
of f, $\lambda x [f^{(x)}(x)]$. Basu builds dense chains of subrecursive classes, where
these classes are closed under the application of a fixed recursive operator.

Machtey [11] has announced universal embedding theorems for both
the "honest" primitive recursive degrees and the "dishonest" primitive
recursive degrees. Both of these theorems follow immediately from our
results.

We also note that Alton [9] has independently announced our embedding theorem.

We leave open the question of the size of the functions in our embedding theorem. That is, given F, what is a reasonable upper bound on the size of p_0 in terms of F (recall that p_0 bounds all the functions $\{p_k\}_{k\in N}$ on all arguments).

The author wishes to acknowledge the generous assistance of Professor Albert R. Meyer in the conception and preparation of this paper.

REFERENCES

1. M. Blum, A machine-independent theory of the complexity of recursive functions, JACM 14, 1967, 322-336.

2. H. Rogers, Jr., Theory of recursive functions and effective computability, McGraw-Hill, 1967.

3. A. Mostowski, Über gewisse universelle relationen, Ann. Soc. Polon. Math. 17, 1938, 117-118.

4. G. Sacks, Degrees of unsolvability, Annuals of Math. Studies, No. 55, Princeton, N.J. 1963.

5. E. McCreight, Classes of computable functions defined by bounds on computation, Doctoral Dissertation, Carnegie-Mellon University, Department of Computer Science, 1969.

6. H. Enderton, Degrees of computational complexity, JCSS No. 6, 1972, 389-396.

7. A. Meyer and D. Ritchie, Classification of functions by computational complexity, Proc. of the Hawaii Internat'l Conf. on Sys. Sciences, 1968, 17-19.

8. S.K. Basu, On classes of computable functions, ACM Symp. on Theory of Computing, 1969, 55-61.

9. D. Alton, Operator embeddability in computational complexity, Notices of the AMS, 1972, A-763.

10. A. Meyer and P. Fischer, Computational speed-up by effective operators, JSL, No. 37, 1972, 55-68.

11. M. Machtey, Augmented loop languages and classes of computable functions, JCSS, to appear.

12. S. Feferman, Classifications of recursive functions by means of hierarchies, Trans. of the AMS, No. 104, 1962, 101-122.

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
MASSACHUSETTS INSTITUTE OF TECHNOLOGY	UNCLASSIFIED
PROJECT MAC	2b. GROUP NONE

3. REPORT TITLE

AN OPERATOR EMBEDDING THEOREM FOR COMPLEXITY CLASSES OF RECURSIVE FUNCTIONS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
INTERIM SCIENTIFIC REPORT

5. AUTHOR(S) *(First name, middle initial, last name)*

ROBERT MOLL

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
MAY 17, 1973	11	12

8a. CONTRACT OR GRANT NO. N00014-70-A-0362-0006	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	MAC TM-32
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	NONE

10. DISTRIBUTION STATEMENT

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	OFFICE OF NAVAL RESEARCH

13. ABSTRACT

Let $\mathcal{F}(t)$ be the set of functions computable by some machine within time bound $t(x)$ for all but finitely many arguments. If $\underset{\sim}{F}$ is any total effective operator and R is any recursive countable partial order, then there is an r.e. sequence of recursive machine running times $T_0, T_1, \ldots, T_k, \ldots$ such that if iRj then $\mathcal{F}(\underset{\sim}{F}(T_i)) \subsetneq \mathcal{F}(T_j)$, and if i and j are incomparable, then $\underset{\sim}{F}(T_i) < T_j$ (i.o.), and $F(T_j) < T_i$ (i.o.).

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
recursive partial order complexity class total effective operator						