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ECONOMY OF DESCRIPTIONS AND MINIMAL INDICES

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ABSTRACT

In Part One sets of minimal indices M_s and M are defined. It is shown that M_s and M are immune and that $M \equiv_T \emptyset''$, M_s join $K \equiv_T \emptyset''$. Subsets of M called M_N and M_F are defined and it is proved that $M_F \equiv_T \emptyset'$ and that $M_N \equiv_T K \cap M \equiv_T \bar{K} \cap M \equiv_T \emptyset''$. M_s is relativized with respect to a set A of integers, and for any two sets A and B of integers such that $A'' \leq_T B'$ and any total function $g \leq_T B''$ and a size function $s \leq_T A$ the following set C is shown to be nonempty

$$C = \{y \mid \exists x [W_x^A = W_y^B \text{ and } x \in M_s^A \text{ and } s(x) > g(y)]\}$$

C however is empty for some total functions $g \leq_T B'''$. Various special cases are considered, e.g. W_y^B in the definition of C is restricted to be finite or a singleton.

INTRODUCTION

It has been known for some time that if the power of a programming language is restricted it very often tends to lose succinctness in its description of programs. A primitive recursive definition scheme for instance is frequently not as concise in describing primitive recursive functions as a double recursive definition scheme. A careful study of the problem has been made by Meyer [7], where he shows that as one increases the power of programming languages, one can obtain economies in program size by any recursive amount for even very simple functions. This parallels a situation in the arithmetic hierarchy where it is possible to get a recursively enumerable set whose smallest recursively enumerable index is much larger than the smallest index for the same set considered, say, as a set recursively enumerable in $\emptyset'$. Parikh has obtained results of this nature using the recursion theorem (see [10], p. 216).

The principal objective of this part of the thesis is the generalization of the results of Meyer to sets of integers A and B related in some recursion theoretic manner, e.g. $A'' \leq_T B'$. As a major tool we have used the concepts of minimal index and size function defined by Manuel Blum (see [1]). We use M_s to denote the set of minimal indices given a size function s, and we write M for M_s when s is the identity function. The first chapter of Part One attempts to place M_s and M in the arithmetic hierarchy for recursive size functions s. In the second chapter we determine the degrees of unsolvability of certain "naturally" defined subsets of M. The main interest here is in getting intuitively interesting immune sets at various arithmetic degree levels. For instance we prove the existence of an immune subset of M of Turing degree equivalent to $\emptyset'$. We come to our main problem in the third chapter. Given two sets of integers A and B such that, for example, $A'' \leq_T B'$, we try to settle problems of the following form: is it the case that for any total function $g \leq_T B''$ there exists a B-r.e. set W_x^B whose smallest A-r.e. index y exceeds g(x)? The answer to the question is in the affirmative. The answer is in the negative however for some total functions $g \leq_T B'''$. At the end of the chapter we enumerate a number of very interesting questions we were unable to resolve.

Topics related to the subject matter of Part One have been treated by various research workers in the past. The work that has been done can be broadly divided into four categories, as follows:

1) First come the papers that deal directly with economy of descriptions and program size. In this group falls the paper by Blum [1] already cited, in which he shows that in order for programs to be economical in size, the programming language must be powerful enough to compute arbitrary general recursive functions, rather than some restricted subset such as the primitive recursive functions. Meyer [7] gives specific instances of this phenomenon when he compares the sizes of Loop and Double Loop programs computing primitive recursive functions. He also shows how the ability to write programs which refer to the universal function of an enumeration enables one to decrease significantly the size of programs. His paper is the first to consider the problem of placing M in the arithmetic hierarchy.

A new approach to the theory of automata and formal grammars which attempts to classify systems by their size rather than their power of description or recognition has been proposed by Meyer and Fischer [8]. Two different types of automata may be equal in their powers of recognition while being of quite different sizes (in the context of a meaningful and sensible definition of machine size). Consider, for instance, nondeterministic and deterministic finite state machines. Meyer and Fischer point out that when it comes to practice, more powerful machines (like Turing machines) tend to be favored over less powerful ones(like pushdown automata), because of the advantage gained in size and simplicity of construction. There is a need for a systematic and detailed study on the subject of economy of descriptions with reference to the hierarchy of automata and formal grammars.

2) In a category by itself is the work of David Pager [9], who has applied recursion theoretic methods to problems of interest to computer scientists. His main concern is in showing that it is not effectively possible to determine minimal length programs for two element decision tables of the form $\{ \langle x,0 \rangle, \langle y,1 \rangle \}$, where x and y are integers, irrespective of whether the given size function is recursive or not. (His definition of minimal index is slightly different from our own. See section 1.2 of Part One.) His proof uses the somewhat obscure fact that maximal sets are strongly inseparable (see [10], p. 125 and p. 250). Inspite of many attempts we have not been able to come up with a simpler proof of his theorem.

3) A third category of papers have dealt with size of functions admitting speed-up in the sense of Blum [2], mainly to answer a question raised by him. He had asked whether, given a sufficiently large recursive function r and a recursive function f with r speed-up, there necessarily existed a recursive function bounding the size of program needed to effect the speed-up? While not actually answering the question in the negative, Helm and Young in [5] came very close. An even better result has been achieved by Meyer and Fischer [12], who have shown the following:

a) Let F be a total effective operator. Then there exists a zero-one valued recursive function f with F speed-up and a recursive function b such that for all i

$$\varphi_i = f \Rightarrow \exists j < b(i)\ [\varphi_j = f \text{ and } F(\Phi_j) < \Phi_i \text{ a.e.}]$$

b) Let a recursive function r be given. Then it is possible to define a total effective operator F in terms of r such that there exist recursive functions f with F speed-up for which the size of programs which are merely faster by r cannot be estimated effectively.

A similar result has been obtained by Constable and Hartmanis [3]. This issue however is only marginally related to the contents of this thesis.

4) Size of programs plays an important part in many of the definitions that have been proposed for random sequences. Kolmogorov, for instance, introduced two measures of complexity for finite sequences in

which he used the lengths of minimal descriptive programs (see Loveland [6]) as the basic measure. An alternative but similar definition has been proposed by Loveland [6]. The aim has been the following:

Let x be an infinite binary sequence and let x^n be the initial subsequence of x of length n. Let $K(x^n)$ represent the complexity of the sequence x^n. Then there would be reason to believe that the infinite sequence x is random in a statistical sense iff $K(x^n)$ is sufficiently complex often enough (e.g. if there exists a constant c such that $K(x^n) > n-c$ for infinitely many n).

In a recent review paper [11], Schnorr has critically analyzed many of the proposed definitions for random sequences. Various open problems remain, and the concept of minimal length of program appears certain to play an important part in discussions on random sequences in the future.

This completes the introductory survey. Through the entirety of Part One, a fixed Godel numbering of the r.e. sets is assumed. The notation, whenever not defined in the body of the thesis, is as in [10]. A list of open problems can be found at the end of each chapter.

Chapter 1

SIZE FUNCTIONS AND MINIMAL INDICES

1.1 Definition: A total function $s: N \to N$ is called a size function if it is finite to one, i.e. $s^{-1}(n)$ is finite or empty for all $x \in N$.

Definition: Given any size function s, we define M_s as follows:

$$M_s = \{z \mid \forall y \, [s(y) < s(z) \Rightarrow \varphi_y \neq \varphi_z]\}$$

If $z \in M_s$, we say that z is an s-minimal index.

When s is the identity function, we write M_s as M, i.e.

$$M = \{z \mid \forall y < z \, [\varphi_y \neq \varphi_z]\}$$

If $z \in M$, we say that z is a minimal index.

Theorem 1: Let s be a recursive size function. Then

$$M_s \text{ join } K \equiv_T \emptyset''$$

Proof: The theorem will follow from the next four lemmas:

Lemma 1A: Let $E = \{z \mid \varphi_z(x) = 0 \text{ for all } x\}$, then

$$\emptyset'' \leq_T E$$

Proof: Using the s-m-n theorem we get a recursive function f such that

$$\varphi_{f(x)}(y) = \begin{cases} 0 \text{ if } \varphi_x(y) \text{ converges} \\ \text{divergent otherwise} \end{cases}$$

Clearly,

$$\varphi_x \text{ total} \Leftrightarrow f(x) \in E$$

Since the set $\{x \mid \varphi_x \text{ total}\}$ is Π_2 complete, it follows that $\emptyset'' \leq_T E$.

Lemma 1B: Let $F = \{ < x,y > \mid \varphi_x \neq \varphi_y\}$

Then $F \leq_T \emptyset''$

Proof: By a Tarski-Kuratowski computation, F is in Σ_2.

Lemma 1C: $M_s \leq \emptyset''$

Proof: To determine whether a given integer z is in M_s, we proceed as follows:

a) Let

$$y_{max} = uy[\forall u > y \; [s(u) \geq s(z)]]$$

By definition of s, y_{max} must exist and its value can be determined with the help of a K-oracle since it has a Π_1 definition.

b) Let $I_z = \{y \mid y \leq y_{max} \text{ and } s(y) < s(z)\}$. Since y_{max} is now known, we can get a canonical index for I_z, which must be a finite set.

c) Since

$$z \in M_s \Leftrightarrow \forall y \in I_z \; [\varphi_y \neq \varphi_z]$$

and since we know the elements of the finite set I_z, we can now settle whether z is in M_s by checking to see if $\forall y \in I_z \; [< y,z > \in F]$. By Lemma 1B, this can be done with a $\emptyset''$ oracle.

Lemma 1D: $E \leq_T M_s$ join K

Proof of Lemma 1D: To determine whether a given integer z is in E, we proceed as follows:

a) If z is in M_s the solution is easy, and we ignore that case. Let

$$Z_{min} = \{z' \mid z' \in M_s \text{ and } \varphi_{z'}(x) = 0 \text{ for all } x\}$$

Then Z_{min} is a finite set, and we can write

$$Z_{min} = \{z_0, z_1, \ldots, z_r\}$$

b) Define y_{max} as in the proof of Lemma 1C and determine I_z, using the K-oracle whenever necessary. Then using the M_s-oracle, get $I'_z = I_z \cap M_s$.

c) Enumerate the set $\{y \mid \varphi_y \neq \varphi_z\}$ with the help of the K-oracle. This is possible since the set is in Σ_2, and hence is r.e. in K. If any element of I'_z appears strike out that element from I'_z. Let I_z^t represent the elements not struck out in I'_z after t steps of the enumeration.

d) There must come a time t when either $I_z^t = Z_{min}$ or $I_z^t \cap Z_{min} = \emptyset$. If the former then $z \in E$. If the latter then $z \notin E$.

This proves the theorem.

Q.E.D.

Theorem 2: Let s be a recursive size function such that there exists a recursive function f,

$$s^{-1}(x) = D_{f(x)} \text{ for all } x.$$

Then $K \leq_T M_s$, and hence $M_s \equiv_T \emptyset''$.

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Proof: To determine whether a given integer z is in K, we proceed as follows:

a) Using the s-m-n theorem, we get a recursive function g such that

$$\varphi_{g(z)}(y) = \begin{cases} n \text{ if } \varphi_z(z) \text{ converges in } n \text{ steps} \\ \\ \text{divergent otherwise} \end{cases}$$

b) Let

$$G_{min} = \{y \mid y \in M_s \text{ and } \varphi_y(x) \text{ is divergent for all } x\}$$

Define a total function h recursive in M_s as follows:

$$h(x) = \max_{\substack{s(y) \le s(x) \\ y \in M_s \\ y \notin G_{min}}} \{\text{the first element in the enumeration of range } \varphi_y\}$$

By convention, we let $h(x) = 0$ if the set on the RHS over which the maximum is being taken is empty. This makes h total always. We also get $h \le_T M_s$ because G_{min} is finite and because by the given condition on s, it is possible to get effectively a canonical index for the set $\{y \mid s(y) \le s(x)\}$.

c) Since by definition $\varphi_{g(z)}$ is either everywhere divergent or a constant function, we have

$$\begin{aligned} z \in K &\Leftrightarrow \varphi_z(z) \text{ converges in } \varphi_{g(z)}(0) \text{ steps} \\ &\Leftrightarrow \varphi_z(z) \text{ converges in } \le hg(z) \text{ steps} \end{aligned}$$

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and the RHS is recursive in M_s.

Q.E.D.

Corollary 2: $M \equiv_T \varphi''$

Proof: Immediate from Theorem 2. A proof can also be found in [7].

Theorem 3: (Blum): Let s be a recursive size function. Then M_s is immune.

Proof: If x and y are integers, let $(x)_y$ be the exponent of the yth prime number in the prime decomposition of x. Also, let W be any given infinite r.e. set. We will show that W intersects the complement of M_s. For any integer x, let W^x be W enumerated to x steps.

Using the s-m-n theorem, we get a recursive function f such that

$$\varphi_{f(z)} = \varphi_{(u)_0} \text{ where } u = \mu x[(x)_0 \in W^{(x)_1} \text{ and } s((x)_0) > s(z)]$$

By the recursion theorem, there exists an integer n such that $\varphi_{f(n)} = \varphi_n$. Thus

$$\varphi_n = \varphi_{(u)_0} \text{ where } u = \mu x[(x)_0 \in W^{(x)_1} \text{ and } s((x)_0) > s(n)],$$

which proves that W contains an integer not in M_s.

Q.E.D.

This completes a discussion of the more interesting recursion theoretic properties of M_s when the size function s is recursive. Some of the problems not yet resolved will be tabulated in section 1.3.

1.2: In section 1.1 we restricted ourselves to considering size functions that were recursive. If we allow ourselves nonrecursive size functions it is possible to get M_s much lower down in the arithmetic hierarchy, as the following theorem shows.

Theorem 4: There exists a one-one nonrecursive size function s such that M_s is recursive and $s \leq_T \emptyset''$.

Proof: By working with the graphs of partial recursive functions, it is possible by a technique due to Friedberg (see [4]) to get an infinite r.e. set W such that W contains one and exactly one index for each partial recursive function. Let f be a one-one recursive function that enumerates W and let p be a recursive function such that for all x and all y,

$$\varphi_x = \varphi_{p(x,y)} \text{ and } p(x,y) < p(x,y+1)$$

Define a one-one increasing recursive function g as follows:

$$g(0) = f(0)$$
$$g(n+1) = p(f(n+1),\ x) \text{ where } x = \mu y[p(f(n+1),\ y) > g(n)]$$

We will define the size function s in such a manner that range g will be M_s. Let

$$s(x) = \begin{cases} x & \text{if } x \in \text{range } g \\ y & \text{if } x \notin \text{range } g, \text{ where } y \text{ is the least number not} \\ & \text{in range } g \text{ such that } y > s(z) \text{ for all } z < x, \\ & \text{and } y > w \text{ where } w \in \text{range } g \text{ and } \varphi_w = \varphi_x. \end{cases}$$

Clearly, s is one-one and M_s = range g. Also, from the definition, $s \leq_T \emptyset''$.

Q.E.D.

Thus it is possible to get M_s to be recursive if we are willing to pay the price and make the size function s sufficiently nonconstructive. In view of Theorem 4, a question that arises is: Given an integer z, how hard is it to get an integer $w \in M_s$ such that $\varphi_z = \varphi_w$? We show below that it is never easy to do so.

<u>Theorem 5</u>: Let a size function s be given. Then there does <u>not</u> exist a total function g with the following properties:

a) $g \leq_T K$

b) $\forall z[g(z) \in M_s \text{ and } \varphi_z = \varphi_{g(z)}]$

<u>Proof</u>: Define Z_{min} as in the proof of Lemma 1D. Then Z_{min} is a finite set. Suppose there does exist a function g with the properties given above. Then given any integer z we can determine whether $z \in E$ with a K-oracle as follows:

Compute g(z) and see if $g(z) \in Z_{min}$. If so, then $z \in E$. If not, then $z \notin E$. This is a contradiction, since by Lemma 1A, $\emptyset'' \leq_T E$.

Q.E.D.

Instead of defining M_s as we did in the last section, we could have done it in the following two different ways:

<u>Definition</u>: Let s be a size function. Define

$$M_s^I = \{z \mid \forall y[s(y) < s(z) \Rightarrow \varphi_z \not\equiv \varphi_y]\}$$

$$M_s^W = \{z \mid \forall y[s(y) < s(z) \Rightarrow W_y \neq W_y]\}$$

It is easy to see that the conclusions of section 1.1 (Theorems 1, 2, 3) hold with the redefined M_s's. Very minor modifications are required in most of the proofs.

We digress briefly here to discuss a problem posed and solved by David Pager [9]. Let s be a size function (possibly nonrecursive) and let c be an integer. Is it possible for any arbitrary choice of c to get a recursive function f such that for all x

$$\varphi_{f(x)}(c) = 0 \text{ and } \varphi_{f(x)}(x) = 1 \text{ and } f(x) \in M_s^I?$$

The question has been answered negatively by Pager in [9]. His result also implies the following: Given a size function s there does <u>not</u> exist a recursive function f such that

$$\forall x \forall y [\varphi_{f(x,y)}(x) = 0 \text{ and } \varphi_{f(x,y)}(y) = 1 \text{ and } f(x,y) \in M_s^I]$$

<u>1.3</u>: In this section we will list some of the problems still unresolved

A) Is the following true or false? For any recursive size function s, $M_s \equiv_T \emptyset''$.

By Theorem 1, M_s join $K \equiv_T \emptyset''$. Hence it follows that $M_s \not\leq_T K$. Thus it must be the case that either $K \leq_T M_s$ (and hence $M_s \equiv_T \emptyset''$) for all recursive size functions s, or that there is a recursive size function s for which K and M_s are Turing incomparable. It should be noted that the proof of Theorem 2 does not work for an arbitrary recursive size function s because it is not possible in general to obtain a canonical index for the finite set $\{y \mid s(y) \leq s(x)\}$ uniformly in x.

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B) Does there exist a recursive size function s such that M_s is hyperimmune? It is easy to see that M is not hyperimmune (cf. [7]), and for any size function s, the set

$$sM_s = \{s(z) \mid z \in M_s\}$$

is not hyperimmune, as we show below.

Using the s-m-n theorem, get a recursive function f such that

$$\varphi_{f(x)}(y) = x \quad \text{for all } y$$

Now define a recursive function g as follows:

$$g(0) = sf(0)$$

$$g(n+1) = sf(u) \text{ where } u = \mu v[sf(v) > g(n)]$$

Then g majorizes sM_s, since M_s contains an index for each constant function.

By Theorem XV of [10], there exists a recursive function θ such that

$$\forall u[D_{\theta(n)} \cap sM_s \neq \emptyset] \text{ and } \forall u \forall v[u \neq v \Rightarrow D_{\theta(u)} \cap D_{\theta(v)} = \emptyset].$$

Let h be a recursive function such that for all u,

$$W_{h(u)} = s^{-1}(D_{\theta(u)}).$$

Then

$$\forall u[W_{h(u)} \cap M_s \neq \emptyset] \text{ and } \forall u \forall v[u \neq v \Rightarrow W_{h(u)} \cap W_{h(v)} = \emptyset] \text{ and}$$

$$\forall u[W_{h(u)} \text{ is finite}]$$

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Hence by [10], p. 144, M_s is not hyper hyperimmune. However, the question whether M_s can be hyperimmune remains open, and we conjecture that it can be.

C) Since M_s is immune, it cannot be the case that $\emptyset'' \equiv_m M_s$, since that would make M_s productive. It has been shown by Paul Young that there exist recursive size functions s_1 and s_2 such that $M_{s_1} \not\equiv M_{s_2}$ (see [7]). The problems still open are:

i) Do there exist recursive size functions s_1 and s_2 such that $M_{s_1} \not\equiv_m M_{s_2}$?

ii) Do there exist recursive size functions s_1 and s_2 such that $M_{s_1} \not\equiv_{tt} M_{s_2}$? We conjecture a positive answer to both i) and ii).

iii) Is $M_s \equiv_{tt} \emptyset''$ for all recursive size functions s?

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Chapter 2

SUBSETS OF M

2.1 This chapter will be concerned with determining the degrees of unsolvability of certain subsets of M. The following subsets will be considered.

a) $M_N = \{z \mid z \in M \text{ and } \varphi_z \text{ is not total}\}$

b) $M_F = \{z \mid z \in M \text{ and } W_z \text{ is a singleton}\}$

c) $K \cap M$

d) $\bar{K} \cap M$

All these subsets of M obviously are or will be shown to be infinite and hence immune.

2.2 Theorem 7: $M_N \equiv_T \emptyset''$

Proof: By a Tarski-Kuratowski computation, $M_N \leq_T \emptyset''$. We will prove $\emptyset'' \leq_T M_N$ in two stages.

a) $\emptyset'' \leq_T M_N$ join K

The proof parallels Theorem 1. Define E as in Lemma 1A and let

$$E' = \{z \mid \varphi_z(x) = 0 \text{ for all } x \neq 0, \text{ and } \varphi_z(0) \text{ divergent}\}$$

It is easy to check that $E' \equiv_T E$. Also let

$$Z'_{min} = \{z \mid z \in M_N \cap E'\}$$

Given any integer z, we have to determine whether $z \in E'$. The procedure is exactly as described in the proof of Lemma 1D except that we work with Z'_{min}

and replace M_s by M_N and s by the identity function.

b) $K \leq_T M_N$

The proof parallels Theorem 2, except that we redefine $\varphi_{g(z)}$ as follows

$$\varphi_{g(z)}(y) = \begin{cases} n & \text{if } y = 0 \text{ and } \varphi_z(z) \text{ converges in n steps} \\ \text{divergent} & \text{otherwise} \end{cases}$$

and replace M_s by M_N and s by the identity function.

Q.E.D.

2.3 Theorem 8: $M_F \equiv_T K$

Proof: The proof is in two steps

a) $M_F \leq_T K$

We will write out a definition for M_F:

$$z \in M_F \Leftrightarrow W_z \text{ is a singleton and } \forall y < z$$
$$[[W_y \text{ is not a singleton}] \text{ or } [W_y \text{ is a singleton and } \varphi_y \neq \varphi_z]]$$

and we note that

$$W_z \text{ is a singleton} \Leftrightarrow \exists u[u \in W_z] \text{ and } \forall u \forall v[u \neq v \Rightarrow u \notin W_z \text{ or } v \notin W_z]$$

and the RHS is recursive in K. It follows that $M_F \leq_T K$, since if we know W_y and W_z to be singletons we can effectively check whether $\varphi_y \neq \varphi_z$.

b) $K \leq_T M_F$

The proof is essentially the same as for part (b) of Theorem 7.

Q.E.D.

We have thus exhibited the existence of a "naturally" defined and intuitively interesting immune set that is Turing equivalent to K.

<u>2.4</u> Theorem 9: $K \cap M \equiv_T \emptyset''$

<u>Proof</u>: The proof is in stages:

a) $K \cap M$ is infinite

This is immediate since if φ_x is total and $x \in M$ then $x \in K \cap M$.

b) $K \leq_T K \cap M$.

The proof given for Theorem 2 works with very minor modifications.

c) $\emptyset'' \leq_T K \cap M$.

let us define

$A = \{<x,y> \mid \varphi_x = \varphi_y$ and $\forall z \leq \max(x,y)$ $[\varphi_x(x)$ convergent$]$

We will show that

$$\emptyset'' \equiv_T A$$

By a Tarski-Kuratowski computation, $A \leq_T \emptyset''$. Using the s-m-n theorem, we define a recursive function f such that

$$\varphi_{f(x)}(y) = \begin{cases} 0 & \text{if } \varphi_x(y) \text{ convergent} \\ \text{divergent} & \text{otherwise} \end{cases}$$

If e is some fixed integer in E, then

$$\varphi_x \text{ total} \Leftrightarrow <f(x), e> \in A$$

hence $\emptyset'' \leq_T A$.

Now we will prove that $A \leq_T K \cap M$. Let $\langle x,y \rangle$ be given. To determine if $\langle x,y \rangle \in A$ we proceed as follows:

Since $K \leq_T K \cap M$, we first check whether $\forall z \leq x\ [\varphi_x(z)$ convergent] and $\forall z \leq y\ [\varphi_y(z)$ convergent]. If not, then $\langle x,y \rangle \notin A$. If so, let $x_0 \in M$ be such that $\varphi_{x_0} = \varphi_x$. Since $x_0 \leq x$, $\varphi_{x_0}(x_0)$ converges, hence $x_0 \in K \cap M$. Similarly there exists $y_0 \in K \cap M$ such that $\varphi_{y_0} = \varphi_y$.

To determine x_0 recursively in $K \cap M$ consider the finite set

$$G_x = \{z \mid z \leq x \text{ and } z \in K \cap M\}$$

Since $K \leq_T K \cap M$, enumerate the set $\{z \mid \varphi_z \neq \varphi_x\}$ with a $K \cap M$ oracle, and strike out elements of G_x as they appear until exactly one element remains in G_x. This element must be x_0.

Similarly, y_0 can be determined recursively in $K \cap M$. Then $\langle x_0,y_0 \rangle \in A$ if $x_0 = y_0$, otherwise not. Since $\emptyset'' \equiv_T M$ this proves $K \cap M \equiv_T M \equiv \emptyset''$.

Q.E.D.

<u>2.5</u> <u>Theorem 10</u>: $\bar{K} \cap M \equiv_T \emptyset''$.

<u>Proof</u>: The proof is in stages.

a) $\bar{K} \cap M$ is infinite.

Using the s-m-n theorem, we get a recursive function f such that

$$\varphi_{f(x)}(y) = \begin{cases} x & \text{if } y = 0 \\ \text{divergent} & \text{otherwise} \end{cases}$$

Let $k_x \in M$ be such that $\varphi_{k_x} = \varphi_{f(x)}$ for each x. Then k_x's are all distinct, hence at most one of them can be 0. Thus at most one of the k_x's is in K. It follows that $\bar{K} \cap M$ is infinite.

b) $K \leq_T \bar{K} \cap M$

Using the s-m-n theorem, we get a recursive g such that

$$\varphi_{g(x)}(y) = \begin{cases} m & \text{if } y = 0 \text{ and } \varphi_x(x) \text{ converges in } m \text{ steps} \\ \text{divergent} & \text{otherwise} \end{cases}$$

Note that if $n_x \in M$ is an index for the function $\varphi_{g(x)}$ (for each x), then $n_x \in (\bar{K} \cap M) \cup \{0\}$.

Let $b \in M$ be an index for the empty function. We define a new set A recursively in $\bar{K} \cap M$ as follows:

$$A = (\bar{K} \cap M) - \{0\} \quad \text{if } b = 0$$

$$A = ((\bar{K} \cap M) - \{b\}) \cup \{0\} \text{ if } b \neq 0$$

Then for each x, $n_x \in A$, and A does not contain an index for the empty function.

We will now define a total function h recursively in A:

$$h(x) = \begin{cases} 0 & \text{if } \forall z \leq x,\ z \notin A \\ \max\{\varphi_z(v) \mid z \leq x \text{ and } z \in A \text{ and } v \text{ is the} \\ \quad \text{first element in an enumeration of } W_z\} & \text{otherwise} \end{cases}$$

clearly,

$x \in K \Leftrightarrow \varphi_x(x)$ converges in $\leq hg(x)$ steps, and the RHS is recursive in A and hence in $\bar{K} \cap M$.

c) $\emptyset'' \leq_T \bar{K} \cap M$

Let us define

$$B = \{\langle x,y\rangle \mid \varphi_x = \varphi_y \text{ and } \forall z \leq \max(x,y)\ [\varphi_x(z) \text{ divergent}]\}$$

We will show that $B \equiv_T \emptyset''$.

By a Tarski-Kuratowski computation, $B \leq_T \emptyset''$. In the other direction, we use the s-m-n theorem to get two recursive functions f' and g':

$$\varphi_{f'(x,z,w)}(y) = \begin{cases} \text{divergent if } y \leq \max(z,w) \\ 0 \quad \text{if } y > \max(z,w) \text{ and} \\ \quad \varphi_x(y-1-\max(f(x),\ g(x))) \text{ converges} \\ \text{divergent otherwise} \end{cases}$$

$$\varphi_{g'(x,z,w)}(y) = \begin{cases} 0 \quad \text{if } \ y > \max(z,w) \\ \text{divergent} \quad \text{otherwise} \end{cases}$$

We can now use the double recursion theorem of Smullyan (see [10], p. 190) to get recursive functions f and g such that

$$\varphi_{f(x)}(y) = \begin{cases} \text{divergent if } y \leq \max(f(x),\ g(x)) \\ 0 \quad \text{if } y > \max(f(x),\ g(x)) \text{ and} \\ \quad \varphi_x(y-1-\max(f(x),\ g(x)) \text{ converges} \\ \text{divergent} \quad \text{otherwise} \end{cases}$$

$$\varphi_{g(x)}(y) = \begin{cases} 0 \quad \text{if } y > \max\ (f(x),\ g(x)) \\ \text{divergent otherwise} \end{cases}$$

Then we have

$$\varphi_x \quad \text{total} \Leftrightarrow \langle f(x),\ g(x)\rangle \in B$$

Hence $\emptyset'' \leq_T B$. To get $B \leq_T \bar{K} \cap M$ we follow the procedure outlined in the proof of Theorem 9, part (c), using B in place of A and $\bar{K}$ in place of K.

Q.E.D.

2

<u>2.6</u> In this section we will describe some problems yet unresolved.

a) Let $M_T = \{z \mid z \in M \text{ and } \varphi_z \text{ is total}\}$

Clearly, $M_T \leq_T \emptyset''$. By techniques described in this chapter, it is easy to get $K \leq_T M_T$. The question we pose is: Is $M_T \equiv_T \emptyset''$?

b) We have proved in section 2.3 the existence of a subset of M Turing equivalent to K. Does there exist an intuitively interesting and easily definable subset of M that is co-simple?

c) While we have shown that

$$K \cap M \equiv_T \bar{K} \cap M \equiv_T M$$

are these sets truth-table or many-one equivalent to one another? They cannot all be one-one equivalent, since, for example, $K \cap M$ is a proper subset of the immune set M. It is known that if B is an immune set and A is a proper subset of B then $A \not\equiv B$ (see [7], lemma 5).

Chapter 3

SMALLER INDICES AT HIGHER HIERARCHY LEVELS

3.1 In [7] Meyer proves the existence of a 0-1 valued primitive recursive function whose smallest primitive recursive index is in some sense "arbitrarily" larger than its smallest double recursive index. This chapter generalizes and extends some of Meyer's results to the arithmetic hierarchy. We describe the major problem below:

Let A and B be sets of integers. If $B \leq_T A$ then any set r.e. in B is alse r.e. in A, and there exists a recursive function f such that given any B-r.e. set W_x^B, the numerically smallest A-r.e. index for W_x^B is $\leq f(x)$. Suppose, however, that $B \not\leq_T A$. In this case not every B-r.e. set is A-r.e. any more. Indeed, B' (the jump of B) cannot be r.e. in A. However, there still exist infinitely many B-r.e. sets which are A-r.e. (consider the finite sets for instance). But does there necessarily exist a B-r.e. set which is also A-r.e. and whose numerically smallest A-r.e. index is (in a sense to be made precise in the following pages) "arbitrarily" larger than its numerically smallest B-r.e. index? Do we need to impose additional conditions on A and B, such as, for example, some relationship in the Turing hierarchy like $A'' \leq_T B'$, in order to be able to answer the above question? If there are such B-r.e. sets, are there infinitely many of them? In the following section we will endeavor to answer these questions in as great a generality as we can. See [10], p. 216, for some related results of Parikh.

3.2: We have been unable to resolve the questions posed in the last section in the general case when all we know about A and B is that $B \not\leq_T A$. In the more restricted situation which arises when $A'' \leq_T B'$, the questions have been answered. We begin by redefining the set M_s in an appropriate way.

Definition: Let A be a set of integers and let s be a size function. Then

$$M_s^A = \{z \mid \forall y\ [s(y) < s(z) \Rightarrow W_y^A \neq W_z^A]\}$$

Remark: If the size function s is A-recursive, then $M_s^A \leq_T A''$. This is a relativization of Lemma 1C.

The next theorem tells us that given two sets of integers A and B such that $A' \leq_T B'$, whenever a B-r.e. set is also A-r.e. its smallest A-r.e. index can be bounded above by a total function recursive in B'''.

Theorem 11: Let A and B be two sets of integers such that $A' \leq_T B'$, and let s be an A-recursive size function. Then there exists a total function g recursive in B''' such that the set

$$\{x \mid \exists y\ [W_y^A = W_x^B \text{ and } y \in M_s^A \text{ and } s(y) > g(x)]\}$$

is empty.

Proof: If $A' \leq_T B'$, then the set R_B where $R_B = \{x \mid W_x^B \text{ is r.e. in } A\}$ is in Σ_3^B by the Tarski-Kuratowski computation shown below.

$$x \in R_B \Leftrightarrow \exists y [W_x^B = W_y^A]$$

$$\Leftrightarrow \exists y \forall z [z \in W_x^B \Leftrightarrow z \in W_y^A]$$

and the term within square brackets on the RHS is clearly recursive in B' under the assumption $A' \leq_T B'$. Then define a total function g as follows:

$$g(x) = \begin{cases} s(y) & \text{where } W_y^A = W_x^B \text{ and } y \in M_s^A \text{ if } x \in R_B \\ 0 & \text{otherwise} \end{cases}$$

Clearly $g \leq_T B'''$, and g satisfies the required conditions.

Q. E. D.

The next question we can ask is whether a total function g with the properties described above exists among only those functions that are recursive in B''. We show in Theorem 12 that given the somewhat stronger relationship $A'' \leq_T B'$, for any function g such that g is total and $g \leq_T B''$, there exist infinitely many B-r.e. sets whose smallest A-r.e. indices are at least g larger than their smallest B-r.e. indices.

<u>Theorem 12</u>: Let A and B be sets of integers such that $A'' \leq_T B'$, let g be a total function recursive in B'', and let s be an A-recursive size function. Then the set

$$C = \{x \mid \exists y[W_y^A = W_x^B \text{ and } y \in M_s^A \text{ and } s(y) > g(x)]\}$$

is <u>not</u> recursive in B''. A fortiori, C is infinite.

Before we prove Theorem 12, we present a couple of lemmas.

<u>Lemma 12A</u>: Let A and B be sets of integers such that $B \not\leq_T A$, and let

$$R_B = \{x \mid W_x^B \text{ is r.e. in } A\}.$$

Then

$$B'' \leq_m R_B.$$

Proof of Lemma 12A: Since $B \not\leq_T A$, it must be the case that B' is not r.e. in A. Using the s-m-n theorem, we get a recursive function f such that

$$\varphi^B_{f(x)}(y) = \begin{cases} 0 & \text{if } y \in B' \text{ and } (\forall z)_{z \leq y}[\varphi^B_x(z) \text{ convergent}] \\ \text{divergent} & \text{otherwise} \end{cases}$$

Then

$$\varphi^B_x \text{ is not total} \Leftrightarrow f(x) \in R_B$$

Hence it follows that $B'' \leq_m R_B$.

Lemma 12B: Let A and B be sets of integers such that $A'' \leq_T B'$. Then

$$B''' \equiv_T R_B$$

where R_B is as in Lemma 12A.

Proof of Lemma 12B: Note that the set

$$Q_B = \{x \mid W^B_x \text{ is recursive}\}$$

is Turing equivalent to B''' by a direct relativization of Theorem XVI, p. 327 of [10]. We will show that $Q_B \leq_T R_B$ join B''. Since $R_B \leq_T B'''$ by a Tarski-Kuratowski computation, this will be enough in view of Lemma 12A.

To determine whether a given integer x is in Q_B, proceed as follows:

a) See if $x \in R_B$. If not, $x \notin Q_B$.

b) If $x \in R_B$, find $z_0 = \mu z[W^A_z = W^B_x]$

This can be done with a B'' oracle.

c) Now see if $W^A_{z_0}$ is recursive. This can be done with an A''' oracle, hence with a B'' oracle, since $A'' \leq_T B'$. If so then $x \in Q_B$.

<u>Proof of Theorem 12</u>: Suppose there is a total function g recursive in B'' for which $C \leq_T B''$. Then

$$x \in R_B \Leftrightarrow \exists y[W^A_y = W^B_x]$$

$$\Leftrightarrow \exists y[W^A_y = W^B_x \text{ and } y \in M^A_s \text{ and } s(y) \leq g(x)] \vee x \in C$$

The RHS is recursive in B'', since with a B''-oracle, we can get a canonical index for the set

$$\{y \mid y \in M^A_s \text{ and } s(y) \leq g(x)\}.$$

This contradicts Lemma 12B.

Q. E. D.

Using the notation of Theorem 12, we can now draw the following corollary:

<u>Corollary 12</u>: There exist Turing incomparable sets A and B such that $C \not\leq_T B''$, where C is defined as in the statement of Theorem 12.

<u>Proof</u>: In p. 266, Corollary IX(b) of [10], Rogers proves the existence of sets A and B such that $A'' \equiv_T B'$ and A is Turing incomparable to B.

Q. E. D.

It should be remarked that in Theorem 12, whether the range of g be finite or infinite, C contains indices for infinitely many distinct B-r.e. sets.

Theorems 11 and 12 answer most of the questions posed in section 3.1 when $A'' \leq_T B'$. When all we know is $B \not\leq_T A$ the questions become more difficult to answer and have not been resolved. These will be listed in section 3.5.

3.3 In section 3.2 we did not place any restrictions on the nature of the B-r.e. sets whose A-r.e. indices we were seeking. How are Theorems 11 and 12 modified when only those B-r.e. sets are considered which are, say, singletons or finite sets? In this section we treat these special cases in a setting of less generality. We begin with a definition.

Definition: Let A be a set of integers. Then

$$M^A = \{x \mid \forall y < x\, [W_y^A \neq W_x^A]\}$$

Remark: $M^A \in \Sigma_2^A$. If $x \in M^A$, we say that "x is A-minimal".

Remark: $\{x \mid W_x^A \text{ is a singleton}\} \equiv_T A'$.

Proof: $\{x \mid W_x^A \text{ is a singleton}\} \leq_T A'$ is a relativization of a result in section 2.3. We leave the proof in the other direction to the reader.

Theorem 13: Let A and B be sets of integers such that $A' \leq_T B$, and let g be any total function recursive in B. Then the set

$$\{x \mid \exists y[W_x^B = W_y^A \text{ and } W_x^B \text{ is a singleton} \text{ and } y \in M^A \text{ and } y > g(x)]\}$$

is infinite.

Proof: Let

$$S = \{x \mid x \in M^A \text{ and } W_x^A \text{ is a singleton}\}$$

Then S is an infinite set r.e. in A' and hence in B. Using the s-m-n theorem we get a recursive function f such that

$$W_{f(x)}^B = \text{first integer in enumeration of } S \text{(with a B-oracle) larger than } g(x).$$

Now consider the infinitely many fixed points of $W_{f(x)}^B$, i.e. consider the set of those x for which $W_{f(x)}^B = W_x^B$.

Q. E. D.

Theorem 14: Let A and B be sets of integers such that $A' \leq_T B$. Then there exists a total function g recursive in B' such that

$$\forall x \forall y [W_x^B = W_y^A \text{ and } W_x^B \text{ is a singleton and } y \text{ is A-minimal} \Rightarrow y \leq g(x)]$$

Proof: Define a total function g recursive in B' as follows:

$$g(x) = \begin{cases} \mu z\, [W_z^A = W_x^B] & \text{if } W_x^B \text{ is a singleton} \\ 0 & \text{otherwise} \end{cases}$$

Note that if W_x^B is a singleton, we can determine $\mu z[W_z^A = W_x^B]$ effectively with a B' oracle by getting an increasing sequence of numbers

$$z_0, z_1, z_2, z_3 \; \ldots\ldots$$

such that $W_{z_k}^A$ is a singleton for all k and then obtaining the least k for which $W_{z_k}^A = W_x^B$.

Q.E.D.

Between them, Theorems 13 and 14 summarize the situation when only singleton B-r.e. sets are being considered. We now proceed to finite B-r.e. sets.

<u>Theorem 15</u>: Let A and B be sets of integers such that $A' \leq_T B$, and let g be a total function recursive in B'. Then the set

$$\{x \mid \exists y[W_x^B = W_y^A \text{ and } W_x^B \text{ is a finite set and } y \in M_A \text{ and } y > g(x)]\}$$

is infinite.

<u>Proof</u>: The proof is in two steps:

a) We will first show that if g is a total function recursive in B' then there exists a B-recursive total function f of two variables such that for all u,

$$g(u) = \lim_{v \to \infty} f(u,v)$$

Let e be an integer such that $g = \varphi_e^{B'}$. Define f as follows:

$$f(x,y) = \begin{cases} \varphi_e^{B'_y}(x) & \text{if it converges in y step} \\ 0 & \text{otherwise} \end{cases}$$

where B'_y is B' enumerated for y steps with a B oracle. Clearly, f is total and $f \leq_T B$.

b) We use the function f obtained in part (a) and using the s-m-n theorem get a recursive function h such that

$$\varphi^B_{h(x)}(y) = \begin{cases} 0 & \text{if } y \in \overline{A'} \text{ and } \exists z[y \le f(x,z)] \\ \text{divergent otherwise} \end{cases}$$

$W^B_{h(x)}$ is finite, but the minimal A-r.e. index for $W^B_{h(x)}$ must exceed $g(x)$. Now consider the infinitely many fixed points for $W^B_{h(x)}$, i.e. consider the set of those x for which

$$W^B_{h(x)} = W^B_x$$ Q. E. D.

Theorem 16: Let A and B be sets of integers such that $A' \le_T B$. Then there exists a total function g recursive in B'' such that

$$\forall x \forall y [W^B_x = W^A_y \text{ and } W^B_x \text{ is finite and } y \in M^A \Rightarrow y \le g(x)]$$

Proof: We note that $\{x \mid W^B_x \text{ is finite}\}$ is Turing reducible to B''. The rest of the proof is similar to that for Theorem 14.

Q. E. D.

We sum up all our results in the table below. A and B are assumed to be sets of integers such that $A' \le_T B$.

B-r.e. sets being considered are	Can minimal A-r.e. index of set be g better than B-r.e. index for all g recursive in B	B'	B''	B'''
singletons	yes	no	no	no
finite	yes	yes	no	no
unrestricted	yes	yes	yes	no

TABLE

2

There is a striking difference between the proofs for Theorem 12 and 13 (or 15). While the proof of Theorem 12 is in some sense nonconstructive, that of Theorem 13 is "effective" in the following way: if we are given a B-r.e. index for the function g, we can uniformly obtain a B-r.e. index x for a singleton set such that the smallest A-r.e. index for W_x^B exceeds g(x). All we need to do is to obtain a fixed point for the function f (in the proof of Theorem 13) in an effective manner. A similar comment applies to Theorem 15. The question arises whether there is an alternative proof of Theorem 12 which would give us the abovementioned uniformity in the general case. (Such uniformities can be found in the proofs of Meyer [7].)

3.4 We will devote this section to a few illustrations from the arithmetic hierarchy.

Example 1: By $\emptyset^{(n)}$ we will mean the n^{th} jump of the empty set. Let n be an integer ≥ 2, and define a total function h as follows

$$h(x) = \min_{0\leq m\leq n-2} \{e_m \mid e_m \text{ is a minimal } \emptyset^{(m)}\text{-r.e. index for } W_x\}$$

It is clear that h has a $\Pi_2^{\emptyset(n-2)}$ definition and is therefore recursive in $\emptyset^{(n)}$. Also, h has infinite range.

Now let g be any total function recursive in $\emptyset^{(n)}$. We will find integers p and r such that

$$W_r^{\emptyset(n)} = W_p \text{ and } h(p) > g(r).$$

Using the s-m-n theorem, we get a recursive function f as follows:

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$$W_{f(x)}^{\phi^{(n)}} = \begin{cases} W_y, \text{ where } h(y) \text{ is the first integer in the} \\ \text{enumeration of range } h \text{ larger than } g(x). \end{cases}$$

Now let r be a fixed point for $W_{f(x)}^{\phi^{(n)}}$ and let p be the corresponding y. Then clearly,

$$W_r^{\phi^{(n)}} = W_p \quad \text{and } h(x) > g(r).$$

Example 2: Suppose we try to generalize the definition of h in the following manner

$$h(x) = \min_{0 \leq m} \{e_m \mid e_m \text{ is a minimal } \phi^{(m)}\text{-r.e. index for } W_x\}$$

Unfortunately, it turns out that h so defined has finite range. To see this, we take a z such that

$$\varphi_z^{\phi^{(n)}}(0) = n \quad \text{for all } n$$

Such a z exists by [10], Problem 13-5. Now define

$$W_{z'}^{\phi^{(n)}} = W_{\varphi_z^{\phi^{(n)}}(0)} = W_n$$

Then z' is independent of n by the unformity of the definition, hence $h(x) \leq z'$ for all x.

We can get around the problem by redefining h as follows:

$$h(x) = \min_{0 \leq m} \{\langle e_m, m\rangle \mid e_m \text{ is a minimal } \phi^{(m)}\text{-r.e. index for } W_x\}$$

Assuming that our pairing function has the property that for all x and y,

$$\langle x,y\rangle \geq \max(x,y)$$

and since we know that $h(x) \leq \langle x,0\rangle$ for all x, in the determination of $h(x)$ only those e_m's need be considered for which $m \leq \langle x,0\rangle$. If A is a set such that $\emptyset^{(n)} \leq_T A$ for all n uniformly in n, then it follows that $h \leq_T A$. Now if g is a total function recursive in A then there will exist integers p and r such that

$$W_r^A = W_p \quad \text{and } h(p) > g(r)$$

The proof is as before.

<u>3.5</u>: In this section we will list some of the problems left unresolved in section 3.2 and 3.3. We will use the notations of those sections.

(1) In section 3.2 we left open the question whether the condition $A'' \leq_T B'$ in the statement of Theorem 12 can be weakened to $B \nleq_T A$ (or perhaps to $A <_T B$). This would be the same as asking the question whether given $B \nleq_T A$ there exists a total function $g \leq_T B''$ such that the set

$$\{x \mid \exists y[W_y^A = W_x^B \text{ and } y \in M^A \text{ and } y > g(x)]$$

is empty. To exhibit the existence of such a function g, it would suffice to get A and B such that $B \nleq_T A$ and $\{x \mid W_x^B \text{ is r.e. in } A\} \leq_T B''$.

(2) Another question we can ask is whether it is possible to get sets A and B with a certain symmetric relationship towards each other in the following sense:

i) A and B are T-incomparable

ii) for any total function g recursive in both A and B, both of the sets defined below are nonempty

$$\{x \mid \exists y[W_y^A = W_x^B \text{ and } y \in M^A \text{ and } y > g(x)]\}$$

$$\{y \mid \exists x[W_y^A = W_x^B \text{ and } x \in M^B \text{ and } x > g(x)]\}$$

In the statements of the problems (1) and (2) we have ignored any A-size functions apart from the identity function. The consideration of other A-size functions may or may not cause any increase in the difficulty of the problems.

(3) We mentioned at the end of section 3.3 that the proof of Theorem 12 lacked a certain effectiveness. This heightens the interest in the following problem:

Is it possible to parallel Theorems 13 and 15 in the case when we wish to consider total functions g recursive in B'' by using cofinite sets for instance? It should be noted that

$$\{x \mid W_x^B \text{ cofinite}\} \equiv_T B'''$$

(see [10], p. 328). The major bottleneck here is in characterizing a function g recursive in B'' in terms of a function recursive in B. It is clear from the proof of Theorem 15, that given a total function $g \leq_T B''$ we can get a total function $f \leq_T B$ such that

$$g(x) = \lim_{y \to \infty} \lim_{z \to \infty} f(x,y,z)$$

Manipulation, however, becomes very difficult with two limit operations to take care of in this case.

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