A TRAFFIC MODEL WITH SERVICE RATE QUADRATIC IN OCCUPANCY

by

ASHA SETH

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Accepted by........ Signature redacted ....
Chairman, Departmental Committee on Graduate Students

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ABSTRACT

The transient behavior of a traffic flow system where the over all service rate is assumed to be a quadratic function of the occupancy is studied in this thesis. The system is viewed as a service station with N servers and the transient probabilities for the system to be in state j at time t are obtained. It can be shown that such a system reaches a state of stagnation or lock up, eventually, regardless of the input rate. The mean time to lock up for small N has been verified with Helly's result. The partial differential equation for time dependent input is also obtained.

Thesis Advisor: Dr. John D. C. Little
Title: Associate Professor of Industrial Management

Professor William C. Greene
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Dear Professor Green:

In accordance with the requirements for graduation, I herewith submit a thesis entitled "A TRAFFIC MODEL WITH SERVICE RATE QUADRATIC IN OCCUPANCY".

Sincerely yours,

Signature redacted

Asha Seth

ACKNOWLEDGEMENT

I take this opportunity to offer my sincere thanks to Prof. J. D. C. Little for his constant advice and the unsparing pains he has taken to guide me throughout this investigation. I also wish to thank Prof. E. F. Bisbee for consenting to be a member of my thesis committee. Many thanks are also due to Prof. F. B. Hilderbrand, Dr. D. Quillen and Mr. A. Malhotra for their help and advice and to Miss Louise Romano for doing an excellent job of typing this thesis. Last but not the least I wish to thank the Government of India for the grant of a scholarship to pursue my studies at M.I.T.

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CHAPTER 1

INTRODUCTION

In a recent paper[2] entitled "Two Stochastic Traffic Systems whose Service Times increase with Occupancy", Walter Helly considers a traffic model characterized by service times that increase with the number of patrons undergoing simultaneous service. A traffic circle is viewed as a service station with N servers handling a Poisson arrival stream of intensity λ . The service times are exponentially distributed with a mean of $1/[c' (N - j)]$. The over all service rate is thus, a quadratic function in j , the number of occupied servers.

$$\mu_j = \begin{cases} c' j (N - j) & (0 < j \leq N) \\ 0 & (j > N) \end{cases}$$

where c' is a constant. Figure 1 is a plot of μ_j as a function of j . The maximum service rate is given by

$$\mu_{max} = \begin{cases} \frac{1}{\mu} c' N^2 & \text{occurring at } j = \frac{N}{2} \text{ ; N even} \\ \frac{1}{\mu} c' (N - 1)(N + 1) & j = \frac{1}{2}(N \pm 1) \text{ ; N odd} \end{cases}$$

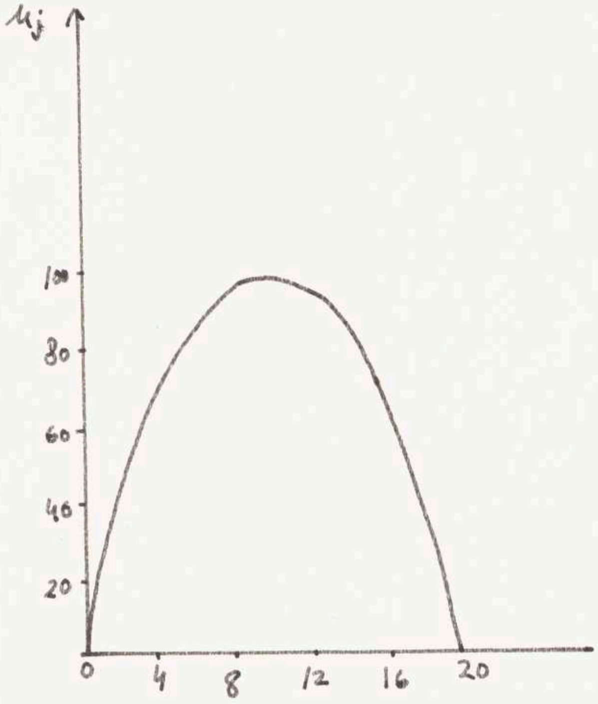

Figure 1

Service rate μ_j as a quadratic function of occupancy j for N = 20, c' = 1

Unfortunately, this maximum cannot be maintained without external control over arrivals. For any $\lambda > 0$, there is some value of j, say k, such that $\mu_j < \lambda$, all $j > k$. Hence the system will ultimately reach the absorbing state N, no matter how small the arrival rate. At that point the system reaches a state of stagnation and no new customers are accepted. A lock up is formed at the traffic island and there are neither new arrivals nor departures from the system. Helly obtains the mean time for the circle to lock up, starting at $t = 0$ from state $j = 0$ and ending at state N.

This thesis studies the transient behavior of the above system and obtains the transient probabilities for the system to be in state j at time t. Assuming that it started at $t = 0$ from state $j = 0$. For small N the mean time to lock up is also obtained and found to be in agreement with Helly's result. The system has also been studied for an arrival rate that varies with time. In chapter III the partial differential equation that describes the system for $\lambda(t) = t - t^2$ is obtained.

CHAPTER II

THE PROBLEM AND ITS SOLUTION

The traffic circle is taken to be a service system in which arrivals follow a Poisson distribution with mean rate λ . N is the number of servers. The maximum number of vehicles that can occupy it simultaneously is equal to the number of servers. The state of the system is defined as the number of vehicles in the circle at any time. Interdeparture intervals are distributed exponentially with a mean service rate $\mu_j = c' j (N - j)$, a quadratic function of occupancy j . There are two reasons for the dependence of the service rate on occupancy. First, when vehicles follow each other, their speed declines with decreased headways. Second, when a circle is crowded, departing vehicles, blocked by others from turning out, cease forward motion thus preventing progress for those behind. It appears that once occupancy is very high, a jam or lock up condition, spontaneous recovery to free flow is exceedingly unlikely unless there is a very substantial reduction in the arrival rate.

Let

$p_j(t) \equiv$ Probability that the system is in state j at time t

The system starts in state $j = 0$ at time $t = 0$.

Thus

$$p_i(0) = 1 \qquad i = 0$$
$$= 0 \qquad i \neq 0$$

Then with the λ and μ_j described above, we have

$$\frac{d\,p_0(t)}{dt} = \mu_1\, p_1(t) - \lambda\, p_0(t)$$

$$\frac{d\,p_j(t)}{dt} = \mu_{j+1}\, p_{j+1}(t) + \lambda\, p_{j-1}(t) - (\mu_j + \lambda)\, p_j(t), \qquad 1 \leq j \leq N-2$$

(2.1)

$$\frac{d\,p_{N-1}(t)}{dt} = \lambda\, p_{N-2}(t) - (\mu_{N-1} + \lambda)\, p_{N-1}(t)$$

and

$$\frac{d\,p_N(t)}{dt} = \lambda\, p_{N-1}(t) \quad .$$

Now define $R(s,t) \equiv \sum_{j=0}^{N} p_j(t)\, s^j$

Multiplying both sides of (2.1) by s^j and summing for all $0 \leq j \leq N$,

$$\frac{dR}{dt} = c' s(s-1) \frac{d^2 R}{ds^2} + c'(s-1)(1-N) \frac{dR}{ds} + \lambda (1-s) R - \lambda p_N(t) s^N (s-1)$$

or

$$\frac{d^2 R}{ds^2} + \frac{1}{c' s(1-s)} \frac{dR}{dt} + \frac{1-N}{s} \frac{dR}{ds} + \frac{\lambda}{c' s} R$$

$$= \frac{\lambda p_N s^{N-1}}{c'} \qquad (2.2)$$

Introducing $R'(s,t) = R(s,t) s^{(1-N)/2}$ and substituting in (2.2) for R, we have

$$\frac{d^2 R'}{ds^2} + \frac{1}{c' s(1-s)} \frac{dR'}{dt} + \left(\frac{1-N^2}{4 s^2} + \frac{\lambda}{c' s} \right) R' = \frac{\lambda}{c'} p_N(t) s^{(N-1)/2}$$

$$(2.3)$$

Multiplying both sides of (2.3) by $e^{-\alpha t}$ and integrating for $t = 0$ to ∞, we get

$$\frac{d^2 x(\alpha, s)}{ds^2} + \frac{1}{c' s(1-s)} \left[-R'(s,0) + \alpha x(\alpha, s) \right]$$

$$+ \left[\frac{1 - N^2}{4 s^2} + \frac{\lambda}{c' s} \right] x \; (a, s) = \frac{\lambda}{c'} s^{(N-1)/2} x^* (a)$$

where x and x^* are defined as the Laplace transforms

$$x (a, s) = \int_0^\infty e^{-at} R' (s, t) \, dt$$

$$x^* (a) = \int_0^\infty e^{-at} p_N(t) \, dt$$

since

$$R' (s, 0) = \left| s^{(1-N)/2} R (s, t) \right|_{t=0} = s^{(1-N)/2} p_0(0) = s^{(1-N)/2}$$

We have

$$\frac{d^2 x (a, s)}{ds^2} + \frac{1}{s^2} \left[\frac{1 - N^2}{4} + \frac{\lambda}{c'} s + \frac{a s}{c' (1 - s)} \right] x(a, s)$$

$$= \frac{\lambda}{c'} s^{(N-1)/2} x^* (a) + \frac{s^{-(1+N)/2}}{c' (1 - s)} \qquad (2.4)$$

Hence the original 2^{nd} order partial differential equation has been reduced to a 2^{nd} order linear differential equation with variable coefficients.

Since $s = 0$ is a regular singular point of the differential equation (2.4), the method of Frobenius can be applied to obtain a power series solution for it.

The complementary function which is the solution of the homogeneous equation obtained by setting R HS of (2.4) to zero, is first obtained. Let $x(\alpha, s) = \sum_{i=0}^{\infty} a_i(\rho)\, s^{i+\rho}$ be a solution of

$$\frac{d^2 x(\alpha, s)}{ds^2} + \frac{1}{s^2}\left[\frac{1 - N^2}{4} + \frac{\lambda}{c'} s + \frac{\alpha s}{c'(1 - s)}\right] x(\alpha, s) = 0.$$

The indicial equation

$$\left(\rho(\rho - 1) + \frac{1 - N^2}{4}\right) a_0(\rho) = 0$$

has roots $\rho_1, \rho_2 = \frac{1 \pm N}{2}$ and $a_0(\rho)$ can be chosen arbitrarily. The difference between ρ_1 and ρ_2 is a positive integer, hence the two solutions corresponding to ρ_1 and ρ_2 respectively are obtained as:

$$\left[s^{\rho} \sum_{0}^{\infty} a_i(\rho)\, s^i \right]_{\rho=\rho_1} \quad \text{and} \quad \left[\frac{d}{dp} \left[s^{\rho} \sum_{0}^{\infty} a_i(\rho)\, s^i \right] \right]_{\rho=\rho_2}$$

The complementary function of (2.4), thus, is

$$x(a, s) = A s^{\rho_1} \sum_0^\infty a_i(\rho_1) s^i + B s^{\rho_2} \left[\log s \sum_N^\infty a_i(\rho_2) s^i + \sum_0^\infty a_i'(\rho_2) s^i \right]$$

where

$$a_i'(\rho_2) = \left[\frac{d}{dp} a_i(\rho) \right]_{\rho=\rho_2}$$

The $a_i(\rho)$ are calculated from the following recursion relations in terms of $a_0(\rho)$.

$$\left[\rho(\rho+1) + \frac{1 - N^2}{4} \right] a_1(\rho) + \frac{a + \lambda}{c'} a_0(\rho) = 0$$

(2.5)

$$\left[(\rho+n)(\rho+n-1) + \frac{1 - N^2}{4} \right] a_n(\rho) - \left[(\rho+n-1)(\rho+n-2) + \frac{1 - N^2}{4} - \frac{(\lambda + a)}{c'} \right] a_{n-1}(\rho) - \frac{\lambda}{c'} a_{n-2}(\rho) = 0 \quad , \quad n \geq 2 .$$

Since $a_0(\rho)$ can be chosen arbitrarily set

$$a_0(\rho_1) = 1$$

and

$$a_0(\rho_2) = \left[(\rho + N)(\rho + N-1) + \frac{1 - N^2}{4} \right]_{\rho=\rho_2}$$

$$\rho_1 = \frac{1 + N}{2} , \quad \rho_2 = \frac{1 - N}{2} .$$

The subsequent a's can now be determined.

Next we determine the particular solution of (2.4). Rewriting (2.4)

$$\frac{s^2 d^2}{ds^2} x(\alpha, s) + [q_0 + q_1 s + q_2 s^2 + \dots\dots] x(\alpha, s)$$

$$= \frac{s^{(3-N)/2}}{c'} [1 + s + s^2 + \dots . s^{N-1} + (\lambda x^* + 1) s^N$$

$$+ s^{N+1} + \dots\dots] \qquad (2.6)$$

Applying the method of variation of parameters it is observed (see appendix A) that the particular solution should be of the form

$$x(a, s) = \log s \sum_{i=0}^{\infty} T_i s^{i+\rho} + \sum_{i=0}^{N-2} U_i s^{i+r} + \sum_{i=1}^{\infty} M_i s^{i+\rho_1}$$

$$r = \frac{3 - N}{2}, \quad q_0 = \frac{1-N^2}{4}, \quad q_1 = \frac{\lambda + a}{c'}, \quad q_i = \frac{\alpha}{c'}, \quad i \geq 2 \qquad (2.7)$$

Here again the T's , U's and M's are obtained by substituting for $x(a, s)$ in (2.6) and equating the coefficients of the various powers of s on both sides. The general solution of (2.4) is found to be

$$x(a, s) = s^{\rho_1} A \sum_{i=0}^{\infty} a_i(\rho_1) s^i + B s^{\rho_2} [\log s \sum_{i=N}^{\infty} a_i(\rho_2) s^i$$

$$+ \sum_{i=0}^{\infty} a_i'(\rho_2) s^i] + s^{\rho_1} \log s \sum_{i=0}^{\infty} T_i s^i + s^r \sum_{0}^{N-2} U_i s^i$$

$$+ s^{\rho_1} \sum_{i=1}^{\infty} M_i s^i.$$

But

$$x(a, s) = \int_0^{\infty} e^{-at} R'(st)\, dt = \int_0^{\infty} e^{-at} R(s, t) s^{(1-N)/2} dt$$

or

$$L(R(s,t)) = s^N A \sum_0^{\infty} a_i(\rho_1) s^i + B \left[\log s \sum_N^{\infty} a_i(\rho_2) s^i + \sum_0^{\infty} a_i'(\rho_2) s^i \right]$$

$$+ s^N \log s \sum_0^{\infty} T_i s^i + s \sum_0^{N-2} U_i s^i + s^N \sum_1^{\infty} M_i s^i \qquad (2.8)$$

$L(R(s,t))$ being an abbreviation for the Laplace transform of $R(s,t)$, where A and B are to be determined from initial conditions. By definition $R(s,t) = \sum_0^{\infty} p_i(t) s^i$, a power series in s not involving any logarithmic terms. Also (see appendix A) $a_N(\rho_2)\, a_{N+1}(\rho_2) \ldots$ satisfy the same recurrence relation as the T_i's . Hence the logarithmic terms in (2.8) must cancel. Thus

$$B = -\frac{T_0}{a_N(\rho_2)} \quad .$$

A can now be determined from the boundary condition

$$L[R(1,t)] = \frac{1}{\alpha} \qquad (2.9)$$

Taking the limit of (2.8) as $s \to 1$

$$\frac{1}{\alpha} = A \sum_{0}^{\infty} a_i(\rho_1) - \frac{T_0}{a_N(\rho_2)} \sum_{0}^{\infty} a_i'(\rho_2) + \sum_{0}^{N-2} U_i + \sum_{1}^{\infty} M_i$$

or

$$A = \left[\frac{1}{\alpha} + \frac{T_0}{a_N(\rho_2)} \sum_{0}^{\infty} a_i(\rho_2) - \sum_{0}^{N-2} U_i - \sum_{1}^{\infty} M_i \right] \frac{1}{\sum_{0}^{\infty} a_i(\rho_1)} \tag{2.10}$$

Hence

$$L(R(s,t)) = s^N A \sum_{0}^{\infty} a_i(\rho_1) - \frac{T_0}{a_N(\rho_2)} \sum_{0}^{\infty} a_i'(\rho_2) s^i$$

$$+ s \sum_{0}^{N-2} U_i s^i + s^N \sum_{1}^{\infty} M_i s^i$$

A being given by (2.10)

$$L(p_0(t)) = \text{coefficient of } s^0 \text{ in RHS}$$

$$= - \frac{T_0}{a_N(\rho_2)} a_0'(\rho_2)$$

or

$$p_0(t) = L^{-1} \left[- \frac{T_0}{a_N(\rho_2)} a_0'(\rho_2) \right]$$

$$p_i(t) = L^{-1} [\text{ coefficient of } s^i \text{ on RHS }], \quad i \leq N-1$$

$$= L^{-1} \left[- \frac{T_0}{a_N(\rho_2)} a_i'(\rho_2) + U_{i-1} \right]$$

L^{-1} denoting the inverse Laplace operator .

$$p_j(t) = L^{-1} \left[A\, a_{j-N}(\rho_1) - \frac{T_0}{a_N(\rho_2)} a_j'(\rho_2) + M_{j-N} \right], \quad j \geq N .$$

Particular cases

(i) $N = 2$

$$a_2(\rho_2) = - \left[\frac{(\lambda + a)^2}{c'^2} + \frac{a}{c'} \right]$$

$$T_0 = \frac{1}{2c'} \left(1 + \frac{\lambda + a}{c'} \right)$$

$$B = \left[\frac{1 + \frac{(\lambda + a)}{c'}}{\frac{(\lambda + a)^2}{c'^2} + \frac{a}{c'}} \right] \frac{1}{2c'}$$

and $a'_0 \ (\rho_2) = 2$.

Therefore

$$L\,(p_0\,(t)) = 2B = \frac{\lambda + a + c'}{(\lambda + a)^2 + a\,c'} \qquad (2.11)$$

By the limit theorems for Laplace transforms

$$\lim_{a \to \infty} a\ L\,(p_0\,(t) = \lim_{t \to 0} p_0\,(t) = 1$$

and

$$\lim_{a \to 0} a\ L\,(p_0\,(t)) = \lim_{t \to \infty} p_0\,(t) = 0$$

Inverting (2.11) w.r.t. a we have

$$p_0\,(t) = \frac{1}{a_2 - a_1} \left[(\lambda + c' - a_1)\,e^{-a_1 t} - (\lambda + c' - a_2)\,e^{-a_2 t} \right]$$

where

$$a_1 = \frac{1}{2} \left[2\lambda + c' - [(2\lambda + c')^2 - 4\lambda^2]^{\frac{1}{2}} \right]$$

$$a_2 = \frac{1}{2} \left[2\lambda + c' + [(2\lambda + c')^2 - 4\lambda^2]^{\frac{1}{2}} \right]$$

From (2.1)

$$p_1(t) = \frac{1}{\mu_1} \left[\frac{d\, p_0(t)}{dt} + \lambda\, p_0(t) \right]$$

$$= \frac{1}{c'(a_2 - a_1)} \left[(2a_2\lambda + a_2 c' - a_2^2 - \lambda^2 - \lambda c')\, e^{-a_2 t} - (2a_1\lambda + a_1 c' - a_1^2 - \lambda^2 - \lambda c')\, e^{-a_1 t} \right]$$

Also

$$p_2(t) = \lambda \int p_1(t)\, dt + \text{constant}$$

$$= \frac{\lambda}{c'(a_2 - a_1)} \left[-\left(\frac{2a_2\lambda + a_2 c' - a_2^2 - \lambda^2 - \lambda c'}{a_2} \right) e^{-a_2 t} + \frac{(2a_1\lambda + a_1 c' - a_1^2 - \lambda^2 - \lambda c')}{a_1}\, e^{-a_1 t} \right] + k$$

Since $p_2(\infty) = 1$, $k = 1$

The mean time to lock up is given by

$$\langle 0 \mid t \mid 2 \rangle = \int_0^\infty t \frac{d}{dt} (p_2(t))$$

using Helly's notation

$$= \frac{2\lambda + c'}{\lambda^2}$$

in agreement with his result.

Figures 2, 3, and 4 represent the behavior of the probabilities $p_0(t)$, $p_1(t)$, and $p_2(t)$ with time. We note that for $c' = 10$, the time to lock up when $\lambda = 5$ is about 42 minutes while for $\lambda = 10$ it is only 15 minutes. This kind of behavior of the system is not surprising in view of increasing the input by a factor of 2.

The time is in the units used for $1/c'$ where c' is the constant in $\mu_j = c'j(N-j)$. Here one supposes j and N to be dimensionless so the units of c' are vehicles per unit time. Thus if $c' = 10$, time is in units of 1/10 hour.

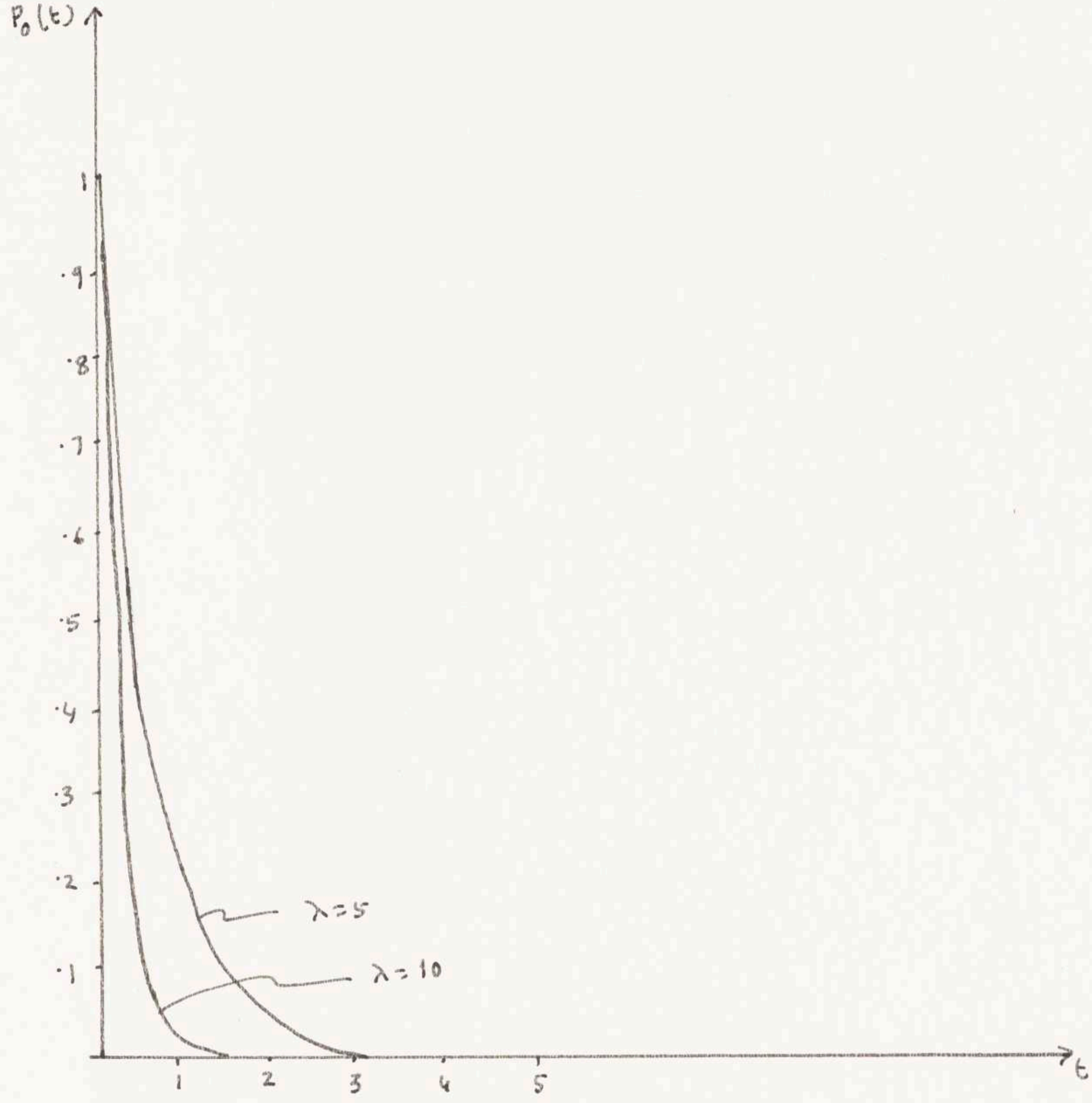

Figure 2

The graphs for $p_0(t)$ in unit of 1/10 hours for $c' = 10$, $N = 2$, $\lambda = 5$ and 10

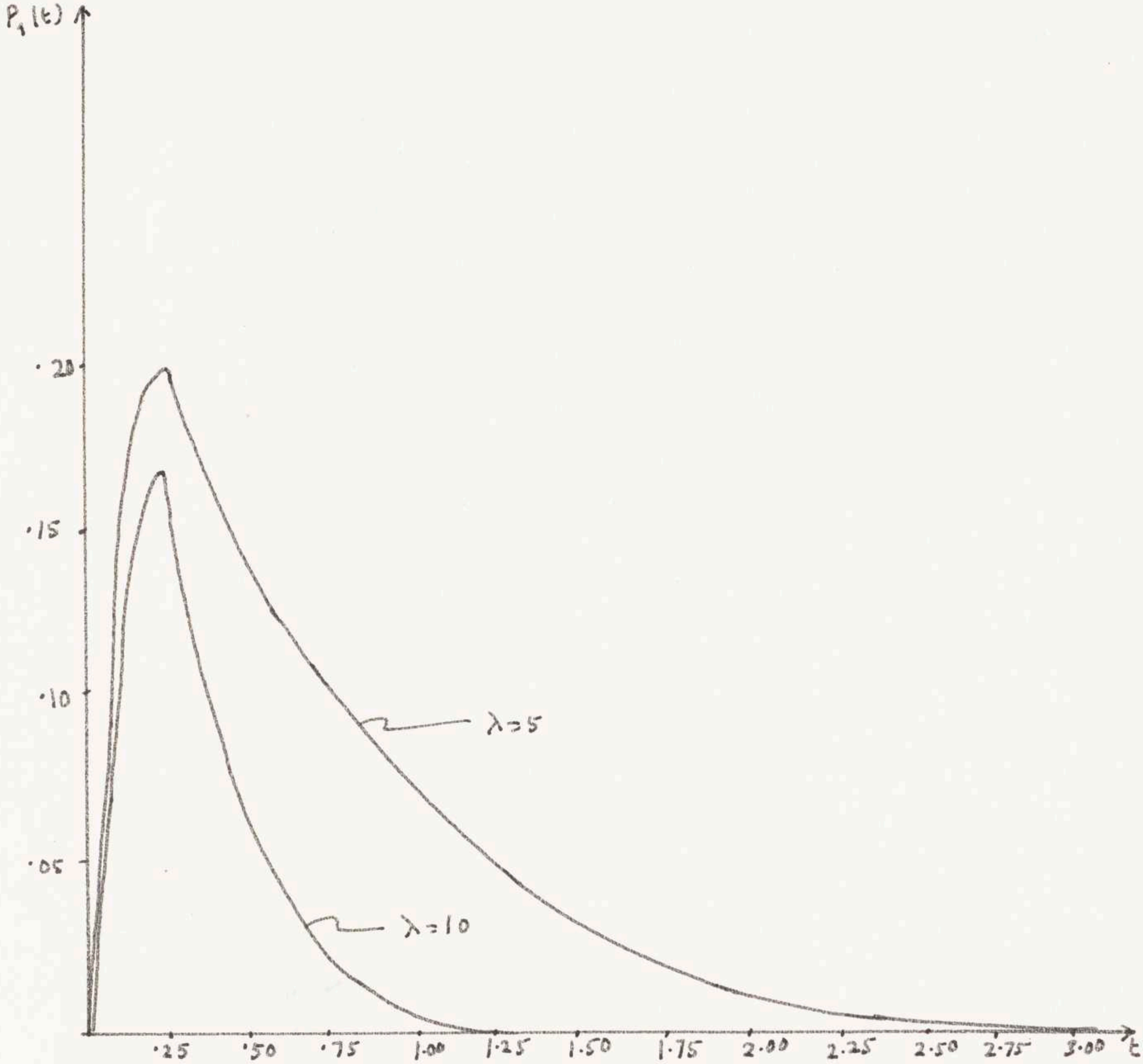

Figure 3

The graphs for $p_1(t)$ in units of 1/10 hours for $c' = 10$, $N = 2$, $\lambda = 5$ and 10

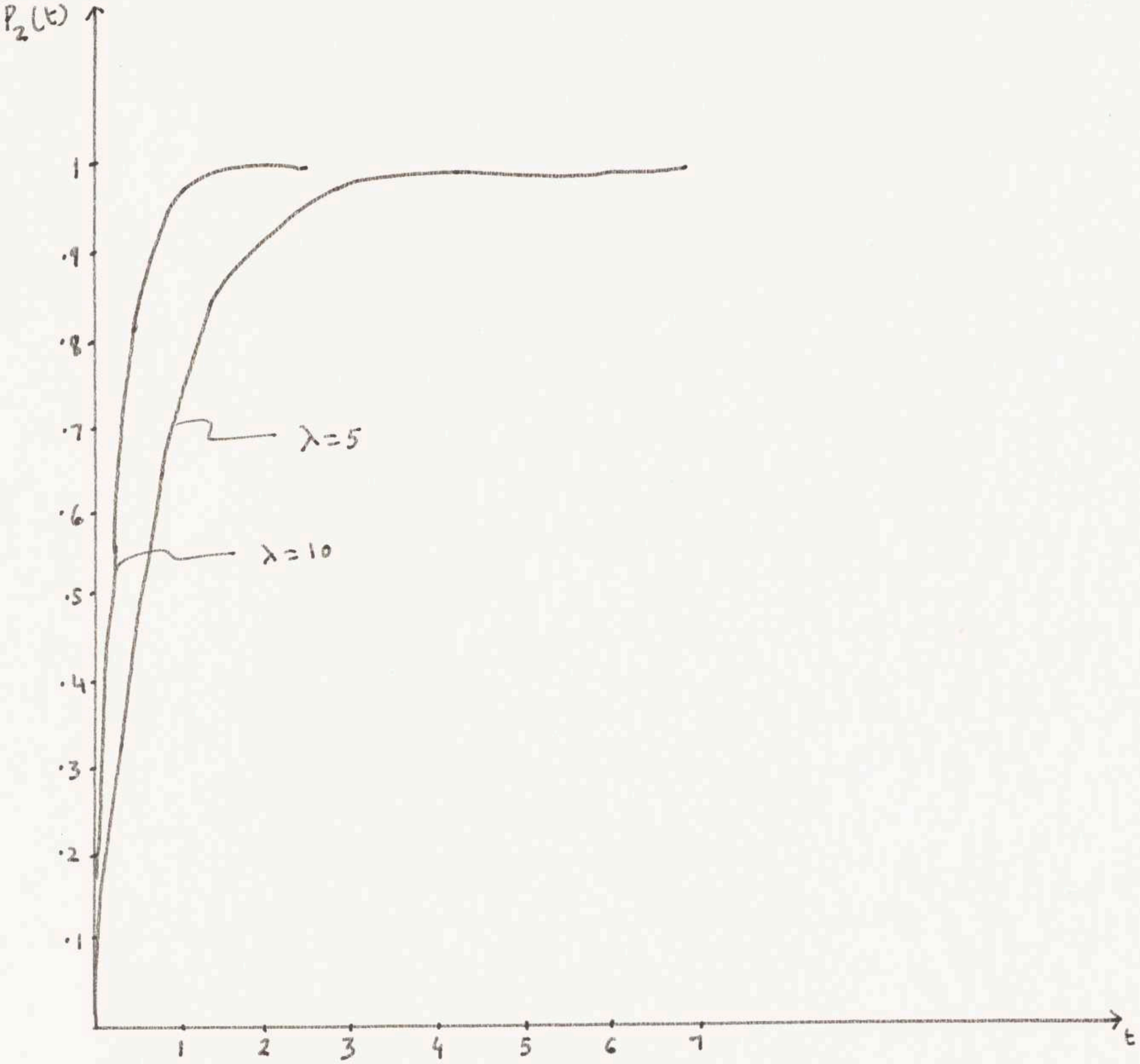

Figure 4

The graphs showing the probabilities to lock up in units of 1/10 hours for $c' = 10$, $N = 2$, $\lambda = 5$ and 10.

(ii) $N = 3$

$$B = -\frac{T_0}{b_3} = \frac{\left[1 + \frac{2\lambda + a}{c'} + \frac{(\lambda + a)^2}{c'^2} \right]}{3 \left[\frac{(\lambda + a)\lambda}{c'} + \frac{(\lambda + a)^3}{4 c'^2} + a \right]}$$

$$= \frac{[c'^2 + \lambda^2 + 2\lambda c' + a(c' + 2\lambda) + a^2]}{3[a^3 + a^2 (4c' + 3\lambda) + a(4\lambda c' + 4c'^2 + 3\lambda^2) + \lambda^3]}$$

$$L(p_0(t)) = \frac{[a^2 + a(c' + 2\lambda) + c'^2 + \lambda^2 + 2\lambda c']}{[a^3 + a^2 (4c' + 3\lambda) + a(4\lambda c' + 4c'^2 + 3\lambda^2) + \lambda^3]} \quad (2.12)$$

Let a_1, a_2, and a_3 be the roots of a in the denominator of (2.12).

For

$$A = \Big[-\frac{1}{54} (-16 c'^3 - 27\lambda^3 + 36 c'^2 \lambda + 27)$$

$$+ \Big\{ \frac{(-16 c'^3 - 27\lambda^3 + 36 c'^2 \lambda + 27)^2}{2916}$$

$$- \frac{16}{729} (c'^2 + 3\lambda c')^3 \Big\}^{1/2} \Big]^{1/3}$$

and

$$B = [-\frac{1}{54} (-16 c'^3 - 27\lambda^3 + 36 c'^2 \lambda + 27)$$

$$+ \{ \frac{(-16 c'^3 - 27\lambda^3 + 36 c'^2 \lambda + 27)^2}{2916}$$

$$- \frac{16}{729} (c'^2 + 3\lambda c')^3 \}^{1/2}]^{1/3} .$$

$$a_1 = A + B - \frac{(4 c' + 3 \lambda)}{3} ,$$

$$a_2 = -\frac{(A + B)}{2} + \frac{i (A - B)\sqrt{3}}{2} - \frac{(4 c' + 3\lambda)}{2} ,$$

and

$$a_3 = -\frac{(A + B)}{2} - \frac{i (A - B)\sqrt{3}}{2} - \frac{(4 c' + 3\lambda)}{2} .$$

Then

$$L (p_0(t)) = \frac{a_1^2 + a_1 k_1 + k_2}{(a_1 - a_2)(a_1 - a_3)} \frac{1}{a - a_1} + \frac{a_2^2 + a_2 k_1 + k_2}{(a_2 - a_1)(a_2 - a_3)} \frac{1}{a - a_2}$$

$$+ \frac{a_3^2 + a_3 k_1 + k_2}{(a_3 - a_1)(a_3 - a_2)} \frac{1}{a - a_3}$$

or

$$p_0(t) = \frac{(a_1^2 + a_1 k_1 + k_2)}{(a_1 - a_2)(a_1 - a_3)} e^{a_1 t} + \frac{(a_2^2 + a_2 k_1 + k_2)}{(a_2 - a_1)(a_2 - a_3)} e^{a_2 t}$$

$$+ \frac{a_3^2 + a_3 k_1 + k_2}{(a_3 - a_1)(a_3 - a_2)} e^{a_3 t}$$

where

$$k_1 = c' + 2\lambda$$

and

$$k_2 = c'^2 + \lambda^2 + 2\lambda c'$$

$p_1(t)$ can be determined as before and is

$$p_1(t) = \frac{1}{2c'} \left[\frac{(a_1^2 + a_1 k_1 + k_2)(\lambda + a_1)}{(a_1 - a_2)(a_1 - a_3)} e^{a_1 t} + \frac{(a_2^2 + a_2 k_1 + k_2)}{(a_2 - a_1)(a_2 - a_3)} (\lambda + a_2) e^{a_2 t} \right.$$

$$\left. + \frac{(a_3^2 + a_3 k_1 + k_2)}{(a_3 - a_1)(a_3 - a_2)} (\lambda + a_3) e^{a_3 t} \right]$$

$$p_2(t) = \frac{1}{2c'} \left[\frac{(a_1^2 + a_1 k_1 + k_2)}{(a_1 - a_2)(a_1 - a_3)} \left\{ a_1 \frac{(\lambda + a_1)}{2c'} - \lambda + \frac{(2c' + \lambda)(\lambda + a_1)}{2c'} \right\} e^{a_1 t} \right.$$

$$+ \frac{a_2^2 + a_2 k_1 + k_2}{(a_2 - a_1)(a_2 - a_3)} \left\{ \frac{a_2(\lambda + a_2)}{2c'} - \lambda + (\frac{(2c' + \lambda)(\lambda + a_2)}{2c'} \right\} e^{a_2 t}$$

$$\left. + \frac{a_3^2 + a_3 k_1 + k_2}{(a_3 - a_1)(a_3 - a_2)} \left\{ a_3 \frac{(\lambda + a_3)}{2c'} - \lambda + \frac{(2c' + \lambda)(\lambda + a_3)}{2c'} \right\} e^{a_3 t} \right]$$

and

$$p_3(t) = 1 + \frac{\lambda}{2c'} \left[\frac{a_1^2 + a_1 k_1 + k_2}{(a_1 - a_2)(a_1 - a_3)} \frac{1}{a_1} \left\{ \frac{a_1(\lambda + a_1)}{2c'} - \lambda + \frac{(2c' + \lambda}{2c'} (\lambda + a_1) \right] e^{a_1 t} \right.$$

$$+ \frac{a_2^2 + a_2 k_1 + k_2}{(a_2 - a_1)(a_2 - a_3)} \frac{1}{a_2} \left\{ \frac{a_2 (\lambda + a_2)}{2\,c'} - \lambda \right.$$

$$\left. + \frac{(2c' + \lambda)(\lambda + a_2)}{2\,c'} \right\} e^{a_2 t}$$

$$+ \frac{a_3^2 + a_3 k_1 + k_2}{(a_3 - a_1)(a_3 - a_2)} \frac{1}{a_3} \left\{ a_3 \frac{(\lambda + a_3)}{2\,c'} - \lambda \right.$$

$$\left. + \frac{(2c' + \lambda)(\lambda + a_3)}{2\,c'} \right\} e^{a_3 t} \quad].$$

CHAPTER III

THE PARTIAL DIFFERENTIAL EQUATION FOR TIME DEPENDENT INPUT

In this chapter the instantaneous arrival rate of vehicles is assumed to depend on time and is taken to equal $t - t^2$. The differential-difference equations describing the system remain the same as in chapter II except that λ is replaced by $t - t^2$.

Using the notation of the last chapter we have,

$$\frac{d^2 R'}{ds^2} + \frac{1}{c' s (1 - s)} \frac{dR'}{dt} + \frac{4 s (t - t^2) + c' (1 - N^2)}{4 c' s^2} R'$$

$$= \frac{(t - t^2) p_N(t) s^{(N-1)/2}}{c'}$$

or

$$\frac{d^2 x}{ds^2} - \frac{2}{c' s} \frac{d^2 x}{da^2} - \frac{2}{c' s} \frac{dx}{da} + \left[\frac{(1 - N^2)}{4 s^2} + \frac{a}{c' s (1 - s)} \right] x$$

$$= \frac{s^{-(1+N)/2}}{c' (1 - s)} - \frac{s^{(N-1)/2}}{c'} \left[\frac{d}{da} x^* + \frac{d^2}{da^2} x^* \right] \tag{3.1}$$

The above equation is a 2^{nd} order partial differential equation of the form[2]

$$Ss + Rr + Tt + Qq + Pp + Zz = U$$

where $S = 0$, $R = 1$, $T = Q = -\dfrac{2}{c' s}$, $Z = \dfrac{1 - N^2}{4 s^2} + \dfrac{a}{c' s (1 - s)}$,

$$P = 0, \quad U = \frac{s^{-(1+N)/2}}{c' (1 - s)} - \frac{s^{(N-1)/2}}{c'} f(a),$$

$$f(a) = \frac{d}{da} x^* + \frac{d^2}{da^2} x^*$$

To solve the linear equation (3.1), the process consists of changing the variables.

Let the independent variables s and a be changed to ξ and η as yet undetermined; then when p', q' .... denote $\dfrac{dx}{d\xi}$, $\dfrac{dx}{d\eta}$ ...... the equation becomes

$$r' \left[R \left(\frac{d\xi}{ds}\right)^2 + T \left(\frac{d\xi}{da}\right)^2 + S \frac{d\xi}{ds} \frac{d\xi}{d\eta} \right]$$

$$+ t' \left[R \left(\frac{d\eta}{ds}\right)^2 + S \frac{d\eta}{ds} \frac{d\eta}{da} + T \left(\frac{d\eta}{da}\right)^2 \right]$$

$$+ s' \left[2R \frac{d\xi}{ds} \frac{d\eta}{ds} + S \left(\frac{d\eta}{ds} \frac{d\xi}{da} + \frac{d\eta}{da} \frac{d\xi}{ds} \right) \right.$$

$$+ 2T \frac{d\xi}{ds} \frac{d\eta}{da} \left. \right] + p' \left[\frac{d^2\xi}{ds^2} + S \frac{d^2\xi}{ds\,da} + T \frac{d^2\xi}{da^2} \right.$$

$$+ P \frac{d\xi}{ds} + Q \frac{d\xi}{da} \left. \right] + q' \left[R \frac{d^2\eta}{ds^2} + S \frac{d^2\eta}{ds\,da} \right.$$

$$+ T \frac{d^2\eta}{da^2} + P \frac{d\eta}{ds} + Q \frac{d\eta}{da} \left. \right] + Zz = U \qquad (3.2)$$

Let m and n be the roots of the quadratic equation in k

$$R k^2 + S k + T = 0$$

Then

$$m, n = \pm \left(\frac{2}{c' s} \right)^{1/2}$$

Since these two roots are unequal, we choose ξ and η so that

$$\frac{d\xi}{ds} = m \frac{d\xi}{da} = \left(\frac{2}{c' s} \right)^{1/2} \frac{d\xi}{da}$$

and

$$\frac{d\eta}{ds} = n \frac{d\eta}{d\alpha} = - \left(\frac{2}{c' s} \right)^{1/2} \frac{d\eta}{d\alpha}$$

which determine ξ and η as

$$\xi = e^{\alpha} e^{2(2/c')^{1/2} s^{1/2}}$$

and

$$\eta = e^{\alpha} e^{-2(2/c)^{1/2} s^{1/2}}$$

Substituting in (3.2) for $\frac{d\xi}{ds}$, $\frac{d\eta}{ds}$ etc., and dividing throughout by

$$\frac{d\xi}{d\alpha} \frac{d\eta}{d\alpha} \left(4T - \frac{S^2}{R} \right) = \frac{-8}{c' s} e^{2\alpha}$$

we have the following reduced form

$$\frac{d^2 x}{d\xi d\eta} + L \frac{dx}{d\xi} + M \frac{dx}{d\eta} + N_1 x = V . \qquad (3.3)$$

Where

$$L = c' \, s \, e^{-\alpha} \, e^{2(2/c')^{1/2} s^{1/2}} (2/c')^{1/2} \left[s^{-1} (2/c')^{1/2} + 1/2 \, s^{-3/2} \right] / 8$$

$$M = c' \, s \, e^{-\alpha} \, e^{-2(2/c')^{1/2} s^{1/2}} (2/c')^{1/2} \left[s^{-1} (2/c')^{1/2} - 1/2 \, s^{-3/2} \right] / 8$$

$$N_1 = - \left[\frac{1 - N^2}{4 s^2} + \frac{\alpha}{c' \, s \, (1 - s)} \right] c' \, s \, e^{-2\alpha} / 8$$

and

$$V = e^{-2\alpha} \left[s^{(N+1)/2} \, f(\alpha) - \frac{s^{(1-N)/2}}{1 - s} \right] / 8$$

(3.3) is a standard partial differential equation and can be solved for x .

APPENDIX A

SOLUTION OF THE DIFFERENTIAL EQUATION

$$\frac{d^2x}{ds^2} + \frac{1}{s^2}\left[\frac{1-N^2}{4} + \frac{\lambda + a}{c'} s + \frac{a}{c'} s^2 + \frac{a}{c'} s^3 + \ldots\right] x$$

$$= \frac{\lambda}{c'} s^{\frac{N-1}{2}} x^* + \frac{s^{-(1+N)/2}}{c'(1-s)}$$

To obtain the complementary function we solve the homogeneous equation:

$$\frac{d^2x}{ds^2} + \frac{1}{s^2}\left[q_0 + q_1 s + q_2 s^2 + q_3 s^3 + \ldots\right] x = 0 \ldots\ldots \qquad (A.1)$$

where

$$q_0 = \frac{1-N^2}{4}, \quad q_1 = \frac{\lambda + a}{c'}, \quad q_i = \frac{a}{c'}, \qquad i \geq 2 .$$

Equation (A. 1) has $s = 0$ for an infinity of degree two. Frobenius's method can therefore be employed to obtain its solution. Let

$$x = \sum_{i=0}^{\infty} a_i s^{i+\rho}$$

Substitution in (A. 1) gives

$$\rho(\rho - 1)\, a_0\, s^{\rho-2} + (\rho + 1)\, \rho\, a_1\, s^{\rho-1} + (\rho + 2)(\rho + 1)\, a_2\, s^{\rho-2} + \ldots$$

$$+ \left[\frac{q_0}{s^2} + \frac{q_1}{s} + q_2 + q_3 s + \ldots \right] \left[a_0\, s^{\rho} + a_1\, s^{\rho+2} \right.$$

$$\left. + \ldots \right] = 0 \qquad \text{(A. 2)}$$

Equating to zero the coefficient of the lowest power of s ,

$$f(\rho)\, q_0 = (\rho(\rho - 1) + q_0)\, a_0 = 0 \ldots\ldots$$

As a_0 is not zero, being the coefficient of the first term in x , (A. 2) shows that the values of ρ to be considered are the roots of $f(\rho)$. i.e.

$$\rho(\rho - 1) + \frac{1 - N^2}{4} = 0$$

which is a quadratic in ρ with roots ρ_1 and ρ_2 equal to $\frac{1 \pm N}{2}$. a_0 is chosen arbitrarily. Since the difference between ρ_1 and ρ_2 is a positive integer N, we take

$$a_0(\rho) = c \left[(\rho + N)(\rho + N - 1) + \frac{1 - N^2}{4} \right]$$

and thus secure that none of the coefficients a_n is infinite; moreover a_0 undetermined and therefore an arbitrary constant, so that c is an arbitrary constant. By equating to 0 the coefficients of various powers of s in (A.2) we have:

$$(\rho(\rho+1)+q_0)\,a_1 + q_1\,a_0 = 0$$

$$\{(\rho+i)(\rho+i-1)+q_0\}\,a_i - \{(\rho+i-1)(\rho+i-2)+q_0-q_1\}\,a_{i-1}$$

$$-\frac{\lambda}{c'}\,a_{i-2} = 0\,, \qquad i \geq 2 \qquad \text{(A.3)}$$

We notice that for $\rho = \rho_1 = \dfrac{1+N}{2}$ all the a_i's are determined in terms of a_0 from the above recurrence relations and the corresponding solution is

$$x = s^{\rho_1} \sum_{i=0}^{\infty} a_i(\rho_1)\,s^i$$

corresponding to the root $\rho_2 = \dfrac{1-N}{2}$, there are two integrals of the equation (A.1) viz

$$x = \left[\sum_0^{\infty} a_i(\rho)\,s^{i+\rho}\right]_{\rho=\rho_2} \qquad \text{and} \quad x = \left[\frac{dx}{d\rho}\right]_{\rho=\rho_2}$$

As regards the first integral

$$x = \sum_0^{\infty} a_i(\rho_2)\, s^{i+\rho_1}$$

the coefficients a_i for all the values $i = 0, 1, \ldots N-1$ contain a factor $\rho - \rho_2$, hence when ρ is made equal to ρ_2 the sum of the first N terms in $s^{\rho_2} \sum_0^{\infty} a_i(\rho_2)\, s^i$ vanishes. As regards the second sum, we write it in the form

$$s^{\rho+N} a_N(\rho_2) + s^{\rho+N-1} a_{N+1}(\rho_2) + \ldots,$$

which when ρ is made equal to ρ_2 $(= \frac{1-N}{2})$, becomes

$$s^{\frac{1+N}{2}} a_N(\rho_2) + s^{\frac{3+N}{2}} a_{N+1}(\rho_2) + \ldots$$

a series that begins with s^{ρ_1} and proceeds in ascending powers of s .

But

$$x = \left[\sum_0^{\infty} a_i(\rho)\, s^{\rho+i} \right]_{\rho=\rho_1}$$

is a series that begins with s^{ρ_1} and proceeds in ascending powers of s ,

hence

$$x = \left[\sum_0^{\infty} a_i(\rho)\, s^{\rho+i} \right]_{\rho=\rho_2}$$

is not an independent integral, it is a constant multiple of

$$x = \left[\sum_{0}^{\infty} a_i(\rho)\, s^{\rho+i} \right]_{\rho=\rho_1}$$

Consequently corresponding to the roots ρ_2 of ρ the solution of (A.1) is

$$x = \left[\frac{d}{d\rho} \sum_{0}^{\infty} a_i(\rho)\, s^{\rho+i} \right]_{\rho=\rho_2}$$

$$= s^{\rho_2} \log s \sum_{i=N}^{\infty} a_i(\rho_2)\, s^i + s^{\rho_2} \sum_{i=0}^{\infty} a_i'(\rho_2)\, s^i$$

where

$$a_i'(\rho_2) = \left[\frac{d}{d\rho} a_i(\rho) \right]_{\rho=\rho_2} .$$

Hence the complementary function of (A.1) is given by

$$x = A\, s^{\rho_1} \sum_{0}^{\infty} a_i(\rho_1)\, s^i + B\, s^{\rho_2} \left[\log s \sum_{i=N}^{\infty} a_i(\rho_2)\, s^i + \sum_{i=0}^{\infty} a_i'(\rho_2)\, s^i \right]$$

A and B are unknown constants to be determined from the boundary conditions.

The Particular Solution

Now we proceed to find the particular solution of the equation

$$\frac{d^2x}{ds^2} + \frac{1}{s^2}\left[q_0 + q_1 s + q_2 s^2 + \ldots\right] x = \frac{s^{\frac{3-N}{2}}}{c'}\left[1 + s + s^2 + (\lambda x^* + 1) s^N + \ldots .\right] \quad (A.4)$$

In order to get an idea of the exact form of the particular solution, the variation of parameter technique is applied. This is a method of determining the particular integral when the complementary function is known. The Wronskian for the complementary function of a 2nd order differential equation is

$$W = u_1 u_2' - u_2 u_1'$$

where the primes represent differentiation w. r. t. s

$$\frac{dW}{ds} = u_1 u_2'' - u_2 u_1'' = \frac{1}{s^2}(q_0 + q_1 st\ldots)(u_1 u_2 - u_2 u_1) = 0.$$

or

$$W = W_c \quad .$$

Hence the Wronskian is a constant W_c .

Let $$h(s) = \frac{1}{c'} \left[\lambda s^{\frac{N-1}{2}} x^* + \frac{s^{-(1+N)/2}}{1 - s} \right]$$

$$u_1(s) = s^{\rho_1} \sum_{0}^{\infty} a_i (\rho_1) s^i$$

$$u_2(s) = s^{\rho_2} \sum_{0}^{\infty} a_i' (\rho_2) s^i + s^{\rho_2} \log s \sum_{i=N}^{\infty} a_i (\rho_2) s^i .$$

Put $a_i = a_i (\rho_1)$, $b_i = a_i (\rho_2)$

Then the particular solution of (A. 4) by variation of parameters is[3]

$$x = c_1(s) u_1(s) + c_2(s) u_2(s)$$

where

$$c_2(s) = \frac{1}{W_c} \int h(s) u_1(s) ds + \text{constant}$$

$$= \frac{1}{W_c} \frac{1}{c'} \left[a_0 s + \frac{(a_0 + a_1) s^2}{2} + \frac{(a_0 + a_1 + a_2) s^3}{3} + \ldots\ldots \right.$$

$$\left. + \frac{(a_0 (1 + \lambda x^*) + a_1 + a_2 + \ldots a_N) s^N}{N + 1} + \ldots\ldots \right]$$

and

$$c_1(s) = - \frac{1}{W_c} \int h(s)\, u_2(s)\, ds + \text{constant}$$

$$= - \frac{1}{W_c c'} \int [\log s \; \{ b_N + (b_{N+1} + b_N) s + \ldots\ldots \}$$

$$+ (\lambda x^* b'_0 + b'_0 + b'_1 + b'_2 + \ldots b'_N) + (\lambda x^* b'_1$$

$$+ b'_0 + b'_1 + \ldots\ldots b'_{N+1}) s + \ldots\ldots b'_0 \; s^{-N}$$

$$+ (b'_0 + b'_1) s^{1-N} + \ldots\ldots$$

$$+ (b'_0 + b'_1 + \ldots\ldots b'_{N-1}) s^{-1} + \ldots\ldots] \, ds$$

$$= - \frac{1}{W_c c'} [\log s \; \{ b'_0 + b'_1 + \ldots b_{N-1} + s\, b_N + \frac{s^2}{2} (b_N + b_{N+1}) + \ldots\ldots \}$$

$$- \frac{b'_0}{N-1} s^{1-N} - \frac{(b'_0 + b'_1)}{N - 2} s^{2-N} \ldots\ldots - (b'_0 + b'_1 + \ldots b'_{N-2}) s^{-1}$$

$$+ \ (\lambda \ x^{*} b'_0 + b'_0 + \dots b'_N - b_N) \ s + \dots\dots$$

$$- \ (b'_0 + \dots b'_{N-2}) \frac{1}{s} - (b'_0 + \dots b'_{N-3}) \frac{1}{2s^2} \dots\dots \frac{b'_0 \ 1}{(N-1) \ s^{N-1}} \ ]$$

The particular solution

$$x = c_1(s) \, u_1(s) + c_2(s) \, u_2(s)$$

is given by

$$x = s^{\rho_1} \log s \sum_{0}^{\infty} T_i \, s^i + s^{\frac{3-N}{2}} \sum_{i=0}^{N-2} u_i \, s^i + s^{\rho_i} \sum_{i=1}^{\infty} M_i \, s^i$$

(A. 5)

substituting for x from (A. 5) in (A. 4)

$$\sum_{0}^{\infty} (i+\rho_1-1) \, T_i \, s^{i+\rho_1} + \sum_{0}^{\infty} (i+\rho_1) \, T_i \, s^{i+\rho_1} + \log s \sum_{0}^{\infty} (i+\rho_1)(i+\rho_1-1) \, T_i \, s^{i+\rho_1}$$

$$+ \sum_{0}^{N-2} (i+r)(i+r-1) \, u_i \, s^{i+r} + \sum_{1}^{\infty} (i+\rho_1)(i+\rho_1-1) \, M_i \, s^{i+\rho_1}$$

$$+ \ (q_0 + q_1 \, s + \dots) \, [\ s^{\rho_1} \log s \sum_{0}^{\infty} T_i \, s^i + \sum_{0}^{N-2} u_i \, s^{i+r} + \sum_{1}^{\infty} M_i \, s^{i+\rho_1}]$$

$$= \frac{1}{c'} \; [\, 1 + s + s^2 + \ldots + (\lambda \, x^* + 1) \, s^N + \ldots \,] \, , \qquad (A.6)$$

Equating coefficients for different values of x

$$u_0 = \frac{1}{(r(r-1) + q_0)\, c'} \, , \quad r = \frac{3 - N}{2}$$

$$[\, r\,(1+r) + q_0]\, u_1 + q_1\, u_0 = \frac{1}{c'}$$

$$[\,(r+i)\,(r+i-1) + q_0]\, u_i - [\,(r+i-1)\,(i+r-2) + q_0 - q_1]\, u_{i-1} - \frac{\lambda}{c'}\, u_{i-2} = 0 \, ,$$

$$2 \leqslant i \leqslant N - 2 \, . \qquad (A.7)$$

The u's can all be determined from above relations. Now equating to zero the coefficient of logarithmic terms on the LHS of (A.6)

$$[\, \rho_1\,(\rho_1 - 1) + q_0\,]\, T_0 = 0$$

$$[\, \rho_1\,(\rho_1 + 1) + q_0\,]\, T_1 + q_1\, T_0 = 0$$

$$[(\rho_1 + i)(\rho_1 + i - 1)\, q_0]\, T_i - [(\rho_1 + i - 1)(\rho_1 + i - 2) + q_0 - q_1]\, T_{i-1} - \frac{\lambda}{c'}\, T_{i-2} = 0$$

$$i \geqslant 2 \, . \qquad (A.8)$$

All the T's are then known in terms of T_0 from (A.8). It is also obvious that the T's satisfy the same recurrence relations as $a_i(\rho_1)$ and $a_i(\rho_2)$. Equating coefficients of $s^{(N+1)/2}$ on both sides of (A.6).

$$((\rho_1 - 1) + \rho_1) T_0 + q_1 u_{N-2} + q_2 u_{N-3} + \dots q_{N-1} u_0 = \frac{1}{c'}$$

Since u's are already known from (A.7), T_0 can be determined from the above equation. Next, equating coefficients of $s^{(N+3)/2}$ etc., on both sides of (A.6) we set

$$(j + \rho_1 - 1) T_j + (j + \rho_1) T_j + \{(j+\rho)(j+\rho_1 - 1) + q_0\} M_j + q_1 M_{j-1} + \dots q_{j-1} M_1$$

$$+ q_{j+1} u_{N-2} + q_{j+2} u_{N-3} + \dots q_{j+N-1} u_0 = \frac{1}{c'} \qquad j \geq 1 .$$

The above relation determines all M's. Hence the particular solution, (A.5) is fully determined.

The required general solution of (A.4) is:

$$x(\alpha, s) = s^{\rho_1} A \sum_{i=0}^{\infty} a_1(\rho_1) s^i + B s^{\rho_2} [\log s \sum_{i=N}^{\infty} a_i(\rho_2) s^i$$

$$+ \sum_{i=0}^{\infty} a_i'(\rho_2) s^i] + s^{\rho_1} \log s \sum_{i=0}^{\infty} T_i s^i + s^r \sum_{i=0}^{N-2} u_i s^i + s^{\rho_1} \sum_{i=1}^{\infty} M_i s^i .$$

Now

$$x(\alpha, s) = L(R'(s,t)) = s^{\frac{1-N}{2}} L(R(s,t))$$

or

$$L(R(s,t)) = s^N A \sum_0^{\infty} a_i s^i + B\left[\log s \sum_N^{\infty} a_i(\rho_2) s^i + \sum_0^{\infty} a_i'(\rho_2) s^i\right]$$

$$+ s^N \log s \sum_0^{\infty} T_i s^i + s \sum_0^{N-2} u_i s^i + s^N \sum_1^{\infty} M_i s^i$$

APPENDIX B

SOLUTION OF THE DIFFERENCE EQUATION

We note that $a_i(\rho_1)$, $a_i(\rho_2)$ and T_i $(i \geq 2)$ satisfy a homogeneous linear difference equation with coefficients which are quadratic in i. The following is a technique for solving such equations. The equation to be solved is

$$[(\rho + i)(\rho+i-1) + \frac{1 - N^2}{4}]\, a_i - [(\rho+i-1)(\rho+i-2) + q_0 - q_1]\, a_{i-1} - \frac{\lambda}{c'}\, a_{i-2} = 0, \quad i \geq 2 \tag{B.1}$$

$$(\rho(\rho + 1) + q_0)\, a_1 = - a_0 q_1$$

Writing

$$(\rho + i)(\rho + i-1) + \frac{1 - N^2}{4} = \alpha_i$$

$$(\rho + i-1)(\rho + i-2) + q_0 - q_1 = \gamma_i$$

and

$$- \frac{\lambda}{c'} = \lambda' .$$

The simultaneous equations corresponding to (B.1) are

$$(\rho(\rho+1) + q_0)\, a_1 = -a_0 q_1$$

$$\gamma_2 a_1 + \alpha_2 a_2 = -\lambda' a_0$$

$$\lambda' + \gamma_3 a_2 + \alpha_3 a_3 = 0$$

. . . .

If the series solutions for a and T are truncated after n terms, the above equations can be written in the following matrix form

$$\begin{bmatrix} (\rho(1+\rho) + q_0 & & & & & & \\ \gamma_2 & \alpha_2 & & & & & \\ \lambda' & \gamma_3 & \alpha_3 & & & & \\ 0 & \lambda' & \gamma_4 & \alpha_4 & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ & & & \lambda' & \gamma_{n-1} & \alpha_{n-1} & \\ & & & & \lambda' & \gamma_n & \alpha_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} -a_0 q_1 \\ -\lambda' q_0 \\ 0 \\ 0 \\ \\ \\ 0 \end{bmatrix}$$

a_0 is assumed to be known and the a_i's $i = 1, 2, \ldots n$ can be evaluated by Cramer's rule. The matrix on the LHS is triangular; hence, its determinant is the product of its diagonal elements viz

$$[\rho(1+\rho) + q_0] \prod_{i=2}^{n} a_i$$

This is the denominator in the expression for all the a_i's and is seen to be independent of a. The numerator for a_i will be the same determinant as the denominators except that the i^{th} column is replaced by the column matrix on the R H S

$$\text{Numerator} = N_i = \begin{vmatrix} \rho(1+\rho) + q_0 & & & & -a_0 q_1 & 0 & & \\ \gamma_2 \ a_2 & & & & -\lambda' a_0 & 0 & & \\ \lambda' \ \gamma_3 \ a_3 & & & & 0 & . & & \\ & \lambda' \ \gamma_i & & & 0 & . & & \\ & & \lambda' & & 0 & a_{i+1} & & \\ & & & & 0 & \gamma_{i+2} \ a_{i+2} & & \\ & & & & 0 & & \lambda' \ \gamma_n & a_n \end{vmatrix}$$

$$\text{or} \quad N_i = a_{i+1} \ a_{i+2} \cdots\cdots a_n \begin{vmatrix} \rho(\rho+1) + q_0 & & & & -a_0 q_1 \\ \gamma_2 & a_2 & & & -\lambda' q_0 \\ \lambda' & \gamma_3 & a_3 & & \\ & & & & \\ & & \lambda' & \gamma_i & 0 \end{vmatrix}$$

$$= a_{i+1} \ a_{i+2} \cdots\cdots a_n \ \Delta_i$$

The determinant Δ_i is an $i \times i$ determinant. Multiplying the i^{th} column by $\frac{\rho(\rho+1) + q_0}{a_0 q_1}$ and adding to the first column we have

$$\Delta_i = \begin{vmatrix} 0 & & & & & -a_0 q_1 \\ \beta_2 & \alpha_2 & & & & -\lambda' q_0 \\ \lambda' & \gamma_3 & \alpha_3 & & & \\ & & & & & \\ & & & \lambda' & \gamma_i & 0 \end{vmatrix}$$

$$\beta_2 = \gamma_2 - \frac{(\rho(1+\rho) + q_0)\lambda'}{q_1}$$

Expanding on the 1^{st} row

$$\Delta_i = -a_0 q_1 \begin{vmatrix} \beta_2 & \alpha_2 & & \\ \lambda' & \gamma_3 & \alpha_3 & \\ & & & \\ & & \lambda' & \gamma_i \end{vmatrix}$$

$$\Delta_i = (-1)^{i+1} (-a_0 q_1) D_{i-1}$$

$$= (-1)^{i+2} a_0 q_1 D_{i-1}$$

$$\therefore \quad a_i = \frac{(-1)^{i+2} a_0 q_1 D_{i-1}}{(\rho(\rho+1) + q_0) \prod_{n=2}^{n} \alpha_n}$$

Where D_{i-1} is a continuant [4] of order $i - 1$ all of whose elements are zero except those in the main diagonal and in the two adjacent diagonal lines parallel to and on either side of the main diagonal.

Writing

$$D_{i-1} = \begin{vmatrix} \bar{a}_1 & \bar{b}_1 & & \\ \bar{c}_1 & \bar{a}_2 & \bar{b}_2 & \\ & \bar{c}_2 & \bar{a}_3 & \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \end{vmatrix}$$

$$D_0 = 1$$

where

$$\bar{c}_i = \lambda' \quad (i \geq 1)$$

$$\bar{a}_1 = B_2 , \; \bar{a}_j = \gamma_{j+1} ; \quad j \geq 2$$

$$\bar{b}_j = \alpha_{j+1} , \quad j \geq 1$$

One term of the continuant D_{i-1} is obviously $\bar{a}_1 \bar{a}_2 \ldots \bar{a}_i$, other terms can be formed from $\bar{a}_1 \bar{a}_2 \ldots \bar{a}_i$ by replacing any pair of consecutive $\bar{a}_i$'s by the product of the $\bar{b}$ and $\bar{c}$ having the same suffix as the first $\bar{a}$ of the pair with a negative sign. For example $\bar{a}_r \bar{a}_{r+1}$ may be replaced by $-\bar{b}_r \bar{c}_r$. This is obvious from the definition and from the fact that to get b_r and c_r into the position of $\bar{a}_r$ and $\bar{a}_{r+1}$ one interchange is necessary.

The numerators for the a_i's are calculated assuming we decide to truncate the a's after N terms. Denoting the numerator of a_i by $N(a_i)$, we have:

$$N(a_1) = - a_0 \, q_1 \, (\alpha_2 \, \alpha_3 \ldots \alpha_N)$$

$$N(a_2) = a_0 \, q_1 \, D_1 = a_0 \, q_1 \, (\bar{a}_1) \, (\alpha_3 \, \alpha_4 \ldots \alpha_N)$$

$$N(a_3) = - a_0 \, q_1 \, D_2 = - a_0 \, q_1 \, (\bar{a}_1 \bar{a}_2 - \bar{b}_1 \bar{c}_1) \, (\alpha_4 \, \alpha_5 \ldots \alpha_N)$$

and so on.

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