Published for SISSA by Springer

Received: *January 14, 2022*
Revised: *February 14, 2022*
Accepted: *February 22, 2022*
Published: *March 9, 2022*

D-instanton superpotential in string theory

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Abstract: We study the non-perturbative superpotential generated by D(-1)-branes in type IIB compactifications on orientifolds of Calabi-Yau threefold hypersurfaces. To compute the D-instanton superpotential, we study F-theory compactification on toric complete intersection elliptic Calabi-Yau fourfolds. We take the Sen-limit, but with finite g_s, in F-theory compactifications with the restriction that all D7-branes are carrying SO(8) gauge groups, which we call the global Sen-limit. In the global Sen-limit, the axio-dilaton is not varying in the compactification manifold. We compute the Picard-Fuchs equations of elliptic Calabi-Yau fourfolds in the global Sen-limit, and show that the Picard-Fuchs equations of the elliptic fourfolds split into that of the underlying Calabi-Yau threefolds and of the elliptic fiber. We then demonstrate that this splitting property of the Picard-Fuchs equation implies that the fourform period of the elliptic Calabi-Yau fourfolds in the global Sen-limit does not contain exponentially suppressed terms $\mathcal{O}(e^{-\pi/g_s})$. With this result, we finally show that in the global Sen-limit, the superpotential of the underlying type IIB compactification does not receive D(-1)-instanton contributions.

Keywords: F-Theory, Flux Compactifications, String and Brane Phenomenology, Superstring Vacua

ArXiv ePrint: 2201.04634

https://doi.org/10.1007/JHEP03(2022)054

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1 Introduction

One of the profound challenges in quantum gravity is to understand vacua of string theory with less supersymmetry. As an intermediate step towards understanding non-supersymmetric vacua of string theory, one can first study four-dimensional $\mathcal{N} = 1$ supersymmetric compactifications of string theory to attain more computational control.

In this context, one of the particularly attractive corners of string compactification is type IIB compactification on O3/O7 orientifolds of Calabi-Yau threefolds X_3. As was pioneered in [1–3], the vacuum structure of $\mathcal{N} = 1$ compactifications of string theory is characterized by the superpotential and Kahler potential of effective supergravity.[1] Therefore, precision computation of the Kahler potential and superpotential of effective supergravity is of great importance. Whereas the computation of the Kahler potential still remains challenging due to the lack of non-renormalization theorem, holomorphy and non-renormalization of the superpotential provide an opportunity to complete the characterization of the superpotential.

It is known that classical terms in the superpotential, including the Gukov-Vafa-Witten flux superpotential [7] and D7-brane superpotential [8, 9], are perturbatively exact [10]. Hence, any corrections to the classical superpotential must arise non-perturbatively via Euclidean D3-branes, D7-brane gaugino condensation, and Euclidean D(-1)-branes. While much is known about non-perturbative corrections to the superpotential from Euclidean D3-branes and D7-brane gaugino condensation [11–15], a systematic computation of Euclidean

[1] For a review on moduli stabilization, see [4–6].

D(-1)-brane superpotential, equivalently D-instanton superpotential, lacks in the literature partly due to its non-perturbative nature.

By definition, the D-instanton superpotential is exponentially suppressed at weak string coupling

$$W_{\text{ED(-1)}} = \mathcal{O}\left(e^{-\pi/g_s}\right). \tag{1.1}$$

Nevertheless, we argue that understanding the D-instanton superpotential is important. First and foremost, it is never warranted that realistic string vacua will lie at parametrically weak string coupling. In fact, string theory vacua are known to suffer from the famous Dine-Seiberg problem [16]. Therefore, to search through all possible corners in moduli space of string theory to find realistic string vacua, a non-perturbative understanding of superpotential is necessary. Even at a relatively weak string coupling, understanding the D-instanton superpotential can be very practical. For KKLT type moduli stabilization to work, one needs an exponentially small vacuum expectation value of the classical superpotential. Recently, a recipe to find flux vacua with exponentially small VEV was proposed with an explicit example [17].[2] The idea of [17] was to first find a perturbatively flat vacuum, then stabilize the perturbatively flat modulus by the non-perturbative corrections to the prepotential, which are determined by Gopakumar-Vafa invariants [27–29]. Importantly, in the perturbatively flat vacua, complex structure moduli and the axio-dilaton are mixed, and therefore the computation of the D-instanton superpotential can be useful for the precise computation of the VEV of the classical superpotential.

In this work, we initiate the study of the D-instanton superpotential in type IIB compactifications by focusing on the Sen-limit [30] in F-theory [31]. As a first step, we study the D-instanton superpotential in F-theory compactification such that all D7-brane stacks are carrying SO(8) gauge groups. Throughout this paper, we will call the Sen-limit with only SO(8) D7-brane stacks the global Sen-limit.[3] Quite surprisingly, we will find that in the global Sen-limit, D-instanton terms in superpotential do not arise. This in turn implies that the D-instanton superpotential at a generic D7-brane configuration takes the form

$$W_{\text{ED(-1)}} = \mathcal{A}e^{-\pi/g_s} + \mathcal{O}(e^{-2\pi/g_s})\,, \tag{1.2}$$

such that the one-loop pfaffian $\mathcal{A}$ vanishes if all D7-branes form SO(8) stacks.

The organization of this paper is as follows. In section 2, we collect ingredients of F-theory that are crucial in the study of the D-instanton superpotential. We explain how the D-instanton superpotential in type IIB string theory arises from the classical flux superpotential in F-theory. Then we will argue that in the global Sen-limit, bare D-instantons don't contribute to superpotential in flux compactification. In section 3, we study the Picard-Fuchs equations of elliptic Calabi-Yau fourfolds in the global Sen-limit and prove that the D-instanton superpotential does not arise in the global Sen-limit. To

[2]For further developments along this line, see for example [18–26].

[3]In [32], it was found that the global Sen-limit appears rarely in a set of elliptic Calabi-Yau fourfolds that are constructed as elliptic fibrations over weak Fano threefolds. However, we don't suffer from this scarcity of the global Sen-limit as in this paper we study elliptic fibrations over orientifolds which by design should admit the global Sen-limit.

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do so, we construct elliptic Calabi-Yau fourfolds as toric complete intersection Calabi-Yau manifolds and we show that the Picard-Fuchs equations of elliptic fourfolds are splitted into the Picard-Fuchs equations of the underlying Calabi-Yau threefold and the Picard-Fuchs equations of the elliptic fiber. We provide an example of this class of Calabi-Yau manifold in section 4. In section 5, we conclude.

2 F-theory compactification

F-theory compactification on an elliptic Calabi-Yau fourfold Y_4 provides a non-perturbative handle on string compactifications [31]. This non-perturbative control is achieved by the geometrization of D7-branes and the running axio-dilaton [31, 33]. In particular, provided that F-theory compactification on Y_4 admits a Sen-limit [30, 34], one can compute non-perturbative g_s corrections to weakly coupled type IIB string compactifications via F-theory.[4]

In this section, we will review the Sen-limit[5] and superpotential in F-theory. In particular, we will study the F-theory superpotential in the Sen-limit to argue that the bare D-instanton superpotential in weakly coupled type IIB string theory is encoded in the classical flux superpotential.

2.1 Elliptic Calabi-Yau and the Sen-limit

Let us define V_5 to be a $\mathbb{P}_{[2,3,1]}$ fibration $\pi_{\mathbb{P}_{[2,3,1]}} : V_5 \to \mathcal{B}_3$, such that $\pi^{-1}_{\mathbb{P}_{[2,3,1]}}(pt) = \mathbb{P}_{[2,3,1]}$ and the three homogeneous coordinates X, Y, and Z of $\mathbb{P}_{[2,3,1]}$ are sections of

$$X \in \Gamma\Big(\overline{K}^2_{B_3} \otimes \mathcal{L}_Z\Big), \quad Y \in \Gamma\Big(\overline{K}^3_{B_3} \otimes \mathcal{L}_Z\Big), \quad Z \in \Gamma(\mathcal{L}_Z)\,. \tag{2.1}$$

As we will explain later $\mathcal{B}_3$ can be regarded as an orientifold of a Calabi-Yau threefold in the underlying type IIB compactification in the Sen-limit. Then, the anti-canonical hypersurface in V_5 defines a Calabi-Yau fourfold Y_4 as an elliptic fibration over $\mathcal{B}_3$

$$Y^2 + a_1 XYZ + a_3 YZ^3 = X^3 + a_2 X^2 Z^2 + a_4 XZ^4 + a_6 Z^6\,. \tag{2.2}$$

Unless noted, we will always treat Y_4 as a maximal projective crepant partial (MPCP) desingularized variety [40]. The equation (2.2) is oftentimes referred to as the Tate-form [41]. One can bring the Tate form into the Weierstrass form by two steps of coordinate redefinitions. First, we redefine Y as

$$Y \mapsto Y - \frac{1}{2}a_1 XZ - \frac{1}{2}a_3 Z^3\,, \tag{2.3}$$

to arrive at

$$Y^2 = X^3 + B_2 X^2 Z^2 + 2B_4 XZ^4 + B_6 Z^6\,, \tag{2.4}$$

where

$$B_2 = a_2 - \frac{1}{4}a_1^2, \quad B_4 = \frac{1}{2}a_4 - \frac{1}{4}a_1 a_3, \quad B_6 = a_6 - \frac{1}{4}a_3^2\,. \tag{2.5}$$

[4]For a comprehensive review, see [35].

[5]For discussion on a stable version of the Sen-limit, see [36–39].

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Then, one can further redefine X as

$$X \mapsto X - \frac{1}{3}B_2 Z^2 \,, \tag{2.6}$$

to arrive at the Weierstrass form

$$Y^2 = X^3 + FXZ^4 + GZ^6 \,, \tag{2.7}$$

where

$$F = -\frac{1}{3}B_2^2 + 2B_4, \quad g = \frac{2}{27}B_2^3 - \frac{2}{3}B_2 B_4 + B_6 \,. \tag{2.8}$$

The elliptic fiber degenerates when the discriminant

$$\Delta := 4F^3 + 27G^2 \tag{2.9}$$

vanishes. We will see momentarily that $\Delta = 0$ encodes the location of 7-branes in the base manifold. Similarly, the axio-dilaton τ is conveniently encoded in the j-invariant of the elliptic fiber

$$j(\tau) = 1728 \frac{4F^3}{\Delta} \,. \tag{2.10}$$

At weak string coupling, the j-invariant enjoys an instanton expansion

$$j(\tau) = e^{-2\pi i \tau} + 744 + 196844 e^{2\pi i \tau} + \mathcal{O}(e^{4\pi i \tau}) \,. \tag{2.11}$$

At a generic point in the moduli space, it is not guaranteed that g_s is small. To reproduce a weakly coupled type IIB string theory description, it is therefore necessary to move towards a special subregion in the moduli space in which the string coupling is small. The observation that weak string coupling is identified with large complex structure of the elliptic fiber leads to the celebrated Sen-limit

$$B_{2i} \mapsto B_{2i} t^{i-1} \,, \tag{2.12}$$

where t is taken to be a small parameter. In the Sen-limit, one obtains

$$\Delta = 4B_2^2 \Big(B_2 B_6 - B_4^2\Big) t^2 + \mathcal{O}(t^3) \,, \tag{2.13}$$

and

$$j = \frac{64 B_2^4}{(B_4^2 - B_2 B_6) t^2} + \mathcal{O}(t^{-1}) \,. \tag{2.14}$$

In the Sen-limit, there are two solution branches to the discriminant locus $\Delta = 0$:

$$B_2 = 0 \,, \tag{2.15}$$

and

$$B_2 B_6 - B_4^2 = 0 \,. \tag{2.16}$$

To understand the nature of these solution branches, one can study how the axio-dilaton changes as one encircles a solution branch of the discriminant locus. We will in turn study

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D7-brane loci and O7-plane loci. First, let $z = 0$ be a simple root of $B_2 B_6 - B_4^2$. Then, the change in the axio-dilaton as one encircles $z = 0$ is

$$\delta\tau = -\frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{dj}{j} = 1\,. \tag{2.17}$$

(2.17) indicates that τ undergoes a monodromy transformation

$$\tau \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \tau\,, \tag{2.18}$$

which implies that $B_2 B_6 - B_4^2 = 0$ is a D7-brane locus in weakly coupled type IIB string theory. Similarly, as one goes around a simple root of $B_2 = 0$, one obtains a monodromy action

$$\tau \mapsto \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \cdot \tau\,, \tag{2.19}$$

which implies that $B_2 = 0$ is an O7-plane locus. It is very useful to note that $B_2 = \xi^2$ describes the underlying Calabi-Yau threefold, whose orientifold is $\mathcal{B}_3$.

Now we introduce one of the central objects of this paper, the holomorphic 4-form $\Omega^{4,0}_{Y_4}$ defined on Y_4. Much of the data on moduli encoded in $\Omega^{4,0}_{Y_4}$ is encoded in the period vector in an integral basis

$$\Pi_I := \int_{\gamma^I} \Omega^{4,0} = \int_{Y_4} \Omega^{4,0} \wedge \gamma_I\,, \tag{2.20}$$

where $\{\gamma^I\}$ is an integral basis of $H_4^{\rm hor}(Y_4, \mathbb{Z})$ and its dual basis is $\{\gamma_I\}$. The intersection pairing

$$\eta_{IJ} := \int_{Y_4} \gamma_I \wedge \gamma_J \tag{2.21}$$

is difficult to compute directly, but can be computed with the help of mirror symmetry.

We now study the asymptotic behaviour of the period vector $\vec{\Pi}$ near $t = 0$. Let us consider a loop γ in the moduli space that encircles $t = 0$ once. As one changes t along γ, the axio-dilaton undergoes a monodromy transformation

$$\tau \mapsto \tau + 2\,, \tag{2.22}$$

which corresponds to an element

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.23}$$

in $\mathrm{SL}(2,\mathbb{Z})$. Because

$$(M - I)^2 = 0\,, \tag{2.24}$$

due to Schmid's nilpotent orbit theorem [42],[6] we have

$$\Omega(t) = e^{\frac{1}{2\pi i}\log(t) N} \cdot \Omega_0 + \mathcal{O}(t)\,, \tag{2.25}$$

$$= \Omega_0 + \frac{1}{2\pi i} \log(t) N \cdot \Omega_0 + \mathcal{O}(t)\,, \tag{2.26}$$

[6]For recent applications of Schmid's nipotent orbit theorem, see for example [43–46].

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where we defined $N := \log(\mathcal{U}(M))$ and $\mathcal{U}(M)$ is a group action of the $\mathrm{SL}(2,\mathbb{Z})$ element M acting on the period vector. Likewise, the period vector in integral basis $\vec{\Pi}$ enjoys the expansion

$$\vec{\Pi}(t) = e^{\frac{1}{2\pi i}\log(t)N} \cdot \vec{\Pi}_0 + \mathcal{O}(t)\,, \tag{2.27}$$

$$= \vec{\Pi}_0 + \frac{1}{2\pi i}\log(t)N \cdot \vec{\Pi}_0 + \mathcal{O}(t)\,. \tag{2.28}$$

As we will explain in further detail in the next section, (2.27) implies that the period vector receives contributions from D-instantons in weakly coupled type IIB string theory.

2.2 Superpotential in F-theory

The curvature of a Calabi-Yau fourfold induces a D3-brane tadpole. To find a consistent F-theory compactification, one therefore needs to include fourform flux G_4 and D3-branes to satisfy the tadpole cancellation condition [47]

$$N_{D3} + \frac{1}{2}\int_{Y_4} G_4 \wedge G_4 = \frac{\chi(Y_4)}{24}\,. \tag{2.29}$$

The fourform flux should satisfy the quantization condition [48]

$$G_4 + \frac{1}{2}c_2(Y_4) \in H^{2,2}(Y_4) \cap H^4(Y_4,\mathbb{Z})\,. \tag{2.30}$$

For the fourform flux G_4 to respect Poincare invariance in type IIB string theory, either one and only one leg wraps a cycle in the elliptic fiber or the fourform flux is localized at a discriminant locus in the base manifold. The former corresponds to a bulk threeform flux and the latter corresponds to a two form gauge flux on a seven brane in the weakly coupled type IIB limit.

A non-trivial fourform flux G_4 induces the classical flux superpotential [7]

$$W^{\mathrm{F}}_{\mathrm{flux}} = \int_{Y_4} G_4 \wedge \Omega \tag{2.31}$$

which is perturbatively exact due to the non-renormalization theorem. Corrections to W_{flux} can arise non-perturbatively from Euclidean M-branes wrapping a non-trivial cycle in homology. Of these, contributions that survive the F-theory limit are Euclidean M5-branes wrapping a vertical divisor with two zero modes in Y_4 [11],

$$W^{\mathrm{F}}_{\mathrm{np}} = \sum_D \mathcal{A}_D(z) e^{-2\pi T_D}\,, \tag{2.32}$$

where to leading order in α' and g_s we have $T_D := \int_D J^3/3! + iC_6$.

Let us analyze the full F-theory superpotential

$$W^{\mathrm{F}} = W^{\mathrm{F}}_{\mathrm{flux}} + W^{\mathrm{F}}_{\mathrm{np}} \tag{2.33}$$

in the Sen-limit. A Euclidean M5-brane wrapping a vertical divisor maps to a Euclidean D3-brane wrapping a divisor in B_3. Similarly, a Euclidean M5-brane on a vertical divisor

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with a non-trivial threeform flux maps to a Euclidean D3-brane with a non-trivial two form flux, which can be understood as a bound state of an ED3-brane and ED(-1)-brane. Recalling that the F-theory complex structure moduli are mapped to complex structure moduli, D7-brane moduli, and the axio-dilaton, one expects to obtain

$$W_{\text{flux}}^{\text{F}} \mapsto W_{\text{flux}}^{\text{IIB}} + W_{\text{D7}}^{\text{IIB}} \tag{2.34}$$

in the Sen-limit. But this cannot be the complete picture as we will explain more.

We reproduced all but D-instanton terms in superpotential in type IIB compactification with O3/O7-planes. Clearly, there is no extended object in M-theory that can generate the D-instanton superpotential in the F-theory limit. Then, the only remaining possibility is that $W_{\text{flux}}^{\text{F}}$ generates the D-instanton superpotential in the Sen-limit. In fact, this is well expected from Schmid's nilpotent orbit theorem (2.27). The F-theory flux superpotential can be written as

$$W_{\text{flux}}^{\text{F}} = \vec{M} \cdot \eta \cdot \vec{\Pi}(z,t)\,, \tag{2.35}$$

where we represented the fourform flux $[G] = \vec{M} \in H^4(Y_4, \mathbb{Z}_4)$. According to Schmid's nilpotent orbit theorem, near $t = 0$ the asymptotic form of the F-theory flux superpotential is

$$W_{\text{flux}}^{\text{F}} = \vec{M} \cdot \eta \cdot \left(\vec{\Pi}_0(z) + \frac{1}{2\pi i}\log(t) N \cdot \vec{\Pi}_0(z) + \mathcal{O}(t)\right). \tag{2.36}$$

As we studied in (2.22), the axio-dilaton τ shifts

$$\tau \mapsto \tau + 2\,, \tag{2.37}$$

as we make a full loop around $t = 0$. This monodromy dictates that the flat-coordinate τ must take the following form

$$\tau = \frac{1}{2\pi i}\log(t^2) + f(z,t)\,, \tag{2.38}$$

where $f(z,t)$ is a holomorphic function of complex structure moduli in F-theory. As a result, we have

$$W_{\text{flux}}^{\text{F}} = \vec{M} \cdot \eta \cdot \left(\vec{\Pi}_0(z) + \frac{1}{2}\tau N \cdot \vec{\Pi}_0(z) + \mathcal{O}(e^{\pi i \tau})\right). \tag{2.39}$$

As a result, in the Sen limit we obtain the D-instanton superpotential from $W_{\text{flux}}^{\text{F}}$

$$W_{\text{flux}}^{\text{F}} \mapsto W_{\text{flux}}^{\text{IIB}} + W_{\text{D7}}^{\text{IIB}} + W_{\text{ED(-1)}}^{\text{IIB}}\,. \tag{2.40}$$

Of particular interest is the D-instanton superpotential in type IIB string theory in a case where the D7-brane tadpole is canceled locally meaning there are four D7-branes on top of every O7-plane. In the global Sen-limit, generically, one may expect that the superpotential will receive non-perturbative g_s corrections as expected from Schmid's nilpotent orbit theorem. But, it is very important to note that Schmid's nilpotent orbit theorem does not necessarily imply the existence of exponentially suppressed corrections to the period integral. In fact, we will now argue that in the global Sen-limit the D-instanton superpotential is not generated, which we will prove in the next section.

A single D-instanton has too many zero modes to generate a term in superpotential. It has 6 bosonic deformation moduli which describe a point in the Calabi-Yau threefold, and their fermionic superpartners. Therefore for a D-instanton to contribute to the superpotential, either the extra zero-modes other than the universal zero-modes should get a mass or the path-integral of the D-instanton over moduli space should nevertheless yield a non-vanishing contribution.

The D(-1)-brane's position moduli are stabilized at which the DBI action, $-2\pi i\tau$, is minimized.[7] In the global Sen-limit, the axio-dilaton does not vary in the compactification manifold. Hence, to stabilize the D(-1)-brane position moduli, we need more ingredients such as soft-supersymmetry breaking terms. In type IIB string theory, the bulk three form flux does not generate a mass for the deformation moduli of a D(-1)-brane. This can be understood from the T-dual picture. It was known that bulk threeform fluxes do not lift D3-brane position moduli [8], and the effective action of D3-brane position moduli are structurally equivalent to deformation moduli of a D(-1)-brane via T-duality. Hence, the bulk fluxes cannot generate a mass for the deformation moduli of a D(-1)-brane.

We can look at the absence of the D-instanton superpotential from a different angle.[8] If one or more D7-branes are displaced from an O7-plane stack, due to the perturbative one-loop running of the axio-dilaton, there appears to be a region around the O7-plane in which the string coupling becomes negative [34]. String theory naturally resolves this putative singularity, as the O7-plane in question non-perturbatively splits into B and C 7-branes which are separated by $ze^{2\pi i\tau}$, where $z = 0$ is the SO(8) configuration. Hence, one can understand the generation of D-instantons as a stringy mechanism to fix the perturbative singularity in the axio-dilaton which vanishes in the global Sen-limit, which points to the absence of the D-instanton superpotential in the global Sen-limit.

Geometrically, it is also natural to expect that $W^{\mathrm{F}}_{\mathrm{flux}}$ doesn't contain exponentially suppressed terms $\mathcal{O}(e^{\pi i\tau})$. Consider a blowdown of the elliptic fibration $\pi'_{\mathbb{E}} : Y'_4 \to \mathcal{B}_3$ such that the elliptic fiber is singular at the discriminant locus. If all the D7-brane configurations are in the SO(8) configuration, then the complex structure of the elliptic fiber does not change along the base manifold. On an SO(8) stack, the elliptic fiber develops a point-like singularity at $X = Y = 0$. But this point-like singularity does not change the complex structure of the elliptic fiber. This is why as far as the period integral is concerned, Y'_4 can be treated as a product space of $\mathcal{B}_3$ and $\mathbb{E}$

$$Y'_4 \simeq X_3/\mathbb{Z}_2 \times \mathbb{E}\,. \tag{2.41}$$

Note that this observation was previously made in [34, 49]. As a result, a non-trivial horizontal γ four-cycle in Y'_4 is topologically equivalent to $\alpha \times \beta$ where α is a three-cycle in $H^-_3(X_3, \mathbb{Z})$ and β is a non-trivial one-cycle in $\mathbb{E}$. Analogously, the holomorphic fourform is a product of the holomorphic three form of X_3 and the holomorphic one form of $\mathbb{E}$

$$\Omega^{4,0} = \Omega^{3,0} \times dz\,. \tag{2.42}$$

[7] We thank Jakob Moritz for insightful comments on this point.

[8] We thank Jakob Moritz and Timo Weigand for insightful comments on this point.

This is why a period integral in Y_4' is also a product of period integrals

$$\int_\gamma \Omega^{4,0} = \left(\int_\alpha \Omega^{3,0}\right) \times \left(\int_\beta dz\right). \tag{2.43}$$

Now note that threefold periods in the *integral basis* do not depend on the axio-dilaton,

$$\partial_\tau \int_\alpha \Omega^{3,0} = 0\,, \tag{2.44}$$

and the torus period in the *integral basis* does not receive any exponentially suppressed correction,

$$\partial_\tau^2 \int_\beta dz = 0\,. \tag{2.45}$$

Because the period integral does not change under blow-ups, and the D-instanton superpotential is encoded in the F-theory flux superpotential which is purely geometric, we now conclude that there is no bare D-instanton superpotential if the only D7-brane configuration is SO(8). But of course, an argument of this sort alone isn't fully satisfactory. We will prove this claim in section 3 by showing that the Picard-Fuchs equations of Y_4 split into the Picard-Fuchs equations of B_3 and of $\mathbb{E}$.

3 Picard-Fuchs equations

In this section, we compute the Picard-Fuchs equations of elliptic fibrations over orientifolds of toric hypersurface Calabi-Yau threefolds in the global Sen-limit.

Let ω be a period of Y_4 defined as an integral of the holomorphic fourform $\Omega^{4,0}$ over an integral homology cycle $\gamma \in H_4(Y_4, \mathbb{Z})$. The period ω is known to satisfy a set of differential equations, the Picard-Fuchs equations,

$$\mathcal{L}^{(a)}(y)\omega = 0\,, \tag{3.1}$$

where $\mathcal{L}^{(a)}(y)$ is a diffential operator which we call a Picard-Fuchs operator, and y is a short-handed notation for the complex structure moduli of Y_4.

A very important point to note is that the Picard-Fuchs equations are linear partial differential equations. Hence, if for a given Calabi-Yau Y_4 there are two sets of the Picard-Fuchs operators $\mathcal{L}_z^{(a)}(z,\tau)$ and $\mathcal{L}_\tau^{(a)}(z,\tau)$, and two sets of complex structure moduli $\{z,\tau\}$ such that

$$[\mathcal{L}_z^{(a)}(z,\tau), f(z)] = 0\,, \tag{3.2}$$

and

$$[\mathcal{L}_\tau^{(a)}(z,\tau), g(\tau)] = 0\,, \tag{3.3}$$

for arbitrary functions f and g that are not solutions of the PF equations, then by separation of variables the period integral $\omega(z,\tau)$ can be written as

$$\omega(z,\tau) = \omega_z(z) \times \omega_\tau(\tau)\,. \tag{3.4}$$

If the Picard-Fuchs equations satisfy the conditions (3.2) and (3.3), we say that the Picard-Fuchs equations are splitted. Given the solutions (3.4), following [50, 51], one can compute the period in an integral basis,

$$\Pi(z,\tau) = \Pi_z(z) \times \Pi_\tau(\tau) \tag{3.5}$$

by matching the asymptotic behavior of $\Pi(z,\tau)$ around LCS points to the asymptotic behaviors of volumes of even-homology cycles of the mirror Calabi-Yau.

As we mentioned in (2.45), near a large complex structure point, the period of an elliptic curve in *integral basis* in the flat coordinate τ is free from exponentially suppressed corrections.[9] This absence of non-perturbative corrections can be traced back to the large amount of supersymmetry in toroidal compactifications of string theory, whose Yukawa-coupling in Kahler moduli sector does not receive worldsheet instanton corrections.

As a result, if the Picard-Fuchs equations split into that of an orientifold and of an elliptic fiber, one can arrive at the result that the period vector of Y_4 does not contain non-perturbative in g_s terms. Although this idea is very clear, one encounters a technical challenge in separating complex structure moduli of Y_4 into that of the underlying orientifold and that of the elliptic fiber. The subtlety arises because the complex structure moduli of the Weierstrass model contain both the complex structure moduli of the orientifold and the elliptic fiber. To alleviate the subtlety, we construct an elliptic Calabi-Yau Y_4 as a toric complete intersection such that the splitting of complex structure moduli is manifest. With this trick in section 3.3 we will establish that in the global Sen limit, the Picard-Fuchs equations split into those of the underlying orientifold and those of the elliptic fiber.

3.1 Griffith-Dwork method

To obtain the period vector in the integral basis in the global Sen-limit, it is necessary to first compute the Picard-Fuchs equations. Because the sub-moduli space in which all D7-branes are in SO(8) configurations is far away from an LCS point, it is difficult to directly apply the Frobenius method around an LCS point to compute the period vector in integral basis [50–52]. This is why in section 3.3 we will extend the Griffith-Dwork method [53–56] to toric complete intersection Calabi-Yau manifolds to compute the Picard-Fuchs equations.[10] Before we compute the Picard-Fuchs equations, let us first explain the Griffith-Dwork method for toric hypersurface Calabi-Yau manifolds.

Let us start with a toric variety V of dimension $d+1$. By x_i we will denote homogeneous coordinates in the Cox ring of V [58].[11] Let the anti-canonical hypersurface X,[12] be defined by the defining equation

$$f(x,z) = f_0(x) + z_a h^a(x)\,, \tag{3.6}$$

where h^a is a monomial, and the index a runs from 1 to the number of monomial deformations. For each cohomology group $H^{d-i,i}(X)$ we choose a basis

$$\mathrm{Span}\Big\{x^{\mu^i_1}, \ldots, x^{\mu^i_{h^{d-i,i}}}\Big\} \equiv H^{d-i,i}(X). \tag{3.7}$$

[9] Similarly, periods of K3 manifolds also do not receive exponentially suppressed corrections.
[10] For a diagrammatic approach to the Griffith-Dwork method, see [57].
[11] For a review on toric geometry, see [59].
[12] For the mirror construction of Calabi-Yau hypersurfaces in toric varieties, see [40].

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A convenient choice for the basis of $H^{d-1,1}(X)$ is $\mathrm{Span}\{h^1, \ldots, h^{d-1,1}\}$. Let there be an integral d-cycle $\gamma \in H_d(X, \mathbb{Z})$. Then, we define a period vector

$$\omega^i_j = \int_\gamma \frac{(-1)^i i! x^{\mu^i_j}}{f(x,z)^{i+1}}. \tag{3.8}$$

The Picard-Fuchs equations are given by a set of equations

$$\partial_{z_a} \omega^i_j = \omega^l_k B^{(a)ik}_{jl}. \tag{3.9}$$

The computation of the Picard-Fuchs equation therefore boils down to the determination of $B^{(a)ik}_{jl}$. How does one determine the matrix B? The computation proceeds by reduction of the pole order. Let us first compute

$$\partial_{z_a} \omega^i_j = \int_\gamma (-1)^{i+1}(i+1)! \frac{x^{\mu^i_j}}{f(x,z)^{i+2}} h^a. \tag{3.10}$$

Then we find a relation

$$(-1)^{i+1}(i+1)! \frac{x^{\mu^i_j} h^a}{f(x,z)^{i+2}} = \sum_{l,k} (-1)^l l! \frac{x^{\mu^l_k}}{f(x,z)^{l+1}} B^{(a)ik}_{jl} + d(g), \tag{3.11}$$

where g is a rational function in x. Because $\int_\gamma dg = 0$, we can relate $\partial_{z_a} \omega^i_j$ to $\omega^l_k B^{(a)ik}_{jl}$.

3.2 PF equations of an elliptic curve

We define an elliptic curve $\mathbb{E}$ in $\mathbb{P}_{[2,3,1]}$ by a defining equation

$$X^3 - Y^2 - \frac{1}{3} a^2 X Z^4 + \left(\frac{2}{27} - t\right) a^3 Z^6 = 0, \tag{3.12}$$

where we will treat a as a constant and t as a variable. The parameter a models a divisor in the base manifold that hosts an SO(8) D7-brane stack. To check the choice of the parameter a, we compute

$$\Delta = t(-4 + 27t) a^6, \tag{3.13}$$

and

$$j(\tau) = -\frac{256}{t(-4+27t)}. \tag{3.14}$$

As a result, $a = 0$ correctly models an SO(8) D7-brane stack.

We choose bases

$$\mathrm{Span}\{1\} \equiv H^{1,0}(\mathbb{E}), \quad \mathrm{Span}\{a^3 Z^6\} \equiv H^{0,1}(\mathbb{E}), \tag{3.15}$$

and

$$\begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix} = \begin{pmatrix} \int_\gamma 1/f(x,t) \\ -\int_\gamma a^3 Z^6/f(x,t)^2 \end{pmatrix}. \tag{3.16}$$

A comment is in order. Although strictly speaking a is a parameter in the elliptic curve, we will assign a spurious GLSM charge 2 to the parameter a under the spurious GLSM gauge group $\mathrm{U}(1)^s$. To homogeneous coordinates $\{X, Y, Z\}$ we assign $\mathrm{U}(1)^s$ charges $\{2, 3, 0\}$, respectively. Importantly, to the Weierstrass form (3.12), we assign $\mathrm{U}(1)^s$ charge 6. This charge assignment will be explained in the next section in further detail. In fact, this charge assignment is chosen such that upon promoting a to a section in $\Gamma(-2K_{\mathcal{B}_3})$, where $\mathcal{B}_3$ is an orientifold of a Calabi-Yau threefold, the Weierstrass model (3.12) describes an elliptic fibration over $\mathcal{B}_3$ in the global Sen limit.

We first compute

$$\partial_t \omega^0 = \int_\gamma \frac{a^3 Z^6}{f(x,t)^2} \tag{3.17}$$

$$= -\omega^1 . \tag{3.18}$$

Similarly,

$$\partial_t \omega^1 = -2 \int_\gamma \frac{a^6 Z^{12}}{f(x,t)^3} . \tag{3.19}$$

To evaluate (3.19), we need to do a bit of work. We observe

$$\alpha_1 := \int_\gamma \partial_X \left(\frac{a^4 Z^8}{f(x,t)^2} \right) = - \int_\gamma \frac{2 Z^8 a^4 \left(3X^2 - \frac{1}{3} a^2 Z^4\right)}{f(x,t)^3} , \tag{3.20}$$

$$\alpha_2 := \int_\gamma \partial_X \left(\frac{a^2 X^2 Z^4}{f(x,t)^2} \right) = \int_\gamma a^2 \left[\frac{2XZ^4}{f(x,t)^2} - \frac{2X^2 Z^4}{f(x,t)^3} \left(3X^2 - \frac{1}{3} a^2 Z^4\right) \right] , \tag{3.21}$$

$$\alpha_3 := \int_\gamma \partial_Z \left(\frac{a X^2 Z^3}{f(x,t)^2} \right) = \int_\gamma a \left[\frac{3X^2 Z^2}{f(x,t)^2} - \frac{2X^2 Z^2}{f(x,t)^3} \left(-\frac{4}{3} a^2 X Z^4 + 6 \left(\frac{2}{27} - t\right) a^3 Z^6 \right) \right] , \tag{3.22}$$

$$\alpha_4 := \int_\gamma \partial_Z \left(\frac{X^3 Z}{f(x,t)^2} \right) = \int_\gamma \left[\frac{X^3}{f(x,t)^2} - \frac{2X^3}{f(x,t)^3} \left(-\frac{4}{3} a^2 X Z^4 + 6 \left(\frac{2}{27} - t\right) a^3 Z^6 \right) \right] . \tag{3.23}$$

Then, we obtain a relation

$$-3\alpha_1 + \frac{4}{t(-4+27t)} \alpha_2 - \frac{3(27t-2)}{2t(27t-4)} \alpha_3 + \frac{9}{t(27t-4)} \alpha_4 = \int_\gamma \frac{-2 a^6 Z^{12}}{f(x,t)^3} + \alpha_5 , \tag{3.24}$$

where we define

$$\alpha_5 := \int_\gamma \left[\frac{(18X^3 - 9a(27t-2) X^2 Z^2 + 16 a^2 X Z^4)}{2t(27t-4) f(x,t)^2} \right] . \tag{3.25}$$

Hence, we obtain

$$\partial_t \omega^1 = - \int_\gamma \left[\frac{(18X^3 - 9a(27t-2) X^2 Z^2 + 16 a^2 X Z^4)}{2t(27t-4) f(x,t)^2} \right] . \tag{3.26}$$

To bring (3.26) to the final form, we observe

$$\beta_1 := - \int_\gamma \partial_X \left(\frac{a Z^2}{f(x,t)} \right) = \int_\gamma \frac{a Z^2}{f(x,t)^2} \left(3X^2 - \frac{1}{3} a^2 Z^4\right) , \tag{3.27}$$

$$\beta_2 := - \int_\gamma \partial_X \left(\frac{X}{f(x,t)} \right) = \int_\gamma \left[-\frac{1}{f(x,t)} + \frac{1}{f(x,t)^2} \left(3X^3 - \frac{1}{3} a^2 X Z^4\right) \right] , \tag{3.28}$$

$$\beta_3 := - \int_\gamma \partial_Z \left(\frac{Z}{f(x,t)} \right) = \int_\gamma \left[-\frac{1}{f(x,t)} + \frac{1}{f(x,t)^2} \left(-\frac{4}{3} a^2 X Z^4 + 6 \left(\frac{2}{27} - t\right) a^3 Z^6 \right) \right] , \tag{3.29}$$

and

$$\frac{3(27t-2)}{2t(27t-4)}\beta_1-\frac{3}{t(27t-4)}\beta_2+\frac{27}{4t(27t-4)}\beta_3=-\int_\gamma\frac{15}{4t(27t-4)}\frac{1}{f(x,t)}$$
$$-\int_\gamma\frac{2a^3(27t-2)}{t(27t-4)}\frac{Z^6}{f(x,t)^2}+\partial_t\omega^1\,. \tag{3.30}$$

As a result, we obtain

$$\partial_t\omega^1=\frac{15}{4t(27t-4)}\omega^0-\frac{2(27t-2)}{t(27t-4)}\omega^1\,. \tag{3.31}$$

This completes the computation of the Picard-Fuchs equation

$$\frac{d}{dt}\begin{pmatrix}\omega^0\\ \omega^1\end{pmatrix}=\begin{pmatrix}0 & -1\\ \frac{15}{4t(27t-4)} & -\frac{2(27t-2)}{t(27t-4)}\end{pmatrix}\begin{pmatrix}\omega^0\\ \omega^1\end{pmatrix}. \tag{3.32}$$

We find two linearly independent solutions for $\omega_t(t)$,

$$\omega_t=c_1\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)+c_2\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right). \tag{3.33}$$

We perform series expansion around $t=0$ to obtain

$$\omega_t^{(0)}:={}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)=1+\frac{15}{16}t+\frac{3465}{1024}t^2+\frac{255255}{16384}t^3+\ldots\,, \tag{3.34}$$

$$\omega_t^{(1)}:={}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right)=-\frac{1}{2\pi}\omega_t^{(0)}\left(\log(2^{-6}t)+\frac{39}{8}t+\frac{14733}{1024}t^2+\ldots\right). \tag{3.35}$$

ω_t is not the period vector in *integral* basis.

First, it is important to note that an identification

$$\tau:=i\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)} \tag{3.36}$$

provides the inverse series of the j-invariant [60, 61]

$$j(\tau)=-\frac{256}{t(-4+27t)}=e^{-2\pi i\tau}+744+196884e^{2\pi i\tau}+\ldots\,. \tag{3.37}$$

This identification implies that $t=\mathcal{O}(e^{2\pi i\tau})$.

To determine an integral basis, one can in general use mirror symmetry [50, 51]. We first define a symplectic basis $\{A,B\}$ of $H_1(\mathbb{E},\mathbb{Z})$, whose symplectic pairing is given by

$$A\cap A=0,\quad A\cap B=1,\quad B\cap B=0\,. \tag{3.38}$$

The mirror manifold of an elliptic curve with complex structure τ is an elliptic curve with complexified volume τ. Henceforth, guided by mirror symmetry, we identify A and B cycle periods with volume of a point and complexified volume of the mirror elliptic curve, respectively. To summarize, the asymptotic form of integral periods we want to obtain are

$$\int_A\Omega=1+\mathcal{O}(e^{2\pi i\tau})\,, \tag{3.39}$$

and

$$\int_B \Omega = \tau + \mathcal{O}(e^{2\pi i\tau})\,. \tag{3.40}$$

Combining (3.39), (3.40), (3.34), and (3.35), we find that a natural choice for periods in integral basis is

$$\omega_A := {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right), \tag{3.41}$$

and

$$\omega_B := i\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right) = \tau\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right). \tag{3.42}$$

Because ${}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)$ contains exponentially suppressed terms $\mathcal{O}(e^{2\pi i\tau})$, one can be tempted to conclude that period vectors in fact receive exponentially suppressed corrections. This conclusion is too quick because a period vector alone is not a good physical observable but the combination

$$e^{\mathcal{K}/2}\left|\int\Omega\right| \tag{3.43}$$

is, where we define $\mathcal{K}$ as

$$\mathcal{K} := -\log\left(\int_{\mathbb{E}}\Omega\wedge\overline{\Omega}\right) = -\log\left(\int_A\Omega\int_B\overline{\Omega} - \int_B\Omega\int_A\overline{\Omega}\right). \tag{3.44}$$

This implies that there is a freedom to rescale the holomorphic one-form while keeping the physics invariant

$$\Omega \mapsto e^{\mathcal{F}}\Omega\,, \tag{3.45}$$

and

$$\overline{\Omega} \mapsto e^{\overline{\mathcal{F}}}\overline{\Omega}\,. \tag{3.46}$$

In fact, this transformation is precisely Kahler transformation. Because ω_A and ω_B contain the same factor ${}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)$, to make the absence of exponentially suppressed corrections more manifest we rescale the holomorphic one-form

$$\Omega \mapsto \frac{1}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)}\Omega\,. \tag{3.47}$$

With the rescaled holomorphic one-form, the period vector in integral basis

$$\Pi_A := \frac{1}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)}\omega_A = 1\,, \tag{3.48}$$

and

$$\Pi_B := \frac{1}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)}\omega_B = \tau\,. \tag{3.49}$$

The integral basis of the period vector, in the flat coordinate, is therefore

$$\int_A\Omega = 1\,, \tag{3.50}$$

$$\int_B\Omega = \tau\,. \tag{3.51}$$

A very important point should be mentioned. In the Picard-Fuchs equation, the parameter a is completely decoupled. This means that the Picard-Fuchs equation for the elliptic fiber in general decouples from Picard-Fuchs equations for the base manifold if the only D7-brane configurations are SO(8) stacks. This does not yet prove that the Picard-Fuchs equations for the base manifold do not receive any corrections from the axio-dilaton. For this, we will need more analysis.

3.3 PF equations of ellitipc fibration over orientifolds

In this section, we consider elliptic fibrations over orientifolds of a Calabi-Yau threefold.[13] To prove that the fourfold period vector does not receive the D-instanton corrections in the global Sen limit, in this section, we will show that the Picard-Fuchs equations split into the Picard-Fuchs equations of the underlying Calabi-Yau threefolds and those of the elliptic fiber.

We will only study toric hypersurface Calabi-Yau manifolds [40] explicitly, but the conclusion can be easily generalized to complete intersection Calabi-Yau manifolds as well [65, 66]. Let Δ be a reflexive polytope of dimension four in the $M \equiv \mathbb{Z}^4$ lattice. We define $N := \mathrm{Hom}(M, \mathbb{Z})$ via the polar duality, and we define Δ° correspondingly. Given an MPCP desingularization $\hat{\mathbb{P}}_\Delta$ of the toric variety $\mathbb{P}_\Delta$, which is obtained by a fine, regular, start triangulation $\mathcal{T}$ of Δ°, to each point $p \in \Delta^\circ$ we associate a homogeneous coordinate x_p and a divisor $D_p \subset \hat{\mathbb{P}}_\Delta$. The anti-canonical class of $\hat{\mathbb{P}}_\Delta$ is given by

$$-K_{\hat{\mathbb{P}}_\Delta} = \sum_{p \in \partial\Delta^\circ} [D_p] \,. \tag{3.52}$$

Hence, Δ is the Newton polytope for the anti-canonical class. We then define a three-dimensional Calabi-Yau manifold X_3 as an anti-canonical hypersurface in $\hat{\mathbb{P}}_\Delta$. We will oftentimes denote $\hat{\mathbb{P}}_\Delta$ by V_4.

Let the anti-canonical hypersurface X_3 be defined by the defining equation

$$f(x,z) = f_0(x) + z_a h^a(x) \,, \tag{3.53}$$

where h^a is a monomial, and the index a runs from 1 to the number of monomial deformations. For each cohomology group $H^{3-i,i}(X)$ we choose a basis

$$\mathrm{Span}\Big\{x^{\mu^i_1}, \dots, x^{\mu^i_{h^{3-i,i}}}\Big\} \equiv H^{3-i,i}(X). \tag{3.54}$$

A convenient choice for the basis of $H^{2,1}(X)$ is $\mathrm{Span}\{h^1, \dots, h^{2,1}\}$. Let there be an integral 3-cycle $\gamma \in H_3(X, \mathbb{Z})$. Then, we define a period vector

$$\omega^i_j = \int_\gamma \frac{(-1)^i i! x^{\mu^i_j}}{f(x,z)^{i+1}}. \tag{3.55}$$

Let there be a relation

$$(-1)^{i+1}(i+1)!\frac{x^{\mu^i_j} h^a}{f(x,z)^{i+2}} = \sum_{l,k} (-1)^l l! \frac{x^{\mu^l_k}}{f(x,z)^{l+1}} B^{(a)ik}_{jl} + C^{(a)i}_{jkl}\partial_{x_k}\left(\frac{x_k x^{\nu^l}}{f(x,z)^{\rho^l+1}}\right), \tag{3.56}$$

[13] For earlier work on orientifolds of toric Calabi-Yau manifolds, see [62–64].

which implies that the PF equations for the Calabi-Yau threefold X are

$$\partial_{z_a}\omega^i_j = \omega^l_k B^{(a)ik}_{jl}\,. \tag{3.57}$$

Now we proceed to find an orientifold $\mathcal{B}_3 \simeq X_3/\mathcal{I}$ of X_3. We first take a representation of the orientifold involution $\mathcal{I}_p : x_p \mapsto -x_p$. Different representations of the same orientifold involution are related via toric actions. It is straigtfoward to show $\mathcal{I}_p \equiv \mathcal{I}_{p'}$ iff $p + p' \equiv 0 \mod 2$. We define the equivalence class of the orientifold action to be $\mathcal{I}$ and I_p the set of points p' that satisfy $p + p' \equiv 0 \mod 2$. For simplicity, we assume that every monomial in $f(x, z)$ is even under the orientifold action $\mathcal{I}_p$. Note that this assumption guarantees that there is a choice of relations (3.56) that are even under the orientifold action $\mathcal{I}_p$, such that the PF equations (3.57) are covariant under the orientifolding.

To embed $\mathcal{B}_3$ into a toric variety, we define $\varphi : V_4 \to \tilde{V}_4$ by a two to one map with fixed loci $\varphi(x^2_{p_i}) = \tilde{y}_{p_i}$ for $p_i \in I_p$ and $\varphi(x_{p_i}) = y_{p_i}$ for $p_i \notin I_p$. The fixed loci of φ are the orbifold singularities induced by the orientifold involution $\mathcal{I}$. Phrased differently, the fixed loci of φ are the O7-plane loci. This two to one map φ is equivalent to a refinement of the lattice N via $\varphi : N \to N'$, such that $\varphi(p)$ for $p \in I_p$ is divisible by 2 in N'.

The anti-canonical class of $\tilde{V}_4$ is therefore

$$-K_{\tilde{V}_4} = -K_{V_4} + \sum_{v\in I_p} [D_v]\,. \tag{3.58}$$

Because the vanishing locus of $f(x, z)$ is a divisor of the class $-K_{V_4}$, the anti-canonical class of the orientifold $\mathcal{B}_3$ is

$$-K_{B_3} = \sum_{v\in I_p} [D_v]\,. \tag{3.59}$$

If there is a point $v \in I_p$ such that $-K_{V_4} = 2[D_v]$, then B_3 is a toric variety. As a result of (3.59), $\mathcal{B}_3$ is not a Calabi-Yau manifold. The defining equations for $\mathcal{B}_3$ is

$$\tilde{f}(y, \tilde{y}, z) = \varphi(f(x, z))\,. \tag{3.60}$$

We now consider $\mathbb{P}_{[2,3,1]}$ fibration over $\tilde{V}_4$ such that $X \in \Gamma\Big(\mathcal{L}^2_Z \otimes \overline{K}^2_{\mathcal{B}_3}\Big)$, $Y \in \Gamma\Big(\mathcal{L}^3_Z \otimes \overline{K}^3_{\mathcal{B}_3}\Big)$, $Z \in \Gamma(\mathcal{L}_Z)$. The elliptic fibration over $\mathcal{B}_3$, which we call Y_4, is defined by a defining equation

$$g := -Y^2 + X^3 + fXZ^4 + gZ^6 = 0\,, \tag{3.61}$$

where $f \in \Gamma(\overline{K}^4_{\mathcal{B}_3})$ and $g \in \Gamma(\overline{K}^6_{\mathcal{B}_3})$. The elliptic curve at the SO(8) configuration is written as

$$g = -Y^2 + X^3 - \frac{1}{3}\xi^2 XZ^4 + \left(\frac{2}{27} - t\right)\xi^3 Z^6\,, \tag{3.62}$$

where $\xi := \prod_{v\in I_p} \tilde{y}_v$. Note that t parametrizes the axio-dilaton. We will give an example in section 4 to explain how this construction can be implemented.

To study the period integral and the associated PF equations for Y_4 at the SO(8) configuration, we first choose bases

$$\mathrm{Span}\{1\} \equiv H^{4,0}(Y_4), \quad \mathrm{Span}\left\{\varphi\left(x^{\mu^1_1}\right), \dots, \varphi\left(x^{\mu^1_{h^{2,1}_X}}\right), \xi^3 Z^6\right\} \subset H^{3,1}(Y_4)\,, \tag{3.63}$$

$$\mathrm{Span}\left\{\varphi\left(x^{\mu^2_1}\right), \dots, \varphi\left(x^{\mu^2_{h^{1,2}_X}}\right), \xi^3 Z^6 \varphi\left(x^{\mu^1_1}\right), \dots, \xi^3 Z^6 \varphi\left(x^{\mu^1_{h^{2,1}_X}}\right)\right\} \subset H^{2,2}(Y_4)\,, \tag{3.64}$$

$$\mathrm{Span}\left\{\varphi\left(x^{\mu^3_1}\right), \xi^3 Z^6 \varphi\left(x^{\mu^2_1}\right), \dots, \xi^3 Z^6 \varphi\left(x^{\mu^2_{h^{1,2}_X}}\right)\right\} \subset H^{1,3}(Y_4)\,, \tag{3.65}$$

$$\mathrm{Span}\left\{\xi^3 Z^6 \varphi\left(x^{\mu^3_1}\right)\right\} \equiv H^{0,4}(Y_4)\,, \tag{3.66}$$

and

$$\omega^{(0,0)} = \int_\gamma \frac{1}{\tilde{f}(y, \tilde{y}, z) g}\,, \tag{3.67}$$

$$\omega_i^{(1,0)} = -\int_\gamma \frac{\varphi(x^{\mu^1_i})}{\tilde{f}^2 g}\,, \tag{3.68}$$

for $1 \le i \le h^{2,1}_X$, and

$$\omega^{(0,1)} = -\int_\gamma \frac{\xi^3 Z^6}{\tilde{f} g^2}\,, \tag{3.69}$$

$$\omega_i^{(2,0)} = \int_\gamma \frac{2\varphi(x^{\mu^2_i})}{\tilde{f}^3 g}\,, \tag{3.70}$$

for $1 \le i \le h^{2,1}_X$, and

$$\omega_i^{(1,1)} = \int_\gamma \frac{\xi^3 Z^6 \varphi(x^{\mu^1_i})}{\tilde{f}^2 g^2}\,, \tag{3.71}$$

for $1 \le i \le h^{2,1}_X$,

$$\omega^{(3,0)} = -\int_\gamma \frac{6\varphi(x^{\mu^3_1})}{\tilde{f}^4 g}\,, \tag{3.72}$$

$$\omega_i^{(2,1)} = -\int_\gamma \frac{2\xi^3 Z^6 \varphi(x^{\mu^2_i})}{\tilde{f}^3 g^2}\,, \tag{3.73}$$

for $1 \le i \le h^{1,2}_X$,

$$\omega^{(3,1)} = \int_\gamma \frac{6\xi^3 Z^6 \varphi(x^{\mu^3_1})}{\tilde{f}^4 g^2}\,. \tag{3.74}$$

The goal is to prove the following two equations

$$\partial_{z_a} \omega_j^{(i,n)} = \omega_k^{(l,n)} B^{(a)ik}_{jl}\,, \tag{3.75}$$

$$\frac{d}{dt} \begin{pmatrix} \omega^{(i,0)} \\ \omega^{(i,1)} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{15}{4t(27t-4)} & -\frac{2(27t-2)}{t(27t-4)} \end{pmatrix} \begin{pmatrix} \omega^{(i,0)} \\ \omega^{(i,1)} \end{pmatrix}. \tag{3.76}$$

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To prove the equation (3.75), we can simply show the following identities

$$\varphi\left(\frac{(\xi^3 Z^6)^{\lambda^i-1}}{g^{\lambda^i}}\partial_{x_k}\left(\frac{x_k x^{\nu^i}}{f(x,z)^{\rho^i}}\right)\right) \propto \partial_{\tilde{y}_k}\left(\frac{\tilde{y}_k \varphi(x^{\nu^i})(\xi^3 Z^6)^{\lambda^i-1}}{\tilde{f}^{\rho^i} g^{\lambda^i}}\right) + d(H), \tag{3.77}$$

for x_k such that $\varphi(x_k^2) = \tilde{y}_k$, and

$$\varphi\left(\frac{(\xi^3 Z^6)^{\lambda^i-1}}{g^{\lambda^i}}\partial_{x_k}\left(\frac{x_k x^{\nu^i}}{f(x,z)^{\rho^i}}\right)\right) \propto \partial_{y_k}\left(\frac{y_k \varphi(x^{\nu^i})(\xi^3 Z^6)^{\lambda^i-1}}{\tilde{f}^{\rho^i} g^{\lambda^i}}\right) + d(H'), \tag{3.78}$$

for x_k such that $\varphi(x_k) = y_k$. By definition, (3.78) is true. So, we only need to show (3.77). We first compute the left hand side of (3.77)

$$\varphi\left(\frac{(\xi^3 Z^6)^{\lambda^i-1}}{g^{\lambda^i}}\partial_{x_k}\left(\frac{x_k x^{\nu^i}}{f(x,z)^{\rho^i}}\right)\right) = \varphi\left(\frac{(\xi^3 Z^6)^{\lambda^i-1}}{g^{\lambda^i}}\frac{(\nu_k^i+1)x^{\nu^i}f(x,z) - \rho^i x^{\nu^i} x_k \partial_{x_k} f(x,z)}{f(x,z)^{\rho^i+1}}\right) \tag{3.79}$$

$$= \frac{(\nu_k^i+1)\varphi(x^{\nu^i})\tilde{f} - 2\rho^i \varphi(x^{\nu^i})\tilde{y}_k \partial_{\tilde{y}_k}\tilde{f}}{\tilde{f}^{\rho^i+1} g^{\lambda^i}}\left(\xi^3 Z^6\right)^{\lambda^i-1}. \tag{3.80}$$

The right hand side of (3.77) is then

$$\begin{aligned}\partial_{\tilde{y}_k}\left(\frac{\tilde{y}_k \varphi(x^{\nu^i})(\xi^3 Z^6)^{\lambda^i-1}}{\tilde{f}^{\rho^i} g^{\lambda^i}}\right) &= \frac{(\nu_k^i/2 + 1 + 3(\lambda^i - 1))\varphi(x^{\nu^i})\tilde{f} - \rho^i \varphi(x^{\nu^i})\tilde{y}_k \partial_{\tilde{y}_k}\tilde{f}}{\tilde{f}^{\rho^i+1} g^{\lambda^i}}\left(\xi^3 Z^6\right)^{\lambda^i-1} \\ &\quad - \frac{\lambda^i \tilde{y}_k \partial_{\tilde{y}_k} g}{\tilde{f}^{\rho^i} g^{\lambda^i+1}}\left(\xi^3 Z^6\right)^{\lambda^i-1}\varphi(x^{\nu^i}).\end{aligned} \tag{3.81}$$

By using an identity

$$\tilde{y}_k \partial_{\tilde{y}_k} g = \frac{1}{2} Z \partial_Z g\,, \tag{3.82}$$

we find

$$-\frac{\lambda^i \tilde{y}_k \partial_{\tilde{y}_k} g}{\tilde{f}^{\rho^i} g^{\lambda^i+1}}\left(\xi^3 Z^6\right)^{\lambda^i-1}\varphi(x^{\nu^i}) = -\frac{\lambda^i Z \partial_Z g}{2\tilde{f}^{\rho^i} g^{\lambda^i+1}}\left(\xi^3 Z^6\right)^{\lambda^i-1}\varphi(x^{\nu^i})\,, \tag{3.83}$$

$$= \partial_Z\left(\frac{Z(\xi^3 Z^6)^{\lambda^i-1}\varphi(x^{\nu^i})}{2\tilde{f}^{\rho^i} g^{\lambda^i}}\right) - \frac{6\lambda^i - 5}{2}\frac{(\xi^3 Z^6)^{\lambda^i-1}\varphi(x^{\nu^i})}{\tilde{f}^{\rho^i} g^{\lambda^i}}. \tag{3.84}$$

As a result, we finally obtain

$$\begin{aligned}\partial_{\tilde{y}_k}\left(\frac{\tilde{y}_k \varphi(x^{\nu^i})(\xi^3 Z^6)^{\lambda^i-1}}{\tilde{f}^{\rho^i} g^{\lambda^i}}\right) &= \frac{(\nu_k^i + 1)\varphi(x^{\nu^i})\tilde{f} - 2\rho^i \varphi(x^{\nu^i})\tilde{y}_k \partial_{\tilde{y}_k}\tilde{f}}{2\tilde{f}^{\rho^i+1} g^{\lambda^i}}\left(\xi^3 Z^6\right)^{\lambda^i-1} \\ &\quad + \partial_Z\left(\frac{Z(\xi^3 Z^6)^{\lambda^i-1}\varphi(x^{\nu^i})}{2\tilde{f}^{\rho^i} g^{\lambda^i}}\right),\end{aligned} \tag{3.85}$$

which confirms (3.77). As one can prove (3.76) by extending results in section 3.2, we complete the proof that the Picard-Fuchs equation split into (3.75) and (3.76).

As a result, we establish that in the global Sen-limit the fourfom period does not receive $e^{-\pi/g_s}$ corrections

$$\partial_\tau^2 \int_\gamma \Omega^{4,0} = 0\,, \tag{3.86}$$

for an arbitrary integral 4-cycle $\gamma \in H_4(Y_4, \mathbb{Z})$. Consequently, we also conclude that the Gukov-Vafa-Witten superpotential is linear in τ in the global Sen-limit

$$\partial_\tau^2 \int G_4 \wedge \Omega^{4,0} = 0\,. \tag{3.87}$$

Hence, we arrive at the main conclusion that the superpotential does not receive D-instanton corrections in the global Sen-limit.

4 An example

In this section, we construct a simple elliptic Calabi-Yau fourfold as an elliptic fibration over an orientifold of the Octet Calabi-Yau $X := \mathbb{P}_{[1,1,1,1,4]}[8]$ to illuminate a few steps in the proof in section 3.3.

The Octet Calabi-Yau manifold is known to admit the Greene-Plesser mirror symmetry [67]. The existence of the Greene-Plesser mirror construction implies that there is a discrete group $G := Z_8^3$ such that a blow-up of X/G is the mirror Calabi-Yau $\tilde{X}$ as studied in depth in [68]. To simplify the discussion, we will consider X at a symmetric point in the moduli space such that defining polynomial of X is invariant under G

$$x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 - \psi x_1^2 x_2^2 x_3^2 x_4^2 = 0\,. \tag{4.1}$$

We consider an orientifold involution $\mathcal{I}$

$$\mathcal{I} : x_5 \mapsto x_5\,. \tag{4.2}$$

In the ambient variety $\mathbb{P}_{[1,1,1,1,4]}$ there are two fixed loci of $\mathcal{I}$

$$\{x_5 = 0\} \cup \{x_1 = x_2 = x_3 = x_4 = 0\}\,. \tag{4.3}$$

Because for a generic value of ψ, $x_1 = x_2 = x_3 = x_4 = 0$ does not intersect X we conclude that the only O-plane in the orientifold $X/\mathcal{I}$ is an O7-plane at $x_5 = 0$.

Now as in the previous section, we define a new toric variety $\mathbb{P}_{[1,1,1,1,8]}$ with homogeneous coordinates y_i such that y_i is identified with x_i for $i = 1, \ldots, 4$, and y_5 is identified with x_5^2. Then in the new homogeneous coordinates, the defining polynomial of $X/\mathcal{I} \equiv \mathbb{P}_{[1,1,1,1,8]}[8]$ is given by

$$y_1^8 + y_2^8 + y_3^8 + y_4^8 + y_5 - \psi y_1^2 y_2^2 y_3^2 y_4^2 = 0\,. \tag{4.4}$$

Because $X/\mathcal{I}$ is a degree 8 hyperplane in $\mathbb{P}_{[1,1,1,1,8]}$, one can consider an automorphism group action

$$y_5 \mapsto y_5 + \psi y_1^2 y_2^2 y_3^2 y_4^2 - (y_1^8 + y_2^8 + y_3^8 + y_4^8)\,, \tag{4.5}$$

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to treat $X/\mathcal{I}$ as the vanishing locus $\{y_5=0\}\in\mathbb{P}_{[1,1,1,1,8]}$, which is equivalent to $\mathbb{P}^3$. But we will not do so to make the complex structure moduli of X manifest, and split the axio-dilaton from the complex structure moduli of X.

Now we construct $\mathbb{P}_{[2,3,1]}$ fibration over $X/\mathcal{I}$, V_5, which is defined by a GLSM charge matrix

$$\begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & X & Y & Z \\ 1 & 1 & 1 & 1 & 8 & 8 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{pmatrix}, \tag{4.6}$$

and the defining equation

$$y_1^8+y_2^8+y_3^8+y_4^8+y_5-\psi y_1^2y_2^2y_3^2y_4^2=0\,. \tag{4.7}$$

We now want to find a Calabi-Yau hypersurface Y_4 in V_5, which is by definition an elliptic fibration over $X_3/\mathcal{I}$. To do so, we can take a vanishing locus of the Weierstrass form

$$Y^2=X^3+F(y)XZ^4+G(y)Z^6\,, \tag{4.8}$$

where $F(y)$ is a degree 16 polynomial and $G(y)$ is a degree 24 polynomial.

As was studied in section 3.2, we choose $F(y)$ and $G(y)$ as

$$F(y)=-\frac{1}{3}y_5^2\,, \tag{4.9}$$

and

$$G(y)=\left(\frac{2}{27}-t\right)y_5^3\,. \tag{4.10}$$

The discriminant is therefore given by

$$\Delta=t(-4+27t)y_5^6\,, \tag{4.11}$$

and the j-invariant reads

$$j(\tau)=-\frac{256}{t(-4+27t)}=\frac{1728}{4x(1-x)}\,, \tag{4.12}$$

where we defined $x:=27t/4$. From the discriminant, it is evident that $y_5=0$ supports an SO(8) D7-brane stack. Hence, $t\to 0$ describes the global Sen-limit. We finally note that the underlying Calabi-Yau threefold can be identified as $y_5=\xi^2$ in the Sen limit, where ξ can be treated as a weight 4 homogeneous coordinate in $\mathbb{P}_{[1,1,1,1,4,8]}$. Essentially, this identification $y_5=\xi^2$ can be rephrased as $y_5=x_5^2$, which is the identification we used to construct $\mathbb{P}_{[1,1,1,1,8]}$.

After a laborious computation, we obtain two Picard-Fuchs equations

$$\Big(\theta_z^4-2^5(8\theta_z-1)(8\theta_z-3)(8\theta_z-5)(8\theta_z-7)z\Big)\tilde{\omega}(z,t)=0\,, \tag{4.13}$$

and

$$\left(\partial_t^2-\frac{54t-4}{t(27t-4)}\partial_t+\frac{15}{4t(27t-4)}\right)\tilde{\omega}(z,t)=0\,, \tag{4.14}$$

where $z = 2^{24}\psi^{-4}$, $\theta_z := z\partial_z$. As the Picard-Fuchs equations are splitted, we take an ansatz

$$\tilde{\omega}(z,t) = \omega_z(z) \times \omega_t(t)\,. \tag{4.15}$$

We find four linearly independent solutions for $\omega_z(z)$

$$\begin{aligned}\omega_z(z) &= c_1\, {}_4F_3\left(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8};1,1,1;2^{16}z\right) + c_2\, G^{4,4}_{4,4}\left(\begin{matrix}\frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8}\\ 0 & 0 & 0 & 0\end{matrix}\middle| 2^{16}z\right)\\ &+ c_3\, G^{4,3}_{4,4}\left(\begin{matrix}\frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8}\\ 0 & 0 & 0 & 0\end{matrix}\middle| -2^{16}z\right) + c_4\, G^{4,2}_{4,4}\left(\begin{matrix}\frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8}\\ 0 & 0 & 0 & 0\end{matrix}\middle| 2^{16}z\right),\end{aligned} \tag{4.16}$$

which can be expanded around $z = 0$ to yield series expansions

$$\omega_z^{(0)} = 1 + 1680z + 32432400z^2 + 999456057600z^3 + \dots\,, \tag{4.17}$$

$$(2\pi i)\omega_z^{(1)} = \log(z)\omega_z^{(0)} + 15808z + 329980320z^2 + \frac{31367396784640}{3}z^3 + \dots\,, \tag{4.18}$$

$$(2\pi i)^2\omega_z^{(2)} = \log(z)^2\omega_z^{(0)} + 2\log(z)\Big((2\pi i)\omega_z^{(1)} - \log(z)\omega_z^{(0)}\Big) + 29504z + 973969296z^2 + \dots\,. \tag{4.19}$$

We omit the series expansion for the remaining solution to (4.13), as it is too lengthy.

We find two linearly independent solutions for $\omega_t(t)$,

$$\omega_t = c_1\, {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right) + c_2\, {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right). \tag{4.20}$$

We perform series expansion around $t = 0$ to obtain

$$\omega_t^{(0)} := {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right) = 1 + \frac{15}{16}t + \frac{3465}{1024}t^2 + \frac{255255}{16384}t^3 + \dots\,, \tag{4.21}$$

$$\omega_t^{(1)} := {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right) = -\frac{1}{2\pi}\omega_t^{(0)}\left(\log(2^{-6}t) + \frac{39}{8}t + \frac{14733}{1024}t^2 + \dots\right). \tag{4.22}$$

It is important to note that an identification

$$\tau := i\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)} \tag{4.23}$$

provides the inverse series of the j-invariant [60, 61]

$$j(\tau) = -\frac{256}{t(-4+27t)} = e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + \dots\,. \tag{4.24}$$

We define two flat coordinates around $t = 0$ and $z = 0$

$$\mathsf{t} := \frac{1}{2\pi i}\frac{\omega_z^{(1)}}{\omega_z^{(0)}}\,, \tag{4.25}$$

and

$$\tau := i\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-\frac{27}{4}t\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{27}{4}t\right)}\,. \tag{4.26}$$

Then as a result, following [50, 51], we obtain the period in integral basis

$$\Pi_A = \begin{pmatrix} 1 \\ \mathfrak{t} \\ -\mathfrak{t}^2 + \mathfrak{t} + \frac{11}{12} - \frac{29504}{(2\pi i)^2} e^{2\pi i \mathfrak{t}} + \mathcal{O}(e^{4\pi i \mathfrak{t}}) \\ \frac{2}{3}\mathfrak{t}^3 + \frac{11}{12} - \frac{296\zeta(3)}{(2\pi i)^3} - \frac{29504}{(2\pi i)^3}(2 - 2\pi i \mathfrak{t}) e^{2\pi i \mathfrak{t}} + \mathcal{O}(e^{4\pi i \mathfrak{t}}) \end{pmatrix}, \tag{4.27}$$

and

$$\Pi_B = \tau \Pi_A \,. \tag{4.28}$$

As a result, we conclude that there is no $\mathcal{O}(e^{\pi i \tau})$ terms in the period.

5 Conclusions

In this work, we studied the Picard-Fuchs equations of elliptic Calabi-Yau fourfolds in the global Sen-limit, in which all D7-brane stacks are carrying SO(8) gauge groups, to show that the F-theory superpotential in the global Sen-limit does not contain the D-instanton corrections. The D-instanton superpotential in a more generic D7-brane configuration will be the subject of [69].

The common wisdom is that in the F-theory description, type IIB complex structure moduli, D7-brane moduli, and the axio-dilaton all mix with each other as all these moduli are described as complex structure moduli of the F-theory compactification. This mixing poses a significant challenge to compute the D-instanton superpotential in F-theory in the weakly coupled type IIB limit. In order to clearly separate type IIB complex structure moduli from the axio-dilaton in the defining equation of elliptic Calabi-Yau fourfolds, we constructed elliptic Calabi-Yau fourfolds as toric complete intersections. With the description of elliptic Calabi-Yau fourfolds in hand, we generalized the Griffith-Dwork method to prove that the Picard-Fuchs equations are splitted into that of Calabi-Yau threefolds and that of the elliptic fiber, proving that the period integral is linear in the axio-dilaton.

It would be interesting to directly confirm the result presented in this paper via worldsheet CFT and string field theory techniques along the lines of [70–72].

Acknowledgments

We thank Andres Rios-Tascon for collaboration in the early stages of this work. We thank Jakob Moritz, Liam McAllister, Kepa Sousa, and Andreas Braun for useful discussions. We thank Timo Weigand for comments on the draft. The work of MK was supported by the Pappalardo Fellowship.

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