TECHNICAL MEMORANDUM Copy No.____

ESL-TM-357

NONLINEAR HARMONIC ANALYSIS

by

Jan Willems

July, 1968

NASA Research Grant NGR-22-009-124

This technical memorandum consists of the unaltered thesis of Jan Willems, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Massachusetts Institute of Technology in June, 1968. This research was carried out at the M.I.T. Electronic Systems Laboratory with support extended by the National Aeronautic and Space Administration under Research Grant No. NGR-22-009-124, M.I.T. DSR Project No. 76265.

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June, 1968

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Submitted to the Department of Electrical Engineering on May 10, 1968 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Some properties of the mapping of the spectrum of the input to a nonlinearity into the spectrum of the output are given. The results are presented mainly in terms of positive operators. Special attention is given to nonlinear time-invariant nonlinearities, to convolution operators, to periodic gains and to monotone or odd-monotone nonlinearities. A general theorem is proven which allows to factor a large class of operators in a causal operator and an operator whose adjoint is causal. This then allows to obtain a causal positive operator from a noncausal positive operator. The results are applied to the operator equations governing a feedback loop and some general stability theorems are obtained. Two important examples are included and frequency-domain stability criteria are given. The merit of using linearization techniques to conclude stability for feedback systems is discussed and a class of counterexamples to the Aizerman Conjecture is presented. Some techniques pertaining to the design of optimal nonlinearities are included.

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ACKNOWLEDGEMENT

The author is glad to have the opportunity to express his sincere appreciation to Professor Roger W. Brockett for his encouragement, his friendship and his helpful criticism. The research presented here is greatly influenced by his teaching and work. In particular, his cooperation was instrumental in obtaining the rearrangement inequalities, the instability criteria of Chapter IV and the techniques for designing optimal nonlinearities.

Many useful comments were received from Roberto Canales, Michael Gruber, Ronald Skoog and several other members of the Control Theory Group of the Electronic Systems Laboratory.

The encouragement given by the thesis readers, Professors W. M. Siebert and R. N. Spann, and by his Faculty Counselor, Professor L. A. Gould is greatly appreciated.

The author is indebted to the members of the Technical Staff of the Electronic Systems Laboratory for the excellent typing and drafting.

The research presented in this document was made possible through the support extended by the National Aeronautic and Space Administration under Research Grant No. NGR-22-009(124) with the M.I.T. Electronic Systems Laboratory.

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CHAPTER I

INTRODUCTION

1.1 Generalities

The material presented here under the general heading of "Nonlinear Harmonic Analysis" constitutes an attempt to analyze some of the properties of the mapping of the spectrum of the input to a nonlinearity into its output.

The results are applied mainly to the problem of finding sufficient conditions for the stability or instability of feedback systems.

It is safe to state that there is probably no single notion more familiar to systems engineers than the notion of a transfer function and that no mathematical tool has found wider application than the transform techniques. It is also realized that these ideas are useful mainly if one is dealing with time-invariant systems. The research given here presents some relations between the spectrum of the input to a nonlinearity and its output.

In linear time-invariant systems defined by convolution operators the spectrum of the output is simply equal to the spectrum of the input multiplied by the transfer function and the mapping of the input spectrum into the output spectrum is hence very simple. It is thus in general advantageous to specify all quantities in terms of their spectra rather than as functions of time. In other words, one prefers to do the analysis in the frequency-domain rather than in the time-domain. If the system contains time-variant and/or nonlinear elements, the simplicity of this mapping disappears and very often the analysis will then be done in the time-domain. In particular consider the system where the input, x(t), and the output, y(t), are simply related by $y(t) = f(x(t), t)$. This characterization is very simple in the time-domain, but unknown in the frequency-domain. To obtain qualitative features of the behavior of nonlinear systems one thus generally uses time-domain methods as, for example, if the desired feature is stability, the direct method of Lyapunov.

In at least two situations it would be advantageous to characterize nonlinear systems in the frequency domain; first if the input or inputs are given in terms of their spectra and properties of the output or outputs are sought in terms of their spectra and, second, if the system contains "much more" linear time-invariant elements than nonlinear or time-varying elements. In the former case it might be true that if some simple properties of the frequency-domain characterization of the nonlinear system were known the desired features of the output would follow immediately. As an example, suppose that one wanted to decide some features of the behavior of the output of a particular nonlinear system with respect to all bandlimited signals with a certain cut-off frequency. Clearly this is a very poor characterization of the inputs in the time-domain and some simple properties of the mapping into the spectrum of the output could be sufficient to derive the desired features about the outputs. In the latter case the simplicity of the frequency-domain description of the linear time-invariant part of the system in conjunction with a general idea about the frequency-domain characteristics of the nonlinear or time-varying elements might more likely yield the desired information than a time-domain analysis.

As an example, the first situation arises in the design of frequency converters where one tries to choose a nonlinear system in a certain class and which will transform a given input spectrum in some desired fashion. The second situation arises for instance when a simple nonlinear system is followed by a linear time-invariant system or in a feedback control system with a linear time-invariant element in the forward loop and a simple nonlinear element in the feedback loop.

The results obtained in the second, third, and fourth chapters which are concerned with positive operators and the stability and instability of feedback systems follow the lines of previously studied research topics. The fifth chapter however touches a problem which is new and quite promising. Indeed an attempt is made there to design nonlinearities using optimal control. The techniques presented in this chapter are felt to be important although not many specific results have been obtained. Indeed at all stages of the design of control systems a great deal of electronic devices are used and this brings with it the need for design procedures of filters, of frequency up- and down-converters, of a-c to d-c and d-c to a-c converters, etc. This chapter outlines some ideas regarding design procedures for systems containing nonlinear elements and the results can be viewed as useful at the level of designing individual parts, similar to the Bode-Nyquist and sort like criteria which have proven their usefulness at this level of the design as well as for the design of the overall system.

1.2 Contents

In the second chapter a number of positive operators are derived. A precise definition of a positive operator will be given later.

Roughly speaking an operator will be called positive if the inner product of any element and its image under the operation is positive. Thus for example a linear transformation from a finite dimensional linear vector space into itself will define a positive operator if and only if the matrix associated with this linear transformation plus the transpose of this matrix is positive definite. The Sylvester test thus yields a simple necessary and sufficient condition for a finite dimensional linear transformation to define a positive operator. For nonlinear transformations or operators defined on infinite dimensional spaces the situation is quite different and this is where the techniques and results developed in Chapter II are useful.

Why are positive operators important? There are several areas both in engineering and in applied mathematics where positive operators play a central role. Here are some examples:

(i) Many techniques, e.g., in the theory of optimal control, in prediction theory and in stability theory require at a certain point establishing that a certain function or functional is positive definite, e.g., second variations in optimization theory and Lyapunov functions and their derivatives in stability theory. This verification can often be reduced to the verification that a certain appropriately chosen operator is positive. In this context, it suffices to recall how often the positive definiteness of certain matrices is invoked.

(ii) Another area of research where positive operators have played an essential role is in network synthesis. Recall that a ratio of polynomials in s is the driving point impedance of a two-terminal network that can be realized using a finite number of positive resistors, inductors, capacitors and ideal transformers if and only if this ratio of polynomials is a positive real function of s (see e.g., (27)). This result thus identifies with the input-output relation of these passive networks a class of positive operators. There is no doubt that positive operators will also play an essential role in the synthesis of nonlinear and time-varying networks using certain passive devices.

(iii) An important application of positive operators is in establishing the stability of feedback systems. Roughly speaking stability is the property of systems in which small inputs or initial conditions produce small responses. The technique for generating stability criteria for feedback systems from knowledge of positive operators will be examined in detail in Chapter III but the basic idea is simple and states that the interconnection of passive systems (positive operators) yields a stable system.

(iv) The so-called frequency-power formulas have found wide application in the design of parametric amplifiers. They are formulas which constrain weighted sums of real and reactive powers entering a device at various frequencies to be either zero, positive or negative. This device could for instance be a nonlinear resistor, inductor or capacitor. This work was initiated by Manley and Rowe who analyzed the power flow at various frequencies in a nonlinear capacitor. Their conclusions were the now famous Manley-Rowe frequency-power formulas. Their work has been extended in several directions and the resulting formulas have found wide application in the design of frequency converters. Frequency-power formulas establish fundamental limits on the efficiency of such devices. Other fields of interest where these formulas have been applied are in energy conversion using parametric devices, in hydrodynamic and magnetohydrodynamic stability, and in many other areas. In trying to bring certain methods and results in these areas into harmony, it became apparent that these frequency-power formulas are essentially particular classes of positive operators and can be most easily understood as

(v) Another important area of application where positive operators play an important role is <u>to determine bounds on the optimal performance</u> of nonlinear time-varying systems. One of the most important problems in optimal control theory appears to be, paradoxically, to design suboptimal systems. Indeed either because of computational feasibility or because of simpler or more convenient implementation it is in many cases necessary to resort to suboptimal systems. Little or no attention has been paid to the problem of a priori predicting how far a suboptimal system is from being optimal. In his forthcoming dissertation, R. Canales (15) shows that the requirement that a given system has a better performance than another system with respect to some performance criterion can in many important cases be reduced to requiring that a certain suitably chosen operator be positive. This then allows to estimate a priori bounds for the performance of certain systems and to design feasible suboptimal controls. The basic idea to introduce a positive operator is this: if the inner product of the optimal control and the difference in the derivative of the state of the first and second system is positive (i.e., the state of the second system changes in the right (optimal) direction when no control is applied), then the performance of the second system will be better. In this respect it is also worthwhile to mention that optimal control provides a way of verifying the positivity of an operator O. Indeed if $\inf_x \langle x, Ox \rangle \geq 0$ then the operator is clearly positive. However, it ought to be mentioned that in general optimal control techniques are not too useful in solving the problem this way. The design of suboptimal controls thus appears to be a promising area of positive operators. It also links these techniques further with optimal control theory.

The second chapter thus starts with some mathematical preliminaries and definitions and then establishes some simple positive operators involving convolution operators and memoryless linear or nonlinear gains. These results lead to the Manley-Rowe frequency-power formulas and the positive operators which yield the Popov Criterion and the Circle Criterion for the stability of feedback systems: Then a positive operator formed by the interconnection of a periodically time-varying gain and a linear time-invariant convolution operator is presented. This positive operator leads to a rather elegant frequency-domain stability criterion which is discussed in the third chapter.

In the next section of the second chapter the following problem is completely resolved: What is the most general linear system which when composed with a monotone or an odd-monotone nonlinearity yields a positive operator? The solution to this problem presents in a sense the answer to a question which has been studied by many previous researchers both in connection with frequency-power formulas and with the stability of feedback systems with a monotone or an odd-monotone nonlinearity in the feedback loop. The results require a considerable generalization of a classical rearrangement inequality due to the Hardy, Littlewood, and Polya. The rearrangment inequality thus obtained is felt to be of great interest in its own right.

The last section of the second chapter is devoted to the problem of adjoining to a positive operator a causal positive operator. Roughly speaking, an operator is causal if the output at some time depends only on the values of the input before that time. It is apparent that causality will be a basic property of physical systems. Thus in many problems in system theory e.g., in stability theory, in optimal control theory, in prediction theory or in network synthesis, causal operators are of particular interest. For instance in network synthesis it is clear that causality will be, together with passivity, one of the basic properties of systems which could be realized using passive devices. The question thus arises whether or not the positive operators discovered in the previous sections have an analogue which is in addition causal. The answer to this question is in the affirmative provided the operator admits a suitable factorization. Whether a particular operator satisfies this condition appears to have no general answer and the problem is one of considerable interest and importance. Similar factorizations have received a great deal of attention in the past particularly in the classical prediction theory. In this section a general factorization theorem is presented which is felt to be quite general and of intrinsic importance. Unfortunately the result which is based on contraction arguments does not offer a necessary condition and is rather conservative in some particular cases.

Most of these positive operators give essentially properties of the output spectrum of a nonlinearity in terms of the spectrum of the input. In fact, since most of the positive operators derived here are the composition of a nonlinear possible time-varying memoryless element and a convolution operator which merely represents, if its kernel is time-invariant, a multiplication in the frequency-domain, the resulting formulas simply express the positivity a certain bilinear functionals involving the input and the output spectrum, of the types as in the frequency-power formulas and the Manley-Rowe equations.

The third chapter is devoted to the stability of feedback loops. The type of stability which is considered here is not very common but rather strong and essentially requires that small inputs to the feedback loop produce small responses. The definition of small signals is very simple if the notion of extended space and truncated signals is introduced. A truncated signal is the original signal but replaced by zero from some time on and a signal is said to belong to the extended space if all its truncations belong to the space. The stability theorems essentially put conditions on the forward loop and the feedback loop which result in the fact that all solutions which belong to the extended space actually belong to the space itself. This type of stability together with a basic theorem is used to obtain some general stability conditions. In particular the intuitive ideas that stability follows if the open loop gain is less than unity or if the feedback loop is the interconnection of passive systems (positive operators) are proven. As a refinement to these results the method of using multipliers or factoring the forward loop in two factors one of which is then lumped with the feedback loop is presented. The resulting theorem is then used in two interesting examples. These stability results also require a factorization as the one discussed in the previous chapter.

The first practical stability theorem applies to a feedback loop with a linear time-invariant convolution operator in the forward loop and a linear periodically time-varying gain in the feedback loop. As most recent stability criteria, the criterion requires the existence of a multiplier having certain properties. However, a necessary and sufficient condition for this multiplier to exist is given and puts conditions on the variation of the Nyquist locus of the forward loop. Essentially it requires that the phase of the transfer function of the forward loop should not change too drastically when the frequency is increased by an amount equal to the frequency of the periodic gain in the feedback loop, thus requiring a certain filtering effect.

The second practical stability theorem treats feedback systems which have a linear convolution operator in the forward loop whose kernel may be time-variant and a monotone or an odd-monotone nonlinearity in the feedback loop. The resulting stability theorem requires the existence of a multiplier having certain properties. This multiplier is less restrictive than the multipliers required in existing criteria but more research is required to obtain conditions which can be stated in terms of the forward loop.

The fourth chapter in a sense motivates the third chapter and takes a critical look at some linearization methods which are commonly used to obtain stability conditions for feedback loops with one nonlinear element. A particular system is presented in which these linearization techniques all predict stability but which nonetheless allows periodic solutions. These conclusions are derived using the Averaging Theory of Cesari and Hale and the example provides a class of simple counterexamples to the well-known Aizerman conjecture.

These examples provide a case where the mapping of the input spectrum into the output spectrum can be quite different for a linear and a strictly nonlinear characteristic and this then accounts for the existence of oscillations which are not expected from consideration of a linearized behavior.

The fifth chapter discusses the optimal design of nonlinearities. An algorithm for choosing the nonlinearity in a certain class which maximizes a linear functional is given and the problem of generating a nonlinearity which yields a given set of Fourier coefficients at the output is discussed in some detail.

1.3 Historical Note

The study of positive operators has found a great deal of interest and application in the study of network synthesis and related areas. These investigations however generally limit themselves to the study of particular classes of positive operators, namely the input-output relations of finite dimensional constant lumped networks (27). Some extension to nonlumped networks have been made (61).

The application of positive operators to the stability of feedback loops was introduced by Sandberg (54) and Zames (62), and was exposed in its full generality by the latter author in (63). The exposition and the analysis presented in the third chapter are greatly influenced by this reference which can, in the present author's opinion, be considered a basic paper in stability theory. It is however apparent that the ideas of positive operators are present, although not very explicitly, in the construction of Lyapunov functions and the resulting frequency-domain stability criteria due to Brockett and Willems (10). The research and the success of frequency domain stability criteria for nonlinear time-varying systems was initiated by Popov (47) and the most impressive results are surveyed in (11).

The most widely known results are the Popov Criterion and the Circle Criterion which is due to Sandberg (52).

The search for frequency-power formulas was initiated by the discovery in 1956 of the now famous Manley-Rowe frequency-power formulas (36) which are compiled in the book by Penfield (45). The frequency-power formulas which are closely related to those obtained here and which apply to nonlinear resistors are due to Pantell (44), Page (43) and Black (7). The latter author uses the rearrangement inequalities of Hardy-Littlewood and Polya (29) to obtain the fact that the cross-correlation of the input and the output to a monotone nondecreasing nonlinearity attains its maximum at the origin. This result was originally due to Prosser (48) in a slightly different setting.

The factorization theorem obtained at the end of the second chapter is original. Its setting using projections in a Banach Algebra follows Zames and Falb (64) and its proof is inspired by a paper by Baxter (4). For additional results pertaining to similar factorizations see for instance the book by Wiener (58), and particularly the paper by Krein (34).

The two examples of practical stability theorems given in Chapter III have been studied before in several places. The feedback system with a linear time-invariant convolution operator in the forward loop and a linear periodically time-varying gain in the feedback loop is of the same type as the one studied by Bongiorno (9) and Sandberg (51), but the result given here makes use of the fact that the feedback gain is periodic to obtain an improved stability criterion. The stability theorem pertaining to the stability of feedback systems with a monotone or an odd-monotone nondecreasing nonlinearity in the feedback loop as studied in the second example is a generalization of similar results obtained by several authors. In particular the papers by Brockett and Willems (10), by Zames (63) and Zames and Falb (64), by O'Shea (41), by Narendra and Neuman (39), by Thathachar, Srinath and Ramapryan (55) and by Baker and Desoer (3) treat problems along the same lines.

For the counterexamples to Aizerman's conjecture and their history, see the thesis by Pliss (46) and the thesis by Fitts (21). Particularly the experimental results described in this last reference were instrumental in obtaining the example given in Chapter IV.

CHAPTER II

POSITIVE OPERATORS

2.1 Introduction

This chapter is devoted to positive operators and starts with a number of well-known definitions from functional analysis. These notions will then be used freely in the sequel. The definition of limit-in-the-mean transforms and of almost periodic functions and some of their properties are given for easy reference. For a more extended treatment on these subjects see e.g., (49, 56, and 8).

The first class of operators which are examined for positivity are convolution operators, and operators in which the output is an instantaneous function of the input. The positive operators thus discovered lead to the well-known Manley-Rowe equations and play an important role in stability theory since they are closely connected with the Popov Criterion and the Circle Criterion for the stability of non-linear and time-varying feedback systems.

Next, attention is focused on the question what class of convolution operators can be composed with a positive periodically time-varying linear gain and still yield a positive operator. The answer to this question is that this convolution operator should itself be positive and that the kernel of the convolution should be a string of impulses occurring at multiples of the period of the time-varying gain. It is shown that this result is both necessary and sufficient and the proof relies on the fact that two operators of this type commute.

In the next section of this chapter, an answer to the following question is sought: What is the most general linear operator which when composed with a monotone nondecreasing (or an odd-monotone nondecreasing) nonlinearity yields a positive operator ? This problem has received a great deal of attention in the past, both in connection with frequency-power formulas and with the stability of feedback loops with a monotone nonlinearity in the feedback loop. The resulting class of positive operators is closely related to certain classes of matrices, i.e., the dominant matrices, which play an important role in network synthesis. The reason for this connection however remains vague and deserves further investigation. As an intermediate step in deriving this class of positive operators a considerable generalization of a classical inequality due to Hardy, Littlewood and Polya on the rearrangement of sequences is derived. It is felt that the extension of this rearrangement inequality is of intrinsic importance in itself and is potentially applicable in other areas of system theory.

The last section of this chapter considers the problem of the factorization of linear operators in a part which is causal (a lower-triangular matrix) and a part whose transpose is causal (an upper-triangular matrix). This problem has received a great deal of attention in connection with stability theory, optimal control theory and prediction theory. The factorization theorem obtained here is quite interesting since it applies to time-variant convolution operators as well as to time-invariant convolution operators. It is pointed out however that in the latter case the results are rather conservative.

2.2 Mathematical Preliminaries:

Definitions: Let X and Y be two spaces. The product space, denoted XXY, is the collection of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. A space X and a map, d, from XXX into the reals, R, is called a metric space if (i) $d(x_1, x_2) \geq 0$ for all $x_1, x_2 \in X$ and $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$, if (ii) $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$ and if (iii) $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$ for all $x_1, x_2, x_3 \in X$ (the triangle inequality). A sequence $\{x_n\}$ of elements of a metric space X is said to converge to a point $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = 0$, it is called a Cauchy sequence if for any $\epsilon > 0$ there exists an N such that $d(x_n, x_m) \leq \epsilon$ for all $n, m \geq N$. A metric space is called complete if every Cauchy sequence converges. A subset X_1 of a metric space X is said to be dense if for every $x \in X$ and every $\epsilon > 0$ there exists a $x_1 \in X_1$ such that $d(x, x_1) \leq \epsilon$. A set X is said to be countable if there exists a map from X into the integers, I. A metric space is said to be separable if it has a countable dense subset.

Definitions: Let K denote the real or complex number system, R or C, and let X be a vector space over K. A mapping, $\|\cdot\|$, from X into R is called a norm on X if (i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$, if (ii) $\|cx\| = |c| \; \|x\|$ for all $x \in X$ and $c \in K$, and if (iii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in X$ (the triangle inequality). A normed vector space has a natural metric, i.e., $d(x_1, x_2) = \|x_1 - x_2\|$ for all $x_1, x_2 \in X$. This metric is called the metric induced by its norm. An inner product space over K is a vector space over K and a mapping from XXX into K, called the inner product and denoted by $\langle \cdot \, , \cdot \rangle$ such that (i) $\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$

($^{-}$ denotes complex conjugate) for all $x_1, x_2 \in X$, (ii) $<c_1x_1 + c_2x_2, x_3>$ $= \bar{c}_1 < x_1x_2 > + \bar{c}_2 < x_1, x_3>$ for all $x_1, x_2, x_3 \in X$ and $c_1, c_2 \in K$, and (iii) $<x, x> \geq 0$ for all $x \in X$ and $<x, x> = 0$ if and only if $x=0$. It follows from these definitions that $< x, x >^{1/2}$ is a norm on X. This norm is called the norm induced by the inner product or the natural norm. The metric induced by this norm will be called the metric induced by the inner product. An important relation is the Schwartz inequality which states that $|<x_1, x_2>| \leq \|x_1\| \cdot \|x_2\|$ for all $x_1, x_2 \in X$. (As always, unless explicitly mentioned, the norm on an inner product space will always be taken to be the natural norm.)

Definitions: A Banach space is a normed vector space which is complete in the metric induced by its norm. A Hilbert space is an inner product space which is complete in the metric induced by its inner product.

Examples: A mapping x from the interval $(a, b) \subset R$ into K ($a = -\infty$ and $b = +\infty$ are allowed) is said to belong to $L_p(a, b)$, $p \geq 1$ if $x(t)$ is measurable and if $\int_a^b |x(t)|^p dt < \infty$. It is said to belong to $L_\infty(a, b)$ if it is measurable and if $|x(t)| \leq M$ for some M and almost all $t \in (a, b)$. Two elements of $L_p(a, b)$ or $L_\infty(a, b)$ will be considered equal if they are equal for almost all $t \in (a, b)$, i.e., if $x_1, x_2 \in L_p(a, b)$ or $L_\infty(a, b)$ then $x_1 = x_2$ if $x_1(t) = x_2(t)$ for almost all $t \in (a, b)$. With this equivalence relation, $L_p(a, b)$ and $L_\infty(a, b)$ are Banach spaces with $\|x\|_{L_p} = (\int_a^b |x(t)|^p dt)^{1/p}$ if $x \in L_p$ and the infimum of all numbers M satisfying $|x(t)| \leq M$ for almost all $t \in (a, b)$ for $x \in L_\infty(a, b)$. $L_2(a, b)$ is a Hilbert space with $<x_1, x_2>$ $= \int_a^b \overline{x_1(t)} x_2(t) dt$ for $x_1, x_2 \in L_2(a, b)$. An important inequality (Hölder's Inequality) on L_p-spaces states that if $f \in L_p(a, b)$ and

$g \in L_q(a,b)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L_1(a,b)$ and $\|fg\|_{L_1} \leq \|f\|_{L_p} \cdot \|g\|_{L_q}$, and if $f \in L_1(a,b)$ and $g \in L_\infty(a,b)$ then $fg \in L_1(a,b)$ and $\|fg\|_{L_1} \leq \|f\|_{L_1} \cdot \|g\|_{L_\infty}$. Another useful fact is that if a and b are finite or if $x(t) = 0$ off a bounded set, then $x \in L_{p_1}(a,b)$ if $x \in L_{p_2}(a,b)$ for $p_1 \leq p_2$, and that if $x \in L_\infty(a,b)$ then $x \in L_p(a,b)$ for all p. $L_p(-\infty, +\infty)$ or $L_\infty(-\infty, +\infty)$ will be denoted by L_1 and L_∞.

A mapping x from I into K is said to <u>belong to</u> $\ell_p (p \geq 1)$ if $\sum_{k=-\infty}^{+\infty} |x_k|^p < \infty$. It is said to <u>belong</u> to ℓ_∞ if $|x_k| \leq M$ for some M and all k. ℓ_p forms a Banach space with $\|x\|_{\ell_p} = (\sum_{k=-\infty}^{+\infty} |x_k|^p)^{1/p}$ if $x \in \ell_p$ and the infimum of all numbers M satisfying $|x_k| \leq M$ for all k if $x \in \ell_\infty$. ℓ_2 is a Hilbert space with $\langle x, y \rangle = \sum_{k=-\infty}^{+\infty} \overline{x}_k y_k$ for $x, y \in \ell_2$. <u>Hölder's inequality</u> becomes $\|xy\|_{\ell_1} \leq \|x\|_{\ell_p} \|y\|_{\ell_q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, $x \in \ell_p$ and $y \in \ell_q$ and $\|xy\|_{\ell_1} \leq \|x\|_{\ell_1} \|y\|_{\ell_\infty}$ if $x \in \ell_1$ and $y \in \ell_\infty$. Another useful fact is that $\ell_{p_1} \subset \ell_{p_2}$ if $p_1 \leq p_2$, and that $\ell_p \subset \ell_\infty$ for all p.

<u>Remark</u>: For $p=q=2$, Hölder's inequality becomes the Schwartz inequality. The triangle inequality for L_p, L_∞, ℓ_p or ℓ_∞, is often referred to as <u>Minkowski's inequality</u>.

<u>Definitions:</u> A mapping from a space X into a space Y will be called an <u>operator</u> from X into Y. Thus an operator associates with each element $x \in X$ a unique element $y \in Y$. X is called the <u>domain</u> of O, and is denoted by $Do(O)$. Let O be an operator from X into Y. The image of $x \in X$ under O will be denoted by Ox. Thus $Ox \in Y$ by assumption. Let X and Y be subsets of a real inner product space (i.e., an inner product space over R). An operator O from X into Y will be called a <u>nonnegative operator</u> on X (denoted

$O \geq 0$) if $<x, Ox> \geq 0$ for all $x \in X$. It is said to be a positive operator on X if $O - \epsilon I \geq 0$ (I denotes the identity operator on X, i.e., $Ix = x$ for all $x \in X$) for some $\epsilon > 0$. An operator from a normed linear space X into a normed linear space Y is said to be bounded if there exists a number M such that $\|Ox\| \leq M\|x\|$ for all $x \in X$. The infimum of all numbers M satisfying the above inequality is called the bound of O, denoted $\|O\|$. The range of an operator O, from X into Y, denoted $Ra(O)$ are all members of Y which can be expressed as Ox for some $x \in X$. An operator O from X into Y is said to be invertible if there exists an operator O^{-1} from $Ra(O)$ into X such that the operator from X into itself defined by $O^{-1}O$ equals the identity operator. This implies that the operator from $Ra(O)$ into itself defined by $O^{-1}O$ also equals the identity operator. An operator from a metric space X into a metric space Y is said to be continuous at x if $\{Ox_n\}$ converges to Ox whenever $\{x_n\}$ converges to x. If X, Y and Z are normed linear spaces, if O_1 is a bounded operator from X into Y and if O_2 is a bounded operator from Y into Z, then O_2O_1 is a bounded operator from X into Z, and $\|O_2O_1\| \leq \|O_1\| \cdot \|O_2\|$.

Definitions: Let X and Y be vector spaces over K. An operator T from X into Y is said to be linear if $T(\alpha x_1+\beta x_2) = \alpha Tx_1 + \beta Tx_2$ for all $x_1, x_2 \in X$ and $\alpha, \beta \in K$. Let T be a linear operator from a normed vector space X into a normed vector space Y. Then T is continuous everywhere (i) if and only if it is bounded or (ii) if and only if it is continuous at one point. Also if T is bounded, then $\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\|$. Let X be real inner product space, and let T be a bounded linear transformation from X into itself. Then there exists a bounded linear operator T^* from X into itself such that $< x_1, Tx_2 > = < T^*x_1, x_2 >$ for all $x_1, x_2 \in X$. Moreover $\|T\| = \|T^*\|$, $(T^*)^* = T$, and T is invertible if and only if T^* is, and $(T^*)^{-1} = (T^{-1})^*$.

Theorem 2.1: Let X_1 and X_2 be subsets of a real inner product space and let O_1 and O_2 be nonnegative operators on X_1 and X_2 respectively. Then

(i) $\alpha^2 O_1 \geq 0$ on X_1 for all $\alpha \in R$

(ii) $O_1 + O_2 \geq 0$ on $X_1 \cap X_2$

(iii) If O_1 is invertible then $O_1^{-1} \geq 0$ on $Ra(O_1)$

Proof: Since (i) $< x, \alpha^2 O_1 x > = \alpha^2 < x, O_1 x >$ for all $x \in X$, since (ii) $< x, O_1 x + O_2 x > = < x, O_1 x > + < x, O_2 x >$ for all $x \in X_1 \cap X_2$ and since (iii)

$$
\begin{aligned}
< x, O_1^{-1} x > &= < O_1 y, O_1^{-1} O_1 y > \\
&= < O_1 y, y > \\
&= < y, O_1 y >
\end{aligned}
$$

for all $x \in Ra(O_1)$, the theorem follows.

Theorem 2.2: Let O be a nonnegative operator from a real inner product space X into itself and let T be a bounded linear operator from X into itself. Then

(i) $T^*OT \geq 0$ on X

(ii) $T \geq 0$ on X if and only if $T^* \geq 0$ on X

Proof: Since (i) $<x, T^*OTx > = < Tx, OTx >$ and since (ii)

$$
\begin{aligned}
< x, Tx > &= < T^*x, x > \\
&= < x, T^*x >
\end{aligned}
$$

and $(T^*)^* = T$, the theorem follows.

2.3 Transform Theory

Definitions: Let $x \in L_1$, then the function X defined by

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

is called the Fourier transform of x. Clearly $X \in L_\infty$, $\|X\|_{L_\infty} \leq \|x\|_{L_1}$ and if $x(t)$ is real, then $X(j\omega) = \overline{X(-j\omega)}$. Since this transform need not belong to L_1, it is in general impossible to define the inverse Fourier transform. However if X turns out to belong to L_1 then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}d\omega$$

(As always, this equality is to be taken in the L_1 sense). Thus the need of a slightly more general transform in which the inverse transform can always be defined is apparent. This is done by the limit-in-the-mean transform. It is well-known that if $x, y \in L_2 \cap L_1$ then $\langle x, y\rangle = \frac{1}{2\pi}\langle X, Y\rangle$ (Parseval's Equality). Let $x \in L_2$. Since $L_1 \cap L_2$ is dense in L_2, i.e., any L_2-function can arbitrarily closely be approximated (in the L_2 sense) by a function in $L_1 \cap L_2$, there exists a sequence of functions $\{x_n\}$ in $L_2 \cap L_1$ which is Cauchy (with respect to L_2) and which converges to x (in the L_2 sense). Let X_n be the Fourier transform of x_n. It follows from the Parseval relation that $\|x_n - x_m\| = \frac{1}{(2\pi)^{1/2}}\|X_n - X_m\|$ and that $X_n \in L_2$. Thus since L_2 is complete, these transforms, X_n, converge to an element X of L_2. This element X is called the limit-in-the-mean transform of x. It follows that the limit-in-the-mean-transform maps L_2 into itself and that $\langle x, y\rangle = \frac{1}{2\pi}\langle X, Y\rangle$ for all $x, y \in L_2$, and their limit-in-the-mean transforms X, Y. This equality will be referred to as <u>Parseval's equality</u>. One way of defining a limit in the mean transform is by

$$X(j\omega) = \lim_{T \to \infty} \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

where the limit is to be taken in the L_2-sense (It is easily verified that this constitutes essentially a particular choice for the Cauchy sequence $\{x_n\}$.) The notation that will be used for limit-in-the mean is

$$X(j\omega) = \text{l.i.m.} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

With this definition of transforms, the inversion is always possible and the <u>inverse transform formula</u> states that

$$x(t) = \text{l.i.m.} \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

<u>Definitions:</u> Let $x \in L_2(0, T)$, $T > 0$. Then the sequence $X = \{x_k\}$, $k \in I$, defined by

$$x_k = \frac{1}{T} \int_{0}^{T} x(t) e^{-jk\frac{2\pi}{T}t} dt$$

is well defined since $L_2(0, T) \subset L_1(0, T)$, and is called the <u>Fourier series</u> of $x(t)$. Clearly $X \in \ell_\infty$ and $x_k = \overline{x}_{-k}$ whenever $x(t)$ is real. The <u>Parseval relation</u> states that if $x_1, x_2 \in L_2(0, T)$ and if X_1, X_2 are their Fourier series, then $\langle x_1, x_2 \rangle = 2\pi \langle X_1, X_2 \rangle$. (These inner products are of course with respect to L_2 and ℓ_2 respectively.) In trying to obtain the <u>inverse Fourier series</u> formula, the same difficulties as in the inverse Fourier transform are encountered, and the same type of solution is presented. This leads to

$$x(t) = \text{l.i.m.} \sum_{k=-\infty}^{+\infty} x_k e^{jk\frac{2\pi}{T}t}$$

One way of expressing this l.i.m. summation is by

$$x(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} x_k e^{jk\frac{2\pi}{T}t}$$

where the limit is to be taken in the $L_2(0, T)$ sense.

Definitions: Let $x \in \ell_1$, then the function X defined by

$$X(z) = \sum_{k=-\infty}^{+\infty} x_k z^{-k}$$

exists for all $|z| = 1$ and is called the z-transform of x. In trying to extend this notion to sequences in ℓ_2 the same difficulties and the same solution as in the previous cases present themselves. This leads to the limit-in-the mean z-transform

$$X(z) = \text{l.i.m.} \sum_{k=-\infty}^{+\infty} x_k z^{-k}$$

and the inverse z-transform

$$x_k = \frac{1}{2\pi} \oint_{|z|=1} X(z) z^{-1} dz$$

Definitions: A continuous function, x, from R into K is said to be almost-periodic if for every $\epsilon > 0$ there exists a real number ℓ such that every interval of the real line of length ℓ contains at least one number τ such that

$$|x(t+\tau) - x(t)| \le \epsilon \quad \text{for all } t$$

Some properties of almost-periodic functions are:

(i) Every almost periodic function is bounded and uniformly continuous

(ii) Continuous periodic functions are almost-periodic

(iii) The sums, products and limits of uniformly convergent almost periodic functions are almost periodic

(iv) The limit of the mean value

$$\frac{1}{2T} \int_{-T}^{T} x(t+\tau)\, dt$$

as $T \to \infty$ exists, and is independent of τ for all almost periodic functions x, and the convergence is uniform in τ.

(v) If x_1 and x_2 are almost periodic functions then so is

$$x_1 * x_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_1(t-\tau)\, x_2(\tau)\, d\tau$$

Moreover, $x_1 * x_2 = x_2 * x_1$ and $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$ for all almost periodic functions x_1, x_2, x_3.

(vi)
$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, e^{-j\omega t} dt$$

vanishes for all but a countable number of values of ω.

(vii) The space of almost periodic functions forms an inner product space with $\langle x_1, x_2 \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \overline{x_1}(t)\, x_2(t)\, dt$

for x_1, x_2 almost periodic functions. (This inner product space is however not complete and not separable.) Let x be an almost periodic function and let $\{\omega_k\}$ be the set of values for which the limit in (vii) does not vanish and let x_k be the value of that limit for $\omega = \omega_k$. The sequence $\{x_k\}$ is called the generalized Fourier series of $x(t)$. If $x(t)$ is real then ω belongs to the set $\{\omega_k\}$ if and only if $-\omega$ does and the values x_k associated with ω and $-\omega$ are complex conjugates. The inverse Fourier series is defined as

$$x(t) = \lim_{N \to \infty} \sum_{k=-N}^{N} x_k e^{j\omega_k t}$$

This limit, which exists, is to be taken in the metric induced by the inner product on the space of almost periodic functions.

2.4 Some Simple Positive Operators

In this section a number of well-known positive operators will be discussed and generalized. The results yield the Manley-Rowe equations and the positive operators which led to the Popov Criterion and the Circle Criterion for the stability of feedback systems. The discussion is mainly concerned with positive operators on L_2 but the Manley-Rowe equations will also be stated (without proof) for almost periodic functions.

In this section L_2 is assumed to be taken over the real numbers.

Definitions: Let $\underset{\sim}{G}$ denote the class of operators from L_2 into itself each element, G, of which has associated with it an element $G(j\omega)$ of L_∞, with $G(j\omega) = \overline{G(-j\omega)}$ and which maps an element,

$x(t)$, of L_2 as follows: let $X(j\omega)$ be the limit-in-the-mean transform of $x(t)$ then the function $y(t) = Gx(t)$ is the inverse limit-in-the-mean transform of $G(j\omega)X(j\omega)$.

Let $\underset{\sim}{K}$ denote the class of operators from L_2 into itself each element, K, of which has associated with it a real element of L_∞, $k(t)$, and which maps an element, $x(t)$, of L_2, into $y(t)$ with $y(t) = Kx(t) = k(t)x(t)$.

Let $\underset{\sim}{F}_t$ denote the class of operators from L_2 into itself each element, F_t, of which has associated with it a measurable function, $f(\sigma, t)$ from $R \times R$ into R, satisfying the inequality $|f(\sigma,t)| \leq M |\sigma|$ for some M, all σ and almost all t, and which maps an element, $x(t)$, of L_2 into $y(t)$ with: $y(t) = F_t x(t) = f(x(t), t)$ for all t.

Let $\underset{\sim}{F}$ denote the class of operators from L_2 into itself each element, F, of which has associated with it a measurable function, $f(\sigma)$, from R into itself, satisfying the inequality $|f(\sigma)| \leq M|\sigma|$ for some M and all σ, and which maps an element, $x(t)$, of L_2 into $y(t)$ with $y(t) = Fx(t) = f(x(t))$.

It is a simple matter to verify that the above operators are indeed well-defined, i.e., that they map L_2 into itself. A subclass of operators of the class $\underset{\sim}{G}$ which is particularly important will now be examined more closely. Let $(g(t), \{g_k\})$ be an element of $L_1 \times \ell_1$ and let t_k be a mapping from I into R. Let $y(t) = Gx(t)$ be formally defined as

$$y(t) = \sum_{k=-\infty}^{+\infty} g_k \, x(t-t_k) + \int_{-\infty}^{+\infty} g(t-\tau)x(\tau)\,d\tau$$

Lemma 2.1: The operator G defined formally by the above equation maps L_2 into itself. Moreover $G \in \underset{\sim}{G}$ and the function $G(j\omega)$ associated with G is given by

$$G(j\omega) = \sum_{k=-\infty}^{+\infty} g_k e^{-j\omega t} + \int_{-\infty}^{+\infty} g(t) e^{-j\omega t} dt$$

Proof: This is a standard result from Fourier transform theory (see e.g., (56, p 90).

Remark: Actually if $g(t) \in L_2$ and if its limit-in-the-mean transform of $G(j\omega) \in L_\infty$ then the above lemma remains valid.

The following theorems on positive operators will now be established.

Theorem 2.3: Every element $G \in \underset{\sim}{G}$ defines a bounded linear transformation from L_2 into itself, $\|G\| = \|G(j\omega)\|_{L_\infty}$ and G is a nonnegative (positive) operator on L_2 if and only if $\mathrm{Re}\, G(j\omega) \geq 0$ ($\mathrm{Re}\, G(j\omega) \geq \epsilon$ for some $\epsilon > 0$) for almost all $\omega \geq 0$. Moreover, $G^* \in \underset{\sim}{G}$ and has the function $\overline{G(j\omega)}$ associated with it.

Proof: The theorem is obvious with the possible exception of the positivity condition. This however follows from Parseval's equality. Indeed,

$$< x(t), Gx(t) > = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(j\omega) \, |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} 2\mathrm{Re}\, G(j\omega) \, |X(j\omega)|^2 d\omega$$

Theorem 2.4: Every element $K \in \underset{\sim}{K}$ defines a bounded linear transformation from L_2 into itself. $\| K \| = \|k(t)\|_{L_\infty}$ and K is a nonnegative (positive) operator on L_2 if and only if $k(t) \geq 0$ ($k(t) \geq \epsilon$ for some $\epsilon > 0$) for almost all t. Moreover $K^* = K$, i.e., K is self adjoint.

Proof: This theorem is immediate.

Theorem 2.5: Every element $F_t \in \underset{\sim}{F}_t$ defines a bounded operator from L_2 into itself. $\|F_t\| = K'$ where $K' = \inf K$ over all K such that $|f(\sigma, t)| \leq K|\sigma|$ for all σ and almost all t. F_t is a nonnegative (positive) operator on L_2 if and only if $\sigma f(\sigma, t) \geq 0$ ($\sigma f(\sigma, t) \geq \epsilon\sigma^2$ for some $\epsilon > 0$) for all σ and almost all t.

Proof: This theorem is immediate.

The theorem similar to Theorems 2.3, 2.4, and 2.5 for the class $\underset{\sim}{F}$ is exactly as Theorem 2.5, with $f(\sigma, t)$ replaced by $f(\sigma)$. This follows from the fact that $\underset{\sim}{F} \subset \underset{\sim}{F}_t$. There is however one refinement possible which is due to the fact that the function $f(\sigma)$ does not depend on the variable t explicitly. This refinement leads to the Manley-Rowe frequency-power formulas and the Popov stability criterion.

Definitions: A function x from R into itself is said to be absolutely continuous if $\sum_{k=1}^{N-1} |x(t_k) - x(t_{k+1})| \to 0$ whenever $\sum_{k=1}^{N-1} |t_k - t_{k+1}| \to 0$, for any sequence $\{t_k\}$, $k = 1, 2, \ldots, N$, and any N. A classic result in analysis states that a function is absolutely continuous if and only if $x(t) = x(a) + \int_a^t r(t)\,dt$ for some function $r(t) \in L_1(a,b)$. Naturally $r(t) = \dot{x}(t)$ for almost all t. Let $S_2^1(a,b)$ be the subspace of $L_2(a,b)$ formed by the functions on $[a,b]$ which are absolutely continuous and which, together with their derivatives belong to $L_2(a,b)$. S_2^1 denotes $S_2^1(-\infty, +\infty)$. S_2^1 is an inner product space with the inner product as in L_2. It is however not complete.

Lemma 2.2: If $x \in S_2^1$ then $\lim_{t \to \pm\infty} x(t) = 0$

Proof: Since $\int_{-T_1}^{T_2} x(t)\dot{x}(t)\,dt = \frac{1}{2}\left[x(T_2)^2 - x(-T_1)^2\right]$ it follows that these limits exist since the limit on the left for T_1 or $T_2 \to \infty$ exists by the Schwartz inequality. Since the limits exist and since $x(t) \in L_2$ they must be zero.

Definition: A function f from R into R is said to satisfy a Lipshitz condition on R if $|f(\sigma_1) - f(\sigma_2)| \leq K|\sigma_1 - \sigma_2|$ for all $\sigma_1, \sigma_2 \in R$ and some K. K is called a Lipshitz constant for f. Clearly, if f satisfies a Lipshitz condition and if $x(t)$ is absolutely continuous, then $y(t) = f(x(t))$ is also absolutely continuous.

Theorem 2.6: Assume that $F \in \underset{\sim}{F}$ and that the f which defines F satisfies a Lipschitz condition on R. Then $\langle x, \frac{d}{dt} F x \rangle = 0$ for all $\in S_2^1$.

Proof: Let $y(t) = Fx(t)$ and let K be a Lipshitz constant for f. It is simple to show that $|\dot{y}(t)| \leq K|\dot{x}(t)|$ whenever both exist (and thus almost everywhere). Thus the above inner product is well defined since $y \in S_2^1$. Integration by parts yields

$$\int_{-\infty}^{+\infty} x(t)\frac{d}{dt}y(t)\,dt = -\int_{-\infty}^{+\infty} f(x(t))\frac{d}{dt}x(t)\,dt$$

$$= -\lim_{T\to\infty}\int_{x(-T)}^{x(T)} f(\sigma)\,d\sigma$$

$$= 0$$

The last equality follows from Lemma 2.2.

Remark: If $x \in S_2^1$, then the limit-in-the-mean transform of $\dot{x}$ exists and equals $j\omega X(j\omega)$ where $X(j\omega)$ is the limit-in-the-mean transform of x. Thus Theorem 2.6 merely states that

$$\int_{-\infty}^{+\infty} j\omega X(-j\omega)\,Y(j\omega)\,d\omega = 0$$

which is precisely the Manley-Rowe power-frequency formula for elements of L_2.

Theorems 2.5 and 2.6 combine to:

Theorem 2.7: Let $F \in \underset{\sim}{F}$ and assume that the f which determines F satisfies a Lipshitz condition. Then $(1 + \alpha \frac{d}{dt})F$ is a nonnegative (positive) operator on S_2^1 if and only if $\sigma f(\sigma) \geq 0$ $(\sigma f(\sigma) \geq \epsilon\sigma^2$ for some $\epsilon > 0)$ for all $\sigma \in R$.

Theorem 2.8: Let $F \in \underset{\sim}{F}$, and assume that the function f which determines F satisfies a Lipshitz condition on R. Let $G \in \underset{\sim}{G}$ be determined by $G(j\omega) = \frac{1}{1+\alpha j\omega}$. Then FG is a nonnegative operator on L_2 if and only if $\sigma f(\sigma) \geq 0$, for all $\sigma \in R$.

Proof: The theorem is a particular case of Theorem 2.5 if $\alpha = 0$. Let therefore $\alpha \neq 0$. Since the operator G corresponds to a convolution, Gx is absolutely continuous for all $x \in L_2$. Moreover, since $j\omega/1 + \alpha j\omega \in L_\infty$ for $\alpha \neq 0$, $Gx \in S_2^1$ for all $x \in L_2$. Thus

$$\begin{aligned} \langle x, FGx\rangle &= \langle (1+\alpha \tfrac{d}{dt})\,Gx, FGx\rangle \\ &= \langle Gx, FGx\rangle + \alpha \langle \tfrac{d}{dt}\,Gx, FGx\rangle \\ &\geq \alpha \langle \tfrac{d}{dt}\,Gx, FGx\rangle \end{aligned}$$

This last integral equals zero by Theorem 2.6, which proves the theorem.

Because of their importance, it is worthwhile to state analogues to Theorems 2.5 and 2.6 when $x(t)$ is an almost periodic function.

Theorem 2.5': Let f satisfy a Lipshitz condition on R and let x be almost periodic. Let $\{x_k\}$ and $\{y_k\}$ be the generalized Fourier series of x and y. If $\sigma f(\sigma) \geq 0$ for all σ, then

$$\sum_k \bar{x}_k y_k \geq 0$$

Theorem 2.6': Let f satisfy a Lipshitz condition and let x and $\dot{x}$ be almost periodic. Then $y(t) = f(x(t))$ is almost periodic. Let $\{x_k\}$ and $\{y_k\}$ be the generalized Fourier series of x and y. Then

$$\sum_k j\omega_k \bar{x}_k y_k = 0$$

Proof: The proofs are completely analogous to the proof of Theorems 2.5 and 2.6.

Remark: As pointed out by Penfield (45), the Manley-Rowe are essentially conservation laws and hold for a very wide class of systems.

2.5 Periodic Gain:

The result in this section is novel. It represents a positive operator formed by the interconnection of a periodically time-varying gain and a linear time-invariant convolution-type operator. The proof is very simple and the positive operator will lead to a rather elegant frequency-domain stability criterion which will be discussed in the next chapter.

Definitions: Let T be a positive number, and let $\underset{\sim}{K}_T$ denote the subclass of $\underset{\sim}{K}$ determined by the functions $k(t)$ which in addition satisfy $k(t+T) = k(t)$ for almost all t. Let $\underset{\sim}{G}_T$ denote the subclass of elements of $\underset{\sim}{G}$ determined by the functions $G(j\omega)$ which in addition satisfy $G(j(\omega + 2\pi T^{-1})) = G(j\omega)$ for almost all ω.

Lemma 2.3: Let $K \in \underset{\sim}{K}_T$ and $G \in \underset{\sim}{G}_T$. Then K and G commute on L_2, i.e., $KGx = GKx$ for all $x \in L_2$.

Proof: Since both K and G are bounded linear operators from L_2 into itself, KG and GK are. Thus by continuity of bounded linear operators, it suffices to prove the lemma for a dense set in L_2. Define the sequence $\{g_k\}$, $k \in I$ by

$$g_k = \frac{T}{2\pi} \int_0^{2\pi/T} G(j\omega)\, e^{-jk\omega T} d\omega$$

It follows from the theory of Fourier series that $\{g_k\} \in \ell_2$ and that

$$G(j\omega) = \text{l.i.m.} \sum_{k=-\infty}^{+\infty} g_k\, e^{jk\omega T} = \lim_{N\to\infty} \sum_{k=-N}^{+N} g_k\, e^{jk\omega T}$$

Let v be any element of $L_2 \cap L_1$. Then

$$w_N(t) = \sum_{k=-N}^{N} g_k\, v(t-kT) \in L_2 \cap L_1$$

Let V and W_N be the limit-in-the-mean transforms of v and w_N. Then

$$W_N(j\omega) = \sum_{k=-N}^{N} g_k\, e^{jk\omega T}\, V(j\omega)$$

Thus $\| W_N(j\omega) - G(j\omega)V(j\omega) \|_{L_2} = \| (G(j\omega) - \sum_{k=-N}^{N} g_k\, e^{jk\omega T}) V(j\omega) \|_{L_2}$

Since $V(j\omega) \in L_\infty$ and $\|V(j\omega)\|_{L_\infty} \leq \|v\|_{L_1}$. It follows from Hölder's inequality that

$$\| W_N(j\omega) - G(j\omega)V(j\omega) \|_{L_2} \leq \|v\|_{L_1}\; \| G(j\omega) - \sum_{k=-N}^{N} g_k\, e^{jk\omega T} \|_{L_2}$$

which since $\sum_{k=-N}^{N} g_k\, e^{jk\omega T}$ approaches $G(j\omega)$ in the L_2-sense shows that w_N approaches in the L_2-sense the function whose limit-in-the-mean transform is $G(j\omega)V(j\omega)$. Thus

$$w(t) = \text{l.i.m.} \sum_{k=-\infty}^{+\infty} g_k\, v(t-kT)$$

exists, belongs to L_2, and has $G(j\omega)V(j\omega)$ as limit-in-the-mean transform. This holds for all $v \in L_2 \cap L_1$. The lemma will now be proven for all $x \in L_2 \cap L_1$. Since then $Kx \in L_2 \cap L_1$, the above analysis applies for both x and Kx. However

$$k(t)\, g_k\, x(t-kT) = g_k\, k(t-kT)\, x(t-kT) \text{ for all } k \in I .$$

It follows thus that

$$k(t) \sum_{k=-N}^{N} g_k\, x(t-kT) = \sum_{k=-N}^{N} g_k\, k(t-kT)\, x(t-kT)$$

which after taking the limit-in-the-means of both sides and observing that $k \in L_\infty$ yields the lemma for all $x \in L_2 \cap L_1$. Since $L_2 \cap L_1$ is dense in L_2, the lemma follows.

Remark: The conclusion of Lemma 2.3 is immediate if one is satisfied with the following formal argument:

Since $Fx(t) = \sum_{n=-\infty}^{+\infty} f_n\, x(t-nT)$ and $k(t) = k(t-nT)$

$$KFx(t) = k(t) \sum_{n=-\infty}^{+\infty} f_n\, x(t-nT) = \sum_{n=-\infty}^{+\infty} f_n\, k(t-nT)\, x(t-nT) = FKx(t)$$

Definition: An operator 0 from X into itself is said to possess a square root, denoted by $0^{1/2}$, if there exists an operator, $0^{1/2}$, from X into itself such that $0 = 0^{1/2} 0^{1/2}$.

Lemma 2.4: Let $K \in \underset{\sim}{K}$ be determined by $k(t)$, and assume that $k(t) \geq 0$. The $K^{1/2}$ exists. Moreover $K^{1/2} \in \underset{\sim}{K}$ and $K^{1/2} \in \underset{\sim}{K}_T$ if $K \in \underset{\sim}{K}_T$.

Proof: The element of $\underset{\sim}{K}$ determined by $k(t)^{1/2}$ possesses all the required properties.

Theorem 2.9: Let $K \in \underset{\sim}{K}_T$ and let $G \in \underset{\sim}{G}_T$. Then KG and GK are nonnegative (positive) operators on L_2 if $k(t) > 0$ and if

Re $G(j\omega) \geq 0$ ($k(t) \geq \epsilon$ and Re $G(j\omega) \geq \epsilon$ for some $\epsilon > 0$) for almost all t and $\omega \geq 0$. Moreover, the elements of $\underset{\sim}{G}_T$ which satisfy the above inequality are the most general elements of $\underset{\sim}{G}$ which yield nonnegative (positive) operators KG and GK for all $K \in \underset{\sim}{K}_T$ which are determined by a $k(t)$ satisfying $k(t) \geq 0$ ($k(t) \geq \epsilon$ for some $\epsilon > 0$) for almost all t.

Proof: The first part of the theorem follows from Lemma 2.3 if it is proven for KG. But by Lemmas 2.3 and 2.4

$$< x, KGx > = < x, K^{1/2} G K^{1/2} x >$$

Since $K^{1/2} \in \underset{\sim}{K}$, it is self-adjoint, and thus

$$< x, KGx > = < K^{1/2}x, GK^{1/2}x >$$

which is nonnegative by Theorem 2.3. To prove the positivity condition, write KG as $KG = (K - \epsilon I)G + \epsilon G$ and apply the previous part of this theorem and Theorem 2.3.

For the converse part of the theorem, assume first that Re $G(j\omega) < 0$ for all ω in a set of positive measure. Then picking $K = I$ and applying Theorem 2.3 yields the result by contradiction. Assume next that Re $G(j\omega) \geq 0$ for almost all ω, but that $G(j(\omega + 2\pi T^{-1})) - G(j\omega) \neq 0$ for all ω in a set of positive measure, say Ω. This part of the theorem is proven by choosing particular functions for $k(t)$ and $x(t)$ which lead to $< x, KGx > < 0$. For simplicity assume that $\mathrm{Re}(G(j\omega) - G(j(\omega + 2\pi T^{-1}))) < 0$ on the set Ω. (A similar argument holds for the other cases). Then there exists a $\epsilon > 0$ such that $\mathrm{Re}(G(j\omega) - G(j(\omega + 2\pi T^{-1}))) \leq -\epsilon$ for all $\omega \in \Omega'$ with $\Omega' \subset \Omega$ a set of positive measure. Let Ω''_n be a subset of $[n 2\pi T^{-1}, (n+1)2\pi T^{-1}] \cap \Omega'$ which is a positive measure (such a subset exists since Ω' is of positive measure). Let $\Omega''_n + k2\pi T^{-1}$ denote the set of all points x such that $x - k2\pi T^{-1} \epsilon \Omega''_n$. Pick $X(j\omega) = 1$ for $\omega \epsilon \Omega''_n + k2\pi T^{-1}$, $k \epsilon I$ and $|k| \leq N$, and $X(j\omega) = 0$ otherwise, and pick $k(t) = 1 - \cos 2\pi T^{-1} t$. Clearly $k(t) \geq 0$ and the K corresponding to k(t) belongs to K_T. Let $y = KGx$. Then $Y(j\omega) = G(j\omega)X(j\omega) - \frac{1}{2} G(j(\omega + 2\pi T^{-1})) X(j(\omega + 2\pi T^{-1})) - \frac{1}{2} G(j(\omega - 2\pi T^{-1})) X(j(\omega - 2\pi T^{-1}))$. A simple calculation shows that the inner product $< x, KGx >$ becomes $M + \frac{N}{\pi} \operatorname{Re}(G(j\omega) - G(j(\omega + 2\pi T^{-1}))) \mu(\Omega''_n)$, with M a number independent of N, and $\mu(\Omega''_n)$ the Lebesgue measure of Ω''_n. Thus $< x, KGx >$ can be made negative by choosing N sufficiently large. This ends the proof of Theorem 2.9.

Remark: Theorem 2.9 essentially shows that the operator K can be composed with a class of convolution operators without destroying the positivity. Similar positive operators are, either implicitly or explicitly, the basis of most of the recently discovered frequency-domain stability criteria for feedback loops containing a time-invariant convolution operator in the forward loop and a nonlinear time-varying element in the feedback loop. For the case in which the feedback loop is an operator of the class $\underset{\sim}{K}$ the positive operator obtained by Gruber and Willems (26), and in its full generality by Zames and Freedman (65) seems particularly interesting. By restricting the derivative of k(t), they obtain a class of convolution operators which can be composed with $K \epsilon \underset{\sim}{K}$ such that positivity is not destroyed. This idea is used in the latter reference to obtain a very elegant stability criterion.

2.6 Positive Operators with Monotone or Odd-Monotone Nonlinearities

In this section an answer to the following question is given: What is the most general linear operator which when composed with a monotone nondecreasing (or an odd-monotone nondecreasing) nonlinearity yields a positive operator? The answer to this question represents in some sense the solution of a problem which has been studied by many previous researchers. In particular it is the problem studied by Page (43), Pantell (44), and Black (7) in connection with frequency-power formulas and it plays a central role in the determination of stability criteria for feedback systems with a monotone or an odd-monotone nonlinearity in the feedback loop. In the latter context it has been treated by Brockett and Willems (10), Narendra and Neuman (39), Zames(63), O'Shea (41,42), Zames and Falb (64), Thathachar, Srinath and Ramapriyan (55), and others.

The preliminary result obtained in this section constitutes a considerable extension of a classical rearrangement inequality. This inequality then forms the basis from which the positive operators of this section are derived. It is felt that these rearrangement inequalities are of intrinsic importance and are potentially useful in other areas of system theory. For various technical reasons, the discussion is mainly concerned with sequences. With some modifications, similar results can be obtained for the continuous case.

2.6.1 Generalizations of a Classical Rearrangement Inequality

Chapter X of Hardy, Littlewood and Polya's classic book on inequalities (29) is devoted to questions relating the inner products of similarly ordered sequences to the inner products of rearranged sequences. The simplest result given there states that if $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$ and if $y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n)}$ is any rearrangement of the y-sequence then

$$\sum_{k=1}^{n} x_k\, y_k \geq \sum_{k=1}^{n} x_k y_{\pi(k)}$$

The informal explanation of this fact given in (29) is that given a lever arm with hooks at distances $x_1, x_2, \ldots, x_n$ from a pivot and weights $y_1, y_2, \ldots, y_n$ to hang on the hooks, the largest moment is obtained by hanging the largest weight on the farthest hook, the next largest weight on the next most distant hook, etc.

This result has an interpretation in terms of positive operators. Suppose that f is a function from R into itself, and denote by x and Fx the n-vectors whose components are $x_1, x_2, \ldots, x_n$ and $f(x_1), f(x_2), \ldots f(x_n)$. Then in language of positive operators the Hardy, Littlewood and Polya rearrangement theorem says that the operator on R^n defined by $0x = (I - P)Fx$ is nonnegative if I is the identity matrix, P is any permutation matrix and f is monotone nondecreasing.

It will be shown that this result together with a result of Birkhoff on the decomposition of doubly stochastic matrices permits the derivation of a number of interesting positivity conditions for a class of operators. The results thus represent a test for checking the positivity of a class of nonquadratic forms parallel to the Sylvester test for checking the positive definiteness of a symmetric matrix. This result is less important only because quadratic forms which go hand in hand with linear transformations and linear systems are used more often and can thus be considered to be more important than non-quadratic forms which go hand in hand with nonlinear transformations and nonlinear systems.

Remark: It is interesting to notice that Prosser (48) and Black(7) have used the Hardy, Littlewood and Polya rearrangement inequality as the basis to prove that the crosscorrelation of the input and the output to a monotone nondecreasing nonlinearity attains its maximum value at the origin.

Definitions: Two sequences of real numbers $\{x_1, x_2, \ldots x_n\}$ and $\{y_1, y_2, \ldots y_n\}$ are said to be similarly ordered if the inequality $x_k < x_\ell$ implies that $y_k \leq y_\ell$. Thus two sequences are similarly ordered if and only if they can be rearranged in such a way that the resulting sequences are both monotone nondecreasing, i.e., there exists a permutation $\pi(k)$ of the first n integers ($\pi(k)$ takes on each of the values $1, 2, \ldots, n$ just once as k varies through the values $1, 2, \ldots, n$) such that both the sequences $\{x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}\}$ and $\{y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n)}\}$ are monotone nondecreasing. Two sequences are said to be unbiased if $x_k y_k \geq 0$. Clearly two sequences are similarly ordered and unbiased if and only if the augmented sequences $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ and $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$ with $x_{n+1} = y_{n+1} = 0$ are similarly ordered. Two sequences are said to be similarly ordered and symmetric if they are unbiased and if the sequences $\{|x_1|, |x_2|, \ldots, |x_n|\}$ and $\{|y_1|, |y_2|, \ldots, |y_n|\}$ are similarly ordered.

Example: Let $f(\sigma)$ be a mapping from the real line into itself, and consider the sequences $\{x_1, x_2, \ldots x_n\}$ and $\{f(x_1), f(x_2), \ldots, f(x_n)\}$. These two sequences will be similarly ordered for all sequences $\{x_1, x_2, \ldots, x_n\}$ if and only if $f(\sigma)$ is a monotone nondecreasing function of σ, i.e., if for all σ_1 and σ_2, $(\sigma_1 - \sigma_2)(f(\sigma_1) - f(\sigma_2)) \geq 0$. They will be unbiased if and only if $f(\sigma)$ is a first and third quadrant function, i.e., if for all σ, $\sigma f(\sigma) \geq 0$. They will be similarly ordered and symmetric if and only if $f(\sigma)$ is an odd monotone nondecreasing function of σ, i.e., if $f(\sigma)$ is monotone nondecreasing and $f(\sigma) = -f(-\sigma)$ for all σ.

Definitions*: A real $(n\times n)$ matrix $M = (m_{k\ell})$ is said to be doubly hyperdominant with zero excess if $m_{k\ell} \leq 0$ for $k \neq \ell$, and if $\sum_{k=1}^{n} m_{k\ell} = \sum_{\ell=1}^{n} m_{k\ell} = 0$ for all k, ℓ. It is said to be doubly hyperdominant if $m_{k\ell} \leq 0$ for $k \neq \ell$, and if $\sum_{k=1}^{n} m_{k\ell} \geq 0$ and $\sum_{\ell=1}^{n} m_{k\ell} \geq 0$ for all k, ℓ. A $(n\times n)$ matrix M is said to be doubly dominant if $m_{\ell\ell} \geq \sum_{\substack{k=1 \\ k\neq\ell}}^{n} |m_{k\ell}|$ and $m_{kk} \geq \sum_{\substack{\ell=1 \\ \ell\neq k}}^{n} |m_{k\ell}|$. It is clear that all of the classes of matrices introduced above are subclasses of the class of all matrices whose symmetric part is nonnegative definite and that every doubly hyperdominant matrix is doubly dominant.

Two other classes of matrices which will be used in the sequel and have received ample attention in the past are defined below.

Definitions: A $(n\times n)$ matrix M is said to be doubly stochastic if it is a nonnegative matrix (i.e., $m_{k\ell} \geq 0$ for all k, ℓ) and if its rows and columns sum to one. A $(n\times n)$ matrix is said to be a permutation matrix if every row and column contains $n-1$ zero elements and an element which equals one. The relation between the

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* The term dominant is standard. Hyperdominant is prevalent, at least in the electrical network literature. The term doubly is used by analogy with doubly stochastic where a property of a matrix also holds for its transpose. Beyond this the nomenclature originates with the author.

class of doubly stochastic matrices and permutation matrices is given in the following lemma due to Birkhoff.

Lemma 2.5 (Birkhoff): The set of all doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices, i.e., if M is a doubly stochastic matrix then

$$M = \sum_{i=1}^{N} a_i P_i$$

with $a_i \geq 0$, $\sum_{i=1}^{N} a_i = 1$ and P_i a permutation matrix. This decomposition need not be unique.

Proof: A short proof can be found in (37)

Theorem 2.10 states the main result of this section and constitutes a considerable generalization of a classical rearrangement inequality due to Hardy, Littlewood and Polya (29). This inequality is stated in Lemma 2.6.

Lemma 2.6 (Hardy, Littlewood and Polya): Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ be two similarly ordered sequences, and let $\pi(k)$ be a permutation of the first n integers. Then

$$\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\pi(k)}$$

Proof: A simple proof can be found in (29). A convincing plausibility argument is given in the introduction to this section.

Theorem 2.10: A necessary and sufficient condition for the bilinear form $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell$ to be nonnegative for all similarly ordered sequences $\{x_1, x_2, \ldots x_n\}$ and $\{y_1, y_2, \ldots y_n\}$ is that the matrix $M = (m_{k\ell})$ be doubly hyperdominant with zero excess.

Proof: (i) Sufficiency: Let M be a doubly hyperdominant matrix with zero excess and let r be any positive number such that $r \geq m_{k\ell}$ for all k, ℓ. Clearly $M = r(I - \frac{1}{r}(rI-M))$. Since however $\frac{1}{r}(rI-M)$ is a doubly stochastic matrix, it can, by Lemma 2.5, be decomposed as $\sum_{i=1}^{N} \alpha_i P_i$ with $\alpha_i \geq 0$, $\sum_{i=0}^{N} \alpha_i = 1$ and P_i a permutation matrix. Thus M can be written as

$$M = \sum_{i=1}^{N} \beta_i (I-P_i) \quad \text{with} \quad \beta_i \geq 0$$

This decomposition of doubly hyperdominant matrices with zero excess shows that it is enough to prove the sufficiency part of Theorem 2.10 for the matrices $I-P_i$. This however is precisely what is stated in Lemma 2.6.

(ii) Necessity: The matrix M may fail to be doubly hyperdominant with zero excess because $m_{k\ell} > 0$ for some $k \neq \ell$ in which case the sequences with $n-1$ zero elements except $+1$ and -1 in respectively the k-th and ℓ-th spots lead to $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell = -m_{k\ell} < 0$. Assume next that the matrix M fails to be doubly hyperdominant with zero excess because $\sum_{k=1}^{n} m_{k\ell} \neq 0$ for some ℓ (a similar argument holds if $\sum_{\ell=1}^{n} m_{k\ell} \neq 0$ for some k), and consider the similarly ordered sequences $\{1, \ldots, 1, 1+\epsilon, 1, \ldots, 1\}$ and $\{0, \ldots, 0, \epsilon^{-1}, 0, \ldots, 0\}$ with $\epsilon \neq 0$, and the elements $1+\epsilon$ and ϵ^{-1} in the ℓ-th spot. This leads to $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell = \epsilon^{-1} \sum_{k=1}^{n} m_{k\ell} + m_{\ell\ell}$. By taking ϵ sufficiently small and of an appropriate sign $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell$ can thus be made negative.

The following two theorems are generalizations of Theorem 2.10 to similarly ordered unbiased and to similarly ordered symmetric sequences.

Theorem 2.11: A necessary and sufficient condition for the bilinear form $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell$ to be nonnegative for all similarly ordered unbiased sequences $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ is that the matrix $M = (m_{k\ell})$ be doubly hyperdominant.

Proof: (i) Sufficiency: Let M be a doubly hyperdominant matrix and define $m_{k,n+1} = -\sum_{\ell=1}^{n} m_{k\ell}$, $m_{n+1,\ell} = -\sum_{k=1}^{n} m_{k\ell}$ for $k, \ell \leq n$, and $m_{n+1,n+1} = \sum_{k,\ell=1}^{n} m_{k\ell}$. Then taking $x_{n+1} = y_{n+1} = 0$ it follows from Theorem 2.1 that $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell = \sum_{k,\ell=1}^{n+1} m_{k\ell} x_k y_\ell \geq 0$ since the augmented $(n+1 \times n+1)$ matrix $M_* = (m_{k\ell}), k, \ell = 1, 2, \ldots, n+1$ is doubly hyperdominant with zero excess and since the sequences $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ and $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$ with $x_{n+1} = y_{n+1} = 0$ are similarly ordered.

(ii) Necessity: The same sequences as in Theorem 2.10 can be used if the matrix M fails to be doubly hyperdominant because $m_{k\ell} > 0$ for some $k \neq \ell$. Assume next that the matrix M fails to be doubly hyperdominant because $\sum_{k=1}^{n} m_{k\ell} < 0$ for some ℓ (a similar argument holds if $\sum_{\ell=1}^{n} m_{k\ell} < 0$ for some k), and consider the sequences used in Theorem 2.10 with the additional restriction that $\epsilon > 0$. Notice that these sequences are similarly ordered and unbiased. It follows then that by taking $\epsilon > 0$ sufficiently small $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell = \epsilon^{-1} \sum_{k=1}^{n} m_{k\ell} + m_{\ell\ell}$ can be made negative.

Theorem 2.12: A necessary and sufficient condition for the bilinear form $\sum_{k,\ell=1}^{n} m_{k\ell} x_k y_\ell$ to be nonnegative for all similarly ordered symmetric sequences $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ is that the matrix $M = (m_{k\ell})$ be doubly dominant.

Proof: (i) Sufficiency: Let M be a doubly dominant matrix. Clearly

$$\sum_{k,\ell=1}^{n} m_{k\ell}x_k y_\ell > \sum_{\substack{k,\ell=1\\k=\ell}}^{n} m_{k\ell}\,|x_k|\,|y_\ell| - \sum_{\substack{k,\ell=1\\k\neq\ell}}^{n} |m_{k\ell}|\,|x_k|\,|y_\ell|$$

The right hand side of the above inequality is nonnegative by Theorem 2.11 since the matrix $M_*=(m^*_{k\ell})$ with $m^*_{k\ell}=m_{k\ell}$ when $k=\ell$ and $m^*_{k\ell}=-|m_{k\ell}|$ when k/ℓ is doubly hyperdominant and since the sequences $\{|x_1|, |x_2|, \ldots, |x_n|\}$ and $\{|y_1|\ |y_2|, \ldots |y_n|\}$ are similarly ordered and unbiased. This implies that $\sum_{k,\ell=1}^{n} m_{k\ell}x_k y_\ell \geq 0$.

(ii) Necessity: Assume that the matrix M fails to be doubly dominant because $m_{\ell\ell} - \sum_{\substack{k=1\\k\neq\ell}}^{n} |m_{k\ell}| < 0$ for some ℓ (an analogous argument holds if $m_{kk} - \sum_{\substack{\ell=1\\\ell\neq k}}^{n} |m_{k\ell}| < 0$ for some k), and consider the sequences $\{-\text{sgn}\, m_{1\ell}, \ldots, -\text{sgn}\, m_{\ell-1,\ell}, 1+\epsilon, -\text{sgn}\, m_{\ell+1,\ell}, \ldots, -\text{sgn}\, m_{n\ell}\}$ and $\{0, \ldots, 0, \epsilon^{-1}, 0, \ldots, 0\}$ with $\text{sgn}\, a = \frac{a}{|a|}$ if $a\neq 0$, $\text{sgn}\, 0=0, \epsilon>0$ and $1+\epsilon$ and ϵ^{-1} elements in the ℓ-th spots. These sequences are simarly ordered and symmetric and lead to $\sum_{k,\ell=1}^{n} m_{k\ell}x_k y_\ell = \epsilon^{-1}(m_{\ell\ell} - \sum_{\substack{k=1\\k\neq\ell}}^{n} |m_{k\ell}|) + m_{\ell\ell}$ which by taking ϵ sufficiently small yields $\sum_{k,\ell=1}^{n} m_{k\ell}x_k y_\ell < 0$.

Let f be a mapping from R into R and denote by F the mapping from R_n into itself which takes the element $\text{col}(x_1, x_2, \ldots, x_n)$ into $\text{col}(f(x_1), f(x_2), \ldots, f(x_n))$. Then in terms of positive operators Theorems 2.10 to 2.12 become:

Theorem 2.13: Let M be an $(n\times n)$ matrix and let f be any

(i) monotone nondecreasing function

(ii) monotone nondecreasing first and third quadrant function

(iii) odd-monotone nondecreasing function

Then MF is a nonnegative operator on R^n for all mappings f satisfying the above conditions if and only if the matrix M is

(i) doubly hyperdominant with zero excess

(ii) doubly hyperdominant

(iii) doubly dominant

2.6.2 Extension to ℓ_2-summable Sequences

In this section ℓ_p is taken over the field of real numbers unless otherwise mentioned.

Definitions: Let $\mathcal{L}(\ell_2, \ell_2)$ denote all bounded linear transformations from ℓ_2 into itself. Let $R \in \mathcal{L}(\ell_2, \ell_2)$. Then R determines (see e.g. (2, p.50)) an array of real numbers $\{r_{k\ell}\}, k, \ell \in I$, such that $y=Rx$ is defined by $y_k = \sum_{k=-\infty}^{+\infty} r_{k\ell} x_\ell$ for $x = \{x_k\}$ and $y = \{y_k\}, k \in I$. This infinite sum exists for all $x \in \ell_2$ and the resulting sequence belongs to ℓ_2. A standard result in the theory of bounded linear operators in Hilbert space (see e.g., (2 ; p.52)) states that the array $\{r^*_{k\ell}\}$, $k, \ell \in I$ corresponding to the adjoint of R, R^*, satisfies $r^*_{k\ell} = r_{\ell k}$ for all ℓ, $k \in I$. It is not known what arrays in turn determine elements of $\mathcal{L}(\ell_2, \ell_2)$. The following lemma however covers a wide class.

Lemma 2.7: Let the array $\{r_{k\ell}\} k,\ell \in I$ be such that the sequences $\{r_{k\ell}\}$ belong to ℓ_1 for fixed k and ℓ, uniformly in k and ℓ, i.e., there exists an M such that $\sum_{\ell=-\infty}^{+\infty} |r_{k\ell}| \leq M$ and $\sum_{k=-\infty}^{+\infty} |r_{k\ell}| \leq M$. Then $\{r_{k\ell}\}$ determines an element R of $\mathcal{L}(\ell_2, \ell_2)$ and $\|R\| \leq M$.

Proof: The Schwartz inequality and Fubini's Theorem for sequences (17, p.245) yield the following inequalities

$$\left(\sum_{k=-\infty}^{+\infty} \left| \sum_{\ell=-\infty}^{+\infty} r_{k\ell} x_\ell \right|^2 \right)^{1/2} \leq \left(\sum_{k=-\infty}^{+\infty} \left(\sum_{\ell=-\infty}^{+\infty} |r_{k\ell}| \; |x_\ell| \right)^2 \right)^{1/2}$$

$$\leq \left(\sum_{k=-\infty}^{+\infty} \left(\sum_{\ell=-\infty}^{+\infty} |r_{k\ell}| \right) \left(\sum_{\ell=-\infty}^{+\infty} |r_{k\ell}| \; |x_\ell|^2 \right) \right)^{1/2}$$

$$\leq M \left(\sum_{\ell=-\infty}^{+\infty} |x_\ell|^2 \right)^{1/2}$$

In what follows an important role will be played by some particular elements of $\mathcal{L}(\ell_2, \ell_2)$ and some particular sequences which will now be introduced.

Definitions: The definitions of similararaly ordered, similarly ordered unbiased and similarly ordered symmetric infinite sequences are completely analogous to the case of finite sequences and will not be repeated here. It is possible to show that two sequences in ℓ_2 are similarly ordered if and only if they are similarly ordered and unbiased. Let M be an element of $\mathcal{L}(\ell_2, \ell_2)$, and let $\{m_{k\ell}\}, k, \ell \in I$ be the associated array. M is said to be doubly hyperdominant if $m_{k\ell} \leq 0$ for $k \neq \ell$ and if $\sum_{k=-\infty}^{+\infty} m_{k\ell}$ and $\sum_{\ell=-\infty}^{+\infty} m_{k\ell}$ exist and are nonnegative for all ℓ and k. M is said to be doubly dominant if $m_{\ell\ell} \geq \sum_{\substack{k=-\infty \\ k \neq \ell}}^{+\infty} |m_{k\ell}|$ and $m_{kk} \geq \sum_{\substack{\ell=-\infty \\ \ell \neq k}}^{+\infty} |m_{k\ell}|$.

It is clear from Lemma 2.7 that if an array or real numbers $\{m_{k\ell}\}, k, \ell \in I$ satisfies the doubly dominance condition and if the sequence $\{m_{kk}\} \in \ell_\infty$, then $\{m_{k\ell}\}$ determines an element, M, of

$\mathcal{L}(\ell_2, \ell_2)$ with $\|M\| \leq 2 \sup_{k \in I} m_{kk}$. Thus it is a simple matter to check whether an element of $\mathcal{L}(\ell_2, \ell_2)$ is doubly hyperdominant or doubly dominant.

The following extension of Theorems 2.11 and 2.12 holds:

Theorem 2.14: Let M be an element of $\mathcal{L}(\ell_2, \ell_2)$. Then a necessary and sufficient condition for the inner product $\langle x, My \rangle$ to be nonnegative for all

(i) similarly ordered unbiased ℓ_2-sequences x and y

(ii) similarly ordered symmetric ℓ_2-sequences x and y

is that M be

(i) doubly hyperdominant

(ii) doubly dominant

Proof: It is clear that all finite subsequences of x and y are similarly ordered and unbiased or similarly ordered and symmetric. Hence, by Theorems 2.11 and 2.12 all finite truncations of the infinite sum in the inner product $\langle x, My \rangle$ yield a nonnegative number. Thus the limit, since it exists, is also nonnegative.

Of particular interest are the arrays $\{r_{k\ell}\}$, $k, \ell \in I$ for which the entries depend on the difference of the indices k and ℓ only. These arrays are said to be of the Toeplitz type and have been intensively studied in classical analysis (see e.g., (25)). It follows from Lemma 3.7 that if the array $\{r_{k\ell} = r_{k-\ell}\}$, $k, \ell \in I$ is of the Toeplitz type then it determines an element of $\mathcal{L}(\ell_2, \ell_2)$ if $\{r_k\}, k \in I$, belongs to ℓ_1. (In fact the elements of $\mathcal{L}(\ell_2, \ell_2)$ for which the associated array is of the Toeplitz type stand in one-to-one correspondence to all ℓ_2-summable sequences whose limit-in-the-mean z-transform belongs to L_∞ for $|z| = 1$.) An element of $\mathcal{L}(\ell_2, \ell_2)$ is said to be of the Toeplitz type if the associated array is of the Toeplitz type. An element R of $\mathcal{L}(\ell_2, \ell_2)$ which is of the Toeplitz type determines thus a sequence $\{r_k\}, k \in I$ with $\{r_k\} \in \ell_2$ and whose limit-in-the-mean z-transform belongs to L_∞ for $|z| = 1$. The importance of these linear transformations stems from the fact that they define convolution operators with a time-invariant kernel and are therefore closely associated with time-invariant systems.

Definitions: A sequence of real numbers $\{a_k\}, k \in I$, is said to be hyperdominant if $\{a_k\} \in \ell_1$, if $a_k \leq 0$ for all $k \neq 0$ and if $\sum_{k=-\infty}^{+\infty} a_k \geq 0$. It is said to be dominant if $\{a_k\} \in \ell_1$, and if $2a_0 \geq \sum_{k=-\infty}^{+\infty} |a_k|$.

Theorem 2.15: Let M be an element of $\mathcal{L}(\ell_2, \ell_2)$ which is of the Toeplitz type. Then a necessary and sufficient condition for the inner product $<x, My>$ to be nonnegative for all

(i) similarly ordered unbiased ℓ_2-sequences x and y

(ii) similarly ordered symmetric ℓ_2-sequences x and y

is that the sequence $\{m_k\}$ which is determined by M be

(i) hyperdominant

(ii) dominant

Proof: This theorem is a special case of Theorem 2.14.

Theorems 2.14 and 2.15 have an obvious interpretation in terms of positive operators. Moreover Theorem 2.15 yields some simple properties of the input and the output spectra to (odd) monotone nondecreasing nonlinearities. This is stated explicitly in Theorem 2.15'.

Definitions: Let $\underset{\sim}{A}$ denote the class of operators from ℓ_2 into itself, each element, A, of which has associated with it a function A(z) with $A(z) \in L_\infty$ for $|z| = 1$, with $A(\bar{z}) = \overline{A(z)}$ and which maps an element x of ℓ_2 as follows: let X be the limit-in-the-mean z-transform of x. Then the sequence y is the inverse z-transform of the function $A(z)X(z)$.

Let $\underset{\sim}{F}$ denote the class of operators from ℓ_2 into itself, each element, F, of which has associated with it a function, $f(\sigma)$, from R into itself, satisfying the inequality $|f(\sigma)| \leq M|\sigma|$ for some M and all σ, and which maps the sequence $x=\{x_k\}, k \in I$ of ℓ_2 into the sequence $y=\{y_k\}$ with $y_k=f(x_k)$.

It is a simple matter to verify that these operators are indeed well defined, i.e., that they map ℓ_2 into itself. The class $\underset{\sim}{A}$ stands in one-to-one correspondence with all ℓ_2-sequences whose limit-in-the-mean z-transform belongs to L_∞ for $|z|=1$. Moreover if $\{a_k\} \in \ell_2$ and $A(z) \in L_\infty$ for $|z|=1$ are such a sequence and its limit-in-the-mean z-transform then the element of $\underset{\sim}{A}$ which has the function $A(z)$ corresponding with it maps ℓ_2 into itself by the convolution

$$y_k = \sum_{\ell=-\infty}^{+\infty} a_{k-\ell} x_\ell$$

<u>Theorem 2.15': Let $A \in \underset{\sim}{A}$ and $F \in \underset{\sim}{F}$. Then AF is a non-negative operator on ℓ_2 if</u>

(i) <u>the f corresponding to F is a (odd) monotone non-decreasing first and third quadrant function</u>

(ii) <u>the inverse z-transform of $A(z)$ is hyperdominant (dominant)</u>

<u>Moreover the elements of $\underset{\sim}{A}$ satisfying (ii) are the most general elements of $\underset{\sim}{A}$ which yield a nonnegative operator, AF, on ℓ_2, for any $F \in \underset{\sim}{F}$ satisfying (i). AF is a positive operator on ℓ_2 if</u>

$A-\epsilon I$ and $F-\epsilon I$ satisfy (i) and (ii) for some $\epsilon > 0$.

Proof: The theorem follows from Theorem 2.15.

Theorem 2.15' thus states that if $X(z)$ and $Y(z)$ are the limit-in-the-mean z-transforms of the input and the output of a (odd) monotone nondecreasing nonlinearity then

$$\oint_{|z|=1} A(z)X(z)\overline{Y(z)}\, dz \geq 0$$

where $A(z)$ is the z-transform of any (dominant) hyperdominant sequence.

2.6.3 Frequency-Power Relations for Nonlinear Resistors

In this section a class of positive operators formed by the composition of a linear time-invariant convolution operator and a (odd) monotone nondecreasing nonlinearity will be derived. The analysis is done for operators on L_2 but the results are also stated for almost-periodic functions thus placing the positive operators obtained in this section in the context of the classical frequency-power relations for nonlinear resistors.

In this section L_p is taken over the field of real numbers unless otherwise mentioned.

Definitions: Let $\underset{\sim}{M}$ denote the class of operators from L_2 into itself each element of which belongs to $\underset{\sim}{F}$ and for which the associated function f is a monotone nondecreasing function, i.e., $(\sigma_1-\sigma_2)(f\sigma_1)-f(\sigma_2)) \geq 0$ for all $\sigma_1, \sigma_2 \in R$.

Let $\underset{\sim}{S}$ denote the class of operators from L_2 into itself each element of which belongs to $\underset{\sim}{M}$ and for which the associated function f is in addition an odd function, i.e., $f(\sigma) = -f(-\sigma)$ for all $\sigma \in R$.

Let $x_1, x_2 \in L_2$. Then $x_2(t+\tau) \in L_2$ for all $\tau \in R$, and $\|x_2(t+\tau)\|_{L_2} = \|x_2(t)\|_{L_2}$. The crosscorrelation function of x_1 and x_2 is defined as the function $R_{x_1x_2}(\tau) = \langle x_1(t), x_2(t+\tau)\rangle$. Note that the Schwartz inequality yields that $|R_{x_1x_2}(\tau)| \leq \|x_1\|_{L_2}\|x_2\|_{L_2}$. Moreover, since the limit-in-the-mean transforms of $x(t)$ and $x(t+\tau)$ are given by $X(j\omega)$ and $X(j\omega)e^{j\omega\tau}$ respectively it follows thus from Parseval's relation that $R_{x_1x_2}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{X_1(j\omega)}X_2(j\omega)e^{j\omega\tau}d\omega$.

The theorem which follows is a generalization of a well-known fact about autocorrelation functions: it states that the crosscorrelation function of x and y attains its maximum at the origin provided x and y are related through a monotone nondecreasing nonlinearity.

Theorem 2.16: Let $F \in \underset{\sim}{M}$, $x \in L_2$ and let $y = Fx$. Then $R_{xy}(0) \geq R_{xy}(t)$ for all $t \in R$. If F belongs in addition to $\underset{\sim}{S}$, then $R_{xy}(0) \geq |R_{xy}(t)|$ for all $t \in R$.

Proof: Let $F(\sigma) = \int_0^{\sigma} f(x)dx$. $F(\sigma)$ is a convex function of σ since its derivative exists and is monotone nondecreasing. The convex function inequality (5) yields that $(\sigma_1 - \sigma_2)f(\sigma_1) \geq F(\sigma_1) - F(\sigma_2)$ for all $\sigma_1, \sigma_2 \in R$. (This inequality can simply be obtained by integrating $f(\sigma) - f(\sigma_1)$ versus σ from σ_1 to σ_2.) Taking $\sigma_1 = x(t+\tau)$ and $\sigma_2 = x(t)$ it follows thus that

$$(x(t) - x(t+\tau))\, y(t) \geq F(x(t)) - F(x(t+\tau))$$

which yields, after integration, that

$$R_{xy}(0) - R_{xy}(\tau) \geq \int_{-\infty}^{+\infty} F(x(t))dt - \int_{-\infty}^{+\infty} F(x(t+\tau))dt = 0$$

The integrals on the right hand side exist since by assumption $F \in \underset{\sim}{M}$ and thus $|f(\sigma)| \leq K|\sigma|$ for some K and all $\sigma \in R$, which implies that $|F(\sigma)| \leq \frac{1}{2}K|\sigma|^2$ for all $\sigma \in R$. Hence $R_{xy}(0) \geq R_{xy}(t)$ for all $F \in \underset{\sim}{M}$ and $t \in R$.

If f is in addition odd then the convex function inequality can be rewritten as $(\sigma_1 - (-\sigma_2))f(\sigma_1) \geq F(\sigma_1) - F(-\sigma_2)$, which using the fact that f is an odd function yields that $(\sigma_1 + \sigma_2)f(\sigma_1) \geq F(\sigma_1) - F(\sigma_2)$. Using exactly the same argument as above this then leads to $R_{xy}(0) + R_{xy}(t) \geq 0$ for all $t \in R$. Thus $R_{xy} \geq |R_{xy}(t)|$ for all $F \in \underset{\sim}{S}$ and $t \in R$.

Remark: Using an analogous argument as the one used in (59), it can be shown that the above theorem is also sufficient in the sense that if $y = Fx$ for some $F \in \underset{\sim}{F}$ and if $R_{xy}(0) \geq R_{xy}(t)$ $(R_{xy}(0) \geq |R_{xy}(t)|)$, for all $x \in L_2$ and $t \in R$, then $F \in \underset{\sim}{M}(\underset{\sim}{S})$.

Theorem 2.17: Let $F \in \underset{\sim}{M}(\underset{\sim}{S})$ and let $G \in \underset{\sim}{G}$ be determined by the function $G(j\omega)$ given by the Fourier-Stieltjes integral

$$G(j\omega) = 1 - \int_{-\infty}^{+\infty} e^{-j\omega\tau} dV(\tau)$$

where $V(\tau)$ is any monotone nondecreasing function (any function of bounded variation) of total variation less than or equal to unity. Then GF is a nonnegative operator on L_2.

Proof: Assume first that $F \in \underset{\sim}{F}$. This theorem follows then from the previous theorem if it is noted that $R_{xy}(0) \geq 0$ and that the operator G corresponds to the convolution defined by

$$y = Gx = x(t) - \int_{-\infty}^{+\infty} x(t-\tau)\, dV(\tau)$$

Let $y = Fx$. Thus $<x, Gy> = c^2 R_{xy}(0) + \int_{-\infty}^{+\infty} [R_{xy}(0) - R_{xy}(\tau)]\, dV(\tau)$

where $c^2 = 1 -$ the total variation of V. Note that the above integrals exist since R_{xy} is bounded and since V is of bounded total variation. Thus $<x, Gy> = <x, GFx> \geq 0$ by Theorem 2.16. The odd-monotone case is proven in a similar way.

Remark: GF will be a positive operator on L_2 if $F - \epsilon I \epsilon \underset{\sim}{M}(\underset{\sim}{S})$ for some $\epsilon > 0$ and if the total variation of V is (strictly) less than unity.

Theorem 2.18: Let F and G satisfy the conditions of Theorem 2.17, and assume that the function f which determines F satisfies a Lipshitz condition, then $(G + \alpha \frac{d}{dt})F$ is a nonnegative operator on S_2^1 for all $\alpha \epsilon R$.

Proof: This theorem follows from Theorems 2.6 and 2.17.

Theorem 2.18 states thus that if X and Y are the limit-in-the-mean transforms of x and y=Fx with x and F as in Theorem 2.18, then

$$\int_{-\infty}^{+\infty} M_1(j\omega)\overline{X(j\omega)}\,Y(j\omega)\, d\omega \geq 0$$

for all functions $M_1(j\omega)$ given by the Fourier-Stieltjes integral

$$M_1(j\omega) = 1 + \alpha j\omega - \int_{-\infty}^{+\infty} e^{-j\omega\tau}\, dV_1(\tau)$$

where $\alpha \epsilon R$ and $V_1(\tau)$ satisfies the conditions of Theorem 2.17.

There is however one refinement possible to this result which has no immediate interpretation in terms of positive operators unless additional smoothness assumptions are made on x. Indeed, consider the functions of the form

$$M_2(j\omega) = \int_{-\infty}^{+\infty} \frac{1-e^{j\omega\tau}-j\omega\tau g(\tau)}{\tau^2}\, dV(\tau)$$

with $V_2(\tau)$ a monotone nondecreasing function of τ (any function of τ which is a bounded variation over compact sets) such that

$$\int_{x}^{\infty} \frac{dV_2(\tau)}{\tau^2} \quad \text{and} \quad \int_{-\infty}^{-x} \frac{dV_2(\tau)}{\tau^2}$$

exist for $x > 0$, and $g(\tau)$ is any bounded real-valued function of τ which is continuous at the origin and with $g(0)=1$. (It can be shown that under these conditions $M_2(j\omega)$ is well-defined). It is then possible to show using an argument which is completely analogous to the one used previously that the integral

$$\int_{-\infty}^{+\infty} M(j\omega)\overline{X(j\omega)}\, Y(j\omega)\, d\omega$$

exists and is nonnegative for any $M(j\omega) = M_1(j\omega) + M_2(j\omega)$ with $M_1(j\omega)$ and $M_2(j\omega)$ of the form given above.

Functions of this type have been studied in probability analysis in connection with characteristic functions of (possibly defective) probability distribution functions and infinitely divisible distributions. (See e.g., (20)). It is an interesting and somewhat puzzling fact that they also occur in the present context.

The following simple functions of ω belong to this class (for the monotone case) and are of particular interest:

$M(j\omega) = 1 - \gamma e^{-|\omega|^{\tau}}$ where γ and τ are real numbers satisfying $0 \leq \gamma \leq 1$ and $0 \leq \tau \leq 2$

$M(j\omega) = 1 - e(\omega)$ where $e(\omega)$ is any real valued, non-negative even function of ω which is convex for $\omega \geq 0$ and with $e(0) \leq 1$.

$M(j\omega) = |\omega|^{\tau}[1+j\delta\tan\frac{\pi\tau}{2}]$ for $\omega \geq 0$

$M(-j\omega) = \overline{M(j\omega)}$ for $\omega \leq 0$

$M(j\omega) = |\omega|[1+j\delta \ln \frac{|\omega|}{\omega_o}]$ for $\omega \leq 0$

where τ, δ and ω_o are real numbers satisfying $0 \leq \tau \leq 2$, $\tau \neq 1$, $|\delta| \leq 1$, and $\omega_o > 0$. For the details in the calculations see((20), p.541)

In the remainder of this section these results will be tied in with the classical frequency-power formulas. A nonlinear resistor with an almost periodic input absorbs power at some frequencies and supplies power at others. Using the bounds on the cross-correlation of the input and the output, similar to those obtained in Theorem 2.16, a general relation between the power at the different frequencies follows and some interesting frequency-power formulas are thus obtained.

<u>Definition:</u> A <u>positive nonlinear resistor</u> is a two-terminal device for which the current output is given as an instantaneous function of the voltage input, i.e., the output $y(t)$ is given in terms of the input $x(t)$ by the relation $y(t) = f(x(t))$, where f is mapping from R into itself. Moreover the function f satisfies

(i) $f(0) = 0$

(ii) $\frac{\partial f(\sigma)}{\partial \sigma}$ exists and is nonnegative for all σ

Let x be an almost-periodic function of t. It follows then from the smoothness conditions on f that y is also almost-periodic.

Definitions: Let ω_k be a basic frequency common to both $x(t)$ and $y(t)$ and let x_k and y_k be the corresponding Fourier coefficients. Let $\omega_k \geq 0$. Then the complex power, the active power and the reactive power absorbed by the nonlinear resistor at frequency ω_k are defined as respectively

$$R_k = \frac{1}{2} \bar{x}_k y_k \qquad P_k = \mathrm{Re}\ R_k \qquad Q_k = \mathrm{Im}\ R_k$$

Frequency-power formulas are relations between the active and reactive powers absorbed by the nonlinear resistor at the different frequencies.

Using exactly the same methods as in the previous section the following general frequency-power relation can be obtained in a straightforward fashion

$$\mathrm{Re} \sum_{\omega_k \geq 0} R_k\, M(j\omega_k) \geq 0$$

where $M(j\omega)$ is any function of the type given above. The particular choices of M given above lead to the following simple frequency-power formulas

$$\sum_{\omega_k \geq 0} (1 - \gamma e^{-|\omega_k|^{\tau}}) P_k \geq 0$$ where γ and τ are real numbers

satisfying $0 \leq \gamma \leq 1$ and $0 \leq \tau \leq 2$

$$\sum_{\omega_k \geq 0} (1 - e(\omega_k)) P_k \geq 0$$

where $e(\omega)$ is any real valued, nonnegative, even function of ω which is convex for $\omega \geq 0$ and with $e(0) \leq 1$

$$\sum_{\omega_k \geq 0} |\omega_k|^{\tau} (P_k + Q_k\, \delta \tan \frac{\pi\tau}{2}) \geq 0$$

$$\sum_{\omega_k \geq 0} |\omega_k| (P_k + \Omega_k \, \delta \log \frac{\omega_k}{\omega_o}) \geq 0$$

where τ, δ and ω_o are real numbers satisfying $0 \leq \tau \leq 2$, $\tau \neq 1$, $|\delta| \leq 1$, and $\omega_o > 0$.

Remark: For nonlinear capacitors with voltage versus charge characteristic $v = f(q)$ where f satisfies the same assumptions as above, analogous frequency power formulas can be obtained with R_k replaced by $\frac{jR_k}{\omega_k}$. The same is true for nonlinear inductors with current versus flux characteristic $i = f(\Phi)$ with R_k replaced by $R_k/j\omega_k$.

2.7 Factorization of Operators

Before motivating the analysis which follows one definition is needed which will help to fix the ideas.

Definition: Let S be a subset of R, and let Y be the space consisting of all mappings from S into some space V. Let 0 be an operator from $X \subset Y$ into Y. The operator 0 is said to be a causal operator on X if for any $\tau \in S$ and any $x_1, x_2 \in X$, with $x_1(t) = x_2(t)$ for all $t \in S$ with $t \leq \tau$, then $0x_1(t) = 0x_2(t)$ for all $t \in S$ with $t \leq \tau$. Thus a causal operator is one in which the value of the output at any time t does not depend on the values of the input after that time t. A causal operator is often called nonanticipative.

In many problems in system theory, e.g., in stability theory, in optimal control theory and in prediction theory there is particular interest in causal operators. For instance, in network synthesis it is expected that a synthesis procedure for passive nonlinear networks will require two basic properties of the operator defining the input-output relation, namely positivity and causality. The importance to stability theory of generating positive operators which are also causal will become more apparent in the next chapter. In this section some techniques for generating a causal positive operator from an arbitrary positive operator are developed. The basic idea is simple and is expressed in the next theorem.

<u>Theorem 2.19:</u> <u>Let</u> 0 <u>be a nonnegative operator on an inner product space</u> X <u>and assume that</u> 0 <u>can be factored as</u> $0=0^{-}0^{+}$ <u>with</u> 0^{+} <u>a causal operator on</u> X <u>and</u> 0^{-} <u>a bounded linear operator on</u> X <u>which is invertible and such that</u> $(0^{-*})^{-1}$ <u>is a causal operator on</u> X. <u>Then</u> $0^{+}(0^{-*})^{-1}$ <u>is a nonnegative causal operator on</u> X.

<u>Proof:</u> Let $x \in X$. Then

$$\langle x, 0^{+}(0^{-*})^{-1}x\rangle = \langle (0^{-*})(0^{-*})^{-1}x, 0^{+}(0^{-*})^{-1}x\rangle$$
$$= \langle (0^{-*})^{-1}x, 0^{-}0^{+}(0^{-*})^{-1}x\rangle$$
$$= \langle (0^{-*})^{-1}x, 0(0^{-*})^{-1}x\rangle$$
$$\geq 0$$

Furthermore, since 0^{+} and $(0^{-*})^{-1}$ are causal operators on X, so is $0^{+}(0^{-*})^{-1}$. Thus $0^{+}(0^{-*})^{-1}$ is a nonnegative causal operator on X.

The above theorem and the resulting possibility of generating a causal positive operator from a noncausal positive operator show the importance of obtaining sufficient conditions for a factorization as required in the theorem to be possible. Similar problems have received a great deal of attention in the classical prediction theory, in the theory of linear integral equations and in probability theory. It brings to mind some of the work of Wiener (58) and Krein (34) but the existing results deal almost exclusively with linear time-invariant convolution-type operators in Hilbert spaces and the analysis uses the fact that these operators are commutative in an essential way. The operators which will be considered here, however, need not have this property. The results obtained by these authors are hence not immediately applicable and a factorization theorem which applies to more general operators is required. The factorization theorem obtained in this section is felt to be of great interest in its own right. It applies to linear convolution operators whose kernel might be time-varying and which need therefore not be commutative.

The factorization problem is one of considerable interest and importance and the natural setting for the study of such factorizations appears to be a Banach Algebra (64,34). Assume thus that the operators under consideration form a Banach Algebra. As is easily verified, the causal operators will then form a subalgebra since causal operators are closed under addition, under composition and under multiplication by scalars. This is the reason for the introduction of the projection operators and for stating the theorem in terms of arbitrary projections and elements of a Banach Algebra.

The general factorization theorem thus obtained is then specialized to certain classes of linear operators in Hilbert space. It will also be indicated that in the case of certain convolution operators with a time-invariant kernel the results are rather conservative and that less restrictive factorization theorems due to Krein (34, p. 198) exist. The setting of the factorization problem is the same as used by Zames and Falb (64), but the results are more general. The method of proof is inspired by a paper by Baxter (4).

<u>Definitions:</u> A <u>Banach Algebra</u> is a normed linear vector space, σ, over the real or complex field which is complete in the metric induced by its norm and which has a mapping (<u>multiplication</u>)

from $\sigma \times \sigma$ into σ defined. This multiplication is associative, is distributive with respect to addition, is related to scalar multiplication by $a(AB)=A(aB)=(aA)B$, and to the norm on σ by $\|AB\| \leq \|A\| \ \|B\|$ for all $A, B \in \sigma$ and all scalars a. A Banach Algebra is said to have a unit element if there exists an element $I \in \sigma$ such that $AI=IA=A$ for all $A \in \sigma$. An element A of a Banach Algebra with a unit element is said to be invertible if there exists an element, A^{-1}, of σ such that $AA^{-1}=A^{-1}A=I$. A bounded linear transformation, π, from σ into itself is said to be a projection on σ if $\pi^2=\pi$ and if the range of π forms a subalgebra of σ. Note that the range of a projection is thus assumed to be closed under addition and multiplication. The norm of π, $\|\pi\|$ is defined in the usual way as the greatest lower bound of all numbers M which satisfy $\|\pi A\| \leq M \|A\|$ for all $A \in \sigma$. θ denotes the identity transformation* on σ.

The following factorization theorem states the main result of this section.

Theorem 2.20: Let σ be a Banach Algebra with a unit element and let π^+ and $\pi^- = \theta - \pi^+$ be projections on σ. Let σ^+ and σ^- be the ranges of π^+ and π^-, and assume that $\|\pi^+\| \leq 1$ and that $\|\pi^-\| \leq 1$. Let Z be an element of σ and let ρ be a nonzero scalar. If $\|Z\| < |\rho|$, then there exist elements $Z^+ \in \sigma$ and $Z^- \in \sigma$ such that

(i) $M = \rho I - Z = Z^- Z^+$

(ii) Z^+ and Z^- are invertible.

---

* Not to be confused with I, the unit element of σ.

(iii) Z^+ and $(Z^+)^{-1}$ belong to* $\sigma^+ \oplus I$ and Z^- and $(Z^-)^{-1}$ belong to $\sigma^- \oplus I$.

Proof: Since the proof of the theorem is rather lengthy, it is subdivided into several lemmas.

Lemma 2.8: Let $\{A_k\}$, $\{P_k\}$ and $\{N_k\}$, $k=1,2,\ldots$, be sequences of elements of σ, σ^+ and σ^- respectively and assume that for some $r_o > 0$ and all $|r| \leq r_o$

(i) the series $$A = I + \sum_{k=1}^{\infty} A_k r^k$$

$$P = I + \sum_{k=1}^{\infty} P_k r^k$$

and $$N = I + \sum_{k=1}^{\infty} N_k r^k$$ converge

(ii) $$A = PN$$

Then A uniquely determines the sequences $\{P_k\}$ and $\{N_k\}$.

Proof: Equating coefficients of equal powers in r in the equality $A=PN$ leads to $P_1+N_1=A_1$ and $P_n+N_n=A_n-\sum_{k=1}^{n-1} P_k N_{n-k}$ for $n=2,3,\ldots$. Thus $P_n=\pi^+(A_n-\sum_{k=1}^{n-1} P_k N_{n-k})$ and $N_n=\pi^-(A_n-\sum_{k=1}^{n-1} P_k N_{n-k})$ which shows that A uniquely determines P_n and N_n provided it uniquely determines $P_1,\ldots,P_{n-1}$ and $N_1,\ldots,N_{n-1}$. Since A uniquely determines P_1 and N_1 by $P_1 = \pi^+ A_1$ and $N_1 = \pi^- A_1$, the result follows by induction.

---

* $\sigma^+ \oplus I$ denotes all elements of σ which are of the form $R + aI$ with $R \subset \sigma^+$ and a a scalar. $\sigma^- \oplus I$ is defined analogously.

Lemma 2.9: The equations

$$P = I + r\pi^{+}(ZP)$$

and

$$N = I + r\pi^{-}(NZ)$$

have a unique solution $P \in \sigma$ and $N \in \sigma$ for all $|r| \leq |\rho|^{-1}$. Moreover, these solutions are given by the convergent series

$$P = \sum_{k=0}^{\infty} P_k r^k \quad \text{and} \quad N = \sum_{k=0}^{\infty} N_k r^k$$

with $P_o = N_o = I$, $P_{k+1} = \pi^{+}(ZP_k)$ and $N_{k+1} = \pi^{-}(N_k Z)$. Notice that $P \in \sigma^{+} \oplus I$ and that $N \in \sigma^{-} \oplus I$.

Proof: The result follows from the inequalities

$$\| r\pi^{+}(Z(A-B)) \| \leq |\rho|^{-1} \| Z \| \; \| A-B \|$$

$$\| r\pi^{-}((A-B)Z) \| \leq |\rho|^{-1} \| Z \| \; \| A-B \|$$

and the Contraction Mapping Principle. Moreover, it is easily verified that the successive approximations obtained by this contraction mapping with $P_o = N_o = I$ yield the power series expressions of P and N as claimed in the lemma.

Lemma 2.10: The solutions P and N to the equations of Lemma 2.9 are invertible for all $|r| \leq |\rho|^{-1}$ and

$$P^{-1} = I - r\pi^{+}(NZ)$$

$$N^{-1} = I - r\pi^{-}(ZP)$$

Moreover, $N^{-1}P^{-1} = I - rZ$ for all $|r| \leq |\rho|^{-1}$. Notice that $P^{-1} \in \sigma^{-} \oplus I$ and that $N^{-1} \in \sigma^{+} \oplus I$.

Proof: From the equations defining P and N it follows that for $|r| \leq |\rho|^{-1}$, $\| r\pi^{+}(NZ) \| \leq \frac{|r| \, \|Z\|}{1 - |r| \, \|Z\|}$ and $\| r\pi^{-}(ZP) \| \leq \frac{|r| \, \|Z\|}{1 - |r| \, \|Z\|}$. Since all elements of σ which are of the form $I-B$ with $\|B\| < 1$ are invertible, it follows thus that $I-r\pi^+(NZ)$, $I-r\pi^-(ZP)$ and $I-rZ$ are invertible for $|r| \leq |\rho|^{-1}/2$. Furthermore their inverses are given by the convergent series

$$(I-r\pi^+(NZ))^{-1} = I + \sum_{k=1}^{\infty} (\pi^+(NZ))^k r^k$$

$$(I-r\pi^-(ZP))^{-1} = I + \sum_{k=1}^{\infty} (\pi^-(ZP))^k r^k$$

$$(I-rZ)^{-1} = I + \sum_{k=1}^{\infty} Z^k r^k$$

From the equations of P and N it follows that for $|r| \leq |\rho|^{-1}$ $(I-rZ)\,P=I-r\pi^-(ZP)$ and $N(I-rZ)=I-r\pi^+(NZ)$ and thus that for $|r| < |\rho|^{-1}/2$, $(I-rZ)^{-1}=P(I-r\pi^-(ZP))^{-1}=(I-r\pi^+(NZ))^{-1}N$. Since all factors in the above equalities are given by the convergent series given above and in Lemma 2.9, and since σ^+ and σ^- are closed under multiplication, Lemma 2.8 is thus applicable. This yields for $|r| \leq |\rho|^{-1}/2$ $P=(I-r\pi^+(NZ))^{-1}$, $N=(I-r\pi^-(ZP))^{-1}$ and $PN=(I-rZ)^{-1}$. Thus for $|r| \leq |\rho|^{-1}/2$ the following equalities hold:

$$P(I-r\pi^+(NZ)) = (I-r\pi^+(NZ))\,P=I$$

$$N(I-r\pi^-(ZP)) = (I-r\pi^-(ZP))\,N=I$$

$$(I-r\pi^-(ZP))(I-r\pi^+(NZ)) = I-rZ$$

Since, for $|r| \leq |\rho|^{-1}$, all terms in the above equalities are given by geometrically convergent power series in r, they are analytic functions of r for $|r| \leq |\rho|^{-1}$. Since equality holds for $|r| \leq |\rho|^{-1}/2$ it is thus concluded from analyticity that equality holds for all $|r| \leq |\rho|^{-1}$. This ends the proof of Lemma 2.10.

Proof of Theorem 2.20: Let $r = \rho^{-1}$ in the above lemma. The theorem follows with $Z^- = \rho(I - \rho^{-1}\pi^-(ZP))$, $(Z^-)^{-1} = \rho^{-1}N$, $Z^+ = I - \rho^{-1}\pi^+(NZ)$ and $(Z^+)^{-1} = P$.

Under a suitable choice of the Banach Algebra and the projection operators a number of interesting corollaries to Theorem 2.20 hold, two of which will now be given.

Definitions: Let R be an element of $\mathcal{L}(\ell_2, \ell_2)$ and let $\{r_{k\ell}\}$, $k, \ell \in I$ be the corresponding array. R is said to belong to $\mathcal{L}^+(\ell_2, \ell_2)$ if $r_{k\ell} = 0$ for all $k < \ell$. It is said to belong to $\mathcal{L}^-(\ell_2, \ell_2)$ if R^* belongs to $\mathcal{L}^+(\ell_2, \ell_2)$.

Corollary 2.1: Let Z be an element of $\mathcal{L}(\ell_2, \ell_2)$ which is such that $Z - \epsilon I$ is doubly dominant for some $\epsilon > 0$. Then there exist elements M and N of $\mathcal{L}(\ell_2, \ell_2)$ such that

(i) $Z = MN$

(ii) M and N have a bounded inverses M^{-1} and N^{-1}

(iii) N and N^{-1} belong to $\mathcal{L}^+(\ell_2, \ell_2)$ and M and M^{-1} belong to $\mathcal{L}^-(\ell_2, \ell_2)$

Corollary 2.2: Let $A(z) - \epsilon$ be the z-transform of a sequence which is dominant for some $\epsilon > 0$. Then there exist functions $A^+(z)$ and $A^-(z)$ such that

(i) $A(z) = A^-(z)\, A^+(z)$

(ii) $A^+(z)$ and $(A^+(z))^{-1}$ are the z-transforms of ℓ_1-sequences $\{a_k^+\}$ and $\{b_k^+\}$ with $a_k^+ = b_k^+ = 0$ for $k < 0$ and $A^-(z)$ and $(A^-(z))^{-1}$ are the z-transforms of ℓ_1-sequences $\{a_k^-\}$ and $\{b_k^-\}$ with $a_k^- = b_k^- = 0$ for $k > 0$.

Proof: It will be shown that these corollaries follow from Theorem 2.20 under a suitable choice of the Banach Algebra σ and the projections π^+ and π^-.

Corollary 2.1 follows from Theorem 2.20 with the Banach Algebra σ all members of $\mathcal{L}(\ell_2, \ell_2)$ such that if $A \in \sigma$ and if $\{a_{k\ell}\}, k, \ell \in I$ is the corresponding array, then the sequences $\{a_{k\ell}\}$ belong to ℓ_1 for fixed k and ℓ, uniformly in k and ℓ, i.e., there exists an M such that $\sum_{k=-\infty}^{+\infty} |a_{k\ell}| \leq M$ and $\sum_{\ell=-\infty}^{+\infty} |a_{k\ell}| \leq M$. Multiplication is defined in the usual way as composition of elements of $\mathcal{L}(\ell_2, \ell_2)$. The norm is defined as the greatest lower bound of all numbers M satisfying the above inequalities. The nonobvious elements in the verification of the fact that σ forms a Banach Algebra are that σ is closed under multiplication, that $\|AB\| \leq \|A\| . \|B\|$ for all $A, B \in \sigma$, and that σ is complete. Closedness under multiplication follows from Fubini's Theorem for sequences (17, p.245) and the inequalities

$$\sum_{k=-\infty}^{+\infty} \left|\sum_{i=-\infty}^{+\infty} a_{ki} b_{i\ell}\right| \leq \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} |a_{ki}|\, |b_{i\ell}|$$

$$= \sum_{i=-\infty}^{+\infty} |b_{i\ell}| \sum_{k=-\infty}^{+\infty} |a_{ki}|$$

$$\leq \|A\|\, \|B\| .$$

also
$$\sum_{\ell=-\infty}^{+\infty} \left|\sum_{i=-\infty}^{+\infty} a_{ki}\, b_{i\ell}\right| = \leq \|A\|\, \|B\|$$

These inequalities also show that $\|AB\| \leq \|A\| . \|B\|$. Completeness follows from the fact that ℓ_1 is complete (31). The projection operator π^+ is defined by $\pi^+ A = B$ with if $\{a_{k\ell}\}$ and $\{b_{k\ell}\}$, $k, \ell \in I$ are the corresponding arrays, then $a_{k\ell} = b_{k\ell}$ for all $k \geq \ell$, and $b_{k\ell} = 0$

otherwise π^- is defined by $\pi^- = \theta - \pi^+$. It is clear that $\|\pi^+\| = 1$ and that $\|\pi^-\| = 1$. The only fact that is left to be shown is that if for some $\epsilon > 0$, $Z - \epsilon I$ is doubly dominant then Z can be written as $Z = \rho I - A$ with $\|A\| < \rho$. It is easily verified that any ρ with $|\rho| \geq \sup_{k \in I} z_{kk}$ yields such a decomposition.

The proof of Corollary 2.2 is completely along the lines of the proof of Corollary 2.1 but with a Banach Algebra σ all ℓ_1 sequences, multiplication of $A = \{a_k\}$ and $B = \{b_k\}$ defined by $AB = C = \{c_k\}$ with $c_k = \sum_{\ell=-\infty}^{+\infty} a_{k-\ell} b_\ell$ and $\|A\| = \sum_{k=-\infty}^{+\infty} |a_k|$. The projection operator π^+ is defined by $\pi^+ A = B$ with $A = \{a_k\}$, $B = \{b_k\}$, $b_k = a_k$ for $k \geq 0$, and $b_k = 0$ for $k < 0$. π^- is defined by $\pi^- = \theta - \pi^+$.

Remark: The factorization in Corollary 2.2 is valid under much weaker conditions than stated. Indeed although dominance of the involved sequence is certainly sufficient for the factorization to be possible, it is by no means necessary as is shown by the following theorem due to Krein (34, p. 198).

Theorem 2.21 (Krein): Let $A(z)$ be the z-transform of an ℓ_1-sequence. Then there exist functions $A^+(z)$ and $A^-(z)$ such that

(i) $A(z) = A^-(z)\, A^+(z)$

(ii) $A^+(z)$ and $(A^+(z))^{-1}$ are the z-transforms of ℓ_1-sequences $\{a_k^+\}$ and $\{b_k^+\}$ with $a_k^+ = b_k^+ = 0$ for $k < 0$ and $A^-(z)$ and $(A^-(z))^{-1}$ are the z-transforms of ℓ_1-sequences $\{a_k^-\}$ and $\{b_k^-\}$ with $a_k^- = b_k^- = 0$ for $k > 0$

if and only if $A(z) \neq 0$ for $|z| = 1$ and the increase in the argument of the function $A(z)$ as z moves around the circle $|z| = 1$ is zero.

Moreover, all factorization which satisfies conditions (i) and (ii) differ only by a nonzero multiplicative constant.

Proof: A proof can be found in (34, Theorem 5.1)

It is clear that if $A(z) - \epsilon$ is the z-transform of a dominant sequence from some $\epsilon > 0$ then $A(z)$ satisfies the conditions of the above theorem since then $\mathrm{Re}A(z) \geq \epsilon > 0$ for $|z|=1$.

Remark: An interesting question is if it is possible to extend Corollary 5.1 so that it would cover Theorem 2.21 and become an extension of it. Unfortunately this has not been possible as yet. Two possible avenues of generalization are:

1. Find a class of elements of $\mathcal{L}(\ell_2, \ell_2)$ which can be factored at MZN with M and N invertible elements of $\mathcal{L}(\ell_2, \ell_2)$, $M, M^{-1} \epsilon \mathcal{L}^+(\ell_2, \ell_2)$ and $N, N^{-1} \epsilon \mathcal{L}^-(\ell_2 \ell_2)$ and $Z - \epsilon I$ doubly dominant, for some $\epsilon > 0$. Clearly to find the most general class is just as difficult as to do the original factorization (this can be seen by taking $Z=I$). However it might obtain some results by adding some additional restrictions on M and N.

2. See what the methods used by Krein in (34) have to say about the factorization of generalized Toeplitz-type elements of $\mathcal{L}(\ell_2, \ell_2)$ for which the corresponding array $\{r_{k\ell}\}$, $k, \ell \epsilon I$ is of the form

$$\begin{bmatrix} \cdot & \cdot & \cdot & & & & & & & \\ & \cdot & \cdot & \cdot & & & & & & \\ & & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & A_{-1} & A_o & A_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & A_{-1} & A_o & A_1 & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & A_{-1} & A_o & A_1 & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

where $\{A_k\}$, $k \in I$ is a sequence of $(n \times n)$ matrices, such that $\{\|A_k\|\} \in \ell_1$. The conditions would most likely be in terms of the generalized z-transform of $\{A_k\}$ for $|z| = 1$, i.e., in terms of the matrix

$$A(z) = \sum_{k=-\infty}^{+\infty} A_k z^{-k}$$

for $|z| = 1$.

Remark: The factorization analogous to those obtained in Corollaries 2.1 and 2.2 but for convolution operators on ℓ_2 with time-varying kernels is straightforward and will not be explicitly given. The analogue to Theorem 2.21 for the operators with a time-invariant kernel follows since it gives a necessary and sufficient condition.

Another useful factorization theorem which is due to Krein and which is less restrictive than the analogous factorization obtained in Theorem 2.20 regards another class of convolution operators.

Let $\underset{\sim}{G}_1$ be a class of operators from L_2 into itself each element of which is determined by an element $(g(t), \{g_k\})$ of $L_1 \times \ell_1$ and by a mapping $\{t_k\}$ from I into R. The operator $G \in \underset{\sim}{G}_1$ maps $x \in L_2$ into y with

$$y(t) = \sum_{k=-\infty}^{+\infty} g_k\, x(t-t_k) + \int_{-\infty}^{+\infty} g(t-\tau)x(\tau)\,d\tau$$

It is simple to verify that G is well defined, i.e., that it maps L_2 into itself. Let $\underset{\sim}{G}_1^+$ denote the subclass of $\underset{\sim}{G}_1$ for which the determining element of $L_1 \times \ell_1$ and the mapping $\{t_k\}$ satisfy $g(t) = 0$ for $t < 0$ and $t_k \geq 0$ for all $k \in I$. Let $\underset{\sim}{G}_1^-$ denote the subclass of $\underset{\sim}{G}_1$ for which the determining element of $L_1 \times \ell_1$ and the mapping $\{t_k\}$ satisfy

$g(t) = 0$ for $t > 0$ and $t_k \leq 0$ for all $k \in I$. Clearly $G \in \underset{\sim}{G}_1^+$ if and only if $G^* \in \underset{\sim}{G}_1^-$.

Theorem 2.22 (Krein): Let $G \in \underset{\sim}{G}_1$. Then there exist elements $G^+ \in \underset{\sim}{G}_1$ and $G^- \in \underset{\sim}{G}_1$ such that

(i) $G = G^- G^+$

(ii) G^+ and G^- are invertible

(iii) G^+ and $(G^+)^{-1} \in \underset{\sim}{G}_1^+$, and G^- and $(G^-)^{-1} \in \underset{\sim}{G}_1^-$

if and only if $|G(j\omega)| \geq \epsilon$ for some $\epsilon > 0$ and all $\omega \in R$ and the increase in the argument of the function $G(j\omega)$ as ω varies from $-\infty$ to $+\infty$ is zero.

Proof: A slightly weaker version of this theorem is given by Krein (34 p. 178, Theorem 2.1). However the extension to cover Theorem 2.22 presents no apparent difficulties.

CHAPTER III

STABILITY OF FEEDBACK LOOPS

3.1 Generalities

In this chapter some sufficient conditions for the stability of feedback loops of the type shown in Fig. 3.1 will be derived. The results obtained in this section are along the lines of those obtained by Sandberg (54) and particularly by Zames (63).*

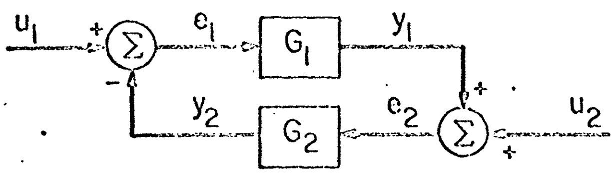

Fig. 3.1 The Feedback System Under Consideration

Before introducing formal definitions of stability it is necessary to define what is meant by a solution.

Definition: Let S be a subset of R and let Y denote the linear vector space of all maps from S into a linear vector space V. Let $u_1, u_2 \in Y$ and let G_1 and G_2 be operators from subsets of Y into Y. (Note that the domain of G_1 and G_2 need not be all of Y.) The quadruple e_1, y_1, e_2, y_2 is said to be a solution of the feedback loop if $e_1, y_1, e_2, y_2 \in Y$, if $e_1 \in Do(G_1)$, $e_2 \in Do(G_2)$ and if the equations

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* Other pertinent references are the papers by Zarantonello (66), Minty (38), Browder (14) and Kolodner (32). For an account of related problems, see the book by Saaty (50).

$$e_1 = u_1 - y_2$$
$$e_2 = u_2 + y_1$$
$$y_1 = G_1 e_1$$
$$y_2 = G_2 e_2$$

are satisfied for all arguments $t \in S$.

Next, the notion of an extended space will be introduced.

Definition: Let $x \in Y$ and let $\tau \in S$. Then the τ-truncation of x, denoted by $P_\tau x$, is defined by $P_\tau x = x$ for all arguments $t \in S$ with $t \leq \tau$ and $P_\tau x = 0$ for all $t \in S$ with $t > \tau$. Let $X \subset Y$ be a normed linear space. The extended space X_e is the space of all elements $x \in Y$ for which $P_\tau x \in X$ for all $\tau \in S$. It is assumed that P_τ is a bounded operator from X into itself, i.e., that $P_\tau x \in X$ for all $x \in X$ and $\tau \in S$, and that the bound of P_τ on X, $\| P_\tau \|$, is less than or equal to unity, i.e., $\| P_\tau x \| \leq \| x \|$ for all $x \in X$ and $\tau \in S$. Since $P_\tau^2 = P_\tau$, P_τ is thus a projection on X for all $\tau \in S$. Let $\sup S$ denote the supremum of all elements of S if S is bounded from above or $+\infty$ if S is not bounded from above. It is assumed that if $\{\tau_k\}$, $k \in I$, is any nondecreasing sequence of elements in S with $\lim_{k \to \infty} \tau_k = \sup S$, then $\lim \| P_{\tau_k} x \| = \| x \|$ for all $x \in X$. Conversely, if $x \in X_e$ and if the sequence of real numbers $\{\| P_{\tau_k} x \|\}$ is bounded then it is assumed that this implies that $x \in X$ and that $\| x \| = \lim_{k \to \infty} \| P_{\tau_k} x \|$. (This limit exists since the sequence $\{\| P_{\tau_k} x \|\}$ is monotone nondecreasing in k.) Thus if $x \in X_e$ then $x \in X$ if and only if $\| P_\tau x \| \leq M$ for all $\tau \in S$ and some constant M and if $x \in X$ then $\| x \| = \sup_{\tau \in S} \| P_\tau x \|$. If X is an inner product space, then P_τ is self-adjoint and $< x_1, P_\tau x_2 > = < P_\tau x_1, P_\tau x_2 >$ for all $x_1, x_2 \in X$ and $\tau \in S$.

All the preliminaries are now available to define the type of stability which will be considered in the sequel.

Definition: The feedback system under consideration is said to be X-stable if $u_1, u_2 \in X$ implies that all solutions with $e_1, y_1, e_2, y_2 \in X_e$ yield $e_1, y_1, e_2, y_2 \in X$ and satisfy the inequality

$$\|e_1\| + \|y_1\| + \|e_2\| + \|y_2\| \leq K_1 \|u_1\| + K_2 \|u_2\|$$

for some constants K_1 and K_2.

At this point, some restrictive assumptions will be made about the operators G_1 and G_2 appearing in the feedback loop. The results obtained below hold under less restrictive conditions. Since however these restrictions are reasonable and satisfied in most practical situations no effort was made to reduce them to their minimality in an attempt to keep the analysis as simple as possible.

Restriction 1: It is assumed that $D_o(G_1)$, $D_o(G_2) \supset X_e$, and that for any $x \in X_e$, $G_1 x$, $G_2 x \in X_e$. Furthermore G_1 and G_2 are assumed to be causal operators on X_e, i.e., $P_\tau x_1 = P_\tau x_2$ implies that $P_\tau G_1 x_1 = P_\tau G_1 x_2$ and $P_\tau G_2 x_1 = P_\tau G_2 x_2$ for all $x_1, x_2 \in X_e$ and $\tau \in S$. An equivalent way of stating this causality assumption is to assume that P_τ commutes on X_e with both $P_\tau G_1$ and $P_\tau G_2$.

Definition: Let 0 be an operator from X_e into itself. 0 is said to be a bounded operator on X_e if there exists a number M such that $\|P_\tau 0x\| \leq M \|P_\tau x\|$ for all $x \in X_e$ and $\tau \in S$. The extended bound of a bounded operator 0 on X_e, denoted $\|0\|_e$, is defined as the infimum of all real numbers M which satisfy the above inequality for all $x \in X_e$ and $\tau \in S$. Recall that if 0 is a bounded operator from X into itself then the <u>bound</u> of 0 on X, denoted $\|0\|$, was defined as the infimum of all real numbers M which satisfy the inequality $\|0x\| \leq M\|x\|$ for all $x \in X$. The lemma which follows shows that the bound and the extended bound of a causal operator are equal.

<u>Lemma 3.1</u>: Let 0 be an operator from X_e into itself. If 0 is causal and bounded on X_e, then 0 maps X into itself, is bounded on X and $\|0\|_e = \|0\|$. Conversely if 0 is causal, maps X into itself and is bounded on X then 0 is bounded on X_e and $\|0\| = \|0\|_e$.

<u>Proof</u>: Let $x \in X$, then $0x \in X_e$ and $\|P_\tau 0x\| \leq \|0\|_e \|P_\tau x\| \leq \|0\|_e \|x\|$. Hence $0x \in X$ and $\|0x\| \leq \|0\|_e \|x\|$. Thus 0 is bounded on X and $\|0\| \leq \|0\|_e$. Let $x \in X_e$, then $\|P_\tau 0x\| = \|P_\tau 0 P_\tau x\| \leq \|0 P_\tau x\| \leq \|0\| \|P_\tau x\|$. Hence $\|0\|_e \leq \|0\|$. Thus 0 maps X into itself and $\|0\|_e = \|0\|$. Conversely, let $x \in X_e$, then $\|P_\tau 0x\| = \|P_\tau 0 P_\tau x\| \leq \|0 P_\tau x\| \leq \|0\| \|P_\tau x\|$. This shows that 0 is bounded on X_e and that $\|0\|_e \leq \|0\|$. Let $x \in X$, then $\|P_\tau 0x\| \leq \|0\|_e \|P_\tau x\| \leq \|0\|_e \|x\|$. Hence $\|0x\| \leq \|0\|_e \|x\|$, and $\|0\| \leq \|0\|_e$. Thus $\|0\| = \|0\|_e$.

<u>Restriction 2:</u> It is assumed that G_1 and G_2 are <u>bounded</u> operators on X_e.

It is thus clear that under these restrictions the feedback system under consideration will be X-stable if and only if for $u_1, u_2 \in X$

$$\|P_\tau e_1\| \leq K_1 \|u_1\| + K_2 \|u_2\|$$

for all solutions with $e_1, y_1, e_2, y_2 \in X_e$, all $\tau \in S$ and some constants K_1 and K_2. Lemma 3.1 and Restriction 1 make the verification of Restriction 2 simpler. Indeed it suffices to verify that G_1 and G_2 map X into itself and that they are bounded operators on X.

Restriction 3: It is assumed that the operator G_2 satisfies a Lipshitz condition on X, i.e., that there exists a constant K (the Lipshitz constant) such that

$$\|G_2(x+y) - G_2 x\| \leq K\|y\|$$

for all $x, y \in X$. Notice that Restriction 2 and linearity of G_2 imply Restriction 3.

Remark: There is of course nothing peculiar in making this restriction on G_2 rather than on G_1, and analogous results as the ones obtained below can be obtained if G_1 satisfies a Lipshitz condition on X.

The following theorem is the basic result from which all other stability criteria will be derived.

Theorem 3.1: If $\|P_\tau(I + G_2G_1)x\| \geq \epsilon\|P_\tau x\|$ for all $x \in X$, all $\tau \in S$ and some $\epsilon > 0$, then the feedback system under consideration is X-stable.

Proof: Let $u_1, u_2 \in X$ and let e_1, y_1, e_2, y_2 be a solution with $e_1, y_1, e_2, y_2 \in X_e$. Since $e_1 = u_1 - G_2(u_2 + G_1e_1)$ it follows that the equality $e_1 + G_2G_1e_1 = u_1 - (G_2(u_2 + G_1e_1)$ holds for all arguments $t \in S$. Hence for all $\tau \in S$

$$P_\tau(I + G_2G_1)\, P_\tau e_1 = P_\tau u_1 - P_\tau(G_2(G_1e_1 + u_2) - G_2G_1e_1)$$

and thus $\|P_\tau(I + G_2G_1)P_\tau e_1\| \leq \|u_1\| + K\|G_2\|\ \|u_2\|$

Using causality and the inequality in the statement of the theorem, it follows that for all $\tau \in S$

$$\|P_\tau e_1\| \leq \epsilon^{-1}\|u_1\| + \epsilon^{-1}K\|G_2\|\ \|u_2\|$$

Hence $e_1 \in X$ and

$$\|e_1\| \leq \epsilon^{-1}\|u_1\| + \epsilon^{-1} K\|G_2\| \; \|u_2\|$$

which implies X-stability.

Theorem 3.1 is graphically illustrated in Fig. 3.2

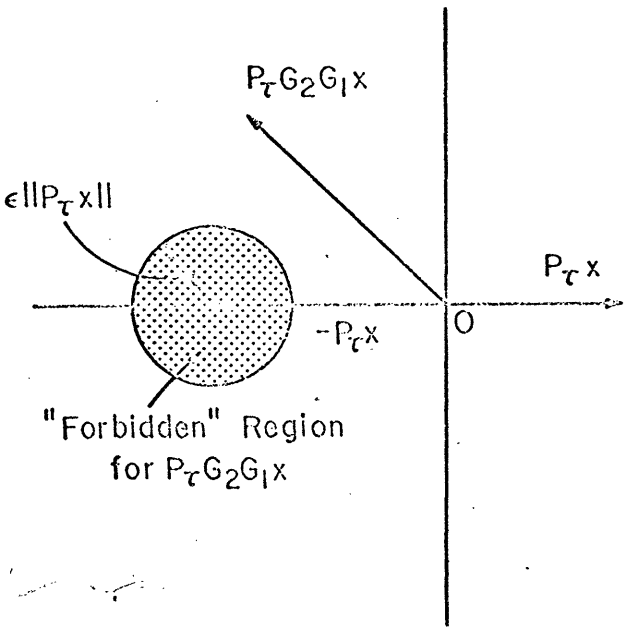

Fig.3.2 Illustration of Theorem 3.1

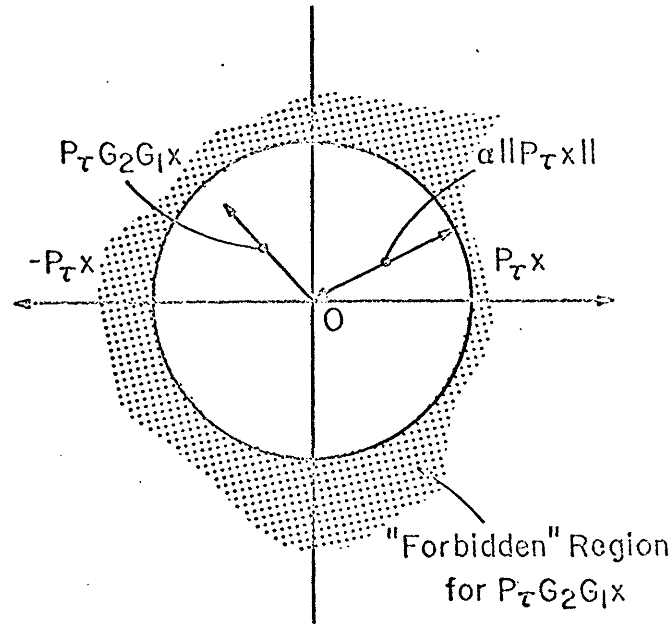

Fig. 3.3 Illustration of Corollary 3.1

Remark: It is very tempting to replace the inequality in the condition of Theorem 3.1 by $\|(I + G_2 G_1)x\| \geq \epsilon\|x\|$ for all $x \in X$ and some $\epsilon > 0$. This however leads to fallacious conclusions. A counter-example is provided by the Nyquist criterion when the Nyquist locus of the forward loop encircles the $-1/k$ point.

The first corollary to Theorem 3.1 provides a proof of the intuitive idea that if the open loop attenuates all signals, then the closed loop is stable. Corollary 3.1 is graphically illustrated in Fig. 3.3.

Corollary 3.1: If $\|G_2 G_1\|$ is less than unity then the feedback system under consideration is X-stable.

Proof: By Lemma 3.1

$$\|P_\tau G_2G_1x\| \leq \|G_2G_1\| \; \|P_\tau x\| \qquad \text{for all } x \in X$$

and thus

$$\|P_\tau(I+G_2G_1)x\| \geq \|P_\tau x\| - \|P_\tau G_2G_1 x\|$$

$$\geq (1 - \|G_2G_1\|)\,\|P_2 x\|$$

which yields the conclusion by Theorem 3.1 with $\epsilon = 1 - \|G_2G_1\|$.

Next, attention is focused on how the interconnection of passive systems leads to a stable system. The outcome will be that the interconnection of a passive system (a nonnegative operator) and a strictly passive system (a positive operator) is stable. This again provides a proof of an intuitive idea.

Lemma 3.2: Let X be a real inner product space, and let $x, y \in X$. If for some $z \in X$ with $\|z\| \neq 0$

(i) $<x, z> \geq 0$

(ii) $<y, z> \geq \epsilon \|y\| \; \|z\|$ for some ϵ with $0 < \epsilon < 1$

Then there exists a real number $c > -1$, depending upon ϵ only, such that

$$<x, y> \geq c \; \|x\| \; \|y\|$$

In fact $c = -\cos \sin^{-1} \epsilon$ satisfies this condition.

Proof: The Grammian matrix

$$G(x,y,z) = \begin{bmatrix} <x,x> & <x,y> & <x,z> \\ <y,x> & <y,y> & <y,z> \\ <z,x> & <z,y> & <z,z> \end{bmatrix}$$

is nonnegative definite (see e.g., (23, p. 247)).

Thus $<x,x><y,y><z,z> + 2<x,y><y,z><z,x> \geq <x,x><y,z>^2$

$$+ <y,y><x,z>^2 + <z,z><x,y>^2$$

Since the lemma is satisfied for any c if $\|x\|$ or $\|y\| = 0$, it is assumed that $\|x\| \neq 0$ and that $\|y\| \neq 0$. After some manipulations the above inequality reduces to

$$\left(\frac{\langle x,y\rangle}{\|x\|\,\|y\|} - \frac{\langle x,z\rangle}{\|x\|\,\|z\|}\cdot\frac{\langle y,z\rangle}{\|y\|\,\|z\|}\right)^2 \leq \left[1-\left(\frac{\langle x,z\rangle}{\|x\|\,\|z\|}\right)^2\right]\left[1-\left(\frac{\langle y,z\rangle}{\|y\|\,\|z\|}\right)^2\right]$$

which implies either that

$$\frac{\langle y,y\rangle}{\|x\|\,\|y\|} - \frac{\langle x,z\rangle}{\|x\|\,\|z\|}\cdot\frac{\langle y,z\rangle}{\|y\|\,\|z\|} \geq 0$$

in which case the lemma is satisfied with $c = 0$

or that $$\frac{\langle x,y\rangle}{\|x\|\,\|y\|} \geq \frac{\langle x,z\rangle}{\|x\|\,\|z\|}\cdot\frac{\langle y,z\rangle}{\|y\|\,\|z\|} - \left(1-\left(\frac{\langle x,z\rangle}{\|x\|\,\|z\|}\right)^2\right)^{1/2}\left(1-\left(\frac{\langle y,z\rangle}{\|y\|\,\|z\|}\right)^2\right)^{1/2}$$

Let α and β be defined by

$$\frac{\langle x,z\rangle}{\|x\|\,\|z\|} = \cos\alpha \qquad |\alpha| \leq \pi/2$$

$$\frac{\langle y,z\rangle}{\|y\|\,\|z\|} = \cos\beta \qquad |\beta| \leq \pi/2$$

By assumption $\cos\beta \geq \epsilon$, and thus $|\beta| \leq \frac{\pi}{2} - \sin^{-1}\epsilon$. The above inequality becomes in terms of α and β

$$\frac{\langle x,y\rangle}{\|x\|\,\|y\|} \geq \cos(|\alpha| + |\beta|)$$

since $$\cos(|\alpha| + |\beta|) \geq \cos(\pi - \sin^{-1}\epsilon)$$
$$= -\cos\sin^{-1}\epsilon$$

the lemma follows thus as claimed with $c = -\cos\sin^{-1}\epsilon$.

<u>Lemma 3.3</u>: Let X be a real inner product space and let 0 be a causal operator on X. Then 0 is a nonnegative operator on X if and only if $\langle P_T x, P_T 0x\rangle \geq 0$ for all $x \in X$ and all $T \in S$.

<u>Proof</u>: (i) <u>Only if</u>: The proof goes by contradition. Assume therefore that $\langle P_T x, P_T 0x\rangle < 0$ for some $x \in X$ and some $T \in S$.

Then $<P_\tau x, P_\tau 0x> = <P_\tau x, P_\tau 0 P_\tau x> = <P_\tau x, 0 P_\tau x> < 0$. This contradicts the fact that 0 is a nonnegative operator on X.

(ii) If: The proof goes again by contradiction. Assume therefore that $<x, 0x> < 0$ for some $x \in X$. Since $<x, 0x> = \frac{1}{4}(\|x+0x\|^2 - \|x-0x\|^2)$ it follows that $\|x + 0x\|^2 > \|x - 0x\|^2$. Since however the norm of any element $x \in X$ can be arbitrarily closely approximated by $P_\tau x$ for a suitable chosen $\tau \in S$ it follows that $\|P_\tau(x+0x)\|^2 > \|P_\tau(x-0x)\|^2$ for some $\tau \in S$. Thus $<P_\tau x, P_\tau 0x> < 0$ for some $x \in X$ which yields the contradiction.

Corollary 3.2: Let X be a real inner product space. If G_1 and G_2 are nonnegative operators on X one of which is positive, then the feedback system under consideration is X-stable.

Proof: Assume that $G_1 \geq \epsilon_1 I$ and that $G_2 \geq 0$ (the other case is proven analogously), then $\|G_1\| \neq 0$. By Lemmas 3.1 and 3.3 $<P_\tau x, P_\tau G_1 x> \geq \epsilon_1 <P_\tau x, P_\tau x> \geq \epsilon_1 \|G_1\|^{-1} \|P_\tau x\| \; \|P_\tau G_1 x\|$ and $<P_\tau G_2 P_\tau G_1 x, P_\tau G_1 x> = <P_\tau G_2 G_1 x, P_\tau G_1 x> \geq 0$. Thus by Lemma 3.2 there are two possibilities: either $<P_\tau x, P_\tau G_2 G_1 x> \geq -\cos \sin^{-1} \epsilon_1 \|G_1\|^{-1} \|P_\tau x\| \; \|P_\tau G_2 G_1 x\|$ or $\|P_\tau G_1 x\| = 0$. The latter case yields $\|P_\tau G_2 G_1 x\| = 0$ since G_2 is bounded and shows that in this case the conditions of Theorem 3.1 are satisfied for any $\epsilon < 1$. Assume therefore that $<P_\tau x, P_\tau G_2 G_1 x> \geq -\cos \sin^{-1} \epsilon_1 \|G_1\|^{-1} \|P_\tau x\| \; \|P_\tau G_2 G_1 x\|$. However, $\|P_\tau x + P_\tau G_2 G_1 x\|^2 = \|P_\tau x\|^2 + 2<P_\tau x, P_\tau G_2 G_1 x> + \|P_\tau G_2 G_1 x\|^2$. There are again two cases to consider: either $<P_\tau x, P_\tau G_2 G_1 x> \geq 0$ in which case the conditions of Theorem 3.1 are satisfied for any $\epsilon < 1$, or $<P_\tau x, P_\tau G_2 G_1 x> \leq 0$ in which case $|<P_\tau x, P_\tau G_2 G_1 x>| \leq |\cos \sin^{-1} \epsilon_1 \|G_1\|^{-1}| \; \|P_\tau x\| \; \|P_\tau G_2 G_1 x\|$. Thus for this case

$$\| P_\tau x + P_\tau G_2 G_1 x \| \geq (1 - |\cos \sin^{-1} \epsilon_1 \| G_1 \|^{-1} |) \| P_\tau x \|$$

Hence the inequality

$$\| P_\tau x + P_\tau G_2 G_1 x \| \geq (1 - |\cos \sin^{-1} \epsilon_1 \| G_1 \|^{-1} |) \| P_\tau x \|$$

is satisfied for all $\tau \in X$ and $x \in X$, which yields the corollary by Theorem 3.1 with $\epsilon = 1 - |\cos \sin^{-1} \epsilon_1 \| G_1 \|^{-1}|$.

Corollary 3.2 and Lemma 3.2 are graphically illustrated in Fig. 3.4

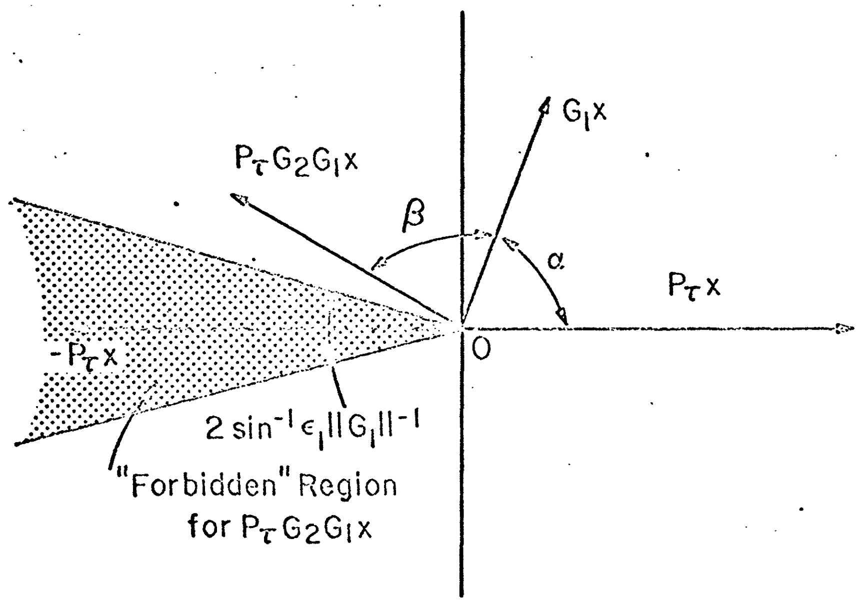

Fig. 3.4 Illustration of Corollary 3.2

In an actual situation it is rather unusual that a system will satisfy the conditions of Corollaries 3.1 or 3.2. This is the motivation for the multiplier theorems of the type used by Popov and which have since widely been used in the literature. These are now introduced. The basic idea again is simple, and is illustrated by the transformation of the original feedback loop to the feedback loop shown in Fig. 3.5.

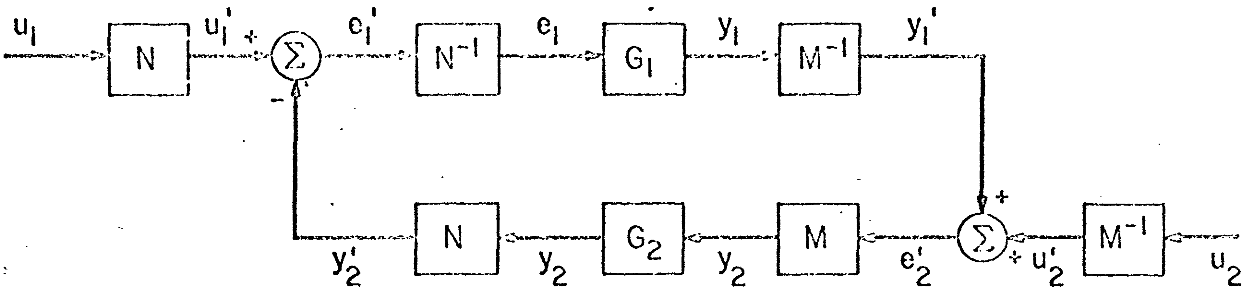

Fig. 3.5 Illustration of the Introduction of Multipliers

Theorem 3.2: Let M be a bounded causal operator on X_e. If M has a bounded inverse on X and if $\|P_\tau(I + MG_2G_1M^{-1})x\| \geq \epsilon\|P_\tau x\|$ for all $x \in X$, all $\tau \in S$ and some $\epsilon > 0$, then the feedback system under consideration is X-stable.

Proof: Let $u_1, u_2 \in X$ and let e_1, y_1, e_2, y_2 be a solution with $e_1, y_1, e_2, y_2 \in X_e$ then, using an argument as in the proof of Theorem 3.1 it follows that for all $\tau \in S$, $P_\tau(I + G_2G_1)P_\tau e_1 = P_\tau(u_1 - G_2(G_1e_1 + u_2) - G_2G_1e_1)$ for all arguments $t \in S$. Since M is causal, it follows that $P_\tau(I + MG_2G_1M^{-1})MP_\tau e_1 = P_\tau M(u_1 - G_2(G_1e_1 + u_2) - G_2G_1e_1)$ which by an argument as in the proof of Theorem 3.1 implies that $\|MP_\tau e_1\| \leq \epsilon^{-1}\|M\|\ \|u_1\| + \epsilon^{-1}K\|M\|\ \|G_2\|\ \|u_2\|$ for all $\tau \in S$. Hence

$$\|P_\tau Me_1\| \leq \|MP_\tau e_1\| \leq \epsilon^{-1}\|M\|\ \|u_1\| + \epsilon^{-1}K\|M\|\ \|G_2\|\ \|u_2\| .$$

Thus $Me_1 \in X$ and since $M(M^{-1}Me_1) = Me_1$, this implies that $e_1 = M^{-1}Me_1 \in X$ and that $\|e_1\| \leq \epsilon^{-1}\|M^{-1}\|\ \|M\|\ \|u_1\| + \epsilon^{-1}K\|M^{-1}\|\ \|M\|\ \|G_2\|\ \|u_2\|$, which implies X-stability.

Since it is in general rather difficult to compute the bound of a composition of two given elements, the following corollary is useful.

Corollary 3.3: If there exist elements M, N, and R such that

(i) M satisfies the conditions of Theorem 3.2

(ii) G_1 can be factored as $G_1 = NR$ and MG_2N and RM^{-1} are bounded operators on X

(iii) $\|MG_2N\| \cdot \|RM^{-1}\| < 1$

then the feedback system under consideration is X-stable.

Proof: Since $\|MG_2G_1M^{-1}\| = \|MG_2NRM^{-1}\| \leq \|MG_2N\| \cdot \|RM^{-1}\| < 1$, the corollary follows from Theorem 3.2 and an argument as in the proof of Corollary 3.1.

Remarks: Notice that Corollary 3.3 does not require N or R to be causal. The corollary similar to the previous one, but using positive operators, is more useful since verifying positivity is in general a simpler task than computing bounds of operators.

Corollary 3.4: Let X be a real inner product space. If there exist operators M, N and R on X such that

(i) M is a bounded causal operator on X_e which has a bounded causal inverse, M^{-1}, on X

(ii) G_1 can be factored as $G_1 = NR$ and MG_2N and RM^{-1} are bounded operators on X

(iii) MG_2N and RM^{-1} are nonnegative operators on X, one of which is positive, and MG_2N is a causal operator on X,

then the feedback system under consideration is X-stable.

Proof: Denote RM^{-1} by Z_1 and MG_2N by Z_2 and assume that $Z_1 \geq c_1 I$ and that $Z_2 \geq 0$ (the other case is proven analogously).

Then

$$< P_\tau x, Z_1 P_\tau x> \geq \epsilon_1 < P_\tau x, P_\tau x> \geq \epsilon_1 \|Z_1\|^{-1} \|P_\tau x\| \, \|Z_1 P_\tau x\|$$

and $< Z_2 P_\tau Z_1 P_\tau x, P_\tau Z_1 P_\tau x> = < P_\tau Z_2 Z_1 P_\tau x, Z_1 P_\tau x> \geq 0$

Thus Lemma 3.2 implies that

$$< P_\tau x, P_\tau Z_2 Z_1 P_\tau x > \geq - \cos \sin^{-1} \epsilon_1 \|Z_1\|^{-1} \|P_\tau x\| \, \|P_\tau Z_2 Z_1 x\|$$

which leads to the conclusion by Theorem 3.4 and the same argument as used in proof of Corollary 3.2.

Remark: The choice of G_1 in the factorization in Corollaries 3.3 and 3.4 is not essential and a similar corollary in which G_2 is factored holds.

Corollary 3.5: Let X be a real inner product space. If there exists an operator Z on X such that

(i) Z can be factored as $Z = MN$ with M a bounded linear operator as X with a bounded linear inverse, M^{-1},

(ii) $(M^*)^{-1}$, M^* and N are causal operators on X

(iii) G_2 can be factored as $G_2 = RZ$

(iv) R and ZG_1 are bounded nonnegative operators as X, one of which is positive;

then the feedback system under consideration is X-stable.

Proof: Assume that $R \geq \epsilon_1 I$ and that $ZG_1 \geq 0$ (the other case is proven analogously). Thus $R \geq \epsilon_1 I$ which implies that $M^*RM \geq \epsilon_1 M^*M$. Similarly $ZG_1 = MNG_1 \geq 0$ which implies that $NG_1(M^*)^{-1} \geq 0$. Since $\|Mx\| \geq \|M^{-1}\|^{-1} \|x\|$ it follows that $M^*M \geq \|M^{-1}\|^{-2} I$ and that $M^*RM \geq \epsilon_1' I = \epsilon_1 \|M^{-1}\|^{-2} I$. All the elements are now available to apply Corollary 3.4 if it can be shown that M^* can be extended to a bounded causal operator on X_e. This however is done simply by defining M^*x for $x \in X_e$ to be the element of X_e, y, such that $P_\tau y = P_\tau M^* P_\tau x$ for all $\tau \in S$. Notice that the right-hand side of this equality is well defined since $P_\tau x \in X$.

3.2 A Standard Modification for Feedback Systems

Since it is generally easier to identify positive operators the question arises whether or not there exist certain transformations which will put the feedback system in a form in which positive operators can be used. This is done by the standard manipulation shown in Fig. 3.6.

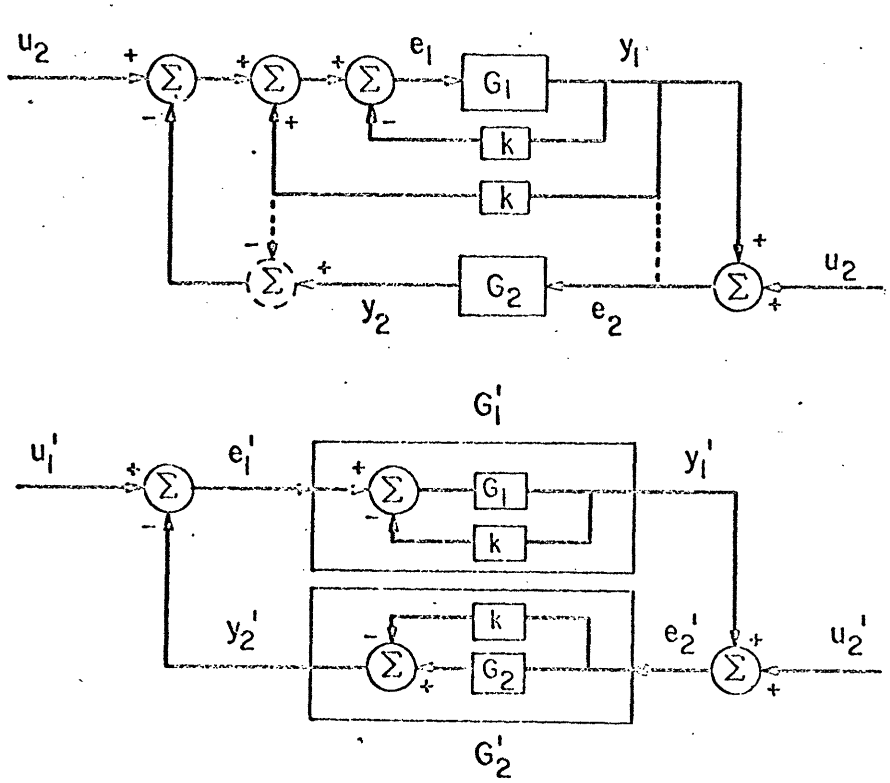

Fig. 3.6 Transformations of the Feedback Loop

Let k be a scalar such that $1 + kG_1$ is invertible on X_e and such that $(1 + kG_1)^{-1}$ is a bounded causal operator on X_e. It will now be shown that under these assumptions it is possible to define a new feedback system such that the stability of the resulting feedback system implies the stability of the original one.

Lemma 3.4: Let $G_1' = G_1(I+kG_1)^{-1}$ and $G_2' = G_2 - kI$. Then every solution $\{e_1, y_1, e_2, y_2\}$ to the original feedback system (with the operators G_1 and G_2) which is such that $e_1, y_1, e_2, y_2 \in X_e$ yields a solution to the feedback system defined above (with the operators G_1' and G_2') with

$$u_1' = u_1 - ku_2 \qquad u_2' = u_2 \qquad e_1' = e_1 + ky_1 \qquad e_2' = e_2$$
$$y_1' = y_1 \qquad y_2' = y_2 - ke_2$$

Furthermore, if the second feedback system is X-stable, then so is the first.

Proof: The verification of the first statement is straightforward and will not be carried out explicitly. The stability part follows from the relations between the solution as given in the lemma.

The unanswered question is of course to determine for what operators G and scalars k the operator $(I+kG)^{-1}$ exists and is a bounded causal operator on X_e. For the operators as in the classes $\underset{\sim}{G}$ and $\underset{\sim}{F}_t$ introduced in the second chapter, it is possible to give at least a partial answer. The first is the well-known Nyquist criterion.

Let $(g(t), \{g_k\}) \in L_1 \times \ell_1$ and let t_k be a mapping from I into R. Let $y(t) = Gx(t)$ be defined by

$$y(t) = \sum_{k=-\infty}^{+\infty} g_k x(t-t_k) + \int_{-\infty}^{+\infty} g(t-\tau)x(\tau)\,d\tau$$

It follows from Lemma 2.1 that G maps L_2 into itself, that $G \in \underset{\sim}{G}$ and that the function $G(j\omega)$ associated with it is given by

$$G(j\omega) = \sum_{k=-\infty}^{+\infty} g_k e^{-j\omega t_k} + \int_{-\infty}^{+\infty} g(t)e^{-j\omega t}\,dt$$

It is clear that if $g(t) = 0$ for $t \leq 0$ and if $t_k \geq 0$ for all $k \in I$, then G as defined above is causal and maps L_{2e} into itself. Indeed, let $x \in L_{2e}$ then $P_\tau Gx = P_\tau GP_\tau x$ which since $P_\tau x \in L_2$ for all τ yields $GP_\tau x \in L_2$ and thus that $P_\tau GP_\tau x \in L_2$ for all τ. Thus $Gx \in L_{2e}$.

Lemma 3.5: Let $g(t) = 0$ for $t < 0$ and let $t_k \geq 0$ for all $k \in I$. Then $(I + kG)^{-1}$ exists and is a bounded causal operator on L_{2e} if and only if the Nyquist locus of G (i.e., the locus of the points in the complex plane defined by $G(j\omega)$, for $\omega \in R$) does not encircle and is bounded away from $-1/k + o.j$. Moreover, $(I + kG)^{-1} \in \underset{\sim}{G}$ if it exists.

Proof: This is a basic result originally due to Nyquist and in its present form and generality to Desoer (18).

Let f be a mapping from $R \times S$ into R such that there exists a number K such that $\|f(\sigma, t)\| \leq K\|\sigma\|$ and define the operator F on Y (as defined in the beginning of this chapter with $V = R$) by: $Fx = f(x(t), t)$ for all $t \in S$ and $x \in Y$. It is easy to verify that under these assumptions F is a bounded causal operator on X_e.

Lemma 3.6: $(I + kF)^{-1}$ exists and is a bounded causal operator on X_e if and only if $\sigma + kf(\sigma, t) - \epsilon\sigma$ is a monotone nondecreasing function of σ for all $t \in S$ and some $\epsilon > 0$. Moreover $(I + kF)^{-1}$, if it exists, is of the same type as F and the corresponding mapping from $R \times S$ into R is given by the inverse of the function $\sigma + kf(\sigma, t)$.

Proof: The lemma is immediate

3.3 A Stability Criterion for Feedback Systems with a Linear Periodic Gain in the Feedback Loop

The first example of a concrete stability criterion deals with a feedback system with a linear time-invariant convolution operator in the forward loop and a linear periodically time-varying gain in the feedback loop. This feedback system is shown in Fig. 3.7.

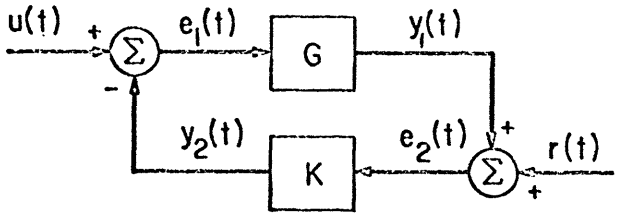

Fig. 3.7 The Feedback System Under Consideration in Section 3.3

Definitions: The operators G and K are formally defined by

$$Gx(t) = \sum_{k=-\infty}^{-\infty} g_k x(t-t_k) + \int_{-\infty}^{+\infty} g(\tau)\, x(t-\tau)\, d\tau$$

and

$$Kx(t) = k(t)x(t)$$

Assumptions: It is assumed that:

(i) t_k is a map from I into R

(ii) $k(t) \in L_\infty$ and $(g(t), \{g_k\}) \in L_1 \times \ell_1$

(iii) $t_k \geq 0$ for all $k \in I$ and $g(t) = 0$ for $t \leq 0$

It has been pointed out previously that under these conditions the operators G and K map L_{2e} into itself and that they are causal and bounded. Furthermore, since they are also linear, they satisfy Lipshitz conditions on L_{2e}.

Definition: The feedback system under consideration is said to be L_2-stable if all $u, r \in L_2$, and $e_1, y_1, e_2, y_2 \in L_{2e}$ which satisfy the equations $e_1(t) = u(t) - y_2(t)$; $e_2(t) = r(t) + y_1(t)$; $y_1(t) = Ge_1(t)$; $y_2(t) = Ke_2(t)$ for all $t \in R$ yield $e_1, y_1, e_2, y_2 \in L_2$ and if there exist constants K_1 and K_2 such that

$$\|e_1\|_{L_2} + \|y_1\|_{L_2} + \|e_2\|_{L_2} + \|f_2\|_{L_2} \leq K_1\|u\|_{L_2} + K_2\|r\|_{L_2}$$

Example: Consider the linear time-invariant differential equation

$$p(D)x(t) + k'(t)q(D)x(t) = 0 \qquad D^i = \frac{d^i}{dt^i}$$

The following assumptions are made:

A.1 $p(s)$ and $q(s)$ are real polynomials in s, i.e.,

$$p(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_0$$

$$q(s) = q_n s^n + q_{n-1}s^{n-1} + \ldots + q_0$$

with p_i and q_i real numbers

A.2 $k'(t)$ is a real-valued piecewise continuous function of t which belongs to L_∞

A.3 Either of the following conditions is satisfied:

(i) $q_n = 0$

(ii) $q_n \neq 0$ and $-1/q_n \notin [\alpha, \beta]$ where α and β are such that $\alpha \leq k'(t) \leq \beta$ for all $t \in R$.

A real valued continuous function $x(t)$ is said to be a solution of this differential equation if it possesses $(n-1)$ continuous derivatives and if it satisfies the above differential equation for all t for which $k'(t)$ is continuous. Clearly $x(t) \equiv 0$ is a solution. This solution is called the null-solution and is said to be asymptotically stable if all solutions approach the null-solution for $t \rightarrow \infty$.

It will now be shown that in many cases asymptotic stability of the null-solution of the above time-varying differential equation can be deduced from L_2-stability of a feedback system of the type which is being considered in this section.

<u>Assumption</u>: It is assumed that there exists a real number α such that the zeros of the polynomial $p(s) + \alpha q(s)$ have a negative real part.

It can be shown without much difficulty (see e.g., (60)) that the differential equation can then be rewritten as

$$p_1(D)x(t) + k_1(t)\, q_1(D)x(t) = 0$$

with $p_1(s)$ a monic Hurwitz polynomial of degree n (i.e., all its zeros have a negative real part, and the coefficient of s^n is one) with the degree of $p_1(s)$ larger than the degree of $q_1(s)$. This n-th order scalar differential equation can then be written as a first order vector differential equation

$$\frac{dz(t)}{dt} = Az(t) + bu(t)$$
$$y(t) = c' z(t)$$
$$u(t) = k_1(t)\, y(t)$$

where $$z(t) = \text{col}\left(x(t), \frac{dx(t)}{dt}, \ldots, \frac{d^{n-1}x(t)}{dt^{n-1}}\right)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -p_{1,0} & -p_{1,1} & \cdots & \cdots & -p'_{1,n-1} \end{bmatrix}$$

$$b = \text{col}(0, 0, \ldots, 0, 1)$$

$$c = \text{col}(q_{1,0}, q_{1,1}, \ldots, q_{1,n-1})$$

$$c'(Is - A)^{-1} b = \frac{q_1(s)}{p_1(s)}$$

The null-solution of the differential equation under consideration will then be asymptotically stable if and only if given any $z(0)$, $\lim_{t \to \infty} |z(t)|$ exists and is zero. It is well-known (see e.g., (16)) that the smoothness conditions on $k'(t)$ are sufficient to ensure the existence of a unique solution which assumes the value $z(0)$ for $t = 0$. Furthermore, the solutions satisfy the integral equation

$$z(t) = e^{At} z(0) - \int_0^t e^{A(t-\tau)} b\, k_1(\tau)\, y(\tau)\, d\tau \qquad \text{for } t \geq 0$$

which implies that

$$y(t) = c' e^{At} z(0) - \int_0^t c' e^{A(t-\tau)} b\, k_1(\tau)\, y(\tau)\, d\tau \qquad \text{for } t \geq 0$$

At this point it is clear that this equation is of the form of the feedback system under consideration with

$$u(t) = 0 \qquad r(t) = c' e^{At} z(0) \quad \text{for } t \geq 0$$
$$= 0 \quad \text{otherwise}$$

$$y(t) = y(t) \qquad y_2(t) = k_1(t)\,(y(t) + r(t))$$

$$e_1(t) = y(t) \qquad e_2(t) = y(t) + r(t)$$

$$g(t) = c' e^{At} b \quad \text{for } t \geq 0 \qquad k(t) = k_1(t)$$
$$= 0 \quad \text{otherwise}$$

$$g_k = 0 \quad \text{for all } k \in I$$

It follows from the assumption on the zeros of $p_1(s)$ that all eigenvalues of A have a negative real part and thus that $c' e^{At} b \in L_p(0,\infty)$ for all $p \geq 1$.

Thus if L_2-stability for this feedback system is proven it follows that all solutions $z(t)$ to this vector differential equation which are such that $y(t) \in L_{2e}(0,\infty)$ also belong to $L_2(0,\infty)$. Since all solutions are continuous, all solutions $y(t)$ do belong to $L_{2e}(0,\infty)$ and hence all solutions yield $y(t) \in L_2(0,\infty)$. Since however

$$z(t) = e^{At} z(0) + \int_0^\infty e^{A(t-\tau)} b k_1(\tau) y(\tau) d\tau$$

Since $e^{At} b \in L_1(0,\infty)$, $k_1(\tau) \in L_\infty(0,\infty)$, $y(t) \in L_2(0,\infty)$ and the convolution of an L_1-function with an L_2-function yields an L_2-function, it follows thus that $z(t) \in L_2(0,\infty)$. Furthermore

$$\frac{dz(t)}{dt} = Az(t) - k_1(t) bc' z(t)$$

hence $\frac{dz(t)}{dt} \in L_2(0,\infty)$. Since $z(t)$ and $\frac{dz(t)}{dt}$ belong to $L_2(0,\infty)$ $\lim_{t\to\infty} z(t)$ exists and is zero. Hence L_2-stability of the above feedback system implies asymptotic stability of the null-solution of the differential equation.

These simple manipulations show that although it might at first glance seem that the type of stability which is obtained in the theorem in the previous section is not as strong as Lyapunov stability, in many circumstances it actually implies it.

Additional Assumption: In addition to the assumptions made in the beginning of this section it will be assumed that $k(t)$ is periodically time-varying, i.e., that

$k(t) = k(t + T)$ for almost all t and a given $T > 0$.

Feedback systems of the resulting type occur frequently in the design of systems containing parametric devices. The stability properties of such systems are of course of primary importance and criteria using frequency-domain conditions similar to the Nyquist criterion have proven to be a particularly feasible tool for the designer. Moreover, the local stability of a periodic solution of a nonlinear differential equation is often equivalent to the stability of the null-solution of a linear time-varying differential equation of the form of the differential equation in the above example.

The stability properties of the feedback system under consideration have received a great deal of attention in the past (see (11) for a survey), and the result that is best known is the Circle Criterion which has evolved out of the work of Sandberg (52) and others. Although the Circle Criterion is applicable under much weaker conditions (the feedback gain need not be linear or periodic) than the ones stated above, it was originally proven making essentially the same assumptions.

In this section a new frequency-domain stability criterion is developed which assumes explicitly that the feedback gain is linear and periodic with a certain given period. This assumption makes it then possible to obtain an improved stability criterion. The result gives, for a particular transfer function of the forward loop, combinations of the lower bound α, the upper bound β, and the period T of $k(t)$ which yield stability. This dependence on the period is of course as expected and has been investigated exhaustively for certain classical types of second order differential equations. The result obtained by Sandberg in (51) is essentially also of this type.

The criterion, which is stated in Theorem 3.3 and 3.4, requires, as most recent frequency-domain stability criteria, the existence of a multiplier having certain properties. With the exception of the Popov criterion however, there is generally no procedure offered to determine whether or not such a multiplier exists for a given transfer function of the forward loop. This is not the case for the criterion presented here since Theorems 3.3 and 3.4 can be completely rephrased in terms of this transfer function. In fact, a simple graphical procedure is given to determine whether or not the multiplier exists.

Stability Criteria: Let

$$G(j\omega) = \sum_{k=-\infty}^{+\infty} g_k e^{-j\omega t_k} + \int_{-\infty}^{+\infty} g(t) e^{-j\omega t} dt$$

Theorem 3.3: The feedback system under consideration is L_2-stable if

(i) $0 \leq k(t) = k(t+T) \leq k_{max} - \epsilon$ for some $\epsilon > 0$ and almost all t

(ii) there exists a real function of s, F(s), such that for almost all $\omega \geq 0$

F.1 $\operatorname{Re} F(j\omega) \geq \epsilon$ for some $\epsilon > 0$,

F.2 $F(j\omega) = F(j(\omega + 2\pi T^{-1})) \in L_\infty$

F.3 $\operatorname{Re} F(j\omega)(G(j\omega) + 1/k_{max}) \geq 0$

Theorem 3.4: The feedback system under consideration is L_2-stable if

(i) $\alpha + \epsilon \leq k(t) = k(t+T) \leq \beta - \epsilon$ for some $\epsilon > 0$, almost all t and $\alpha \neq 0$

(ii) the Nyquist locus of G, i.e., the points in the complex plane determined by $G(j\omega)$, $\omega \in R$, is bounded away from $-\frac{1}{\alpha} + 0.j$ and does not encircle it

(iii) there exists a real function of s, F(s), such that for almost all $\omega \geq 0$ conditions F.1 and F.2 are satisfied and such that

$$F.3' \quad \text{Re } F(j\omega) \frac{\beta(G(j\omega)+1}{\alpha G(j\omega)+1} \geq 0$$

Before interpreting these results and reducing the requirement that the multiplier $F(j\omega)$ exists to a condition on the transfer function $G(j\omega)$ the theorems will be proven using the methods outlined in Section 1 of this chapter and the reduction outlined in Section 2. In a forthcoming paper (60) the author has proven this criterion for feedback systems which can be described by ordinary differential equations. The proof presented there is much more elementary and uses Floquet theory for ordinary differential equations with periodic coefficients and the classical theory of Toeplitz forms. The results however are less general.

Proof of Theorem 3.3: (i) A reduction of the feedback system under consideration with the methods outlined in Section 2 of this chapter shows that it suffices to prove L_2-stability for the feedback system with $G_1' = G + 1/k_{max} I$ in the forward loop and $G_2' = K(I - 1/k_{max} K)^{-1}$ in the feedback loop. Observe that it follows from Lemmas 3.4 and 3.5 that $G_1' \in \underset{\sim}{G}$ and that it has the function $G(j\omega) + 1/k_{max}$ associated with it, and that $G_2' \in \underset{\sim}{K}_T$ and has the function $k(t)(1-k(t)/k_{max})^{-1}$ associated with it.

(ii) Let F be the element of $\underset{\sim}{G}_T$ which has the function $F(j\omega)$ associated with it. By Theorem 2.3, and the assumptions of the theorem, $G_1'F$ is a nonnegative operator on L_2 and $F^{-1}K$ is a positive operator on L_2. (Note that F^{-1} exists on L_2 since $\text{Re } F(j\omega) \geq \epsilon > 0$.) Write K as $K = F^{-1}FK$. Then by Corollary 3.5, it suffices to prove that Z can be factored as $Z = MN$ with M a bounded linear operator from L_2 into itself with a bounded linear inverse and N a bounded operator from L_2 into itself, and with M^*, $(M^*)^{-1}$ and N causal.

(iii) Since $t_k \geq 0$, $g(t) = 0$ for $t < 0$ and since $(g(t), \{g_k\}) \in L_1 \times \ell_1$, $G(j\omega)$ is an analytic function of ω and therefore the multiplier $F(j\omega)$ if it exists can be chosen such that its Fourier series belongs to ℓ_1 (i.e., if there exists a function $F(j\omega)$ satisfying conditions F1-3, then there exists one whose Fourier series belongs to ℓ_1). $F(j\omega)$ thus can be written as the uniformly convergent Fourier series

$$F(j\omega) = \sum_{k=-\infty}^{+\infty} f_k e^{-jk\omega T}$$

with $\{f_k\} \in \ell_1$. Hence

$$Fx = \sum_{k=-\infty}^{+\infty} f_k x(t-kT)$$

Let $F_1(z)$ denote the z-transform of $\{f_k\}$. It is simple to verify that $F_1(e^{j\omega T}) = F(j\omega)$. Since $\text{Re}F(j\omega) \geq \epsilon > 0$, it follows thus that $F_1(z) \neq 0$ for $|z| = 1$ and that the increase in its argument as z moves around the circle $|z| = 1$ is zero. Hence Theorem 2.21 is applicable. This theorem then yields the factorization required to complete the proof of Theorem 3.3.

<u>Proof of Theorem 3.4:</u> The only matter which is different in this proof is the preliminary modification of the feedback loop. It

thus suffices to prove L_2-stability for the feedback system with $G_1' = G_1(I + \alpha G_1)^{-1}$ in the forward loop and $G_2' = K - \alpha I$ in the feedback loop. A similar transformation shows in turn that it suffices to prove L_2-stability for the feedback system with $G_1'' = G_1' + \frac{1}{\beta-\alpha} I$ in the forward loop and $G_2'' = G_2'(I - \frac{1}{\beta-\alpha} G_2')^{-1}$ in the feedback loop. However, $G_1'' = G_1(I+\alpha G_1)^{-1} + \frac{1}{\beta-\alpha} I = \frac{1}{\beta-\alpha} (\beta G_1 + I)(\alpha G_1 + I)^{-1}$ and $G_2'' = (\beta-\alpha)(K-\alpha I)(\beta I - K)^{-1}$. Notice that the above inverses exist on L_{2e} and are causal and bounded by Lemmas 3.4 and 3.5. It is now a simple matter to verify that the conditions of the theorem imply the positivity of $G_1'' F^{-1}$ and FG_2'' and that the same proof as in Theorem 3.3 yields L_2-stability.

Theorems 3.3 and 3.4 are not very useful as they stand since they leave the question unanswered whether or not the multiplier $F(s)$ exists. This question can be resolved however, and this leads to an equivalent formulation of the above theorems.

Let

$$\phi_{max}(\omega) = \sup_{k \in I} \phi(\omega + k\, 2\pi T^{-1})$$

$$\phi_{min}(\omega) = \inf_{k \in I} \phi(\omega + k\, 2\pi T^{-1})$$

where

$$\phi(\omega) = \arg(G(j\omega) + 1/k_{max}) \quad \text{in Theorem 3.3}'$$

and

$$\phi(\omega) = \arg \frac{\beta G(j\omega)+1}{\alpha G(j\omega)+1} \quad \text{in Theorem 3.4}'$$

<u>Theorem 3.3'</u>: <u>The feedback system under consideration is L_2-stable if</u>

(i) $\epsilon \leq k(t) = k(t+T) \leq k_{max} - \epsilon$ <u>for some</u> $\epsilon > 0$ <u>and almost all</u> t

(ii) $|\phi_{max}(\omega) - \phi_{min}(\omega)| < \pi$ <u>for all</u> $|\omega| \leq \pi T^{-1}$

Theorem 3.4': The feedback system under consideration is L_2-stable if

(i) $\alpha + \epsilon \leq k(t) = k(t+T) \leq \beta - \epsilon$ for some $\epsilon > 0$, almost all t and $\alpha \neq 0$

(ii) The Nyquist locus of G, i.e., the points in the complex plane determined by $G(j\omega)$, $\omega \in R$, is bounded away from $-\frac{1}{\alpha} + 0.j$ and does not encircle it

(iii) $|\phi_{max}(\omega) - \phi_{min}(\omega)| < \pi$ for all $|\omega| \leq \pi T^{-1}$

Proof: Since $G(j\omega)$ is uniformly continuous and bounded, the sequence of functions $G(j\omega + k2\pi T^{-1})$, $k \in I$, is equicontinuous and thus $\phi_{max}(\omega)$ and $\phi_{min}(\omega)$ as defined above are continuous functions of ω. Hence, $|\phi_{max}(\omega) - \phi_{min}(\omega)|$ is a continuous function of ω. Because of symmetry ϕ_{max} and ϕ_{min} are periodic and thus $\phi_{max}(\omega) - \phi_{min}(\omega) = \phi_{max}(\omega+2\pi T^{-1}) - \phi_{min}(\omega+2\pi T^{-1})$. Since $|\phi_{max}(\omega) - \phi_{min}(\omega)| < \pi$, there exists an $\epsilon > 0$ such that $|\phi_{max}(\omega) - \phi_{min}(\omega)| \leq \pi - \epsilon$. Let $F(j\omega) = e^{-\frac{1}{2}[\phi_{max}(\omega) + \phi_{min}(\omega)]j}$. It is easily verified that this choice for $F(j\omega)$ yields the conclusion by Theorem 3.3 and 3.4. For the converse part of the equivalence, assume that $\phi_{max}(\omega') - \phi_{min}(\omega') = \pi$ for some $\omega' \in R$. Then since $\mathrm{Re}\, G(j\omega) F(j\omega)$ has to be nonnegative for all ω, this implies that $|\mathrm{Arg}\, F(j\omega')| \geq \pi/2$ which contradicts the condition that $\mathrm{Re}\, F(j\omega) \geq \epsilon > 0$.

The following two corollaries show that the criterion is a trade-off between the Circle Criterion (T arbitrary) and the "local" application of the Nyquist criterion (T small).

Corollary 3.6: The feedback under consideration is L_2-stable if $k(t)$ is periodic and if either

(i) $\epsilon \leq k(t) \leq k_{max} - \epsilon$ for some $\epsilon > 0$ and almost all t,

and $\text{Re}\, G(j\omega) + \frac{1}{k_{max}} \geq 0$ or

(ii) $\alpha + \epsilon \leq k(t) \leq \beta - \epsilon$ for some $\epsilon > 0$, almost all t, $\alpha \neq 0$,

and, $\text{Re}\, \frac{\alpha G(j\omega) + 1}{\beta G(j\omega) + 1} \geq 0$

and the Nyquist locus of G, i.e., the points in the complex plane determined by $G(j\omega)$, $\omega \in R$, is bounded away from $-\frac{1}{\alpha} + 0.j$ and does not encircle it.

Proof: Take $F(s) = 1$ and apply Theorems 3.3 or 3.4.

This corollary is essentially a particular case (since it assumes the feedback gain linear and periodic) of the Circle Criterion.

Consider the stability properties of the linear time-invariant system obtained by replacing $k(t)$ in the feedback loop by $k_t = k(t)$ for some t. If the time-invariant system thus obtained is L_2-stable for all constants k_t, it does not follow in general that the original feedback system is L_2-stable (see e.g. (12)). This fact is closely related to the Aizerman conjecture for time-invariant systems which will be discussed in the next chapter. However, the following corollary states that this procedure is legitimate if the period T is sufficiently small. The corollary essentially states that if the frequency of the feedback gain is sufficiently high compared to the natural frequencies of the forward loop then no instability due to "pumping" can occur.

Corollary 3.7: Assume that in the definition of G, $g_k = 0$ for all $t_k \neq 0$, $k \in I$, and that the feedback system is L_2-stable for any $k(t) = k$ = constant in the feedback loop with $\alpha \leq k \leq \beta$. Then there exists a T_1 such that for all $T < T_1$ the feedback system with any gain $\alpha \leq k(t) = k(t+T) \leq \beta$ in the feedback loop is also L_2-stable.

Proof: Since $\lim_{|\omega| \to \infty} G(j\omega) = c$ exists by the Riemann-Lebesgue lemma, and is real, $\lim_{|\omega| \to \infty} \phi(\omega)$ exists and is zero. Since the feedback system is L_2-stable for constant gains k in the feedback loop with $\alpha \leq k \leq \beta$, there exists a real function of s, Z(s), such that for all ω, $\text{Re}\, Z(j\omega) \geq \epsilon > 0$ and $\text{Re}\ \ Z(j\omega) \frac{\beta G(j\omega) + 1}{\alpha G(j\omega) + 1} \geq 0$. (This follows from the Nyquist diagram and a simple graphical construction, see e.g. Ref. (10)). It thus follows that for ω_o sufficiently large the function $F(j\omega) = Z(j\omega)$ for $|\omega| \leq \omega_o/2$ and $F(j(\omega+\omega_o)) = F(j\omega)$ otherwise, will yield the conclusion by Theorems 3.3 and 3.4.

Application of the Criterion:

Theorems 3.3' and 3.4' suggest an obvious graphical procedure for determining whether or not Theorems 3.3 and 3.4 predict L_2-stability of the feedback loop. Let $\omega_o = 2\pi T^{-1}$. This is illustrated in Fig. 3.8 and requires plotting the curves $\phi_N(\Omega) = \phi(\Omega + N\omega_o)$ versus Ω for $|\Omega| \leq \omega_o/2$ and $N = 0, \pm 1, \pm 2, \ldots$. The upper and lower envelope of these curves give $\phi_{max}(\Omega)$ and $\phi_{min}(\Omega)$. Theorems 3.3 and 3.4 predict L_2-stability if and only if condition (i) of Theorem 3.4 is satisfied and $\phi_{max}(\Omega) - \phi_{min}(\Omega) < \pi$ for all $|\Omega| \leq \omega_o/2$. It is apparent that this procedure, although straightforward, is rather tedious.

In order to facilitate the application of the criterion some simple necessary conditions for the multiplier F(s) to exist are given below for the case $0 < \alpha \leq \beta$:

(i) The Nyquist locus of G(s) should not encircle or intersect the straight line segment $[-1/\alpha, -1/\beta]$ of the negative real axis of the Nyquist plane,

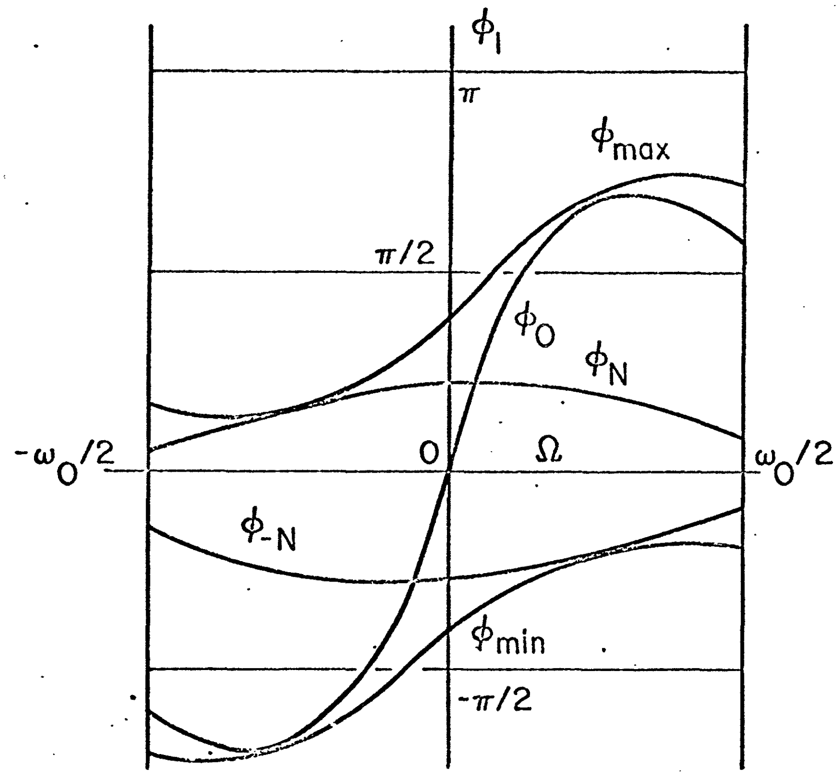

Fig. 3.8 Graphic Procedure for Determining F(s)

(ii) the points $G(jn\omega_o/2)$ $n = 0, 1, 2, \ldots$, should satisfy the conditions of the Circle Criterion, i.e., for $0 < \alpha \leq \beta$, they should not lie inside the closed disc centered on the negative real axis at $-\frac{1}{2}(1/\alpha + 1/\beta)$ with radius $\frac{1}{2}(1/\alpha - 1/\beta)$.

Analogous conditions hold for other ranges of α and β.

The second necessary condition follows from the fact that, since $F(s)$ is a real function of s, and since $F(j(\omega + \omega_o) = F(j\omega)$, $F(jn\omega_o/2 = \text{Re}\, F(jn\omega_o/2)$ for $n = 0, \pm 1, \pm 2, \ldots$. Thus conditions F.1 and F.3 of Theorems 3.3 and 3.4 imply that $\text{Re}\, \frac{\beta G(jn\omega_o/2) + 1}{\alpha G(jn\omega_o/2) + 1} > 0$ for $n = 0, \pm 1, \pm 2, \ldots$, which leads to the second necessary condition.

By choosing particular functions for $F(s)$ it is of course possible to obtain other sufficient conditions for L_2-stability. The next corollary is based upon this idea and gives a quite simple sufficient condition for the multiplier $F(s)$ to exist. It is expressed entirely in terms of the Nyquist locus of $G(s)$, and is stated here for the case $0 < \alpha \leq \beta$.

Corollary 3.8: The null-solution of (1) is asymptotically stable if

(i) the Nyquist locus of $G(s)$ does not encircle the point $-1/\alpha$ on the negative real axis of the Nyquist plane,

(ii) there exists a circle, C, which passes through the points $-1/\alpha$ and $-1/\beta$, such that the Nyquist locus of $G(s)$ for $\omega \geq 0$ does not intersect it.

Let C' be the mirror image of C with respect to the real axis, and consider the following two parts of the Nyquist locus of $G(s)$:

$$S_1: \ \{G(j\omega) \quad \text{for} \quad n\omega_o \leq \omega \leq (n+1/2)\omega_o\}$$
$$S_2: \ \{G(j\omega) \quad \text{for} \quad (n+1/2)\omega_o \leq \omega \leq (n+1)\omega_o\}$$

where $n = 0, 1, 2, \ldots$

(iii) C' does not intersect both S_1 and S_2.

This corollary is illustrated in Fig. 3.9.

Proof: Condition (i) assures that the second condition of Theorem 3.4 is satisfied. Let $|\theta| \leq \pi/2$ be the angle between the positive real axis and the straight line through the origin of the complex plane defined by the points

$$\{\frac{\beta\tau + 1}{\alpha\tau + 1} \ ; \ \tau \in C\}$$

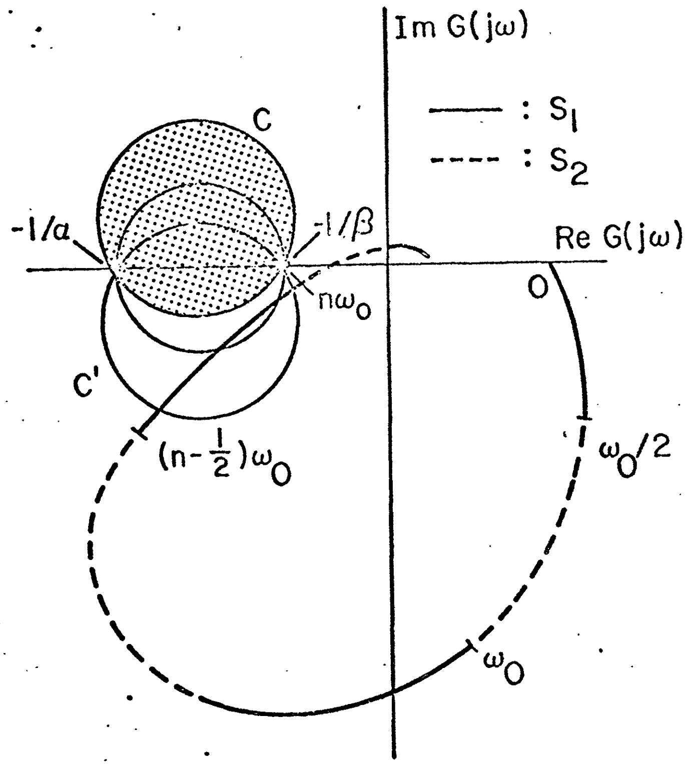

Fig. 3.9 Illustration of Corollary 3.8

Assume that $\theta \geq 0$ and that C' does not intersect S_2. (A similar argument establishes the corollary for the other cases.) Let F(s) be a real function of s such that

$$\arg F(j\omega) = \begin{cases} \pi/2 - \theta & \text{for} \quad n\omega_o < \omega < (n+1/2)\,\omega_o \\ -(\pi/2 - \theta) & \text{for} \quad (n-1/2)\,\omega_o < \omega < n\omega_o \\ 0 & \text{for} \quad \omega = n\omega_o,\ (n+1/2)\,\omega_o \end{cases}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Clearly, F(s) satisfies conditions F. 1 and F. 2 of Theorems 3. 3 and 3.4. From condition (ii) of the corollary it follows that for $\omega \geq 0$

$$-\pi + \theta < \phi(\omega) < \theta$$

and from the fact that C' does not intersect S_2 it follows that

$$-\theta < \phi(\omega) < \pi - \theta$$

for $\omega \geq 0$ and $(n-1/2)\omega_o \leq \omega \leq n\omega_o$.

Thus it follows that for $\omega \geq 0$

$$-\pi/2 < \arg F(j\omega) + \phi(\omega) < \pi/2$$

which establishes condition F.3 of Theorems 3.3 and 3.4 since

$\arg F(-j\omega) + \phi(-\omega) = -\arg F(j\omega) - \phi(\omega)$.

Examples

1. Let

$$G(s) = \frac{s}{(s+10)(s^2+0.4s+1)}$$

$k(t) = k(t+T)$ and $0 \leq k(t) \leq 2$. Determine for which range of $\omega_o = 2\pi/T$ this feedback system is stable. The Nyquist locus of G(s) is shown in Fig. 3.10.

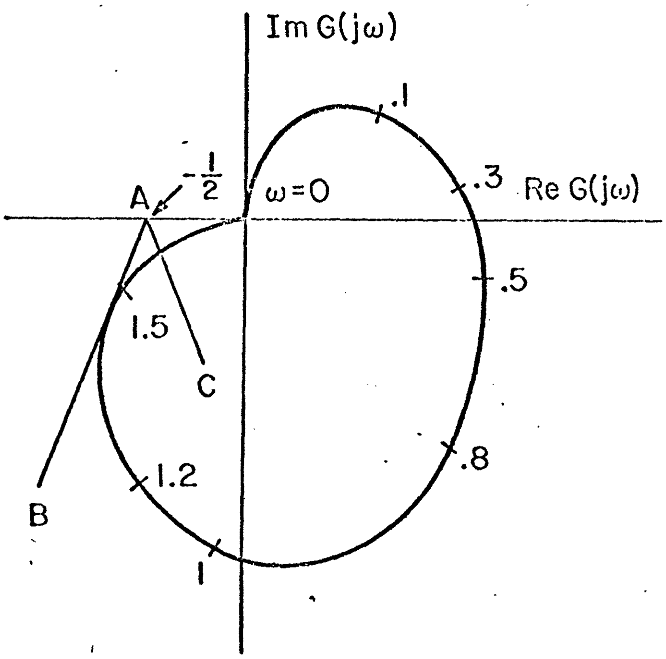

Fig. 3.10 Nyquist Locus of $s/(s+10)(s^2+0.4s+1)$

It is apparent from the Nyquist locus that the Circle Criterion cannot be used to predict L_2-stability. Using the procedure suggested above, Theorem 3.3 shows that this feedback system is L_2-stable for all k(t) in the determined range provided $\omega_o > 1.55$. Using Corollary 3.8 on the other hand this feedback system is found to be L_2-stable for all k(t) in the given range provided $\omega_o > \omega_r = 3.3$. (This number ω_r was obtained as follows: Let AB be the tangent to the Nyquist locus through the point $(-1/2+0j)$; let AC be the line symmetric to AB with respect to the real axis. The intersection of the Nyquist locus and AC then gives $\omega_r/2$.)

This example shows that although Corollary 3.8 did not give an excellent estimate, it is quite simple to apply.

2. Let $G(s) = 1/s(s+2)$. Determine $K(\omega_o)$ such that the feedback system is L_2-stable for all $k(t) = k(t+T)$, $\omega_o = 2\pi/T$ and $0 < \epsilon \leq k(t) \leq K(\omega_o)$. The Nyquist locus of $G(s)$ is shown in Fig. 3.11.

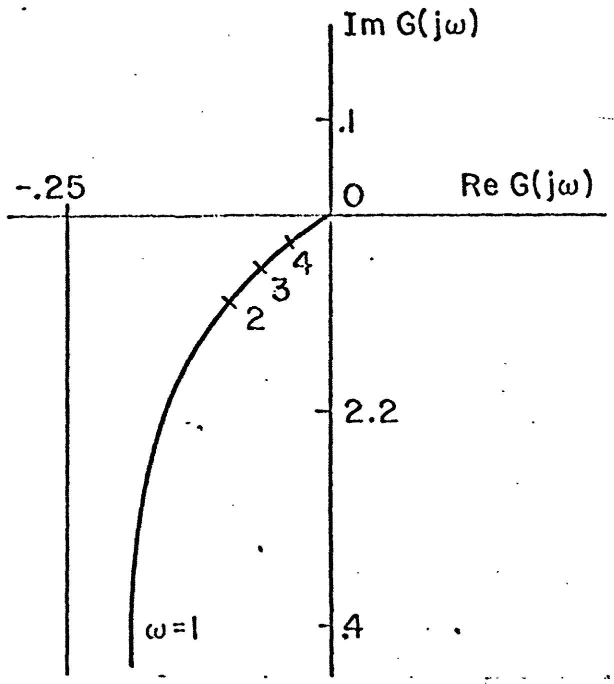

Fig. 3.11 Nyquist Locus of 1/s(s+2)

Using the Circle Criterion one obtains $K(\omega_o) = 4$. Brockett (12) has shown by examining the worst possible variation in $k(t)$ that $K(\omega_o) = 11.6$. Applying Theorem 3.3 and the graphical procedure outlined above results in $K(\omega_o)$ as shown in Fig. 3.12. The same figure also shows the result obtained using Corollary 3.8 and a graphical construction analogous to the one used in Example 1. Thus by restricting the feedback gain to be periodic it was possible by means of Theorem 3.3 to obtain higher values of K as the frequency was increased.

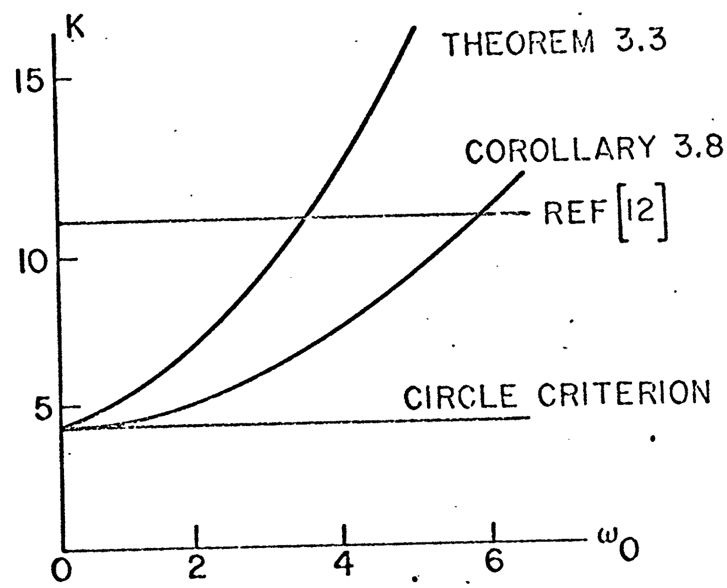

Fig. 3.12. Regions of Stability for Example 2

Remark: It follows from Example 2 that the converse of Theorems 3.3 and 3.4 is false, i.e., if $F(s)$ does not exist then there will in general not necessarily be a $k(t)$ in the required range such that the feedback system is not L_2-stable.

3.4 A Stability Criterion for Feedback Systems with a Monotone or an Odd-Monotone Nonlinearity in the Feedback Loop

As a second example of a stability criterion for feedback systems consider the system with a time-variant convolution operator, G, in the forward loop and a monotone or an odd-monotone nonlinearity in the feedback loop. For convenience and simplicity the analysis will be given for systems described by difference equations. With some modifications similar results can be obtained for the continuous case. The feedback system which will be considered is shown in Fig. 3.13.

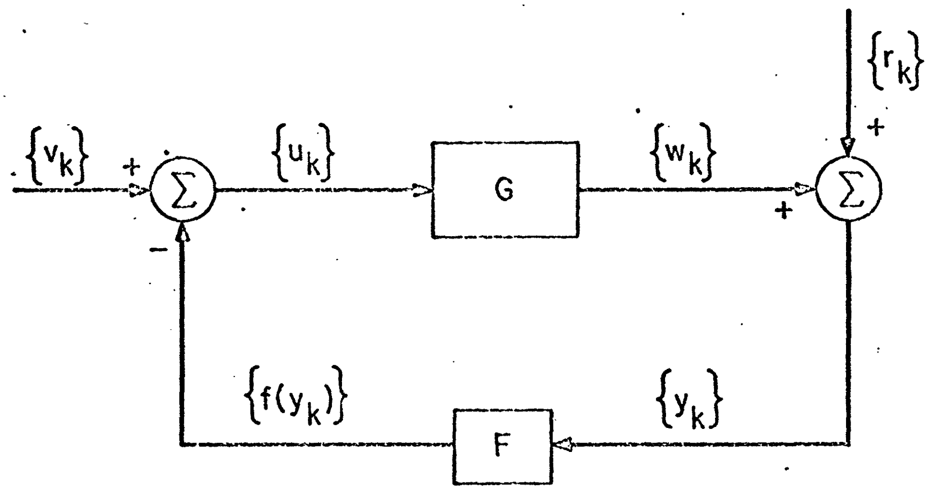

Fig. 3.13 The Feedback Loop Under Consideration in Section 3.4

Definitions: The operators G` and K are formally defined

$$G(\{x_k\})_k = \sum_{\ell=-\infty}^{+\infty} g_{k\ell}\, x_\ell \qquad k \in I$$

and

$$F(\{x_k\})_k = f(x_k) \qquad k \in I$$

Assumptions: It is assumed that

(i) $G \in \mathcal{L}^+(\ell_2, \ell_2)$, i.e., that G maps ℓ_2 into itself and that $g_{k\ell} = 0$ whenever $k < \ell$

(ii) f is a mapping from R into itself for which there exists a k such that $|f(\sigma)| \leq K|\sigma|$ for all $\sigma \in R$

It is simple to verify that under these conditions G and F map ℓ_{2e} into itself and that they are bounded and causal.

The equation describing the forward loop of the feedback system is thus

$$y_k = \sum_{\ell=-\infty}^{+\infty} g_{k\ell} u_\ell + r_k \qquad k \in I$$

The array $\{g_{k\ell}\}$ is often referred to as the <u>weighting pattern</u> of the system. This system is slightly more general than the input-output relation governed by the n-dimensional difference equation

$$\begin{aligned} x_{k+1} &= A_k x_k + b_k u_k \\ y_k &= c_k' x_k + d_k u_k \qquad\qquad k = 0, 1, 2, \ldots \\ x_0 &= \text{given} \end{aligned}$$

where b_k and c_k are n-vectors, d_k is a scalar, A_k is an $(n \times n)$ matrix and x_k is an n-vector called the <u>state</u> of the system. This input-output relation is a particular case of the input-output relation defined above with

$$\begin{aligned} g_{k\ell} &= c_k' A_{k-1} \cdots A_{\ell+1} b_\ell &\quad& \text{for } k \geq \ell + 2 \\ g_{k\ell} &= c_k' b_k && \text{for } k = \ell + 1 \\ g_{k\ell} &= d_k && \text{for } k = \ell \\ g_{k\ell} &= 0 && \text{otherwise} \end{aligned}$$

and

$$r_k = c_k' A_{k-1} \cdots A_o x_o \qquad \text{for } k \leq 1$$

$$r_0 = c_0' x_0$$
$$r_k = u_k = 0 \qquad \text{for } k < 0$$

The case in which the system is time-invariant is of particular interest. The system is then defined by the equation

$$y_k = \sum_{\ell=-\infty}^{+\infty} g_{k-\ell} u_\ell + r_k \qquad k = 0, \pm 1, \pm 2, \ldots$$

where g_k is assumed to be zero for $k < 0$. This system is slightly more general than the input-output relation governed by the n-dimensional difference equation

$$x_{k+1} = Ax_k + bu_k$$
$$y_k = c' x_k + d \qquad k = 0, 1, 2, \ldots$$

where b and c are constant n-vectors, d is a scalar constant, A is a constant $(n \times n)$ matrix and x_k is an n-vector called the <u>state</u> of the system. This input-output relation is a particular case of the input-output relation defined above with

$$g_k = c' A^{k-1} b \qquad \text{for } k > 0$$
$$g_0 = d$$
$$g_k = 0 \qquad \text{for } k < 0$$
$$r_k = c' A^k x_0 \qquad \text{for } k \geq 0$$
$$r_k = u_k = 0 \qquad \text{for } k < 0$$

The equation describing the feedback loop is

$$u_k = f(y_k) + v_k \qquad k \in I$$

and the closed loop equation of motion becomes

$$y_k + \sum_{\ell=-\infty}^{+\infty} g_{k\ell} f(y_\ell) = \sum_{\ell=-\infty}^{+\infty} g_{k\ell} v_\ell + r_k \qquad k \in I$$

Definition: The feedback system under consideration is said to be ℓ_2-stable if for all ℓ_2-sequences $r = \{r_k\}$ and $v = \{v_k\}$, all solutions $\{y_k\}$ which are such that $\left(\sum_{k=-\infty}^{n} y_k^2\right)^{1/2}$ exists for all $n \in I$, belong to ℓ_2 and satisfy the inequality

$$\left(\sum_{k=-\infty}^{+\infty} y_k^2\right)^{1/2} \quad K_1\left(\sum_{k=-\infty}^{+\infty} v_k^2\right)^{1/2} + K_2\left(\sum_{k=-\infty}^{+\infty} r_k^2\right)^{1/2}$$

for some constants K_1 and K_2.

Remark: Notice that ℓ_2-stability implies that $\lim_{k \to \infty} y_k = \lim_{k \to \infty} f(y_k) = 0$, and that for the n-dimensional difference equation described above it implies that if $v_k = 0$ for all k then $\lim_{x_0 \to 0} \sup_{k=0,1,2,\ldots} |y_k| = 0$, which in turn implies asymptotic stability in the sense of Lyapunov provided the system is uniformly completely observable.

Notation and Definitions: F is said to be monotone (or odd-monotone) if $f(\sigma)$ is a monotone (or an odd-monotone) function of σ. F is said to be strictly monotone (or strictly odd-monotone) if $f(\sigma) - \epsilon\sigma$ is a monotone (or an odd-monotone) function of σ for some $\epsilon > 0$.

Application of the principles exposed earlier and the positive operators discovered in Section 2.6, lead to the following stability theorem which is an extension of similar results obtained by O'Shea (42) and Zames and Falb (64).

Theorem 3.5: A sufficient condition for the feedback system under consideration to be ℓ_2-stable is that

(i) G belongs to $\mathcal{L}(\ell_2, \ell_2)$ and F is strictly monotone (strictly odd-monotone), and bounded

(ii) there exists an element, Z, of $\mathcal{L}(\ell_2, \ell_2)$, such that $Z - \epsilon I$ is doubly hyperdominant (doubly dominant) for some $\epsilon > 0$ and such that ZG is nonnegative.

Proof: This theorem is a straightforward application of Corollary 3.5 if it can be shown that Z can be factored as required there. This is, however, precisely what is stated in Corollary 2.1.

The case in which the system is time-invariant and the multiplier is of the Toeplitz type is, of course, of particular interest and yields the stability theorem obtained by O'Shea (42). The positivity condition and the doubly hyperdominance (doubly dominance) condition can then be stated in terms of z-transforms. This is done in the next corollary.

Lemma 3.7: Let $R = \{r_{k-\ell}\}$, $k, \ell \in I$ define an element of $\mathcal{L}(\ell_2, \ell_2)$ which is of the Toeplitz type. Then a necessary and sufficient condition for the inner product $\langle x, Rx \rangle$ to be nonnegative for all ℓ_2-summable sequences x is that the z-transform of $\{r_k\}$, R(z), satisfies $\mathrm{Re}\, R(z) \geq 0$ for almost all z with $|z| = 1$.

Proof: It is well known that

$$\langle x, Rx \rangle = \frac{1}{2\pi} \oint_{|z|=1} R(z) \, |X(z)|^2 \, z^{-1} dz$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\omega}) \, |X(e^{j\omega})|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re } R(e^{j\omega}) \, |X(e^{j\omega})|^2 \, d\omega$$

and the conclusion follows.

Corollary 3.9: A sufficient condition for the feedback system under consideration to be ℓ_2-stable is that

(i) G is an element of $\mathcal{L}(\ell_2, \ell_2)$ which is of the Toeplitz type and F is strictly monotone (strictly odd-monotone) and bounded

(ii) there exists a $Z(z)$ such that $Z(z)-\epsilon$ is the z-transform of a hyperdominant (dominant) sequence for some $\epsilon > 0$ and such that $\text{Re } G(z) Z(z) \geq 0$ for almost all z with $|z| = 1$.

Proof: The theorem follows from Theorem 3.5 and Lemma 3.7.

Remark : For the n-dimensional time-invariant difference equation introduced above it is quite simple to show that G will belong to $\mathcal{L}(\ell_2, \ell_2)$ if all eigenvalues of A have magnitude less than unity.

CHAPTER IV

LINEARIZATION AND STABILITY OF FEEDBACK SYSTEMS

4.1 Introduction

In the previous chapter, a number of stability criteria for nonlinear feedback systems have been derived. The question of whether or not these criteria are conservative cannot be given a general answer, but both from the analysis and from examples one suspects that these criteria are by no means necessary and sufficient (see e.g., (12)). Thus the question arises whether these criteria are indeed or if they are too conservative and if instability and stability can be derived using some approximate methods. There is one case for which necessary and sufficient conditions for the stability of feedback systems is known: namely the Nyquist criterion for feedback systems where the forward loop is linear and time-invariant and where the feedback gain is a constant. Thus by associating with a nonlinear feedback system a class of feedback systems of this type one tries to conclude stability or instability. This chapter takes a critical look at some of these linearization procedures and exposes, by means of an example, unexpected periodic solutions in a nonlinear feedback system. Although the system chosen to obtain this conclusion might seem quite particular, the method of analysis remains applicable to other systems and will expose essentially a similar behavior. The examples also suggest to what extent and for which systems the existing frequency-domain stability criteria can be improved. They also show the need for caution in applying linearization techniques in stability analysis.

4.2 About Linearization

Consider the feedback system shown in Fig. 4.1.

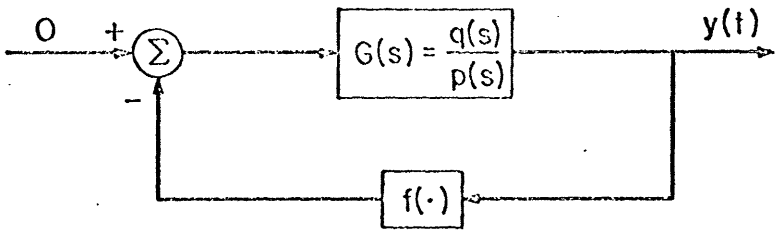

Fig. 4.1 The Feedback System

The relation between the input $u(t)$ and the output $y(t)$ of the element in the forward loop is determined by the ordinary time-invariant differential equation

$$\dot{x}(t) = Ax(t) + b\,u(t)$$

$$y(t) = c'x(t)$$

where A is a constant $(n \times n)$ matrix, and b and c are constant n-vectors. The transfer function of this element is thus given by $G(s) = c'(Is-A)^{-1}b$ and is the ratio of two polynomials in s with the degree of the numerator less than the degree of the denominator. The element $f(\cdot)$ in the feedback loop generates an output $f(\sigma)$ when its input is σ, where f is a mapping from the real line into itself. The differential equation describing the closed loop system is thus

$$\dot{x}(t) = Ax(t) - bf(c'x(t))$$

It is assumed that $f(0) = 0$. The solution $x(t) \equiv 0$ is called the <u>null-solution</u> of this system and is said to be <u>asymptotically stable in the large</u> if it is stable (in the sense of Lyapunov) and if all solutions converge to the null-solution for $t \to \infty$. For convenience the feedback system under consideration is said to be asymptotically stable in the large if this null-solution is.

For the case for which $f(\sigma) = K\sigma$, this stability problem can be completely resolved using root-locus techniques, the Nyquist stability criterion or a Routh-Hurwitz test and thus presents no difficulties. If $f(\sigma)$ is nonlinear however, this is not so, and often in engineering practice the question whether a particular feedback system of the above type is asymptotically stable in the large is answered by considering a linearized model. Three common types of linearization are the d-c type of linearization, the a-c type of linearization and the describing function type of linearization. These are formally defined below:

<u>Definitions:</u> Let f be a mapping from the real line into itself with $f(0) = 0$. The <u>d-c gain</u> or the <u>total gain</u> of the nonlinearity $f(\sigma)$ at $(\sigma \neq 0)$ is defined by $K_t(\sigma) = \frac{f(\sigma)}{\sigma}$. If f is differentiable then the <u>a-c gain</u> or the <u>incremental gain</u> of the nonlinearity $f(\sigma)$ at σ is defined by $K_i(\sigma) = \frac{\partial f(\sigma)}{\partial \sigma}$. If f satisfies the inequality $|f(\sigma)| \leq M|\sigma|$ for some M and all σ then the <u>describing function gain</u> of the nonlinearity $f(\sigma)$ at amplitude $A (A \neq 0)$ is the complex number $K_d(A)$ defined by

$$K_d(A) = \frac{1}{\pi A} \int_0^{2\pi} f(A\cos t) e^{jt} dt$$

The procedure by which linearization is used to conclude stability for the d-c and the a-c types of linearization goes as follows: If the linear system with $f(\sigma) = K\sigma$ is asymptotically stable for all K in the range of the d-c or the a-c gain (i.e., for all $K = K_t(\sigma)$ or $K = K_i(\sigma)$ and all σ) then the nonlinear system is asymptotically stable in the large. For the describing function method of linearization the procedure is analogous but cannot be stated as simply since the equivalent gain needs not be a real number. One way of stating the method in that case is as follows (24): If the Nyquist locus of $G(s)$ for $s = j\omega$ does not intersect the locus of the points $-(K_d(A))^{-1}$ for all A, but encircles it ρ times where ρ is the number of open-loop poles of $G(s)$ in $\text{Re}\, s > 0$ (with the usual assumption for imaginary axis poles of $G(s)$), then the nonlinear feedback system is asymptotically stable in the large.

Both the d-c type and the a-c type of linearization and the resulting conclusions about stability have been the subject of rather well-known conjectures, due to respectively Aizerman (1), and Kalman (30). Particularly the Aizerman conjecture has received a lot of attention. Originally published in 1949, it took till 1958 before Pliss (46) gave a satisfactory counterexample. It is possible to show that for second order systems the conjecture is true with the exception however of some cases where the d-c gain approaches for large values of its argument a gain for which the resulting linear system is not asymptotically stable. The counterexample given by Krasovskii (33) is in fact of this kind. The counterexamples obtained by Pliss however are more satisfactory. The very stringent conditions on the nonlinearity and the involved mathematics kept the work of Pliss from being well known. More recently, Dewey and Jury (19), and Fitts (22) gave numerical counterexamples derived from a computer simulation. The conjecture due to Kalman in which the a-c gain is used predicts stability in the large only for a subclass of the nonlinearities for which Aizerman's conjecture does. Fitts (22) gives counterexamples to this conjecture derived from a computer analysis.

In what is to follow, a simple, rigorous proof of the existence of periodic solutions in a fourth order system will be given. It will be shown that all of the above mentioned linearization techniques however predict asymptotic stability in the large. These oscillations thus constitute a class of counterexamples to both Aizerman's conjecture and Kalman's suggestion. The results are obtained using the perturbation theory of Cesari and Hale (28). Since the ideas behind this technique are basically simple the theorem from which the results follow will be proven. This method of proof is suggested by a paper by Urabe (57).

4.3 Averaging Theory

Consider the differential equation

$$\dot{x}(t) = Ax(t) + \epsilon f(x(t), z, \epsilon)$$

where $x(t)$ is an element of R^n, A a constant $(n \times n)$ matrix, z a parameter (an element of R_m), ϵ a scalar parameter and f a mapping from $R_n \times R_m \times R$ into R_n such that for all R, ϵ_o and M there exists a constant $K(R, \epsilon_o, M)$ (the Lipshitz constant) such that $\|f(x_1, z, \epsilon) - f(x_2, z, \epsilon)\| \leq K\|x_1 - x_2\|$ for all $\|x_1\|, \|x_2\| \leq R$, $|\epsilon| \leq \epsilon_o$ and $\|z\| \leq M$.

Since the function f does not satisfy a global Lipshitz condition, it is not clear at this point whether a solution $x(t)$ to the above equation exists for all t. This problem is resolved in the next lemma. But first a few definitions:

Definition: Let $x(t)$ be a continuous map from $[0, T]$ into a normed linear space. Then $\sup_{t \in [0, T]} \|x(t)\|$ exists and is called the norm induced by the uniform topology. Recall that the Contraction

Mapping Principle states that if F is a map from a complete metric space, X, into itself with $d(F(x), F(y)) \leq \alpha\, d(x, y)$ for all $x, y \in X$ and some $\alpha < 1$ then the equation $x = Fx$ has a unique solution (called a fixed point of the mapping F). Moreover, picking any x_o and defining $x_{k+1} = Fx_k$, $k \in I$, $k \geq 0$ yields a sequence $\{x_k\}$ which converges in the metric on X to the fixed point.

Lemma 4.1: Given any $\tau > 0$, ρ, and M, then the above differential equation has a unique solution $x(t)$ for any $x(0), y$ and t which satisfy $\|x(0)\| \leq \rho$, $0 \leq t \leq \tau$ and $\|z\| \leq M$ provided ϵ is sufficiently small (i.e., for all ϵ with $|\epsilon| \leq \epsilon_1$ and some $\epsilon_1 > 0$). Moreover, this solution can be obtained using the successive approximations

$$x_o(t) = e^{At}x(0)$$

$$x_{k+1}(t) = e^{At}x(0) + \epsilon\int_0^t e^{A(t-\sigma)}f(x_k(\sigma), z, \epsilon)\, d\sigma$$

$$\text{for } k \in I,\ k \geq 0$$

Proof: Let S be the normed linear space of all continuous mappings from $[0, \tau]$ into R^n with the norm induced by the uniform topology and with $\|x(t)\| \leq 2\rho N$ where $N = \sup_{0 \leq t \leq \tau} \|e^{At}\|$. S is a complete metric space (see e.g., (31)) and the mapping $\underset{\sim}{F}$ defined on S by

$$\underset{\sim}{F}x(t) = e^{At}x(0) + \epsilon\int_0^t e^{A(t-\sigma)}f(x(\sigma), z, \epsilon)\, d\sigma$$

maps S into itself for all $|\epsilon| \leq \epsilon_1$ with $\epsilon_1 \leq \min\{\epsilon_o, (KN_\tau)^{-1}, \rho\tau^{-1}(4\rho NK + \|f(0,0,0)\|)^{-1}\}$ where K is the Lipshitz constant associated with $R = 2\rho N, \epsilon_o > 0$, and M. Moreover $\underset{\sim}{F}$ is a contraction on S. The verification of these facts is simple and will not be given explicitly. Thus the equation $x(t) = \underset{\sim}{F}x(t)$ has a unique fixed point, which can be obtained using the above successive approximations. This yields the lemma.

The next lemma exposes the dependence of $x(t)$ on ϵ more explicitly:

Lemma 4.2: Given any $\tau > 0$, ρ and M, then the solution $x(t)$ to the above differential equation for any $x(0)$, z, t which satisfy $\|x(0)\| \leq \rho$, $0 \leq t \leq \tau$ and $\|z\| \leq M$ can be expressed as

$$x(t) = e^{At}x(0) + \epsilon\int_0^t e^{A(t-\sigma)}f(e^{A\sigma}x(0), z, \epsilon)\,d\sigma + \epsilon^2 L(t, x(0), z, \epsilon)$$

for all ϵ sufficiently small (i.e., for all ϵ with $|\epsilon| \leq \epsilon_2$ and some $\epsilon_2 > 0$). Moreover, $L(t, x(0), z, \epsilon)$ is bounded for $0 \leq t \leq \tau$, $\|x(0)\| \leq \rho$, $\|z\| \leq M$ and $|\epsilon| \leq \epsilon_2$.

Proof: It will be shown that the (k+1)th element in the successive approximations introduced above is of this form provided the kth one is, and that the bound on L_k can be taken to be independent of k. Since $x_1(t)$ is clearly of that form the result follows then by induction since the limit for $k \to \infty$ which exists, must then also be of this form. Let K be the Lipshitz constant associated with $2\rho N$, ϵ_1, and M, and let $\epsilon_2 < \min\{\epsilon_1, (2N\tau)^{-1}\}$. A simple calculation then shows that $\|L_{k+1}\| \leq \tau^2 N^2(\|f(0,0,0)\| + KN\rho)$, if $\|L_k\| \leq \tau^2 N^2(\|f(0,0,0)\| + KN\rho)$, which then, in view of the above remarks, yields the lemma.

Lemma 4.2 yields the following theorem on the existence of periodic solutions to the differential equation under consideration:

Theorem 4.1: If for ϵ sufficiently small (i.e., for all ϵ with $|\epsilon| \leq \epsilon_o$ and some $\epsilon_o > 0$) there exist bounded functions $x(0)(\epsilon)$, $T(\epsilon)$ and $z(\epsilon)$ such that

$$x(0)(\epsilon) = e^{AT(\epsilon)}x(0)(\epsilon) + \epsilon\int_0^{T(\epsilon)} e^{A(T(\epsilon)-\sigma)} f(e^{AT(\epsilon)}x(0)(\epsilon), z(\epsilon), \epsilon)d\sigma$$

$$+ \epsilon^2 L(T(\epsilon), x(0)(\epsilon), z(\epsilon), \epsilon)$$

then the differential equation under consideration has a periodic solution for ϵ sufficiently small (i.e., for ϵ with $|\epsilon| \leq \epsilon_1$ and some $\epsilon_1 > 0$).

Proof: Lemma 4.1 shows that for ϵ sufficiently small $x(T(\epsilon)) = x(0)(\epsilon)$ which, since the differential equation under consideration is time-invariant, yields a periodic solution of period $T(\epsilon)$.

The above theorem is not very useful as it stands since it requires computing the function L and solving for the functions $x(0)(\epsilon)$, $T(\epsilon)$ and $z(\epsilon)$. However by using the implicit function theorem it is possible to obtain sufficient conditions for the conditions of Theorem 4.1 to be satisfied. These conditions are very simple to verify and are stated here so as to suit the particular case which will lead to the counterexamples to Aizerman's conjecture.

In the theorem which follows, use will be made of the Implicit Function Theorem (35) which states that if f maps $R_n \times R_m$ into R_n and if

(i) $f(x_o, y_o) = 0$ for some $x_o \in R_n$, $y_o \in R_m$.

(ii) $\frac{\partial f}{\partial y}(x, y)$ exists and is continuous in a neighborhood of the point x_o, y_o

(iii) $\frac{\partial f}{\partial y}(x_o, y_o)$ is of rank n

then there exists a map, ϕ, from R_n into R_m, which is continuous in a neighborhood of x_o and such that $y=\phi(x)$ yields $F(\phi(x), x)=0$ for all x in some neighborhood of x_o. Moreover $y_o=\phi(x_o)$ and ϕ is unique in a neighborhood of x_o.

Theorem 4.2: Assume that $e^{AT_o}=I$ (i.e., that all solutions of $\dot{x}(t)=Ax(t)$ are periodic with period T_o), and that $f(x, z, \epsilon)$ is a continuous function of $x, z,$ and ϵ which has continuous first partial derivatives with respect to x and z, for ϵ sufficiently small (i.e., for all ϵ with $|\epsilon| \leq \epsilon_o$ and some $\epsilon_o > 0$.)

Let
$$F(x, z, \epsilon) \triangleq \int_0^{T_o} e^{-A\sigma} f(e^{A\sigma}x, z, \epsilon)\, d\sigma$$

and assume that (i) $F(x_o, z_o, 0)= 0$

(ii) the matrix $\frac{\partial F}{\partial x, \partial z}(x_o, z_o, 0)$ is of full rank.

Then there exists a continuous function $z(\epsilon)$ such that for ϵ sufficiently small (i.e., for all ϵ with $|\epsilon| \leq \epsilon_1$ and some $\epsilon_1 > 0$) the differential equation under consideration has a periodic solution $x^*(t, \epsilon)$ with $\lim_{\epsilon \to 0} z(\epsilon)=z_o$ and $\lim_{\epsilon \to 0} x^*(t, \epsilon) = e^{At}x_o$.

Proof: The smoothness conditions on f together with the resulting smoothness of the solutions of ordinary differential equations (see e.g., Coddington and Levinson (16)) ensure that the Implicit Function Theorem is applicable. This then in turn shows that the conditions (i) and (ii) of the theorem ensure that Theorem 4.1 is applicable which leads to the conclusion of the theorem.

This method of concluding the existence of periodic solutions for differential equations is known as Averaging Theory since the function F as defined above is indeed an average value.

4.4 Application of Averaging Theory

Consider the differential equation

$$x^{(4)}(t)+10x^{(2)}(t)+9x(t)+\epsilon(\alpha x^{(3)}(t)+\beta x^{(2)}(t)+\gamma x^{(1)}(t)+\delta x(t))+\epsilon f(x^{(2)}(t)) = 0$$

where f maps R into R and is continuously differentiable with respect to its argument. This equation describes the feedback system shown on Fig. 4.2 and is equivalent to the following system of first order differential equations:

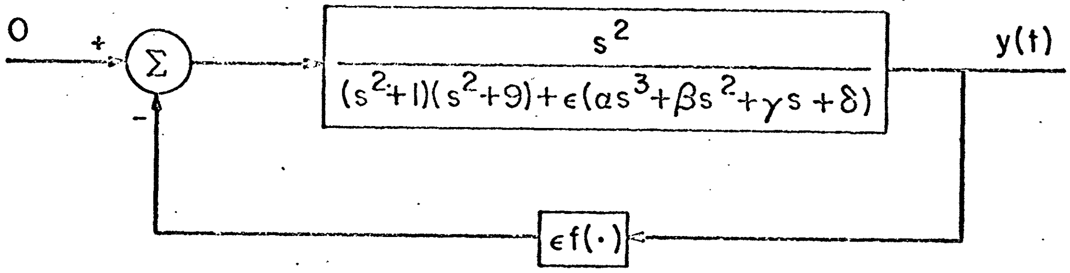

Fig. 4.2 The Fourth Order System to which Averaging Theory is Applied

$$\begin{bmatrix} \dot z_1(t) \\ \dot z_2(t) \\ \dot z_3(t) \\ \dot z_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} + \frac{\epsilon}{8} \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} \delta-\beta & \gamma-\alpha & \delta-9\beta & 3(\gamma-9\alpha) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \end{bmatrix} f(z_1(t)+z_3(t))+0(\epsilon^2)$$

where $0(\epsilon^2)$ denotes a 4-dimensional vector which is such that $\lim_{\epsilon\to 0} \frac{0(\epsilon^2)}{\epsilon}=0$. The application of Theorem 4.2 shows that there exist continuous functions $\alpha(\epsilon)$, $\beta(\epsilon)$, $\gamma(\epsilon)$, and $\delta(\epsilon)$ such that the differential equation under consideration has a periodic solution, $z^*(t,\epsilon)$, with $\lim_{\epsilon\to 0} \alpha(\epsilon),\beta(\epsilon),\gamma(\epsilon),\delta(\epsilon) = \alpha_o,\beta_o,\gamma_o,\delta_o$

and $\lim_{\epsilon\to 0} z^*(t,\epsilon) = e^{At} \begin{bmatrix} z_{1,0} \\ z_{2,0} \\ z_{3,0} \\ z_{4,0} \end{bmatrix}$ where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

if (i) $(\gamma_o-\alpha_o)z_{1,0}+(\beta_o-\delta_0)z_{2,0}+\frac{1}{\pi}\int_0^{2\pi} f(z_{1,0}\cos\sigma-z_{2,0}\sin\sigma+z_{3,0}\cos3\sigma-z_{4,0}\sin3\sigma)\sin\sigma d\sigma = 0$

$$(\beta_o-\delta_o)z_{1,0}-(\gamma_o-\alpha_o)z_{2,0}+\frac{1}{\pi}\int_0^{2\pi} f(z_{1,0}\cos\sigma-z_{2,0}\sin\sigma+z_{3,0}\cos3\sigma-z_{4,0}\sin3\sigma)\cos\sigma d\sigma = 0 .$$

$$(\gamma_o/3-3\alpha_o)z_{3,0}+(\beta_o-\delta_o/9)z_{4,0}+\frac{1}{\pi}\int_0^{2\pi} f(z_{1,0}\cos\sigma-z_{2,0}\sin\sigma+z_{3,0}\cos3\sigma-z_{4,0}\sin3\sigma)\sin3\sigma d\sigma = 0$$

$$(\beta_o-\delta_o/9)z_{3,0}-(\gamma_o/3-3\alpha_o)z_{4,0}+\frac{1}{\pi}\int_0^{2\pi} f(z_{1,0}\cos\sigma-z_{2,0}\sin\sigma+z_{3,0}\cos3\sigma-z_{4,0}\sin3\sigma)\cos3\sigma d\sigma = 0$$

(ii) $(z_{1,0}^2+z_{2,0}^2)(z_{3,0}^2+z_{4,0}^2) \neq 0$

The second equation guarantees that the matrix in Theorem 4.2 is of full rank, and the first equation exposes the requirement that the average be zero.

From these conditions the following theorem which will be central in establishing the counterexamples to Aizerman's conjecture follows:

Theorem 4.3: If $f(\sigma)$ is not identically equal to $k\sigma$ for any constant k, then there exists a nonzero periodic solution to the differential equation under consideration for ϵ sufficiently small (i.e., for all ϵ with $|\epsilon| \leq \epsilon_o$ and $\epsilon_o > 0$). Moreover, the functions $\alpha(\epsilon)$ and $\gamma(\epsilon)$ which yield this periodic solution satisfy the inequality

$$(\gamma(\epsilon) - \alpha(\epsilon))(\gamma(\epsilon) - 9\alpha(\epsilon)) < 0$$

Proof: If $z_{1,0}, z_{2,0}, z_{3,0}$ and $z_{4,0}$ are such that $(z_{1,0}^2 + z_{2,0}^2)$ and $(z_{3,0}^2 + z_{4,0}^2)$ are positive then the equations in (i) above can be solved for $\alpha_o, \beta_o, \gamma_o$ and δ_o. They will yield the following equality for any choice of $z_{1,0}, z_{2,0}, z_{3,0}$ and $z_{4,0}$

$$(\gamma_o - \alpha_o)(z_{1,0}^2 + z_{2,0}^2) + (\gamma_o - 9\alpha_o)(z_{3,0}^2 + z_{4,0}^2) = 0$$

It can also be shown quite easily that if $f(\sigma)$ is not linear, i.e., if $f(\sigma)$ is not identically equal to $k\sigma$ for any constant k, then $z_{1,0}, z_{2,0}, z_{3,0}$ and $z_{4,0}$ can be picked in such a way that $\gamma_0 - \alpha_0 \neq 0$ and $\gamma_0 - 9\alpha_0 \neq 0$. This then yields the conclusion of the theorem.

4.5 Counterexamples to Aizerman's Conjecture

Let $f(\sigma)$ in the above equation be $\tan h\,\sigma$. The linearized gains then satisfy the inequalities $0 \leq K_t(\sigma) \leq 1$, $0 \leq K_i(\sigma) \leq 1$ and $K_d(A)$ is real with $0 \leq K_d(A) \leq 1$. The zeros of the polynomial

$$s^4 + 10s^2 + 9 + \epsilon(\alpha s^3 + \beta s^2 + \gamma s + \delta) + \epsilon K s^2$$

lie, for ϵ sufficiently small and for $0 \leq K \leq 1$

(i) in Re $s < 0$ or if $\epsilon > 0$, $\alpha > 0$, $\gamma > 0$ and $(\gamma-\alpha)(\gamma-9\alpha) < 0$

or if $\epsilon < 0$, $\alpha < 0$, $\gamma < 0$ and $(\gamma-\alpha)(\gamma-9\alpha) < 0$

(ii) in Re $s > 0$ or if $\epsilon < 0$, $\alpha > 0$, $\gamma > 0$ and $(\gamma-\alpha)(\gamma-9\alpha) < 0$

or if $\epsilon > 0$, $\alpha < 0$, $\gamma < 0$ and $(\gamma-\alpha)(\gamma-9\alpha) < 0$

Thus all the linearization techniques would predict that the feedback system under consideration is asymptotically stable in the large provided that $\epsilon > 0$, $\alpha > 0$, $\gamma > 0$ and $(\gamma-\alpha)(\gamma-9\alpha) < 0$ or that $\epsilon < 0$, $\alpha < 0$, $\gamma < 0$ and $(\gamma-\alpha)(\gamma-9\alpha) < 0$. These regions are graphically shown in Fig. 4.3.

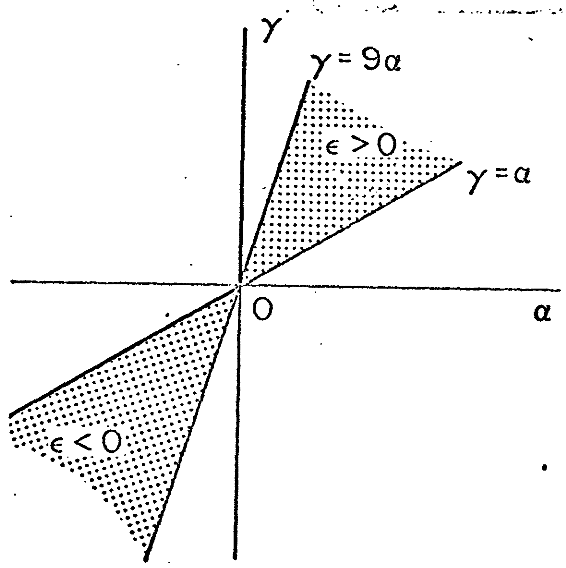

Fig. 4.3 Conditions on ϵ, α, γ to Obtain Counterexamples to Aizerman's Conjecture

It is thus clear that for ϵ sufficiently small and for values of α and γ such that $(\gamma-\alpha)(\gamma-9\alpha) < 0$ the sign of ϵ can be chosen in such a way that the linearization techniques would predict the feedback system under consideration to be asymptotically stable in the large. This however is in direct contradiction with Theorem 4.3 which shows that the feedback system sustains a periodic solution.

Remarks: 1. The choice of the function $f(\sigma) = \tan h\sigma$ is rather irrelevant. In fact, the same conclusion holds for any non-linearity, provided it is sufficiently smooth for Theorem 4.3 to be applicable and provided $|f(\sigma)| \leq K|\sigma|$ for some K and all σ which then yields, for ϵ sufficiently small, the pole locations of the linearized system as given above.

2. The remarkable feature of the periodic solutions discovered in Theorem 4.3 is that (for ϵ sufficiently small), they only occur when the linearized system has all its poles either always in the left half plane or always in the right half plane, contrary to what is to be expected from linearization.

3. The Nyquist locus and the root-locus of the fourth order system under consideration are shown in Fig. 4.4 for the case $\epsilon > 0$, $a > 0$, $\gamma > 0$ and $(\gamma - a)(\gamma - 9a) > 0$ or $\epsilon < 0$, $a < 0$, $\gamma < 0$ and $(\gamma - a)(\gamma - 9a) < 0$

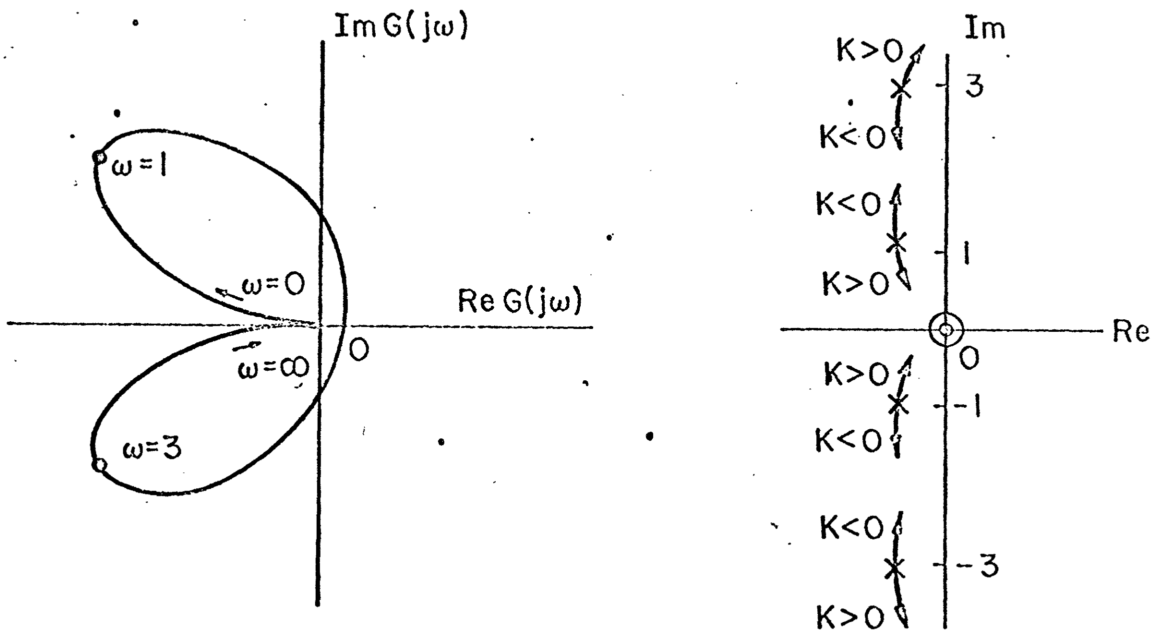

Fig. 4.4 Nyquist-locus of G(jω) and Root-locus of the Linearized Feedback System

4. The local stability properties of these periodic solutions is of course of interest. Variational techniques show that for proper choices of α, β, γ, σ, ϵ and $f(\cdot)$, these periodic solutions can be locally stable.

4.6 A Physical Interpretation of these Oscillations

The existence of the periodic solutions discovered in this chapter will now be given a physical explanation. This will of course be a plausibility argument. Averaging theory essentially allows to conclude that argumentation is correct provided ϵ is sufficiently small.

Assume an input to the nonlinearity $\epsilon f(\cdot)$ which has a first harmonic, a third harmonic and "small" other harmonics. The output to the nonlinearity will thus contain all harmonics, all of comparable magnitudes, and all "small" since they have been multiplied by a small parameter ϵ. Let x_1, x_3, y_1 and y_3 be the Fourier coefficients of the first and the third harmonics of the input and the output to the nonlinearity. It can be shown that for particular choices of x_1 and y_1 the nonlinearity will shift the phases of the first and third harmonics toward one another thus obtaining the situation depicted in Fig. 4.5.

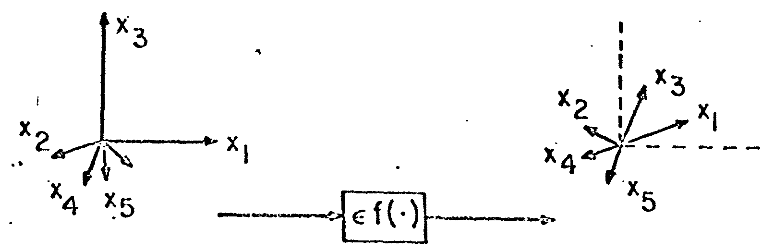

Fig. 4.5 The Spectrum of the Input and the Output of the Element in the Feedback Loop

The negative feedback leads to an input, u, to the forward loop as is shown in Fig. 4.6 which with a Nyquist locus as in Fig. 4.4

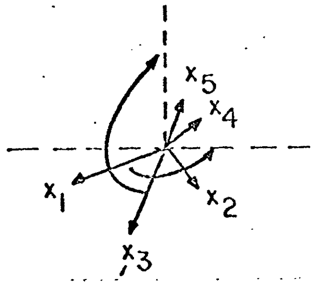

Fig. 4.6 The Spectrum of the Input and the Output of the Element in the Forward Loop

multiplies the 1st and 3rd harmonic by a factor of order ϵ^{-1}, shifts their phases in the right direction, but less than 180° thus obtaining the original situation of x_1 and x_3. The higher harmonics remain of order ϵ. The loop can thus be closed and the feedback system sustains the oscillation.

CHAPTER V

ON THE DESIGN OF NONLINEARITIES ON THE BASIS OF HARMONIC CONTENT

The implementation of Control Systems involves at all stages a great deal of electronic equipment and with it the design of filters, of frequency up- and down-converters, of a-c to d-c converters, etc. In this chapter some ideas and results pertaining to design procedures for systems containing nonlinear elements are outlined and the usefulness of these techniques is to be viewed at the level of designing individual parts to a system, similarly to the Nyquist-Bode design procedures which have proven their use at this level of the design equally well as for the design of the overall system.

The design of nonlinearities is a quite neglected area of research compared to their analysis for which a large amount of material is available. In particular, the previous chapters have essentially all been concerned with analysis problems. The relations obtained there are essentially relations between the spectrum of the input and the spectrum of the output of a certain given nonlinearity. These relations always hold independently of the particular input for which they are applied, i.e., the particular form of the input is not taken into consideration. The types of problems which will be considered in this chapter are of a different nature and the emphasis is on selecting a certain nonlinearity in a given class such that the spectrum of the output meets certain requirements under the assumption that the input is given.

Example: As an example of a problem for which the techniques developed in this chapter are potentially useful, and which requires designing a nonlinearity, consider the feedback system of the form shown in Fig. 5.1.

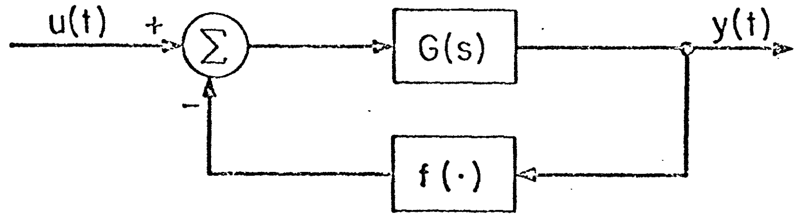

Fig. 5.1 Feedback Configuration

The problem is to select a nonlinearity $f(\cdot)$ in a certain class such that the closed-loop system is optimum in some appropriate sense. Problems as this one cannot be treated directly using the minimum principle of Pontryagin or some other commonly used optimization technique since these techniques require that the controller, $f(\cdot)$, has access to all the state variables, a condition which is not satisfied in the above problem where the controller has only access to the output $y(t)$. From a practical point of view however the above scheme is both simpler to implement and occurs often as an inherent limitation of the allowed controllers. Thus optimization techniques based on the above model can take design requirements into consideration at a much earlier stage of the design.

5.1 Unconstrained Maximization of a Linear Functional

As a first problem related to the optimal design of nonlinearities, the maximization of a linear functional (which could be e.g., a Fourier coefficient) of the output of a nonlinearity will be considered. An explicit algorithm which yields the nonlinearity is obtained.

Problem Statement: Let $x(t)$ be a real-valued function of ϵ which belongs to $L_2(0, T)$, and let $\underset{\sim}{F}_1$ denote the class of measurable functions from R into itself with $0 \leq \sigma f(\sigma) \leq \sigma^2$ for all elements $f \epsilon \underset{\sim}{F}_1$. Let $c(t) \epsilon L_2(0, T)$ and denote $f(x(t))$ by $y(t)$. The "cost-functional" $J(f)$ is defined by $J(f) = \langle c(t), y(t) \rangle = \int_0^T c(t) y(t) dt$, and the problem is to find an element $f_o \epsilon \underset{\sim}{F}_1$, if it exists, such that $J(f_o) \geq J(f)$ for all $f \epsilon \underset{\sim}{F}_1$.

Additional Assumptions: In order to find an explicit algorithm for f_o, a number of simplifying assumptions are made. The above optimization problem can be solved under less stringent conditions, but the solution is somewhat more involved. Since the assumptions are however reasonable and satisfied in most practical situations, the general case will not be pursued. It is thus assumed that $x(t)$ is differentiable on $[0, T]$ and that $\frac{dx(t)}{dt}$ vanishes for at most a finite number of points in $[0, T]$.

Solution of the Optimization Problem: The following algorithm yields the optimum nonlinearity $f_o \epsilon \underset{\sim}{F}_1$:

Let $\{t_i(a)\}$, $i = 1, 2, \ldots, n_a$ be the solution of the equation $x(t) = a$. (It follows from the assumptions on $x(t)$ that n_a will be at most finite for all $a \epsilon R$.) Let $\xi(a)$ be formally defined by:

$$\xi(a) = \sum_{i=1}^{n_a} c(t_i(a)) x(t_i(a)) \left| \frac{dx(t_i(a))}{dt} \right|^{-1}$$

$\xi(a)$ is well-defined for all but a finite number of a's in the range of $x(t)$: namely those corresponding to the values of $x(t)$ at the points where $\frac{dx(t)}{dt} = 0$. Once the function $\xi(a)$ is computed, the nonlinearity f_o follows with

$f_o(a) = a$ whenever $\xi(a) > 0$

$f_o(a) = 0$ whenever $\xi(a) < 0$

and $f_o(a)$ any number between 0 and a whenever $\xi(a) = 0$ and whenever $\xi(a)$ is not defined (i.e., if a is outside the range of $x(t)$ or if the above summation for $\xi(a)$ is not defined).

Remark: It is, in fact, sufficient to find the zeros of the function $\xi(a)$ since $\xi(a)$ is a continuous function of a where it is defined.

Proof of the Algorithm: Let $y(t) = u(t)x(t)$. $u(t)$ exists for all $f \in \underset{\sim}{E}_1$, and the constraints on f require that $0 \leq u(t) \leq 1$ for all $t \in [0, T]$ and that $u(t_1) = u(t_2)$ whenever $x(t_1) = x(t_2)$. Let $t_1 \leq t_2 \leq \ldots \leq t_n$ be the points where $\frac{dx(t)}{dt} = 0$. Hence

$$<c(t), y(t)> = \int_0^{t_1} c(t)y(t)\,dt + \int_{t_1}^{t_2} c(t)y(t)\,dt + \ldots + \int_{t_n}^{T} c(t)y(t)\,dt$$

$$= \int_0^{t_1} c(t)x(t)u(t)\,dt + \int_{t_1}^{t_2} c(t)x(t)u(t)\,dt + \ldots + \int_{t_n}^{T} c(t)x(t)u(t)\,dt$$

Let $\tau = \int_0^t \left| \frac{dx(\sigma)}{d\sigma} \right| d\sigma$ (τ exists since the assumptions on $x(t)$ imply that $x(t)$ is of bounded variation on $[0, T]$). Let $\tau_i = \tau(t_i)$, then $<c(t), y(t)> = \int_0^{\tau_1} c(t)\, x(t) \left|\frac{dx(t)}{dt}\right|^{-1} u(t)\,d\tau + \ldots + \int_{\tau_n}^{\tau(T)} c(t)x(t) \left|\frac{dx(t)}{dt}\right|^{-1} u(t)\,d\tau$

A simple computation shows that $\frac{dx(\tau)}{d\tau} = 1$ whenever $\frac{dx(t)}{dt} > 0$ and that $\frac{dx(\tau)}{d\tau} = -1$ whenever $\frac{dx(t)}{dt} < 0$. Thus $x(\tau)$ versus τ has a sawtooth shape and the constraint that $u(\tau_1) = u(\tau_2)$ whenever $x(\tau_1) = x(\tau_2)$ can readily be taken into consideration at this moment. Let for instance $[\alpha_o, \alpha_1] = [x(0), x(t_1)] \cap [x(t_1), x(t_2)] \cap \ldots \cap [x(t_n), x(T)]$. The above integration restricted to the intervals which map $x(t)$ into $[\alpha_o, \alpha_1]$ leads to

$$\int_{\beta_o}^{\beta_o+\sigma} c(t)\, x(t) \left|\frac{dx(t)}{dt}\right|^{-1} u(t)\, dt + \ldots + \int_{\beta_n}^{\beta_n+\sigma} c(t)\, x(t) \left|\frac{dx(t)}{dt}\right|^{-1} u(t)\, d\tau$$

A change in variables and the condition that $u(t_1) = u(t_2)$ whenever $x(t_1) = x(t_2)$ leads, after some manipulations, to the following expression for the above integration

$$\int_0^{\sigma} \{\sum_{i=0}^{n} c(t_\alpha) x(t_\alpha) \left|\frac{dx(t_\alpha)}{dt}\right|^{-1} u(t_\alpha)\}\, d\tau$$

where t_α is defined above. The choice of $u(t_\alpha)$ as in the statement of the algorithm becomes now apparent. A similar manipulation for the integration over other intervals in the range of $x(t)$ establishes the complete algorithm.

Example: As an example to illustrate the above theory, let

$$x(t) = \sin t \qquad 0 \leq t \leq 2\pi$$

and

$$J(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x(t)) \sin 3t \; dt$$

The above algorithm becomes very straightforward for this example and leads to the following optimum f

$$f_o(\sigma) = \sigma \quad \text{for} \quad |\sigma| \leq 1/2 \text{ and } \sqrt{3}/2 \leq |\sigma| \leq 1$$

$$f_o(\sigma) = 0 \quad \text{for} \quad 1/2 \leq |\sigma| \leq \sqrt{3}/2$$

and $f_o(\sigma)$ any number between 0 and σ otherwise. The resulting nonlinearity and the waveforms of the input and the output to the nonlinearity are shown in Fig. 5.2.

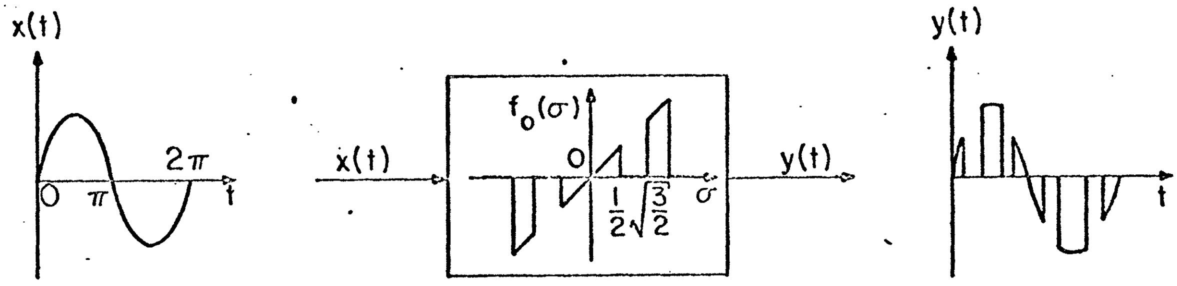

Fig. 5.2 Maximization of the Third Harmonic

. Remark: Consider the following optimization problem: let $x(t)$, $J(f)$ and $\underset{\sim}{F}_1$ be as defined above and let $c_1(t), c_2(t), \ldots, c_n(t) \in L_2(0,T)$. The optimization problem is to find an element $f_o \in \underset{\sim}{F}_1$, if it exists, such that $J(f_o) \geq J(f)$ for all $f \in \underset{\sim}{F}_1$ and such that $\langle c_1(t), f_o(x(t))\rangle = c_1, \ldots, \langle c_n(t), f_o(x(t))\rangle = c_n$, where $c_1, \ldots, c_n$ are given real numbers. An algorithm, similar to, but more involved than the one above can be obtained using similar calculations and involving the Neymann-Pearson lemma (see e.g., (5)). It is more or less apparent how this lemma occurs in connection with this problem: indeed the problem solved by the Neymann-Pearson lemma is precisely the maximization of a linear functional under magnitude constraints and under the additional requirement that other linear functionals yield certain preassigned values.

5.2 Maximization under General Constraints

In this section the maximization of a certain functional of the output of a nonlinearity under given constraints on the nonlinearity and its output is considered. The input to the nonlinearity is again assumed to be known. The interesting feature of the methodology outlined below is that it shows the possibility of transforming a large class of these optimization problems into a form in which Pontryagin's maximum principle and other classical optimization techniques are applicable.

Problem: The problem of generating a nonlinearity which yields a given set of Fourier coefficients and minimizes the distortion is considered. The input is assumed to be given. It will be shown that the question of existence can be reduced to a question about the range of a linear operator. The precise statement of the problem follows:

Let $x(t)$ be given by the uniformly convergent Fourier series

$$x(t) = \sum_{n=0}^{\infty} (a_n \cos 2\pi n \frac{t}{T} + b_n \sin 2\pi n \frac{t}{T})$$

The optimization problem is to find the nonlinearity f (unconstrained) such that

$$y(t) = \sum_{n=0}^{N} (c_n \cos 2\pi n \frac{t}{T} + d_n \sin 2\pi n \frac{t}{T}) + r(t)$$

where $r(t)$ is orthogonal to $\sin 2\pi n \frac{t}{T}$ and $\cos 2\pi n \frac{t}{T}$ for $n=0, 1, \ldots, N$. The first question to be answered is whether there exists such a nonlinearity. If so, the next question is to find the nonlinearity which minimizes the "distortion"

$$\int_0^T r^2(t)\, dt$$

Remark: The above optimization problem is nontrivial mainly due to the constraint that $f(\cdot)$ is required to be a single valued function and the commonly used optimization techniques are not immediately applicable. This example shows one very interesting potential application for the methods outlined here: namely the design of optimal static filters for certain given inputs.

Solution of the Optimization Problem: As mentioned above, this problem does not fall in the class of the usual optimization problems due to the constraint that $y(t) = f(x(t))$ for some nonlinearity $f(\cdot)$. This constraint can however in general be reduced to a set of conditions of the form

$$\begin{aligned}
y(\sigma_{11}(t)) &= y(\sigma_{12}(t)) \quad \text{for } t_{11} \le t \le t_{12} \\
y(\sigma_{21}(t)) &= y(\sigma_{22}(t)) \quad \text{for } t_{21} \le t \le t_{22} \\
&\vdots \\
y(\sigma_{n1}(t)) &= y(\sigma_{n2}(t)) \quad \text{for } t_{n1} \le t \le t_{n2}
\end{aligned}$$

For instance, it is quite easily verified that if $x(t) = \cos 2\pi\frac{t}{T}$, then $y(t) = f(x(t))$ if and only if

$$y(t+T) = y(t)$$

and

$$y(T-t) = y(t)$$

In general if $x(t)$ is assumed to satisfy the conditions given in Section 5.1, then this reduction can always be done, using the following procedure:

Let $t_1, t_2, \ldots, t_n$ be the values of t for which $\frac{dx(t)}{dt}$ vanishes. Assume that $x(t_i)$ is a local maximum of $x(t)$ then the following constraint is obtained for the case that $x(t_{i+1}) \geq x(t_{i-1})$

$$y(t_i - \int_{t_i - t}^{t_i} \left| \frac{dx(t)}{dt} \right| dt) = y(t_i + \int_{t_i}^{t_i + t} \left| \frac{dx(t)}{dt} \right| dt)$$

for all $0 \leq t \leq t_{i+1} - t_i$. This procedure can be pursued until the whole interval $[0, T]$ is thus covered.

The constraint that the N first Fourier coefficients of the output are required to have certain values can be stated as the requirement that the dynamical system

$$x_{2i+1} = y(t) \cos 2\pi i \frac{t}{T}$$

$$x_{2i+2} = y(t) \sin 2\pi i \frac{t}{T} \qquad i = 0, 1, \ldots, N$$

should be driven from the state $(0, 0, \ldots, 0)$ to $(c_o, d_o, c_1, d_1, \ldots, c_N, d_N)$ by a "control" $y(t)$ which satisfies the above constraints. The optimization problem is to find the control, if it exists, which minimizes

$$\int_0^T y^2(t)\, dt$$

This optimization can be further simplified by a change of the time scale as outlined in Section 5.1 which results in a sawtooth form for $x(t)$. In many circumstances this is actually an unnecessary procedure which should be avoided whenever it is possible (an example will be given later). After this change of the time scale has been performed it is easily verified that the problem reduces to a simple particular case of the following optimization problem:

Given a dynamical system

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $A(t)$ and $B(t)$ are $(n \times n)$ and $(n \times m)$ matrices respectively. Find a control $u(t)$ which satisfies $u(t) + T) = u(t)$ and which drives the state x_o at $t=0$ to x_1 at $t=kT$ where k is a positive integer, while minimizing

$$J(u) = \int_0^{kT} L(x(t), u(t))\,dt$$

This problem is not quite in the form for which Pontryagin's maximum principle can be used due to the periodicity constraint on $u(t)$. Therefore the original system is replaced by k copies as follows:

$$\begin{aligned}
\dot{x}_1(t) &= A(t)\,x_1(t) + B(t)\,u(t) \\
\dot{x}_2(t) &= A(t+T)x_2(t) + B(t+T)\,u(t) \\
&\vdots \\
\dot{x}_k(t) &= A(t+(k-1)T)\,x_k(t) + B(t+(k-1)T)u(t)
\end{aligned}$$

and

$$J(u) = \int_0^{T} L(x_1(t), \ldots, x_k(t),\, u(t))\,dt$$

The original transfer is thus possible if and only if there exist elements $x_i(T)$ and a $u(t)$ such that

$$\begin{aligned}
x_1(0) &= x_o \\
x_2(0) &= x_1(T) \\
&\vdots \\
x_k(0) &= x_{k-1}(T) \\
x_k(T) &= x_1
\end{aligned}$$

By a well-known result for controllability of linear systems this transfer is possible if and only if the vector

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_k(0) \end{bmatrix} - \phi(0,T) \begin{bmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_k(T) \end{bmatrix}$$

lies in the range space of the matrix

$$W(0,T) = \int_0^T \phi(0,\sigma) B_1(\sigma) B_1'(\sigma) \phi_1'(0,\sigma)\, d\sigma$$

where ϕ is the transition matrix of the augmented system and $B_1(t)$ equals:

$$B_1(t) = \begin{bmatrix} B(t) \\ B(t+T) \\ \vdots \\ B(t+kT) \end{bmatrix}$$

Using the relations between $x_i(0)$ and $x_i(T)$ this condition requires that for some $x_2(0), x_3(0), \ldots, x_k(0)$ the vector

$$\begin{bmatrix} x_o \\ x_2(0) \\ \vdots \\ x_k(0) \end{bmatrix} - \phi(0,T) \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & \ldots & & 0 & I \\ I & 0 & \ldots & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2(0) \\ \vdots \\ x_k(0) \end{bmatrix}$$

should belong to the range space of $W(0,T)$. If $W(0,T)$ is invertible, then the transfer is always possible. This will however in general not be the case due to the fact that the augmented matrix has a lot of structure to it. In the other case it is necessary to compare the range space of W and of the matrix operating on the vector $\mathrm{col}(0, x_2(0), \ldots, x_k(0))$ in the above expression.

Remark: The above procedure only claims to be an outline by which a particular optimization problem of the type considered in this section could be solved. It is also apparent that many of the assumptions do not have much intrinsic importance and were mainly introduced to fix the ideas. In particular the fact that the Fourier coefficients are required to have certain values could be replaced by any linear functionals. Furthermore although the procedure might seem complicated this is not quite so in most practical situations since the matrix A turns out to be the zero matrix and B is quite simple.

Example: Using the procedure outlined above, the following problem was solved: let $x(t)$ be given by the waveform shown in Fig. 5.3

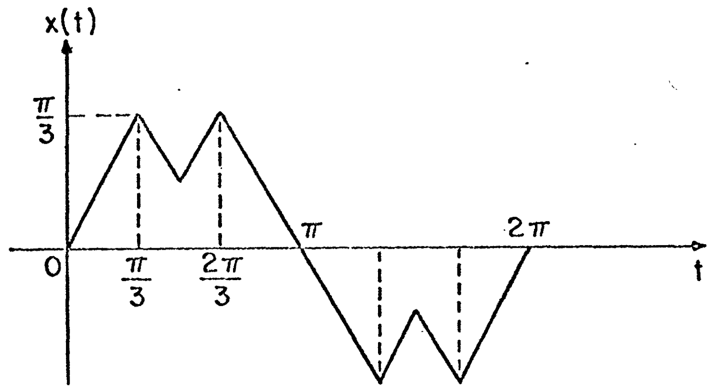

Fig. 5.3 The Input Function

The optimization problem is to choose a nonlinearity $f(\cdot)$ (unrestricted) such that

$$J(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x(t)) \sin t \, dt$$ is a maximum under the constraint

that $\frac{1}{2\pi}\int_0^{2\pi} f(x(t))^2 \, dt = a > 0$

The optimum nonlinearity is given by

$$f_o(\sigma) = \gamma \sin \sigma \qquad \text{for } |\sigma| \leq \pi/6$$

$$f_o(\sigma) = \gamma \frac{\sin \sigma + \sin(\pi/2 - \sigma)}{2} \qquad \text{for } \pi/6 \leq |\sigma| \leq \pi/3$$

$f_o(\sigma)$ arbitrary otherwise

where γ depends on α.

This nonlinearity and the resulting output signal are shown in Fig. 5.4.

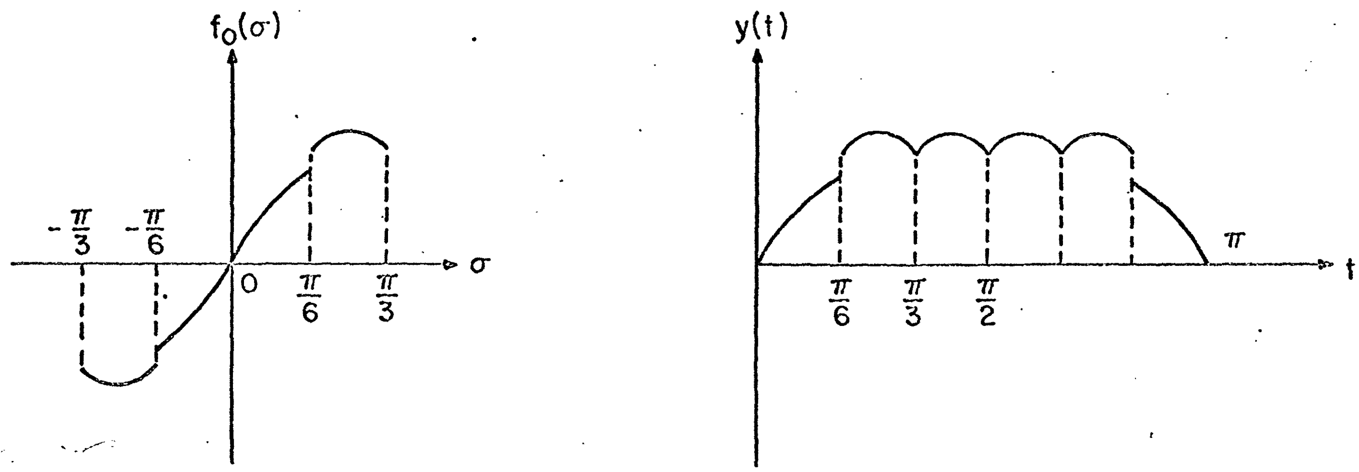

Fig. 5.4 The Optimum Nonlinearity and the Output Signal

This example shows again that the methods and the problems outlined in this chapter are very apt to treat optimization problems related to the design of frequency converters.

5.3 Conclusion and Suggestions for Further Research

In this chapter some techniques for the design of optimal nonlinearities for given inputs were described. This theory as it stands is far from complete and although some interesting problems pertaining to the design of static filters and frequency converters can be solved, the breakthrough which is needed is to apply these methods and solve some problems which also involve dynamics and for which the nonlinearity appears for instance as a feedback gain and with a linear dynamical system in the forward loop.

APPENDIX

In this appendix a general stability and instability theorem is proven which pertains to feedback systems described by the operator equations introduced in Chapter III.

<u>Additional Assumption</u>: In addition to the assumptions made in Chapter III, it is assumed that the operator $I+G_2G_1$ is invertible on X_e i.e., there <u>exists</u> an operator from X_e into itself, $(I+G_2G_1)^{-1}$, such that $(I+G_2G_1)^{-1}(I+G_2G_1)x=x$ for all $x\epsilon X_e$, and that this inverse is <u>causal</u>. This condition is not always satisfied not even for stable systems for which the feedback loop has a unique solution: as an example consider the feedback loop with the identity operator in the forward loop and the identity minus a time delay in the feedback loop. The assumption is satisfied if there is an infinitesimal delay present in the loop or if a filtering condition is satisfied. For instance if $X=L_2(0,\infty)$ it suffices that $\|(P_{t+\tau}-P_t)(G_2G_1x_1-G_2G_1x_2)\| \leq \alpha\| P_{t+\tau} - P_t)(x_1-x_2)\|$ for some $\tau > 0$, some $\alpha < 1$, all $t \geq 0$ and all $x_1,x_2\epsilon\ L_2(0,\infty)$.

Since $I+G_2G_1$ also defines a relation from X into itself, the question arises what the inverse of this relation, $(I+G_2G_1)^{-1}$, implies about the stability of the feedback loop under consideration.

<u>Theorem:</u> <u>A necessary and sufficient condition for the feedback system under consideration to be</u> X-<u>stable is that</u> $(I+G_2G_1)^{-1}$ <u>be bounded and causal on</u> X.

<u>Outline of the Proof:</u> (i) if $(I+G_2G_1)^{-1}$ is causal and bounded then $P_\tau e=P_\tau(I+G_2G_1)^{-1}P_\tau(I+G_2G_1)P_\tau e=P_\tau(I+G_2G_1)^{-1}P_\tau(u_1+G_2(u_2+G_1e_1)-G_2G_1e_1)$ and thus $\|P_\tau e\| \leq \|(I+G_2G_1)^{-1}\|\ \|u_1\|+\|I+G_2G_1)^{-1}\|K\|u_2\|$ which yields the conclusion

(ii) it is simple to show that, if $(I+G_2G_1)^{-1}$ is unbounded, then no constants K_1 and K_2, as required in the definition of stability, can exist

(iii) if $(I+G_2G_1)^{-1}$ is bounded but not causal on X, then the proof goes by contraction as follows: Assume that the system is stable. Then $(I+G_2G_1)^{-1}$ (the inverse on X_e), restricted to X, is bounded. Since $(I+G_2G_1)^{-1}$ is thus bounded on X and is causal, a contradiction follows.

This theorem is being applied to prove the converse of the Circle Criterion as obtained in (13) and in (6).

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Some properties of the mapping of the spectrum of the input to a nonlinearity into the spectrum of the output are given. The results are presented mainly in terms of positive operators. Special attention is given to nonlinear time-invariant nonlinearities, to convolution operators, to periodic gains and to monotone or odd-monotone nonlinearities. A general theorem is proven which allows to factor a large class of operators in a causal operator and an operator whose adjoint is causal. This then allows to obtain a causal positive operator from a noncausal positive operator. The results are applied to the operator equations governing a feedback loop and some general stability theorems are obtained. Two important examples are included and frequency-domain stability criteria are given. The merit of using
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