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EFFICIENCY AND PRODUCTION RATE OF A TRANSFER LINE WITH TWO MACHINES AND A FINITE STORAGE BUFFER

by

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solution for the steady state probabilities of the reliable three-stage model with exponential service time. Buzacott [2] obtained an approximate numerical solution for a discrete time model for two machines involving random failure and repair. Schick and Gershwin [20] obtained a closed form solution to a similar model. In [8] Gershwin and Schick extended their results to three machine transfer lines. Only recently Gershwin and Berman [7] analyzed analytically and obtained a compact efficient solution for a two machine transfer line in which service times as well as failure and repair times are exponential. Other related studies are Buzacott [3], [4], Hillier and Boling [13], Knott [18], Groover [11] and Buchan and Koenisberg [5].

The exponential service time assumption that is typical to many of the studies mentioned above, may become quite troublesome for the majority of actual transfer lines. In this paper the service time for the two machines is assumed to be Erlang with K ($K \geq 1$) phases. The advantage of this assumption is that very large classes of distributions can be approximated very closely by Erlang distributions [17]. We also assume in this paper that the machines are unreliable with exponential distributions for the failure times as well as for the repair times.

First we describe the model and develop the detailed balance equations to obtain all steady state probabilities. Next, based on the detailed balance equations, theoretical results for some important performance measures as efficiency of each one of the machines and production rate of the system are obtained. Next a compact and efficient method for obtaining all the steady state probabilities is presented. Finaly, some limit cases are analyzed and their results indicate good agreement with intuition.

2. THE MODEL

The transfer line is sketched in Figure 1. Parts (workpieces) enter the first machine from outside the system. Each part is operated upon in Machine 1, then proceeds to Machine 2. After being operated on in Machine 2 the part leaves the system. It is assumed that a large reservoir of parts is available to Machine 1.

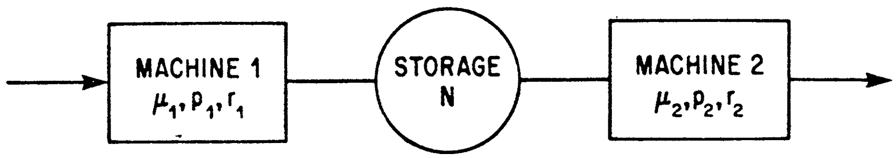

Fig. 1 Two Machine Transfer Line

Failure and repair times for Machine i are assumed to be exponential random variables with parameters p_i, r_i; i=1,2 respectively. The service time distributions for both machines i is Erlang with K($K \geq 1$) phases and parameter μ_i; i=1,2. (We let K be the same for the two machines only for convenience). We also assume that when a machine fails the piece that was being operated when the machine failed must start its service from the beginning, that is from the first phase.

The capacity of the storage buffer is N units. A consequence of the Erlang distribution assumption is that we can now find each one of the two machines in K+1 states, since in addition to being under repair the machines can also be operational in any one of the K phases of the Erlang distribution.

Let i and j represent the states of each of the two machines;

$i,j=0,1,\ldots,K$. By $i=0$ we mean that Machine 1 is under repair and by $i=m$ ($1\leq m\leq K$) we mean that Machine 1 is operational and ready to perform the m'th Erlang phase. There are however conditions on the storage buffer under which a machine cannot operate even if it is operational (in any one of the Erlang phases). Machine 1 can operate on a part only if it is operational and storage is not full, otherwise there is no place for parts from Machine 1 to go. Machine 2 can operate on a part only if it is operational and storage is not empty, otherwise there are no pieces for Machine 2 to operate on. We also assume that if Machine i fails the parts go back to the reservoir or the storage for $i=1,2$ respectively. We consider the system in the steady state. Due to our assumptions we have a Markovian model.

Let n denote the number of units in the storage plus the number of units in Machine 2 (which can be zero or one). Let (n,i,j) be the state of the system; $n=0,1,\ldots,N$; $i,j=0,1,\ldots,K$; $K\geq 1$. By our assumptions Machine 2 cannot operate on a part unless $n>0$ and Machine 1 cannot operate on a part unless $n<N$. Therefore, the probability of any state with $n=0$ and $j>1$ or $n=N$ and $i>1$ is zero. That is,

$$P(0,i,j) = 0 \quad j=2,\ldots,K; \quad i=0,1,\ldots,K \tag{2.1}$$

$$P(N,i,j) = 0 \quad i=2,\ldots,K; \quad j=0,1,\ldots,K \tag{2.2}$$

It is important to observe that (2.2) includes also $j\geq 1$. The reason for this is that we don't want the first machine to operate on a piece when there are N-1 units in the storage, one unit in Machine 2 and Machine 1 is operational since the second machine may fail before the first machine completes its service which would result in N units in the storage and therefore stop of production in Machine 1.

Now we can derive all the detailed balance equations of the system. We distinguish between four sets of detailed balance equations corresponding to the values of i and j.

For i=j=0 we have

$$P(n,0,0)\,(r_1+r_2) = \sum_{i=1}^{K} P(n,i,0)p_1 + \sum_{j=1}^{K} P(n,0,j)p_2\ ,\quad 1\leq n\leq N-1 \tag{2.3}$$

$$P(0,0,0)\,(r_1+r_2) = \sum_{i=1}^{K} P(0,i,0)p_1 \tag{2.4}$$

$$P(N,0,0)\,(r_1+r_2) = \sum_{j=1}^{K} P(N,0,j)p_2 \tag{2.5}$$

These equations represent the fact that the system enters state (n,0,0) either from state (n,i,0) ($n\neq N$, $i\neq 0$) if Machine 1 fails or from state (n,0,j) ($n\neq 0$, $j\neq 0$) if Machine 2 fails.

For i=0, $j\neq 0$,

$$P(n,0,j)\,(r_1+\mu_2+p_2) = P(n,0,j-1)\mu_2 + \sum_{i=1}^{K} P(n,i,j)p_1\ ,\quad 2\leq j\leq K,\ 1\leq n\leq N-1 \tag{2.6}$$

$$P(n,0,1)\,(r_1+\mu_2+p_2) = P(n+1,0,K)\mu_2 + \sum_{i=1}^{K} P(n,i,1)p_1 + P(n,0,0)r_2,\quad 1\leq n\leq N-1 \tag{2.7}$$

$$P(0,0,1)r_1 = P(1,0,K)\mu_2 + \sum_{i=1}^{K} P(0,i,1)p_1 + P(0,0,0)r_2 \tag{2.8}$$

$$P(N,0,j)\,(r_1+\mu_2+p_2) = P(N,0,j-1)\mu_2,\quad 2\leq j\leq k \tag{2.9}$$

$$P(N,0,1)\,(r_1+\mu_2+p_2) = P(N,0,0)r_2 \tag{2.10}$$

For $j=0$, $i \neq 0$,

$$P(n,i,0)\ (p_1+\mu_1+r_2) = P(n,i-1,0)\mu_1 + \sum_{j=1}^{K} P(n,i,j)p_2,$$
$$2 \leq i \leq K,\ 1 \leq n \leq N-1 \qquad (2.11)$$

$$P(n,1,0)\ (p_1+\mu_1+r_2) = P(n-1,K,0)\mu_1 + \sum_{j=1}^{K} P(n,1,j)p_2 + P(n,0,0)r_1,$$
$$1 \leq n \leq N-1 \qquad (2.12)$$

$$P(0,i,0)\ (p_1+\mu_1+r_2) = P(0,i-1,0)\mu_1,\ 2 \leq i \leq K \qquad (2.13)$$

$$P(0,1,0)\ (p_1+\mu_1+r_2) = P(0,0,0)r_1 \qquad (2.14)$$

$$P(N,1,0)r_2 = P(N-1,K,0)\mu_1 + \sum_{j=1}^{K} P(N,1,j)p_2 + P(N,0,0)r_1 \qquad (2.15)$$

For $i \neq 0$, $j \neq 0$,

$$P(n,i,j)\ (p_1+p_2+\mu_1+\mu_2) = P(n,i-1,j)\mu_1 + P(n,i,j-1)\mu_2,$$
$$2 \leq i \leq K;\quad 2 \leq j \leq K,\ 1 \leq n \leq N-1 \qquad (2.16)$$

$$P(n,1,j)\ (p_1+p_2+\mu_1+\mu_2) = P(n-1,K,j)\mu_1 + P(n,1,j-1)\mu_2$$
$$+ P(n,0,j)r_1,\ 2 \leq j \leq K,\ 1 \leq n \leq N-1 \qquad (2.17)$$

$$P(n,i,1)\ (p_1+p_2+\mu_1+\mu_2) = P(n,i-1,1)\mu_1 + P(n+1,i,K)\mu_2$$
$$+ P(n,i,0)r_2,\ 2 \leq i \leq K,\ 1 \leq n \leq N-1 \qquad (2.18)$$

$$P(n,1,1)\ (p_1+p_2+\mu_1+\mu_2) = P(n-1,K,1)\mu_1 + P(n+1,1,K)\mu_2$$
$$+ P(n,0,1)r_1 + P(n,1,0)r_2,\ 1 \leq n \leq N-1 \qquad (2.19)$$

$$P(0,i,1)\ (p_1+\mu_1) = P(0,i-1,1)\mu_1 + P(1,i,K)\mu_2 + P(0,i,0)r_2$$
$$2 \leq i \leq K \qquad (2.20)$$

$$P(0,1,1)\,(p_1+\mu_1) = P(1,1,K)\mu_2 + P(0,0,1)r_1 + P(0,1,0)r_2 \tag{2.21}$$

$$P(N,1,j)\,(p_2+\mu_2) = P(N-1,K,j)\mu_1 + P(N,1,j-1)\mu_2 + P(N,0,j)r_1, \quad 2\leq j\leq K \tag{2.22}$$

$$P(N,1,1)\,(p_2+\mu_2) = P(N-1,K,1)\mu_1 + P(N,0,1)r_1 + P(N,1,0)r_2 \tag{2.23}$$

$$\sum_{n=0}^{N}\sum_{i=0}^{K}\sum_{j=0}^{K} P(n,i,j) = 1 \tag{2.24}$$

The total number of these detailed balance equations is $N(K+1)^2-2K^2+3$. Obviously when K or in particular N are large, computational effort becomes very great. Later in the paper we present an efficient algorithm for obtaining the steady state probabilities. In the next section we derive some theoretical results based on the detailed balance equations.

3. THEORETICAL RESULTS

In this section we derive some theoretical results based on the detailed balance equations. These results, which are an extension of similar results in [7] help us to gain more understanding of the system.

In the following lemma we prove that some of the steady state probabilities are zero.

Lemma 1

$$P(0,i,0) = P(N,0,j) = 0 \quad \text{For } i,j = 0,1,\ldots,K \tag{3.1}$$

Proof Equation (2.13) and (2.14) imply

$$P(0,i,0) = \left(\frac{\mu_1}{p_1+\mu_1+r_2}\right)^{i-1} \frac{r_1}{p_1+\mu_1+r_2} \cdot P(0,0,0) \quad i=1,\ldots,K \tag{3.2}$$

Equation (2.4) can be written as

$$P(0,0,0)(r_1+r_2) = P(0,0,0)\,\frac{p_1 r_1}{p_1+\mu_1+r_2} \sum_{i=1}^{K} \left(\frac{\mu_1}{p_1+\mu_1+r_2}\right)^{i-1} \tag{3.3}$$

or

$$\frac{P(0,0,0)}{p_1+r_2}\left(p_1 r_2 + r_1 r_2 + r_2^{\,2} + p_1 r_1 \left(\frac{\mu_1}{p_1+\mu_1+r_2}\right)^{K}\right) = 0 \tag{3.4}$$

This implies that $P(0,0,0)=0$ and (3.2) implies that $P(0,i,0)=0$ for $i=1,\ldots,K$.

In a similar way from equations (2.9) and (2.10) we can derive $P(N,0,j) = 0$ For $j = 1,\ldots,K$.

Lemma 2 asserts that the rate of transitions from the set of states in which Machine 2 is under repair to the set of states in which Machine 2 can produce a piece is equal to the rate of transitions in the opposite direction.

Lemma 2

$$r_2 \underbrace{\sum_{n=0}^{N}\sum_{i=0}^{K} P(n,i,0)}_{\substack{\text{Probability that}\\ \text{machine 2 is under}\\ \text{repair}}} = p_2 \underbrace{\sum_{n=1}^{N}\sum_{i=0}^{K}\sum_{j=1}^{K} P(n,i,j)}_{\substack{\text{Probability that machine}\\ \text{2 can operate on a piece}}} \tag{3.5}$$

Proof

Let us add equations (2.3) - (2.5) and (2.11) - (2.15).

$$\sum_{n=0}^{N} P(n,0,0)(r_1+r_2) + \sum_{n=0}^{N-1}\sum_{i=1}^{K} P(n,i,0)(p_1+\mu_1+r_2) + P(N,1,0)r_2 \qquad (3.6)$$

$$= \sum_{n=0}^{N-1}\sum_{i=1}^{K} P(n,i,0)p_1 + \sum_{n=1}^{N}\sum_{j=1}^{K} P(n,0,j)p_2$$

$$+ \sum_{n=0}^{N-2}\sum_{i=1}^{K} P(n,i,0)\mu_1 + \sum_{i=1}^{K-1} P(N-1,i,0)\mu_1 + \sum_{n=1}^{N-1}\sum_{i=1}^{K}\sum_{j=1}^{K} P(n,i,j)p_2$$

$$+ \sum_{n=0}^{N-1} P(n,0,0)r_1 + P(N-1,K,0)\mu_1 + \sum_{j=1}^{K} P(N,1,j)p_2 + p(N,0,0)r_1$$

This can be reduced to (3.5).

Lemma 3 establishes a corresponding result for Machine 1.

Lemma 3

$$r_1 \underbrace{\sum_{n=0}^{N}\sum_{j=0}^{K} P(n,0,j)}_{\substack{\text{Probability that}\\ \text{Machine 1 is under}\\ \text{repair}}} = p_1 \underbrace{\sum_{n=0}^{N-1}\sum_{i=1}^{K}\sum_{j=0}^{K} P(n,i,j)}_{\substack{\text{Probability that Machine}\\ \text{1 can operate on a piece}}} \qquad (3.7)$$

Proof: Let us add equations (2.6) - (2.10) and (2.3) - (2.5).

$$\sum_{n=0}^{N} P(n,0,0)(r_1+r_2) + \sum_{n=1}^{N} \sum_{j=1}^{K} P(n,0,j)(r_1+\mu_2+p_2) + P(0,0,1)r_1$$

$$= \sum_{n=0}^{N-1} \sum_{i=1}^{K} P(n,i,0)p_1 + \sum_{n=1}^{N} \sum_{j=1}^{K} P(n,0,j)p_2 + \sum_{n=2}^{N} \sum_{j=1}^{K} P(n,0,j)\mu_2$$

$$+ \sum_{j=1}^{K-1} P(1,0,j)\mu_2 + \sum_{n=1}^{N-1} \sum_{i=1}^{K} \sum_{j=1}^{K} P(n,i,j)p_1 + \sum_{n=1}^{N} P(n,0,0)r_2$$

$$+ P(1,0,K)\mu_2 + \sum_{i=1}^{K} P(0,i,1)p_1 + P(0,0,0)r_2 \tag{3.8}$$

This can be reduced to (3.7)

Lemma 4 shows that the rate of transitions between the set of states with Machine 1 in the K'th phase and n pieces in the line and the set of states with Machine 2 in the K'th phase and n+1 pieces in the line are equal for $0\leq n\leq N-1$.

Lemma 4

$$\mu_1 \sum_{j=0}^{K} P(n,K,j) = \mu_2 \sum_{i=0}^{K} P(n+1,i,K) \qquad 0\leq n\leq N-1 \tag{3.9}$$

Proof: First for n=0 let us add all the detailed balance equations with n=0. Using (2.1) and the results of Lemma 1 we derive (3.9) for n=0. Let us assume now that (3.9) holds for n=m, $0\leq m\leq N-2$. We now prove for n=m+1. Let us add all the detailed balance equations with n=m+1; $0\leq m<N-2$. This can be reduced to (3.10).

$$P(m+1,K,0)\mu_1 + \sum_{i=o}^{K} P(m+1,i,K)\mu_2 + \sum_{j=1}^{K} P(m+1,K,j)\mu_1$$
$$= \sum_{j=0}^{K} P(m,K,j)\mu_1 + \sum_{i=0}^{K} P(m+2,i,K)\mu_2 \tag{3.10}$$

But by the induction assumption

$$\mu_1 \sum_{j=0}^{K} P(m,K,j) = \mu_2 \sum_{i=0}^{K} P(m+1,i,K) \tag{3.11}$$

and therefore by substitution (3.11) in (3.10) we obtain (3.9) for $n=m+1$. Finally, for $n=N-1$, by adding all the detailed balance equations with $n=N$ and by using (2.2) and Lemma 1 we derive (3.9) for $n=N-1$.

Lemma 5 shows that the rate of transitions between the set of states in which Machine 1 is in the K'th phase and the storage is not full, and the set of states in which Machine 2 is in the K'th phase and storage is not empty are equal.

There is an important interpretation of Lemma 5. Let us define E_i to be the probability that Machine i can produce a piece, then

$$E_1 = \sum_{n=0}^{N-1} \sum_{j=0}^{K} P(n,K,j) \tag{3.12}$$

$$E_2 = \sum_{n=1}^{N} \sum_{i=0}^{K} P(n,i,K) \tag{3.13}$$

The quantity $\mu_i E_i$ can be interpreted as the rate at which pieces emerge from Machine i. Lemma 5 says that the rates are equal, so that equation (3.14) below is then a conservation of flow law.

Lemma 5

$$\mu_1 E_1 = \mu_2 E_2 \tag{3.14}$$

Proof: Let us sum (3.9) from n=0 to n=N-1. We get:

$$\mu_1 \sum_{n=0}^{N-1} \sum_{j=0}^{K} P(n,K,j) = \mu_2 \sum_{n=0}^{N-1} \sum_{i=0}^{K} P(n+1,i,K) \tag{3.15}$$

which is (3.14).

E_i defined in (3.12) and (3.13) for i=1,2 can be interpreted as the efficiency of Machine i, since it is the fraction of time in which the i'th machine produce pieces. The production rate of the system, p, can be defined as the rate at which pieces emerge from Machine i, so that

$$p = \mu_2 E_2 = \mu_1 E_1 \tag{3.16}$$

using the result of Lemma 5. In other words the production rate of the system is equal to the production rate of each one of the two machines.

4. EFFICIENT METHOD TO CALCULATE STEADY STATE PROBABILITIES

4a. ANALYSIS OF INTERNAL EQUATIONS

We define internal states (n,S_1,S_2) as states with $1 \leq n \leq N-1$ and $S_1, S_2 = 0,1,\ldots,K$. We define internal equations as all the detailed balance equations that do not include boundary states (states with n=0 or n=N). We guess a solution to the steady state probabilities in the internal equations, of the form

$$P(n,S_1,S_2) = CX^n Y_{11}^{\gamma_1} Y_{12}^{\beta_1} Y_{21}^{\gamma_2} Y_{22}^{\beta_2} \tag{4.1}$$

where for i=1,2,

$$\beta_i = \begin{cases} 0 & \text{if} \quad S_i = 0 \\ 1 & \text{if} \quad S_i \geq 1 \end{cases} \tag{4.2}$$

$$\gamma_i = \begin{cases} 0 & \text{if} \quad S_i = 0 \\ S_i-1 & \text{if} \quad S_i \geq 1 \end{cases} \tag{4.3}$$

By substituting (4.1) - (4.3) in the internal equations (2.6), (2.11), (2.16), (2.17) and (2.18) we get the following five nonlinear equations in the five unknowns X, Y_{11}, Y_{12}, Y_{21}, Y_{22}.

$$Y_{11}Y_{21}(p_1+p_2+\mu_1+\mu_2) = Y_{21}\mu_1+Y_{11}\mu_2 \tag{4.4}$$

$$Y_{11}(p_1+\mu_1+r_2) = \mu_1 + p_2Y_{22}Y_{11}\frac{1-Y_{21}^K}{1-Y_{21}} \tag{4.5}$$

$$Y_{21}(r_1+\mu_2+p_2) = \mu_2 + p_1Y_{12}Y_{21}\frac{1-Y_{11}^K}{1-Y_{11}} \tag{4.6}$$

$$XY_{12}Y_{21}(p_1+p_2+\mu_1+\mu_2) = Y_{12}Y_{11}^{K-1}Y_{21}\mu_1 + XY_{12}\mu_2+ XY_{21}r_1 \tag{4.7}$$

$$Y_{11}Y_{22}(p_1+p_2+\mu_1+\mu_2) = Y_{22}\mu_1 + XY_{11}Y_{22}Y_{21}^{K-1}\mu_2 + Y_{11}r_2 \tag{4.8}$$

These five equations in five unknowns can be reduced to a single non linear equation in Y_{11} (Equation (A.1) in the Appendix 1). It is possible to verify from (4.4) - (4.8) and (A.1) that one of the solutions of the non linear equation is always $Y_{11} = \frac{\mu_1}{\mu_1+p_1}$, $Y_{21} = \frac{\mu_2}{\mu_2+p_2}$ and X=1. This equation has M($M \leq 2K+2$) solutions.

Thus the internal probabilities are expected to be of the form:

$$P(n,S_1,S_2) = \sum_{\ell=1}^{M} C_\ell X_\ell^n Y_{11\ell}^{\gamma_1} Y_{12\ell}^{\beta_1} Y_{21\ell}^{\gamma_2} Y_{22\ell}^{\beta_2} \tag{4.9}$$

$n=0,1,\ldots,N;\quad S_1,S_2 = 0,1,\ldots,K$

In the next subsection we analyze the boundary equations. Before that let us refer again to the equation (A.1). It has been mentioned that the number of distinct solutions of this equation is M($M \leq 2+2K$). Clearly the condition whether M is less or equal to 2+2K depends on the relationship between the parameters of the model. For example, for K=2 let us consider equations (4.4) - (4.8). From equations (4.4) and (4.6) we can obtain that Y_{12} can be written:

$$Y_{12} = \frac{(1-Y_{11})\,[Y_{11}\,(r_1 - p_1 - \mu_1) + \mu_1]}{Y_{11}\,p_1\,(1-{}_{11}^{2})} \tag{4.10}$$

Let us now compute the product of $Y_{11}Y_{12}$ using (4.10),

$$Y_{11}Y_{12} = \frac{Y_{11}\,(r_1 - p_1 - \mu_1) + \mu_1}{p_1\,(1+Y_{11})} \tag{4.11}$$

Obviously if we have

$$r_1 - p_1 - \mu_1 = \mu_1 \quad \text{or} \quad \mu_1 = \frac{r_1-p_1}{2} \tag{4.12}$$

then $\gamma_{11}\gamma_{12} = \frac{\mu_1}{p_1}$, and we must have therefore $M \leq 1+2K = 5$. The conclusion of this discussion is that assuming that (4.9) indeed presents the form of the internal probabilities we must have the following interesting result if (4.12) holds:

$$P(n,0,j) = (\frac{\mu_1}{p_1}) P(n,2,j) \quad j=0,1,2; \ n=1,2,\ldots,N-1 \tag{4.13}$$

Furthermore, in the next subsection we derive an expression also for the boundary probabilities and by examining them we can get that (4.13) satisfies also n=0 and n=N. In Appendix 2 we show an example in which (4.12) holds and $(\frac{\mu_1}{p_1}) = 1$. By following a similar discussion we can get another condition to $M \leq 1+2K$ for K=2:

$$\mu_2 = \frac{r_2 - p_2}{2} \tag{4.14}$$

and then we also have

$$P(n,i,0) = (\frac{\mu_2}{p_2}) \ P(n,i,2) \quad i=0,1,2, \ n=0,1,\ldots,N \tag{4.15}$$

Finally it is important to note that many queuing theory problems yield product form solutions. For example see Jackson [16], Gordon and Newell [10], Basket, Chandy, Muntz and Palacious [1] and others. It is also interesting to observe that when K=1 in (4.9) both γ_{11} and γ_{21} disappear and the analysis gets to be less complicated; see Gershwin and Berman [7].

4b. ANALYSIS OF BOUNDARY EQUATIONS

There are a total of 2(2K+2) boundary states. The probabilities of 2(K+1) of them are specified by Lemma 1. The other 2(K+1) boundary probabilities are characterized by the following theorem.

THEOREM 1

(a) The steady state probabilities for the boundary states (0,K,1) and (N,1,K) are in the internal form, i.e.,

$$P(0,K,1) = \sum_{\ell=1}^{M} C_\ell \, Y_{11\ell}^{K-1} \, Y_{12\ell} \, Y_{22\ell} \tag{4.16}$$

$$P(N,1,K) = \sum_{\ell=1}^{M} C_\ell \, X_\ell^{N} \, Y_{12\ell} \, Y_{21\ell}^{K-1} \, Y_{22\ell} \tag{4.17}$$

(b) The rest of the boundary states have the following steady state probabilities:

$$P(0,i,1) = \left(\frac{p_1+\mu_1}{\mu_1}\right)^{K-i} P(0,K,1) - \frac{\mu_2}{\mu_1} \sum_{\ell=1}^{M} C_\ell X_\ell \, Y_{11\ell}^{i} \, Y_{12\ell} \, Y_{21\ell}^{K-1} \, Y_{22\ell} \tag{4.18}$$

$$\cdot \left[Y_{11\ell}^{K-1-i} \left(\frac{p_1+\mu_1}{\mu_1}\right)^{K-1-i} + \sum_{s=i}^{K-2} \left(Y_{11\ell} \frac{p_1+\mu_1}{\mu_1}\right)^{K-2-s} \right]$$

For $i=1,\ldots, K-1$

$$P(0,0,1) = \frac{(p_1+\mu_1)^K}{\mu_1^{K-2} r_1} P(0,K,1) - \frac{\mu_2}{\mu_1} \sum_{\ell=1}^{M} C_\ell X_\ell Y_{12\ell} Y_{21\ell}^{K-1} Y_{22\ell} \tag{4.19}$$

$$\left[Y_{11\ell}^{K-1} \frac{(p_1+\mu_1)^{K-1}}{\mu_1^{K-2} r_1} + \sum_{s=1}^{K-1} \frac{\mu_1}{r_1} \left(Y_{11\ell} \frac{p_1+\mu_1}{\mu_1}\right)^{K-1-s} \right]$$

$$P(N,1,i) = \left(\frac{p_2+\mu_2}{\mu_2}\right)^{K-i} P(N,1,K) - \frac{\mu_1}{\mu_2} \sum_{\ell=1}^{M} C_\ell X_\ell^{N-1} Y_{11\ell}^{K-1} \, Y_{12\ell} \, Y_{21\ell}^{i} \, Y_{22\ell}. \tag{4.20}$$

$$\left[\left(\frac{p_2+\mu_2}{\mu_2}\right)^{K-1-i} Y_{21\ell}^{K-1-i} + \sum_{s=i}^{K-2} \left(Y_{21\ell} \frac{p_2+\mu_2}{\mu_2}\right)^{K-2-s} \right]$$

For i=1,...,K-1

$$P(N,1,0) = \frac{(p_2+\mu_2)^K}{\mu_2^{K-1} r_2} P(N,1,K) - \frac{\mu_1}{\mu_2} \sum_{\ell=1}^{M} C_\ell X_\ell^{N-1} Y_{11\ell}^{K-1} Y_{12\ell} Y_{22\ell} \cdot \tag{4.21}$$

$$\left[\frac{(p_2+\mu_2)^{K-1}}{\mu_2^{K-2} r_2} Y_{21\ell}^{K-1} + \sum_{s=1}^{K-1} \frac{\mu_2}{r_2} (Y_{12\ell} \frac{p_2+\mu_2}{\mu_2})^{K-1-s} \right]$$

(c) The coefficients C_ℓ; ℓ=1,...,M satisfy the following 2K+3 linear equations:

$$\sum_{\ell=1}^{M} C_\ell Y_{11\ell}^{K-1} Y_{12\ell} = 0 \tag{4.22}$$

$$\sum_{\ell=1}^{M} C_\ell Y_{11\ell}^{K-1} Y_{12\ell} Y_{21\ell}^{j-1} Y_{22\ell} = 0 \qquad j = 2,\ldots,K \tag{4.23}$$

$$\sum_{\ell=1}^{M} C_\ell X_\ell^{N} Y_{21\ell}^{K-1} Y_{22\ell} = 0 \tag{4.24}$$

$$\sum_{\ell=1}^{M} C_\ell X_\ell^{N} Y_{11\ell}^{i-1} Y_{12\ell} Y_{21\ell}^{K-1} Y_{22\ell} = 0 \qquad i = 2,\ldots,K \tag{4.25}$$

$$\mu_1 P(0,K,1) = \mu_2 \sum_{i=0}^{K} P(1,i,K) \tag{4.26}$$

$$\mu_2 P(N,1,K) = \mu_1 \sum_{j=0}^{K} P(N-1,K,j) \tag{4.27}$$

and the normalization equation

$$\sum_{n=0}^{N} \sum_{s_1=0}^{K} \sum_{s_2=0}^{K} P(N,S_1,S_2) = 1 \tag{4.28}$$

Proof

The expressions (4.16) - (4.21) and (3.1) satisfy all the detailed balance equations (2.3) - (2.24) except for the equations (2.8), (2.12) for n=1, (2.17) for n=1, (2.7) for n=N-1, (2.15), and (2.18) for n=N-1.

Equation (2.12) for n=1, can be expressed using (4.9) and (3.1) as

$$\sum_{\ell=1}^{M} C_\ell X_\ell \left\{ (p_1+\mu_1+r_2)Y_{12\ell}-r_1-p_2 Y_{12\ell}Y_{22\ell}\left[\frac{1-Y_{21\ell}^{K}}{1-Y_{21\ell}}\right] \right\} = 0 \tag{4.29}$$

By substituting (4.5) into (4.29) we can rewrite (4.29)

$$\sum_{\ell=1}^{M} C_\ell \frac{X_\ell}{y_{11\ell}} (Y_{12\ell}\mu_1 - Y_{11\ell}r_1) = 0 \tag{4.30}$$

Equations (4.4) and (4.7) yield that

$$\frac{X_\ell}{Y_{11\ell}} (Y_{12\ell}\mu_1 - Y_{11\ell}r_1) = \mu_1 Y_{11\ell}^{K-1} Y_{12\ell} \qquad \text{For } \ell = 1,\ldots,M. \tag{4.31}$$

By substituting (4.31) into (4.30) we get (4.22). In a similar way, equation (2.7) for n=N-1 using (4.6), (4.4) and (4.8) implies equation (4.24). Equation (2.17) for n=1 can be expressed using (4.9), (2.1) and (3.1) for j = 2,...,K as

$$\sum_{\ell=1}^{M} C_\ell Y_{21\ell}^{j-2} Y_{22\ell} \left[X_\ell Y_{12\ell}Y_{21\ell}(p_1+p_2+\mu_1+\mu_2) - \mu_2 X_\ell Y_{12\ell} - r_1 X_\ell Y_{21\ell}\right] = 0 \tag{4.32}$$

But (4.32) can be expressed using (4.7) as

$$\mu_1 \sum_{\ell=1}^{M} C_\ell Y_{11\ell}^{K-1} Y_{12\ell} Y_{21\ell}^{j-1} Y_{22\ell} = 0 \qquad j = 2,\ldots,K \tag{4.33}$$

which implies (4.23) from part (c) of the Lemma. In a similar way by using (4.8), equation (2.18) for $n=N-1$ implies equation (4.25). Finally equations (4.26) and (4.27) can be obtained by substituting (4.16), (4.18) and (4.19) into equation (2.8) and (4.17), (4.20) and (4.21) into equation (2.15). The number of equations specified by part (c) of the Lemma is $3+2K$ whereas the number of the unknowns C_ℓ is $M \leq 2+2K$. Therefore only M equations from the set (4.22) - (4.28) are required to obtain all $C_\ell; \ell = 1,2,\ldots,M$. When $M=2+2K$ then any one of the $2+2K$ equations (4.22) - (4.27) can be ignored. When $M<2+2K$ the choice of the M equation should be done more carefully. For example when $K=2$ and (4.12) holds then (4.24) and (4.25) are linearly dependent (equation (4.24) is (μ_1/p_1) times equation (4.25)). In this case any two equations from the set (4.22) - (4.27) such that at least **one** of them is (4.24) or (4.25) can be ignored.

4c. THE METHOD

Now we can suggest the following method to obtain all the steady state probabilities:

STEP 1: Find $Y_{11\ell}$, $\ell=1,2,\ldots,M$ using (A.1) in Appendix 1.
STEP 2: Obtain $Y_{12\ell}$, $Y_{21\ell}$, $Y_{22\ell}$, X_ℓ, $\ell=1,2,\ldots,M$ using (4.4) - (4.8).
STEP 3: Use (4.9), (3.1) and (4.16) - (4.21) to solve equation (4.28) <u>and</u> M-1 equations of the set (4.22) - (4.27). (Refer to the discussion in section 4.b), to obtain C_ℓ; $\ell=1,2,\ldots M$.
STEP 4: Generate all steady state probabilities using (4.9) for internal states, (3.1) and (4.16) - (4.22) for all the rest.

The reduction in the number of computations is tremendous. For example when $K=5$, $N=100$, the number of detailed balance equations is 3553 whereas $2+2K=12$!!!

It is easy to verify that the production rate of the system can be derived easily by using the coefficients C_ℓ; $\ell=1,2,\ldots M$ without the need to obtain all steady state probabilities first. By substituting (3.1) and (4.9) in (3.13) we can obtain

$$E_2 = \sum_{\ell=1}^{M} C_\ell Y_{22\ell} Y_{21\ell}^{K-1} X_\ell \left(A_\ell + Y_{12\ell} A_\ell B_\ell + Y_{12\ell}(A_\ell + X_\ell^N)\right) \tag{4.34}$$

where for $\ell = 1,2,\ldots,M$

$$A_\ell = \begin{cases} \dfrac{X_\ell - X_\ell^N}{1-X_\ell} & \text{IF} \quad X_\ell \neq 1 \\ N-1 & \text{IF} \quad X_\ell = 1 \end{cases} \tag{4.35}$$

and

$$B_\ell = \frac{Y_{11\ell} - Y_{11\ell}^K}{1-Y_{11\ell}} \tag{4.36}$$

In the next section we analyze some limit cases in order to gain better understanding of the model.

5. LIMIT CASES

In this section we analyze the behavior of the system when we let various parameters of the system approach their limits.

First we show that the efficiency of the two machines E_1 and E_2 that were defined in Section 3 can be rewritten in a different way.

<u>Lemma 6</u>

$$E_1 = \frac{\left(\dfrac{r_1}{r_1+p_1}\right) \cdot \left(\dfrac{\mu_1}{\mu_1+p_1}\right)^K}{\sum_{i=1}^{K} \left(\dfrac{\mu_1}{\mu_1+p_1}\right)^i} P(n\neq N) \tag{5.1}$$

$$E_2 = \frac{\left(\dfrac{r_2}{r_2+p_2}\right) \left(\dfrac{\mu_2}{\mu_2+p_2}\right)^K}{\sum_{i=1}^{K} \left(\dfrac{\mu_2}{\mu_2+p_2}\right)^i} P(n\neq 0) \tag{5.2}$$

Proof Let S_i be the state of machine i, i=1,2. First we show that:

$$P(S_i=0 \text{ and } n\neq N) = \left(\frac{p_1}{p_1+r_1}\right)P(n\neq N) \tag{5.3}$$

By definition:

$$P(S_1=0/n\neq N) = \frac{P(S_1=0 \text{ and } n\neq N)}{P(S_1=0 \text{ and } n\neq N) + P(S_1\neq 0 \text{ and } n\neq N)} \tag{5.4}$$

Applying Lemma 3 in (5.4) yields (5.3).

From (2.11) - (2.14) and (2.16) - (2.21) we can obtain:

$$P(S_1=i \text{ and } n\neq N) = \left(\frac{\mu_1}{\mu_1+p_1}\right) P(S_1=i-1 \text{ and } n\neq N) \qquad i = 2,\ldots,K \tag{5.5}$$

or that,

$$P(S_1=i \text{ and } n\neq N) = \left(\frac{\mu_1}{\mu_1+p_1}\right)^{i-1} P(S_1=1 \text{ and } n\neq N) \qquad i = 2,\ldots,K \tag{5.6}$$

Finally, (5.1) can be derived from (5.3) and (5.6). In a similar way (5.2) can be also obtained.

Expressions (5.1) and (5.2) are obvious due to the conditions under which the machines can produce a piece. For the first machine the requirement is that the machine is operational (not under repair) and the storage is not full which are expressed in the left and right terms of (5.1). In addition to that given that the first machine is not under repair it should be also in the last phase of the Erlang distribution which is expressed in the middle term of (5.1). In a similar way we can explain (5.2).

Let us define the isolated production rate of machine i as the production rate machine i would have if it were not part of a system with other machines and storage; i=1,2. Let ρ_i denote the isolated production rate of machine i;

$$\rho_i = \mu_i e_i \tag{5.7}$$

where e_i is the isolated efficiency of machine i: (e_i is the fraction of time that an isolated machine i is in the K'th Erlang phase).

$$e_i = \frac{\left(\frac{r_i}{r_i+p_i}\right)\left(\frac{\mu_i}{\mu_i+p_i}\right)^K}{\sum_{j=1}^{K}\left(\frac{\mu_i}{\mu_i+p_i}\right)^j} \tag{5.8}$$

The following lemma is the basis for the analysis.

Lemma 7

(a) $\rho_i \longrightarrow 0$ implies $P \longrightarrow 0$, $E_i \longrightarrow e_i$, $E_j \longrightarrow 0$, $j \neq i$,

(b) $\rho_i \longrightarrow \infty$ implies $P \longrightarrow \rho_j$, $E_i \longrightarrow 0$, $E_j \longrightarrow e_j$, $j \neq i$.

Proof From (5.7) and Lemma 5:

$$\rho_1 P(n \neq N) = \rho_2 P(n \neq 0) \tag{5.9}$$

(a) $\rho_1 \rightarrow 0$ implies $P(n \neq 0) \rightarrow 0$ and therefore $P \rightarrow 0$ and $E_2 \rightarrow 0$. But $P(n \neq 0) \rightarrow 0$ implies $P(n \neq N) \rightarrow 1$ and therefore $E_1 \rightarrow e_1$. The same proof holds when ρ_2 approaches zero.

(b) $\rho_1 \rightarrow \infty$ implies $P(n \neq N) \rightarrow 0$ and therefore $E_1 \rightarrow 0$. But $P(n \neq N) \rightarrow 0$ implies $P(n \neq 0) \rightarrow 1$ and therefore $E_2 \rightarrow e_2$ and $P \rightarrow \rho_2$. The same proof holds when ρ_2 approaches infinity.

A direct consequence of Lemma 7 (b) is:

$$P \xrightarrow[\mu_j \rightarrow \infty]{} \rho_i \qquad i \neq j = 1,2 \tag{5.10}$$

This result is intuitive since as μ_1 gets larger the buffer tends to be never empty and $P(n \neq 0)$ in (5.2) approaches 1. When μ_2 gets larger the buffer tends to be never full and $P(n \neq N)$ in (5.1) approaches 1. These are shown in Figures 2 and 3 for a simple example in which $\mu_1=\mu_2=2$, $p_1=9$, $p_2=7$, $r_1=3$, $r_2=6$, K=2 and N=3. We can also observe from the figures that as the service rate for either one of the two machines approaches zero the production rate approaches zero as well. But this is a direct consequence of Lemma 7(a). It is also clear since once a machine stops its production the storage will be affected in such a way that the other machine will soon stop its production as well.

Another consequence of Lemma 7(a) is that when the repair rate of any machine approaches zero the production rate approaches zero. Figures 4 and 5 show the production rate as a function of the repair rate for machine 1 and 2 respectively. It is interesting to observe that as the repair rate for machine i approaches infinity, the production rate approaches a constant. This constant is the result of a similar model in which repairs for machine i are instantaneous.

A reverse situation occurs when the failure rate of a machine approaches its limit. This is shown in Figures 6 and 7 for the same simple example. As the failure rate of machine i approaches infinity the production rate approaches zero; i=1,2 which is a direct consequence of Lemma 7(a). When the failure rate approaches zero the production rate approaches a constant. The constant for machine i is the result of a similar model in which no failures for machine i can occur; i=1,2.

Finally from our computational experience the following result was also observed:

$$P \xrightarrow[N \to \infty]{} \min_{i=1,2} \{\rho_i\} \tag{5.11}$$

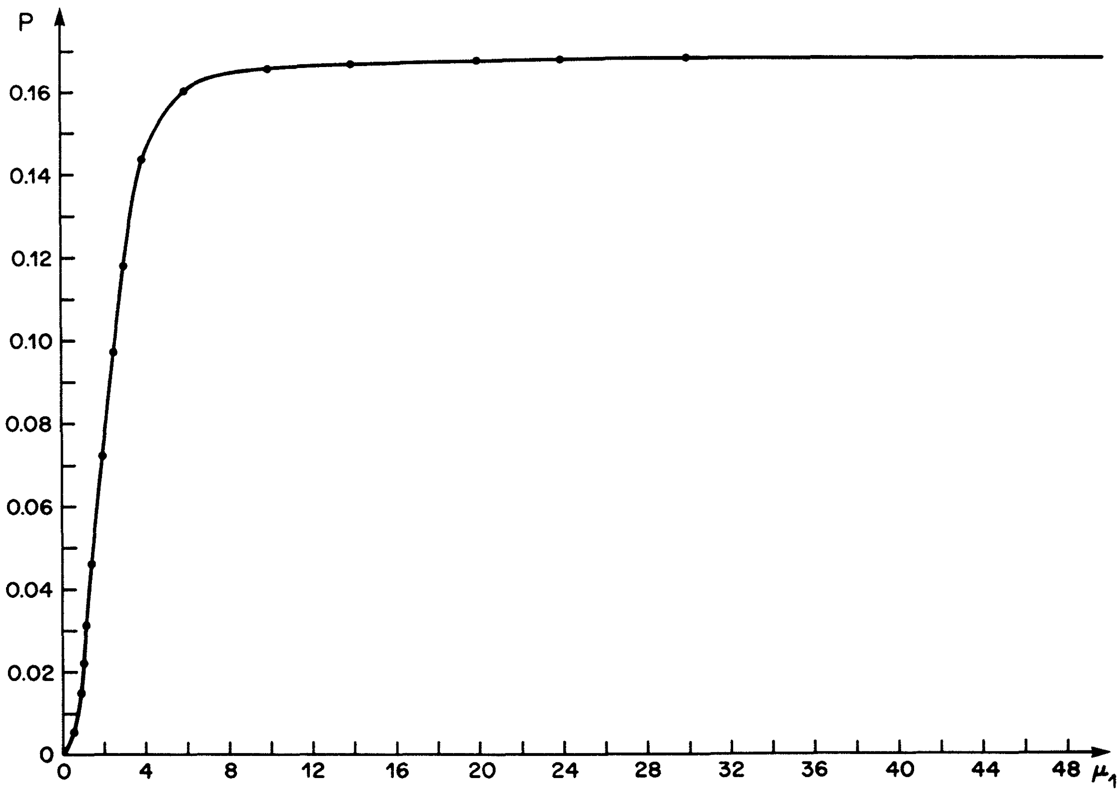

Fig. 2 Production Rate vs Service Rate of the First Machine

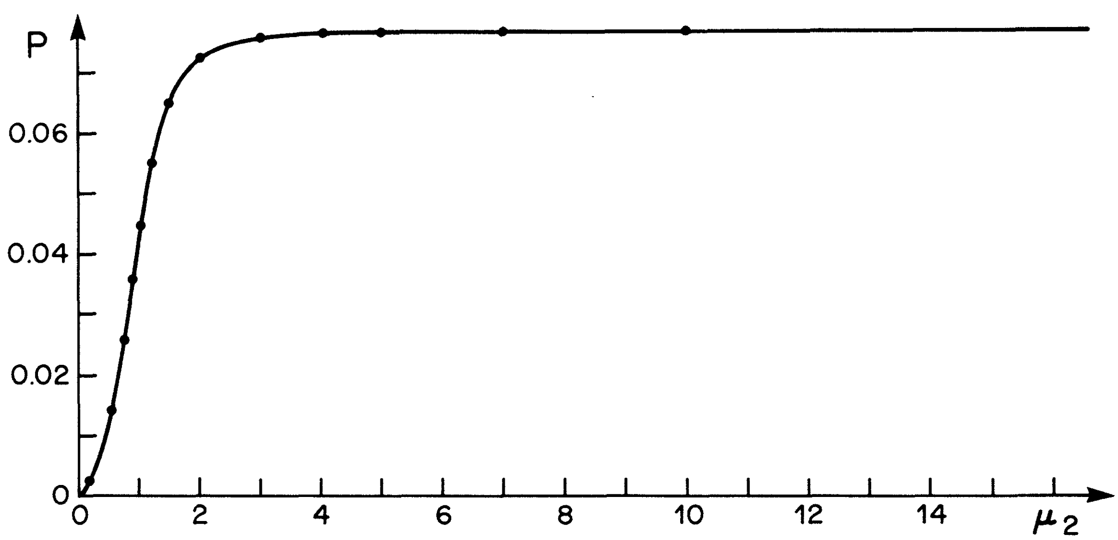

Fig. 3 Production Rate vs Service Rate of the Second Machine

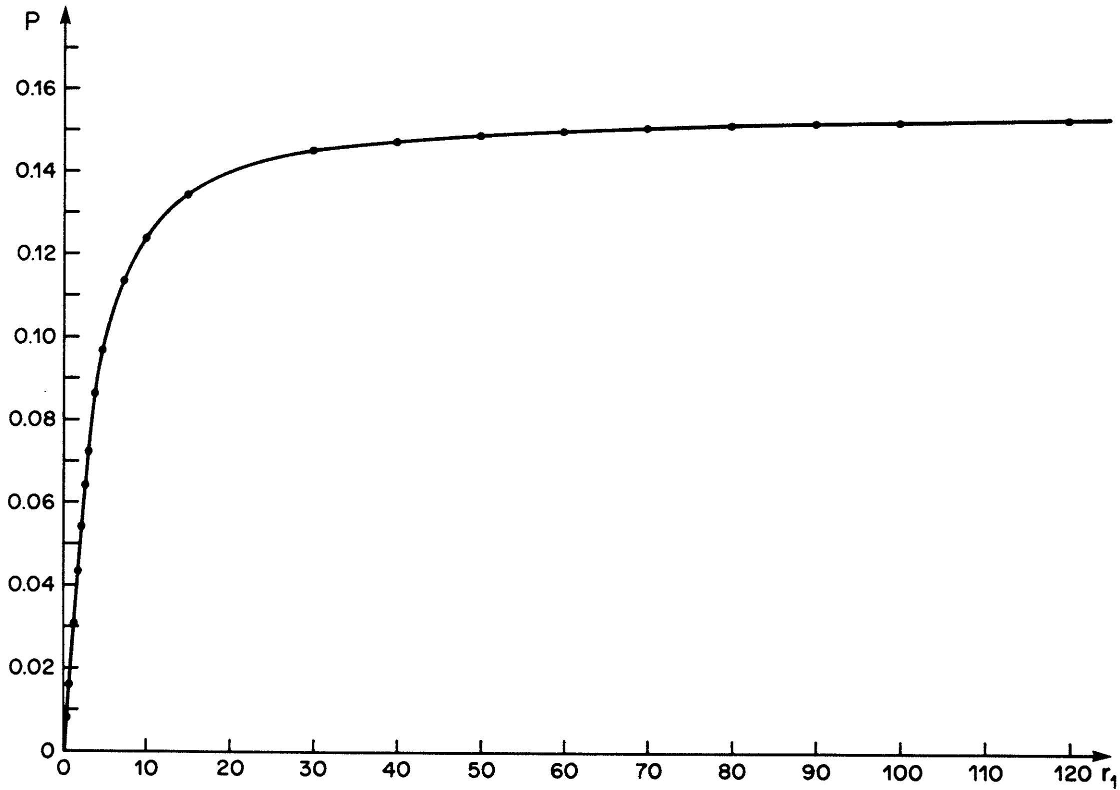

Fig. 4 Production Rate vs Repair Rate of the First Machine

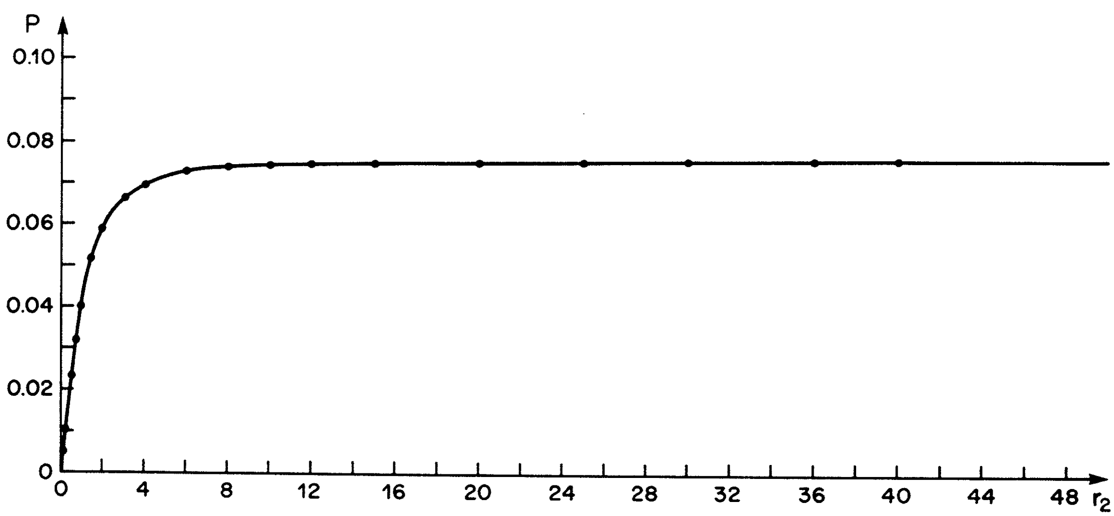

Fig. 5 Production Rate vs Repair Rate of the Second Machine

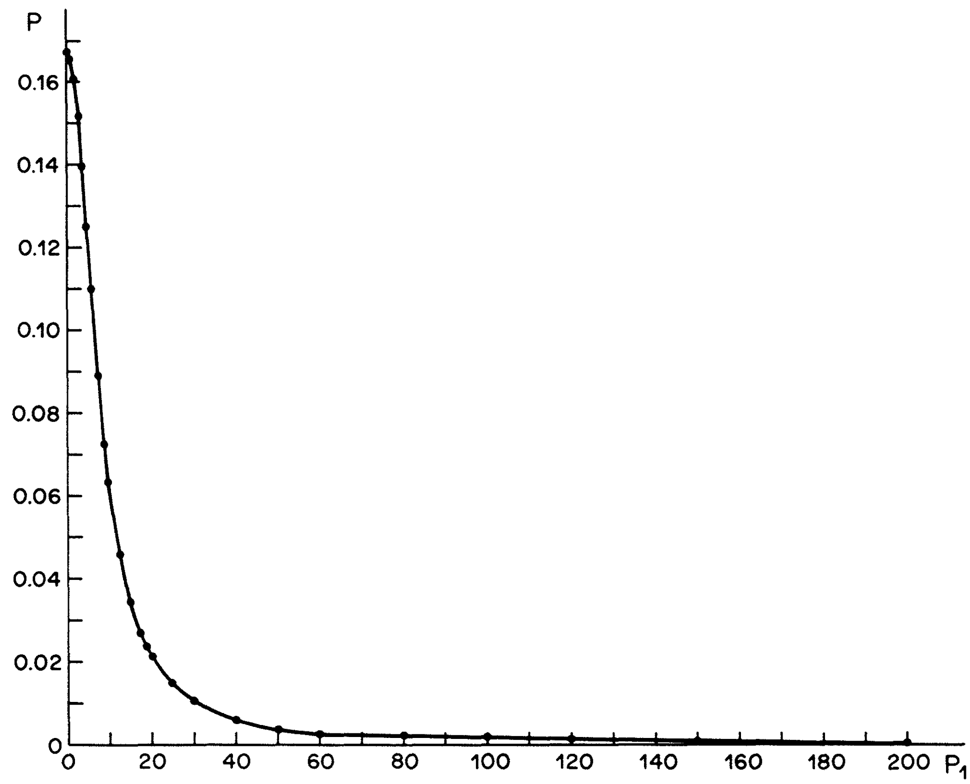

Fig. 6 Production Rate vs Failure Rate of the First Machine

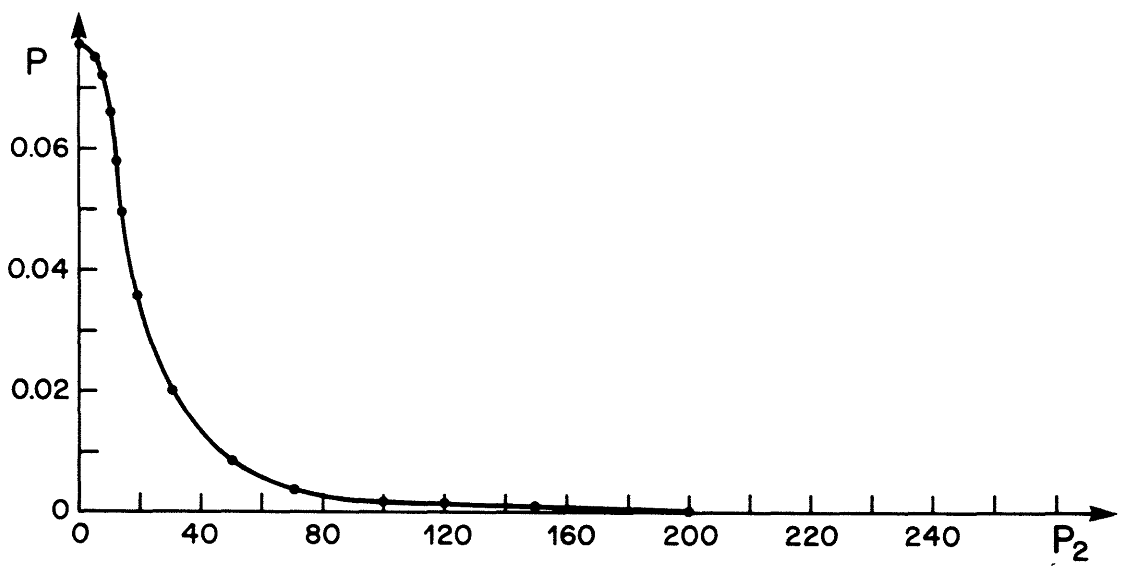

Fig. 7 Production Rate vs Failure Rate of the Second Machine

This result is also intuitive. As N gets larger the buffer can be affected in two different ways:

(i) If the first machine is less productive than the second one (smaller isolated production rate) the storage will tend to be never full in (5.1).

(ii) If the first machine is more productive than the second one the storage will tend to be never empty in (5.2). For our simple example the first machine is less productive and indeed as shown in Figure 8 the production rate of the system approaches the isolated production rate of the first machine. If we refer again to (5.10) we can conclude that instantaneous service rate for the more productive machine or infinite buffer storage size leads to the same production rate.

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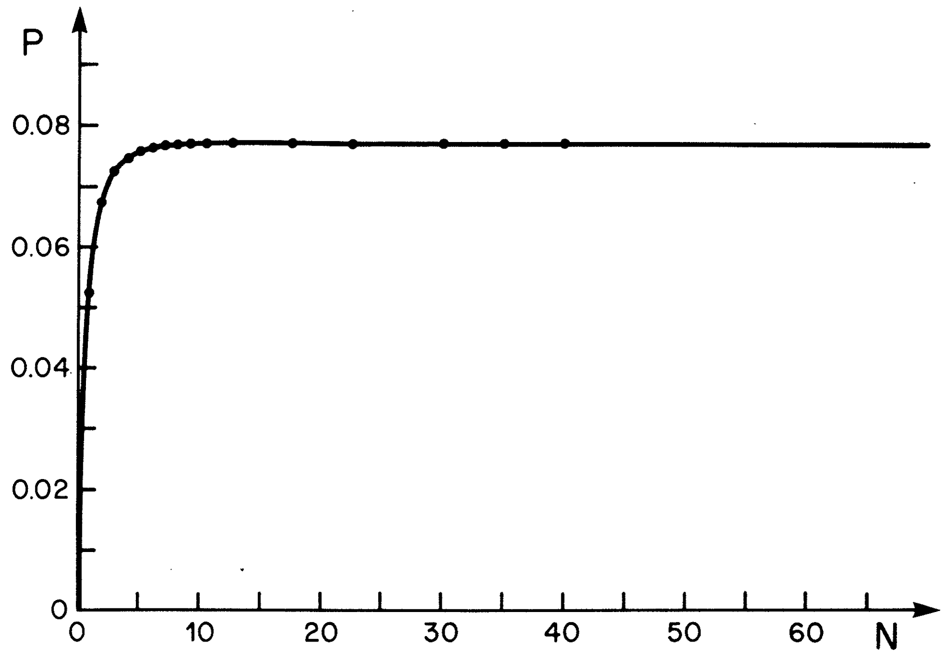

Fig. 8 Production Rate vs Buffer Storage Size

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APPENDIX 1

EQUATION FOR Y_{11}

Equation (A.1) for Y_{11} was obtained from equations (4.4) - (4.8) by means of the MACSYMA system (MACSYMA, 1977). The following notation applies to this equation.

$A = Y_{11}$

$M1 = \mu_1$

$M2 = \mu_2$

$P1 = p_1$

$P2 = p_2$

$R1 = r_1$

$R2 = r_2$

The computer printout of the equation is shown on the page following.

$$(1 - A)\, A^{K - 1}\, M1\, M2\, (A\, (R1 - P1 - M1) + M1)$$

$$/((1 - A^{K})\, P1\, (A\, (P2 + P1 + M2 + M1) - M1)$$

$$\left(\frac{(1 - A)\, M2\, (P2 + P1 + M2 + M1)\, (A\, (R1 - P1 - M1) + M1)}{(1 - A^{K})\, P1\, (A\, (P2 + P1 + M2 + M1) - M1)}\right.$$

$$\left. - \frac{(1 - A)\, M2\, (A\, (R1 - P1 - M1) + M1)}{A\, (1 - A^{K})\, P1} - \frac{A\, M2\, R1}{A\, (P2 + P1 + M2 + M1) - M1}\right))$$

$$- P2 \left(\frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right)^{1 - K} \left(1 - \left(\frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right)^{K}\right)$$

$$((P2 + P1 + M2 + M1)\, (A\, (P2 + P1 + M2 + M1) - M1)$$

$$\left(1 - \frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right) \left(\frac{A\, M2\, (R2 - P2 - M2)}{A\, (P2 + P1 + M2 + M1) - M1} + M2\right)$$

$$/\left(M2\, P2\, \left(1 - \left(\frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right)^{K}\right)\right)$$

$$- M1\, (A\, (P2 + P1 + M2 + M1) - M1)\, \left(1 - \frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right)$$

$$\left(\frac{A\, M2\, (R2 - P2 - M2)}{A\, (P2 + P1 + M2 + M1) - M1} + M2\right)/(A\, M2\, P2$$

$$\left(1 - \left(\frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right)^{K}\right)) - A\, R2)$$

$$/((A\, (P2 + P1 + M2 + M1) - M1)\, \left(1 - \frac{A\, M2}{A\, (P2 + P1 + M2 + M1) - M1}\right)$$

$$\left(\frac{A\, M2\, (R2 - P2 - M2)}{A\, (P2 + P1 + M2 + M1) - M1} + M2\right))$$

$$= 0 \qquad \text{(A.1)}$$

APPENDIX 2

In the following example K=2, N=8, μ_1=1, μ_2=2, p_1=1, p_2=3, r_1=3, r_2=2. If we observe Table 1 we can see that p(n,0,j) = p(n,2,j) j=0,1,2; n=1,2,...,8. But this is obvious if we apply the condition stated in equation (4.12): $\mu_1 = \frac{r_1 - p_1}{2} = 1$

TABLE 1: STEADY STATE PROBABILITIES FOR THE CASE:

$\mu_1 = 1,\ \mu_2 = 2,\ p_1 = 1,\ p_2 = 3,\ r_1 = 3,\ r_2 = 2,\ K = 2,\ N = 8$

n/(i,j)	00	10	20	01	02	11	12	21	22
0	0	0	0	0.0196	0	0.0344	0	0.0196	0
1	0.0122	0.0229	0.0122	0.00615	0.00246	0.0135	0.0049	0.00615	0.00246
2	0.0136	0.0268	0.0136	0.00655	0.00261	0.013	0.00518	0.00655	0.00261
3	0.0149	0.0295	0.0149	0.00718	0.0286	0.0142	0.00565	0.00718	0.00286
4	0.0163	0.0323	0.0163	0.00787	0.00313	0.0156	0.0062	0.00787	0.00313
5	0.0179	0.0354	0.0179	0.00862	0.00343	0.0171	0.00679	0.00862	.00343
6	0.0196	0.0389	0.0196	0.00938	0.00375	0.0188	0.00747	0.00938	0.00375
7	0.0207	0.0439	0.0207	0.00915	0.00385	0.0226	0.00864	0.00915	0.00385
8	0	0.096	0	0	0	0.0402	0.0169	0	0