Massachusetts Institute of Technology
Department of Physics
8.962 Spring 2006

Problem Set 10

Post date: Thursday, May 4th
Due date: Thursday, May 11th

1. Isotropic representation of the Schwarzschild metric

   (a) The Schwarzschild line element, written in Schwarzschild coordinates, takes the form

   $$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{(1 - 2GM/r)} + r^2 d\Omega^2 .$$

   Show that changing to the radial coordinate $\bar{r}$, defined by

   $$r = \bar{r}\left(1 + \frac{GM}{2\bar{r}}\right)^2 ,$$

   puts the Schwarzschild line element into the form

   $$ds^2 = g_{tt}(\bar{r})dt^2 + g(\bar{r})(d\bar{r}^2 + \bar{r}^2 d\Omega^2) .$$

   This new coordinate system is called "isotropic coordinates", since it emphasizes the fundamental local isotropy of the three spatial directions.

   Compute the metric functions $g_{tt}(\bar{r})$ and $g(\bar{r})$.

   (b) Take the limit $\bar{r} \gg GM$. Show that the line element then reduces to a form appropriate for describing the exterior of a spherical body in the linearized limit of general relativity.

2. Numerical construction of neutron star models in GR

   A moderately accurate approximation to the equation of state of the material which makes up a neutron star is given by the polytropic form

   $$P = K\rho_0^\Gamma$$

   where P is the pressure, ρ_0 is the *rest* matter density, and the constants are given by

   $$\begin{aligned} \Gamma &= 5/3 , \\ K &= \frac{3^{2/3}\pi^{4/3}}{5}\frac{\hbar^2}{m_n^{8/3}} \\ &= 5.38 \times 10^9 \,(\mathrm{dyne/cm^2})(\mathrm{gm/cm^3})^{-5/3} . \\ &= 5.38 \times 10^9 \,\mathrm{gm^{-2/3}\,cm^4\,sec^{-2}} . \end{aligned}$$

   In this problem, you will numerically integrate the TOV equations of stellar structure to build models of neutron stars in general relativity. Your goal will be to examine the total mass M_* and radius R_* as a function of central density $\rho_{0,c}$. As such, you *must* familiarize yourself with some system for numerically solving a system of ordinary differential equations. My personal experience has shown that the routine `NDSolve` built into the package MATHEMATICA is well suited to this problem. (An example of using `NDSolve` will be posted to the 8.962 website, to illustrate how it works.) More ambitious students with sufficient programming and numerical methods expertise may prefer to write their own code in `C` or `C++`, or whatever you are most comfortable with.

During your calculations, don't forget to take into account the difference between rest mass density ρ_0 and relativistic energy density ρ:

$$\rho = \rho_0 + \frac{P}{\Gamma - 1} = \rho_0 + \frac{K\rho_0^\Gamma}{\Gamma - 1} \; .$$

The equations you will integrate take the form

$$\begin{aligned} \frac{dm}{dr} &= 4\pi\rho r^2 \\ \frac{dP}{dr} &= -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)} \; . \end{aligned}$$

We have here written these equations in units in which both G and c are set equal to 1. The initial conditions you need to apply are

$$\begin{aligned} m(r = 0) &= 0 \, , \\ P(r = 0) &= P_c = P(\rho_{0,c}) = K\rho_{0,c}^\Gamma \; . \end{aligned}$$

One integrates these equations until the pressure drops to zero: $P(R_*) = 0$ defines the star's surface. (In fact, you will probably find in your numerical integration that MATHEMATICA or your code attempts to take an infinite number of infinitesimally small steps in the vicinity of R_*. The radius at which this numerical inaccuracy sets in will still define the surface of the star to very high accuracy.) The total mass of the star is then defined as $m(R_*) \equiv M_*$.

(a) Experience has shown that numerical calculations of this kind behave best when the units are chosen such that many key quantities are within an order of magnitude or so of unity. A good choice is to put $G = 1$ and $c = 1$; we then pick the units of all dimensionful quantities to be powers of kilometers. With this choice, the order of magnitude of the total mass will be roughly 1 kilometer, and the order of magnitude of the stellar radius will be roughly 10 kilometers.

Using this unit system, convert

(i) $\rho = 1\,\mathrm{gm/cm}^3$ to km^{-2}.

(ii) $P = 1\,\mathrm{dyne/cm}^2 = 1\,\mathrm{gm\,cm^{-1}\,sec^{-2}}$ to km^{-2}.

(iii) $K = 1\,\mathrm{gm}^{-2/3}\,\mathrm{cm}^4\,\mathrm{sec}^{-2}$ to $\mathrm{km}^{4/3}$. Use this conversion factor to convert the K that appears in the polytropic equation of state to $\mathrm{km}^{4/3}$.

(b) Pick a central density $\rho_{0,c} = 10^{15}\ \mathrm{gm/cm}^3$. Inserting appropriate conversion factors such that all quantities are in units of km (or powers of km), integrate the TOV equations to compute the radius R_* and mass M_* of the star that has this central density.

(c) If a photon is emitted radially with energy $E_{\rm em}$ from the surface of this star, what is the energy $E_{\rm ob}$ with which this photon is observed by distant ($r \to \infty$) observers? Using these energies, compute the surface redshift

$$z_{\rm surf} = \frac{E_{\rm ob} - E_{\rm em}}{E_{\rm ob}} \ .$$

(d) As described in lecture, the mass M_* is *not* what one would get by integrating all fluid density elements over the proper volume of the star's interior. Let us define M_p as the mass that would be obtained by this integration. Re-integrate the TOV equations for the central density 10^{15} gm/cm^3, but add the equation

$$\frac{dm_p}{dr} = 4\pi\rho r^2\sqrt{g_{rr}} = 4\pi\rho r^2\left(1 - \frac{2m(r)}{r}\right)^{-1/2} \ .$$

Compute the mass M_P, which is $m_p(R_*)$.

(e) The difference between M_p and M_* reflects the fact that the self-gravity of the star contributes to the star's mass. These differences can be regarded as the "gravitational binding energy" of the star. Compute

$$\Delta \equiv \frac{M_p - M_*}{M_*} \ .$$

Δ is the fraction of the star's mass that is due to this binding energy. What is this fraction for this choice of central density?

3. Stability of a TOV star

   By computing a range of TOV models, we can assess whether a star is stable against radial (i.e., purely spherical) perturbations. Detailed analysis shows that a stable star satisfies

$$\frac{dM}{d\rho_c} > 0 \; ;$$

   an unstable model satisfies

$$\frac{dM}{d\rho_c} < 0 \; .$$

   (Notice it is ρ_c that appears in this criterion, not $\rho_{0,c}$.) A detailed explanation and derivation of this stability criterion can be found in *Black holes, white dwarfs, and neutron stars: The physics of compact objects*, by S. L. Shapiro and S. A. Teukolsky, §6.8 - 6.9. An intuitive explanation goes as follows:

   Suppose we construct a TOV model, and then squeeze it, decreasing its radius by some small amount δR_*. Clearly, the mean density, and thus the central density, will be augmented, since we are decreasing the volume of the object. If the star resists the squeezing, its mass will also increase: It does work to resist the squeezing, work is energy, and mass and energy are equivalent. On the other hand, if the star does *not* resist the squeezing, its mass will decrease: A small squeeze puts it into an energetically more favorable smaller radius. This clearly leads to a runaway, causing the star to collapse into a black hole.

   (a) Compute a range of TOV models with central densities $\rho_{0,c} = 10^{14}, 10^{15}, 10^{16}, 10^{17}$ and 10^{18} gm/cm^3. Make a plot of M_* vs ρ_c for these models. Do you see any evidence for a change from stable to unstable behavior?

   (b) Zoom in on any unstable region you find and locate, as accurately as possible, the marginally stable star ($dM/d\rho_c = 0$). This star has the maximum possible mass for this equation of state in general relativity. What is this maximum mass? Convert your result from km to solar masses.