MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
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PROBLEM SET 7

Post date: Thursday, April 6th
Due date: Thursday, April 13th

1. Gravitomagnetism

   In lecture and working in Lorentz gauge, we examined the linearized Einstein field equations for a static source,

   $$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \quad \to \quad \nabla^2\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ ,$$

   where ∇^2 is the ordinary Euclidean 3-space Laplacian operator. For a static, non-relativistic source, the only non-zero stress-energy component is (to sufficient accuracy for our purposes)

   $$T_{00} = \rho \ .$$

   Using this, we found

   $$\bar{h}_{00} = -4\Phi \to h_{\mu\nu} = -2\Phi\,\mathrm{diag}(1,1,1,1) \ ,$$

   where $\Phi = -GM/r$ is the Newtonian gravitational potential.

   We will now modify this slightly by imagining that the source rotates, and thus is characterized by a spin angular momentum with spatial components S^i as well as a mass M.

   (a) Consider the source to be spherically symmetric, with uniform density ρ and radius R. Take it to be rotating rigidly about the $x^3 \equiv z$ axis with constant angular velocity Ω. Working in a Lorentz frame that is at rest with respect to the center of mass of the source, work out all components of the stress energy tensor $T_{\mu\nu}$ to first order in Ω. (Assume ρ, R, and Ω are constant.) Indicate which components would change if you included terms to second order in Ω, but don't calculate those second order corrections. (You may neglect pressure terms throughout your calculation.)

   (b) Solve for the Cartesian off-diagonal components h_{0x}, h_{0y}, h_{0z}. (Note that $h_{0i} = \bar{h}_{0i}$ since trace reversal has no effect on off-diagonal components.)

   This is a moderately challenging calculation. The following tips should help:

   - Recall that the formal solution to the Poisson-type equation for h_{0i} is

     $$h_{0i}(\mathbf{x}) = 4G\int \frac{T_{0i}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

     where $\mathbf{x}$ is the "field point", the location of the point at which h_{0i} is to be evaluated, and $\mathbf{x}'$ is the "source point", a coordinate within the source over which the integral is taken. [Boldface quantities denote 3-vectors: $\mathbf{x} \doteq (x, y, z)$.]

- The following expansion for the factor $1/|\mathbf{x}-\mathbf{x}'|$ is very useful:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{x^j x^{j'}}{r^3} + \dots$$

  You may assume this identity in your solution. Note also that a sum over j is implied here; we are allowed to be sloppy about the placement of indices since the spatial metric is δ_{ij} to leading order. [This identity is more often seen as an expansion in spherical harmonics; see, for example, J. D. Jackson, Sec. 3.6 (2nd edition). This form in terms of Cartesian coordinates is equivalent.]

- After you have set up your integral, convert the primed integration variable to spherical coordinates to do the integration:

$$\begin{aligned} x^{1'} = x' &\quad\rightarrow\quad r'\sin\theta'\cos\phi' \\ x^{2'} = y' &\quad\rightarrow\quad r'\sin\theta'\sin\phi' \\ x^{3'} = z' &\quad\rightarrow\quad r'\cos\theta' \end{aligned}$$

Your final metric components should be proportional to $\rho R^5/r^3$.

(c) Using the identity $S^i = I\Omega^i$ where I is moment of inertia and Ω^i is the ith component of the angular velocity vector, rewrite your answer in terms of the angular momentum S^i.

Although we derived this result for a special situation (uniform density, spherical body, rigid rotation), the result we obtain in terms of S^i is completely general; see, for example, MTW Sec. 19.1.

(d) Converting to spherical coordinates, find h_{0r}, $h_{0\theta}$, $h_{0\phi}$.

Hint: Only one of these components is non-zero. After changing coordinates, you should find that this non-zero component is $\propto S^z\sin^2\theta/r$.

2. Comparison of linearized GR and Maxwell's theory

Consider the line element

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) - 2\beta^i\,dx^i\,dt\ ;$$

in other words, the usual weak field line element on the diagonal with $h^{0i} = -\beta^i$.

(a) Show that the geodesic equation for a particle moving in this spacetime gives the following equation of motion to first order in the particle's velocity $\mathbf{v}$:

$$m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{g} + m(\mathbf{v}\times\mathbf{H})\ .$$

Here, $\mathbf{x}$ is a 3-vector representing the position of the particle, and

$$\begin{aligned} \mathbf{g} &= -\boldsymbol{\nabla}\Phi\ , \\ \mathbf{H} &= \boldsymbol{\nabla}\times\boldsymbol{\beta}\ , \end{aligned}$$

where $\boldsymbol{\nabla}$ represents the ordinary gradient operator in Euclidean 3-space.

(b) Show that for stationary sources (i.e., no component of the stress energy tensor shows time variation) the Einstein field equations may be written

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{g} &= -4\pi G\rho \,, \\ \boldsymbol{\nabla} \times \boldsymbol{H} &= -16\pi G \boldsymbol{J} \\ \boldsymbol{\nabla} \cdot \boldsymbol{H} &= 0 \,, \\ \boldsymbol{\nabla} \times \boldsymbol{g} &= 0 \,. \end{aligned}$$

The current $\mathbf{J} = \rho\mathbf{v}$, where $\mathbf{v}$ is the velocity of fluid flow in the source. (Note that the second two equations follow from the definitions of $\mathbf{g}$ and $\mathbf{H}$, so the only labor is in working out the first two.)

(c) These equations clearly bear a strong resemblance to Maxwell's equations in the limit $\partial_t\mathbf{E} = \partial_t\mathbf{B} = 0$; the main differences are the reversed sign in both equations, and the extra factor of 4 (compared to Maxwell) in the curl equation. Can you give a simple explanation for these differences?

3. Carroll: Chapter 7, Problem 1.
4. Carroll: Chapter 7, Problem 3.
5. Carroll: Chapter 7, Problem 4.