ON THE CALCULUS OF SYMBOLS FOR PSEUDO-DIFFERENTIAL OPERATORS

by

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ABSTRACT

Let M be a smooth paracompact manifold, let E and F be smooth complex vector bundles over M, and let $\mathcal{P}(E,F)$ be the module of pseudo-differential operators $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ in the sense of Hörmander, i.e. P is $\mathbb{C}$-linear and continuous and there exists a sequence of complex numbers $(z_k)_{k=0,1,\ldots}$ with $\mathrm{Re}\, z_{k+1} \leq \mathrm{Re}\, z_k$ and $\lim\sup \mathrm{Re}\, z_k = -\infty$, such that if $s \in \Gamma_c^\infty(E)$ then there is an asymptotic expansion

$$e^{-i\lambda g} P(e^{i\lambda g} s) \sim \sum_{k=0}^{\infty} P_k(s,g)\, \lambda^{z_k} \qquad (\lambda \to +\infty)$$

in $\Gamma^\infty(F)$, uniformly for g in compact subsets K of $C^\infty(M:\mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp s. If $n_k = [\mathrm{Re}(z_0 - z_k)]$ it may be shown that $P_k(s,g)$ depends only on the n_k-jets of s and dg, and hence relative to the splittings of the exact jet bundle sequences for E and 1 induced by a covariant derivative D on E and a covariant derivative ∇ on T^* the 'local parts' P_k of the operator P induce positively homogeneous smooth fibre preserving maps τ_k of degree z_k of $T^*-(0)$ into Hom(E,F). The corresponding formal sum $\Sigma\, \tau_k$ is called the formal symbol of P relative to the pair (D,∇). It is shown that the mapping which takes $P \in \mathcal{P}(E,F)$ to its formal symbol relative to (D,∇) is onto the module $\mathcal{F}(E,F)$ of such formal sums and has kernel $P_{-\infty}(E,F)$, the module of operators that have smooth distribution-kernels. Coordinate free formulas for the formal symbols of the composition of pseudo-differential operators (and hence for the parametrix of an elliptic operator) and for the transpose of a pseudo-differential operator are found. Also a coordinate free expression for the local parts P_k is given in terms of the τ_k.

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TABLE OF CONTENTS

§1 Introduction

In [8] L. Hörmander considers a class of pseudo-differential operators defined directly on a manifold by means of certain asymptotic series which are analogous to the total characteristic polynomial associated to a differential operator. Among other things, he obtains an intricate calculus, relative to coordinates, for the coefficients (symbols) of these series under natural operations on the pseudo-differential operators (eg. composition, inverse, and transpose). The definition of pseudo-differential operators may be extended to operators acting on C^∞ sections of vector bundles and the same results obtain. ([1,19]).

Our purpose here is to obtain a coordinate free description of the calculus of the symbols. Since it is not obvious how the symbols of pseudo-differential operators transform under coordinate changes we begin by considering the lower order symbols of differential operators (§2) and obtain their transformation law (theorem 2.5). Motivated by these results we introduce a class of objects (generalized symbols, §3 and §4) which we prove includes the symbols of pseudo-differential operators (§9). In coordinates Hörmander obtains expressions for the symbols in terms of certain simpler objects (which we call the singular parts) and in §9 we obtain similar expressions globally in terms of a pair of covariant derivatives. In §9 we prove a global composition formula for these singular parts, and in §11 we prove a global transpose formula.

§5 is devoted to defining the pseudo-differential operators that we consider, and to proving a few facts. There is nothing really new but proofs have been given to show that considering operators with symbols that allow complex degrees of homogenuity does not materially alter the standard arguments.

In §8 we introduce asymptotic sums of pseudo-differential operators. The name asymptotic sum was chosen since the proof of existence of an asymptotic sum depends on Hörmander's main existence theorem [8], which in turn is obtained essentially by the classical summation of an asymptotic series (see A. Erdélyi [5, §1.7]). In §12 we make use of an asymptotic sum to prove the existence of a parametrix for an elliptic operator. Essentially the same notion of an asymptotic sum as here is used in J. J. Kohn and L. Nirenberg [10].

There is a recent paper by W. Shih [18] similar to some parts of this work. Shih considers pseudo-differential operators over a compact manifold and obtains a description of the generalized symbols as in theorem 9.2 here and a description of the composition as in theorem 9.3. He also points out how to obtain the singular parts of the generalized symbols relative to a pair of covariant derivatives, but does not give the inverse operation (theorem 9.4). The composition formula for the singular parts (theorem 9.5), the isomorphism of theorem 9.9, and the transpose formula (theorem 11.5) are apparently still new.

The treatment of differential operators in terms of vector bundles is given in R. S. Palais [13] and D. G. Quillen [14].

The bibliography contains only references mentioned in the text. A very extensive bibliography of pseudo-differential operators and related topics may be found in the survey paper, A. P. Calderón [3], and also in L. Hörmander [9].

M will denote a paracompact Hausdorff C^∞ manifold of dimension m, T^* will be the cotangent bundle of M, and E, F and G will be complex or occasionally real C^∞ vector bundles on M. $\Gamma^\infty(E)$ will denote the C^∞ sections of E, and $\Gamma_c^\infty(E)$ is the submodule of $\Gamma^\infty(E)$ consisting of compactly supported sections, and these spaces are equipped with the standard topologies. $J^n(E)$ will denote the n^{th} jet bundle of E (see [13]) and $j_n : \Gamma^\infty(E) \to \Gamma^\infty(J^n(E))$ will be the n^{th} jet extension map. Finally $Diff_n(E,F)$ will be the module of smooth differential operators from E to F of order $\leq n$, so we have a natural isomorphism

$$Diff_n(E,F) = \Gamma^\infty Hom(J^n(E), F).$$

§2. Lower Order Symbols of Differential Operators

In this section we introduce certain subbundles of the jet bundles of the cotangent bundle and obtain a description of the lower order symbols of a differential operator in terms of these bundles.

We begin by considering real vector bundles. In particular $1 = M \times \mathbb{R}$. Let $d : \Gamma^{\infty}(1) \to \Gamma^{\infty}(T^*)$ be the exterior derivative and for each integer $n \geq 0$, and each $x \in M$ let $Z_x^n = \{ f \in C^{\infty}(M;\mathbb{R}) : f$ vanishes of order $\geq n$ at $x \}$.

Since $j_n d \in \mathrm{Diff}_{n+1}(1, J^n(T^*))$ there exists a unique morphism $\psi_n : J^{n+1}(1) \to J^n(T^*)$ such that

$$\psi_n j_{n+1} = j_n d$$

Let $\iota : 1 \to J^{n+1}(1)$ be the canonical inclusion (induced by the constant sections). Then if $g \in C^{\infty}(M;\mathbb{R})$ we have

$$\psi_n \iota(g) = \psi_n(g\, j_{n+1}(1)) = g\, j_n d(1) = 0$$

since d annihilates constants. Thus $\psi_n \iota = 0$. Suppose now that $v \in J^{n+1}(1)_x$ and $\psi_n(v) = 0$. Choose $f \in C^{\infty}(M;\mathbb{R})$ such that $v = j_{n+1}(f)(x)$. Then

$$j_n(df)(x) = \psi_n\, j_{n+1}(f)(x) = \psi_n(v) = 0$$

and hence $df \in Z_x^{n+1} \cdot \Gamma^{\infty}(T^*)$ which implies $f - f(x) \in Z_x^{n+2}$. Thus $j_{n+1}(f - f(x))(x) = 0$ and so

$$v = j_{n+1}(f)(x) = f(x) j_{n+1}(1)(x) = \iota f(x)$$

which implies $\ker \psi_n \subseteq \operatorname{im} \iota$. It follows that the sequence

$$0 \to 1 \xrightarrow{\iota} J^{n+1}(1) \xrightarrow{\psi_n} J^n(T^*)$$

is exact and therefore the image Ψ^n of ψ_n is a C^∞ vector subbundle of $J^n(T^*)$. Thus we have a short exact sequence

$$0 \to 1 \xrightarrow{\iota} J^{n+1}(1) \xrightarrow{\psi_n} \Psi^n \longrightarrow 0$$

It is clear that this sequence is split by the canonical projection $J^{n+1}(1) \to 1$. Also note since $\psi_0 j_1 = d$, ψ_0 maps onto T^* and hence $\Psi^0 = T^*$.

$(\Psi^n)^*$ may be identified with the bundle $J_{n+1}(M)$, where $J_{n+1}(M)$ is the subbundle of $\operatorname{Hom}(J^{n+1}(1),1)$ whose C^∞ sections are the differential operators in $\operatorname{Diff}_{n+1}(1,1)$ which annihilate constants. (see J.T. Schwartz [15] for properties of $J_{n+1}(M)$). In particular $J_1(M) = T$ and $J_0(M) = 0$. Hence we make the convention that $\Psi^{-1} = 0$.

<u>lemma 2.1</u>

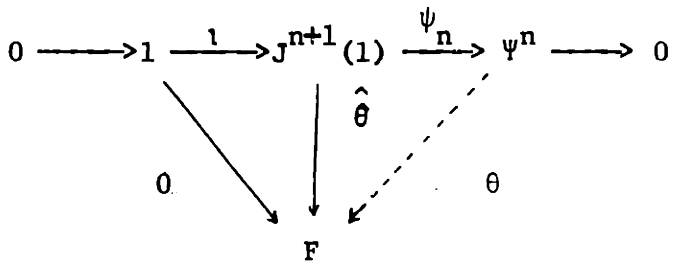
Given any $\mathbb{R}$-linear map $D : \Gamma^\infty(1) \to \Gamma^\infty(F)$ such that $j_n(df)(x) = 0$ implies $(Df)(x) = 0$, then there exists a unique morphism

$$\theta : \Psi^n \to F$$

such that

$$D(g) = \theta\, \psi_n\, j_{n+1}(g) = \theta(j_n dg) \quad , \quad g \in C^\infty(M:\mathbb{R})$$

<u>Proof</u>: The hypotheses clearly imply $D \in \mathrm{Diff}_{n+1}(1,F)$ and hence there is a unique morphism $\hat{\theta} : J^{n+1}(1) \to F$ such that $D(g) = \hat{\theta}\, j_{n+1}(g)$, $g \in C^{\infty}(M;\mathbb{R})$. Now if $g \in C^{\infty}(M;\mathbb{R})$ we have $\hat{\theta}\iota(g) = \hat{\theta}(g\, j_{n+1}(1) = g\, D(1) = 0$ and so we have a commutative diagram



Since the row is exact $\hat{\theta}$ factors uniquely through ψ_n and so induces the desired morphism θ.

The hypotheses on D in lemma 2.1 may be stated in the equivalent form, $D \in \mathrm{Diff}_{n+1}(1,F)$ and D annihilates constants. For $n = 0$, the lemma is the standard universal property of d and T^*.

In terms of the bundles $J_{n+1}(M)$ lemma 2.1 asserts that $J_{n+1}(M) \otimes F$ is canonically isomorphic to the subbundle of $\mathrm{Hom}(J^{n+1}(1),F)$ whose C^{∞} sections are the differential operators which annihilate constants.

<u>lemma 2.2</u>

Given any k-multi-$\mathbb{R}$-linear map $D : \Gamma^{\infty}(1) \times \ldots \times \Gamma^{\infty}(1) \to \Gamma^{\infty}(F)$ such that if $g_1, \ldots, g_k \in C^{\infty}(M;\mathbb{R})$ and for some i, $1 \le i \le k$, $j_n(dg_i)(x) = 0$, then $D(g_1, \ldots, g_k)(x) = 0$, then there exists a unique morphism

$$\theta : \bigotimes^k \Psi^n \to F$$

such that

$$D(f_1,\dots,f_k) = \theta(j_n(df_1)\otimes\cdots\otimes j_n(df_k))$$

for any $f_1,\dots,f_k \in C^\infty(M;\mathbb{R})$.

<u>Proof</u>: If we fix $f_1,\dots,f_{k-1}$ and define $P_k : \Gamma^\infty(1) \to \Gamma^\infty(F)$ by $P_k(f) = D(f_1,\dots,f_{k-1},f)$, $f \in C^\infty(M;\mathbb{R})$, then by hypothesis and by lemma 2.1 there exists a unique morphism

$$\theta_{k-1}(f_1,\dots,f_{k-1}) : \Psi^n \to F$$

such that $P_k(f) = \theta_{k-1}(f_1,\dots,f_{k-1}) \cdot j_n(f)$. Then

$$\theta_{k-1} : \Gamma^\infty(1)_{\times\dots\times}\,\Gamma^\infty(1) \to \Gamma^\infty(\mathrm{Hom}(\Psi^n,F))$$

satisfies the hypotheses of the lemma and is (k-1)-multi-linear. Hence by induction we obtain

$$\theta_1 : \Psi^n \to \mathrm{Hom}(\Psi^n, \mathrm{Hom}(\dots \mathrm{Hom}(\Psi^n,F)\dots))$$

such that

$$\theta_1 \circ j_n(df_1)\cdot j_n(df_2)\dots j_n(df_k) = D(f_1,\dots,f_k)$$

Then θ_1 induces the desired morphism θ.

<u>lemma 2.3</u>

There exists a unique morphism

$$\upsilon_n^k : S^k\Psi^n \otimes J^n(E) \to J^{n+k}(E)$$

such that if $g_1,\dots,g_k \in C^\infty(M;\mathbb{R})$, $s \in \Gamma^\infty(E)$ then

$$\upsilon_n^k \left(j_n(dg_1) \odot \cdots \odot j_n(dg_k) \otimes j_n(s) \right)(x)$$

$$= \frac{1}{k!} j_{n+k}\left((g_1 - g_1(x)) \ldots (g_k - g_k(x))\, s \right)(x)$$

for each $x \in M$. Moreover the sequence

$$S^k \Psi^n \otimes J^n(E) \xrightarrow{\upsilon_n^k} J^{n+k}(E) \xrightarrow{\pi} J^{k-1}(E) \longrightarrow 0$$

is exact, where π is the natural projection.

<u>Proof</u>: If $g_1, \ldots, g_k \in C^\infty(M;\mathbb{R})$ define

$$P(g_1, \ldots, g_k) : \Gamma^\infty(E) \to \Gamma^\infty(J^{n+k}(E))$$

by $P(g_1, \ldots, g_k)(s)(x) = \frac{1}{k!} j_{n+k}\left((g_1 - g_1(x)) \ldots (g_k - g_k(x)) s \right)(x)$.

If $s \in Z_x^{n+1} \cdot \Gamma^\infty(E)$ then $(g_1 - g_1(x)) \ldots (g_k - g_k(x)) s \in Z_x^{n+k+1} \cdot \Gamma^\infty(E)$ and so $P(g_1, \ldots, g_k)(s)(x) = 0$. Thus $P(g_1, \ldots, g_k) \in \mathrm{Diff}_n(E, J^{n+k}(E))$ and hence there is a unique morphism

$$D(g_1, \ldots, g_k) : J^n(E) \to J^{n+k}(E)$$

such that

$$P(g_1, \ldots, g_k)(s) = D(g_1, \ldots, g_k) \cdot j_n(s) \quad , \; s \in \Gamma^\infty(E).$$

Now $D : \Gamma^\infty(1) \times \ldots \times \Gamma^\infty(1) \to \Gamma^\infty \operatorname{Hom}(J^n(E), J^{n+k}(E))$ satisfies the hypotheses of lemma 2.2, since if for some i, $1 \le i \le k$, $j_n(dg_i)(x) = 0$ then $dg_i \in Z_x^{n+1} \cdot \Gamma^\infty(T^*)$ whence $g_i - g_i(x) \in Z_x^{n+2}$ and so $(g_1 - g_1(x)) \ldots (g_k - g_k(x)) \in Z_x^{n+k+1}$ which implies $D(g_1, \ldots, g_k)(x) = 0$, by construction of D. Hence by lemma 2.2 we have a unique morphism

$$\hat{\upsilon}_n^k : \bigotimes^k \Psi^n \to \mathrm{Hom}(J^n(E), J^{n+k}(E))$$

such that

$$D(g_1,\dots,g_k) = \hat{\upsilon}_n^k (j_n(dg_1) \otimes \dots \otimes j_n(dg_k))$$

But clearly P and hence $\hat{\upsilon}_n^k$ is symmetric in the g's and so induces υ_n^k with the required properties. Now if $g_1,\dots g_k \in C^\infty(M;\mathbb{R})$, $s \in \Gamma^\infty(E)$ then

$$\pi\, \upsilon_n^k (j_n(dg_1) \odot \dots \odot j_n(dg_k) \otimes j_n(s))(x)$$

$$= \frac{1}{k!} j_{k-1}((g_1-g_1(x))\dots(g_k-g_k(x))s)(x)$$

$$= 0$$

and so $\pi\, \upsilon_n^k = 0$.

Suppose $v \in J^{n+k}(E)_x$ and $\pi v = 0$. Let $s \in \Gamma^\infty(E)$ be such that $j_{n+k}(s)(x) = v$. Then $j_{k-1}(s)(x) = \pi v = 0$, i.e. $s \in Z_x^k \cdot \Gamma^\infty(E)$. Thus there exist functions $g_1^i,\dots,g_k^i$ in Z_x and sections $s_i \in \Gamma^\infty(E)$ such that

$$s = \sum_{i=1}^N g_1^i,\dots,g_k^i\, s_i$$

Then $v = j_{n+k}(s)(x) = \sum_{i=1}^N j_{n+k}(g_1^i \dots g_k^i\, s^i)(x)$

$$= k! \sum_{i=1}^N \upsilon_n^k(j_n(dg_1^i) \odot \dots \odot j_n(dg_k^i) \otimes j_n(s_i))(x)$$

Thus $\ker \pi \subseteq \mathrm{im}\, \upsilon_n^k$ and so we have exactness, since π is known to be surjective.

For $n = 0$ the sequence in lemma 2.3 is a short exact sequence, the jet bundle sequence

$$0 \to S^k T^* \otimes E \xrightarrow{\upsilon_0^k} J^k(E) \longrightarrow J^{k-1}(E) \to 0$$

but in general υ_n^k is not injective.

When E is complex then $J^n(E)$ is the complex bundle associated to J^n(E, regarded as real) by the complex structure induced by multiplication by i on E, and $S^k\,\Psi^n \otimes J^n(E)$ is the complex bundle associated to $S^k\,\Psi^n \otimes$ ($J^n(E)$ regarded as real) by the complex structure induced by multiplication by i on $J^n(E)$. In this case the morphism υ_n^k in lemma 2.3 is $\mathbb{C}$-linear relative to these natural complex structures. Hence lemma 2.3 holds in the complex case also.

It will be convenient to regard υ_n^k as a C^∞ section of the bundle of homogeneous polynomial maps of degree k

$$HP^k\,(\,\Psi^n,\ \mathrm{Hom}\,(\,J^n(E),\ J^{n+k}(E)\,)\,)$$

We pass from one interpretation to the other without comment.

<u>lemma 2.4</u>

For $n \geq 0$ we have a natural short exact sequence

$$0 \to S^{n+1}\,T^* \xrightarrow{i_n} \Psi^n \xrightarrow{\omega_n} \Psi^{n-1} \to 0$$

<u>**Proof:**</u> By the definition of Ψ^n and by the $(n+1)^{st}$ exact jet bundle sequence of 1 we have a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \longrightarrow & S^{n+1}\,T^* & = & S^{n+1}\,T^* & \rightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow \iota_{n+1} & & \vdots\; i_n & & \\
0 & \rightarrow & 1 & \longrightarrow & J^{n+1}(1) & \xrightarrow{\psi_n} & \Psi^n & \rightarrow & 0 \\
 & & \Big\| & & \Big\downarrow \pi_{n+1} & & \vdots\; \omega_n & & \\
0 & \rightarrow & 1 & \longrightarrow & J^n(1) & \xrightarrow{\psi_{n-1}} & \Psi^{n-1} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By the nine lemma (applied fibrewise) there are induced morphisms at the dotted arrows such that the diagram remains commutative and moreover the resulting column is exact.

Since M is paracompact all short exact sequences of vector bundles over M are split, and hence additive functors of vector bundles preserve short exact sequences. In particular applying the functor Hom $(\cdot\ , 1)$ to the short exact sequence in lemma 2.4 there results a short exact sequence

$$0 \rightarrow J_n(M) \longrightarrow J_{n+1}(M) \longrightarrow S^{n+1}\ T \rightarrow 0$$

The first morphism in this sequence is the natural inclusion, and the second is essentially the symbol, (sections of $J_{n+1}(M)$ are differential operators).

Now let E and F be complex. Then we have the following description of the symbols of a differential operator from E to F.

Theorem 2.5

If $P \in \mathrm{Diff}_n(E,F)$ then for $0 \le k \le n$ there exist unique homogeneous polynomial maps

$$\sigma_k(P) \in \Gamma^\infty(HP^{n-k}(\Psi^k, \mathrm{Hom}(J^k(E),F)))$$

such that if $g \in C^\infty(M:\mathbb{R})$, $s \in \Gamma^\infty(E)$ then

$$e^{-i\lambda g} P(e^{i\lambda g} s) = \sum_{k=0}^{n} \lambda^{n-k} \sigma_k(P) \cdot j_k(dg) \cdot j_k(s) \qquad (\lambda \in \mathbb{R})$$

In particular $\sigma_0(P)$ is the symbol of P in the usual sense. ($\sigma_0(P)$ is the symbol given in R. S. Palais [13] multiplied by i^n.)

Proof: For $x \in M$, $\lambda \in \mathbb{R}$

$$e^{i\lambda(g-g(x))} = \sum_{k=0}^{n} \frac{(i\lambda)^{n-k}}{(n-k)!} (g-g(x))^{n-k} + h_{x,\lambda}$$

where $h_{x,\lambda} \in Z_x^{n+1}$. Thus

$$e^{-i\lambda g(x)} P(e^{i\lambda g} s)(x) = \sum_{k=0}^{n} \lambda^{n-k} \frac{i^{n-k}}{(n-k)!} P((g-g(x))^{n-k} s)(x)$$

Now let $\theta: J^n(E) \to F$ in the unique morphism such that $P = \theta j_n$. Then by lemma 2.3

$$\frac{i^{n-k}}{(n-k)!} P((g-g(x))^{n-k} s)(x) = i^{n-k} \theta \upsilon_k^{n-k} (\delta^{n-k} j_k(dg) \otimes j_k(s))(x)$$

where

$$\delta^{n-k} dg = dg \odot \ldots \odot dg \qquad (n-k \text{ times})$$

We define $\sigma_k(P) \cdot j_k(dg) \cdot j_k(s) = i^{n-k}\,\theta\upsilon_k^{n-k}\,(\delta^{n-k}\,j_k(dg) \otimes j_k(s))$.
Then uniqueness is clear and the last statement follows from the fact that

$$\sigma_0(P) \cdot dg(x) \cdot s(x) = \frac{i^n}{n!}\,P(\,(g-g(x))^n s\,)(x)\ .$$

In order for our notation in theorem 2.5 to be unambiguous it is convenient to regard a differential operator as being a pair (n,P) where n is the order assigned to P.

Remark 2.6

If $P \in \mathrm{Diff}_n(E,F)$ and $0 \leq k \leq n$ then $P \in \mathrm{Diff}_{k-1}(E,F)$ if and only if $\sigma_{n-k}(P) = 0$. This fact is an immediate consequence of the construction of $\sigma_{n-k}(P)$ and the exact sequence of lemma 2.3.

We now consider covariant derivatives on vector bundles and state a few properties which will be useful later. Proofs not given here may be found in R. S. Palais [13]. Other discussions of covariant derivatives are J. Koszul [11] and E. Nelson [12].

A covariant derivative D on E may be described as an element of $\mathrm{Diff}_1(E, T^*\otimes E)$ such that the associated morphism $J^1(E) \to T^*\otimes E$ is a splitting of the exact jet bundle sequence

$$0 \longrightarrow T^*\otimes E \longrightarrow J^1(E) \longrightarrow E \longrightarrow 0$$

Let ∇ be a covariant derivative on T^* and let D be a covariant derivative on E. Then the pair (D,∇) induces in a canonical way, [13], k^{th}

order differential operators

$$D^{(k)} : \Gamma^{\infty}(E) \to \Gamma^{\infty}(S^k\, T^*_{\otimes}E) \quad , \quad k = 0,1,2\ldots$$

which are k^{th} total differentials in the sense that the associated morphism

$$\mu_k : J^k(E) \to S^k\, T^*_{\otimes}E$$

is a splitting of the exact jet bundle sequence

$$0 \longrightarrow S^k\, T^*_{\otimes}E \xrightarrow{\iota_k} J^k(E) \xrightarrow{\pi_k} J^{k-1}(E) \longrightarrow 0$$

Let $\nu_k : J^{k-1}(E) \to J^k(E)$ be the injection associated to this splitting. Then we obtain a direct sum decomposition

$$J^n(E) = \bigoplus_{k=0}^{n} S^k\, T^*_{\otimes}E$$

with injections $\nu_n \ldots \nu_{k+1}\, \iota_{k+1} : S^k\, T^*_{\otimes}E \to J^n(E)$ and

projections $\mu_k\, \pi_{k+1} \ldots \pi_n : J^n(E) \to S^k\, T^*_{\otimes}E$

and relative to this direct sum decomposition $j_n : \Gamma^{\infty}(E) \to \Gamma^{\infty}(J^n(E))$ is given by $(D^{(0)}, D^{(1)}, \ldots, D^{(n)})$.

The exterior derivative $d : \Gamma^{\infty}(1) \to \Gamma^{\infty}(T^*)$ is a covariant derivative on 1 and hence the pair (d,∇) induces k^{th} total differentials

$$\partial^{(k)} : \Gamma^{\infty}(1) \to \Gamma^{\infty}(S^k\, T^*)$$

where $\partial^{(1)} = d$ and $\partial^{(2)} = \mathcal{S}^2\nabla d$ ($\mathcal{S}^2$ is symmetrization). Let

$$u_k : J^k(1) \to S^k T^*$$

$$\nu_k : J^{k-1}(1) \to J^k(1)$$

be the morphisms corresponding to μ_k and ν_k, respectively.

The total differentials $D^{(k)}$ induced by the pair (D,∇) and the total differentials $\partial^{(k)}$ induced by the pair (d,∇) are related by the following "Leibnitz rule".

Theorem 2.7

If $g \in C^\infty(M;\mathbb{R})$ and $s \in \Gamma^\infty(E)$ then

$$D^{(n)}(gs) = \sum_{k=0}^{n} \binom{n}{k} \partial^{(k)} g \odot D^{(n-k)} s$$

Proof: The proof is a simple induction using directly the construction of the $D^{(n)}$ and $\partial^{(n)}$ as given in [13].

Corollary 2.8

If $g_1,\ldots,g_k \in C^\infty(M;\mathbb{R})$ then

$$\partial^{(n)}(g_1\ldots g_k) = \sum \frac{n!}{i_1!\ldots i_k!} \partial^{(i_1)} g_1 \odot\cdots\odot \partial^{(i_k)} g_k$$

where we sum over $i_1+\ldots+i_k = n$. In particular if $\phi \in C^\infty(M;\mathbb{R}^m)$ then

$$\partial^{(n)}(\phi^\alpha) = \sum_{|\beta|=n} \frac{n!}{\beta!} \partial^{(\beta_1)} \phi_1^{\alpha_1} \odot\cdots\odot \partial^{(\beta_m)} \phi_m^{\alpha_m}$$

where α,β are m-multi-indexes, $\phi^\alpha = \phi_1^{\alpha_1}\cdots\phi_m^{\alpha_m}$

lemma 2.9

There exist unique morphisms

$$\dot{u}_k : \Psi^k \to S^{k+1} T^*$$

$$\dot{v}_k : \Psi^{k-1} \to \Psi^k$$

such that

$$\dot{u}_k \psi_k = u_{k+1}$$

$$\dot{v}_k \psi_{k-1} = \psi_k v_{k+1}$$

Moreover, $\dot{u}_k$ and $\dot{v}_k$ are the morphisms associated to a splitting of the short exact sequence

$$0 \longrightarrow S^{k+1} T^* \xrightarrow{\dot{\iota}_k} \Psi^k \xrightarrow{\omega_k} \Psi^{k-1} \longrightarrow 0$$

Proof: Uniqueness is clear since ψ_k and ψ_{k-1} are surjective. Recall that the short exact sequence

$$0 \longrightarrow 1 \longrightarrow J^{k+1}(1) \longrightarrow \Psi^k \longrightarrow 0$$

is split by the canonical projection $\pi : J^{k+1}(1) \to 1$. If we let $\rho : \Psi^k \to J^{k+1}(1)$ be the associated injection then an easy computation using the fact that $\partial^{(k+1)}$ annihilates constants shows that we may define $\dot{u}_k = u_{k+1} \rho$. Then $\dot{u}_k$ is a splitting as required and we let $\dot{v}_k : \Psi^{k-1} \to \Psi^k$ be the corresponding injection. An easy computation shows that $\dot{v}_k \psi_{k-1} \pi_{k+1} = \psi_k v_{k+1} \pi_{k+1}$ whence the lemma follows since $\pi_{k+1} : J^{k+1}(1) \to J^k(1)$ is surjective.

It follows that we have a direct sum decomposition

$$\Psi^n = \bigoplus_{k=0}^{n} S^{k+1} T^*$$

with injections $\dot{\upsilon}_n \dots \dot{\upsilon}_{k+1} \, i_k : S^{k+1} T^* \to \Psi^n$

and projections $\dot{u}_k \, \omega_{k+1} \dots \omega_n : \Psi^n \to S^{k+1} T^*$

and from the commutativity relations of lemma 2.9 we may verify that

$$j_n d : \Gamma^\infty(1) \to \Gamma^\infty(\Psi^n)$$

relative to this direct sum decomposition is given by $(\partial^{(1)},\dots,\partial^{(n+1)})$.

<u>lemma 2.10</u>

Let $n > 0$ be an integer and let $s \in \Gamma^\infty(E)$. For each $x \in M$ let U_x be an open neighborhood of x. Then for each $x \in M$ we can find $s_x \in \Gamma^\infty_c(E|_{U_x})$ such that

(1) $s_x(x) = s(x)$

(2) $(D^{(k)} s_x)(x) = 0 \;, \; 1 \le k \le n$

(3) $x \to j_n(s_x)(x)$ is a C^∞ section of $J^n(E)$.

<u>Proof</u>: Let $h(x) = \nu_n \dots \nu_1 \, s(x)$, so $h \in \Gamma^\infty(J^n(E))$ and for each $x \in M$ choose $s_x \in \Gamma^\infty_c(E|_{U_x})$ such that $j_n(s_x)(x) = h(x)$. Then (3) clearly holds, and (1) holds since $s_x(x) = \pi_1 \dots \pi_n j_n(s_x)(x) = \pi_1 \dots \pi_n \, \nu_n \dots \nu_1 \, s(x) = s(x)$. Finally if $1 \le k \le n$ then

$$
\begin{aligned}
(D^{(k)} s_x)(x) &= \mu_k \, j_k(s_x)(x) \\
&= \mu_k \, \pi_{k+1} \cdots \pi_n \, j_n(s_x)(x) \\
&= \mu_k \, \pi_{k+1} \cdots \pi_n \, \nu_n \cdots \nu_1 \, s(x) \\
&= \mu_k \, \nu_k \cdots \nu_1 \, s(x) = 0 .
\end{aligned}
$$

Hence (2) also holds.

Corollary 2.11

If $x \in M$, U is an open neighborhood of x, $n > 0$ is an integer, then there exists an open neighborhood V of x such that $\bar{V} \subseteq U$, and sections $s_1 \ldots s_p \in \Gamma_c^\infty(E|_U)$ such that $s_1, \ldots s_p$ is a local frame for E over V, and such that

$$(D^{(k)} s_j)(x) = 0 , \quad 1 \leq k \leq n , \quad 1 \leq j \leq p .$$

lemma 2.12

Let $n > 0$ be an integer and let $g \in C^\infty(M;\mathbb{R})$. Then for each $x \in M$ we can find $g_x \in C^\infty(M;\mathbb{R})$ such that

(1) $g_x(x) = 0$

(2) $dg_x(x) = dg(x)$

(3) $(\partial^{(k)} g_x)(x) = 0 \quad$ if $2 \leq k \leq n+1$

(4) $x \to j_n(dg_x)(x) \quad$ is a C^∞ section of Ψ^n .

__Proof:__ Let $h(x) = \nu_{n+1} \dots \nu_2 \iota_1 \, dg(x)$ so $h \in \Gamma^{\infty}(J^{n+1}(1))$ and for each $x \in M$ choose $g_x \in C^{\infty}(M;\mathbb{R})$ such that $j_{n+1}(g_x)(x) = h(x)$. Then $j_n(dg_x)(x) = \psi_n \, j_{n+1}(g_x)(x) = (\psi_n \, h)(x)$ which implies (4) since $\psi_n \, h \in \Gamma^{\infty}(\Psi^n)$.

If $1 \le k \le n+1$ then we have

$$
\begin{aligned}
(\partial^{(k)} g_x)(x) &= u_k \, j_k(g_x)(x) \\
&= u_k \, \pi_{k+1} \dots \pi_{n+1} \, j_{n+1}(g_x)(x) \\
&= u_k \, \pi_{k+1} \dots \pi_{n+1} \, \nu_{n+1} \dots \nu_2 \, \iota_1 \, dg(x) \\
&= u_k \, \nu_k \dots \nu_2 \, \iota_1 \, dg(x) \\
&= \begin{cases} 0 & \text{if } k \ge 2 \\ dg(x) & \text{if } k = 1 \quad \text{since } \partial^{(1)} = d \end{cases}
\end{aligned}
$$

Thus (2) and (3) follow. Finally for (1) we have

$$
\begin{aligned}
g_x(x) &= \pi_1 \dots \pi_{n+1} \, j_{n+1}(g_x)(x) \\
&= \pi_1 \dots \pi_{n+1} \, \nu_{n+1} \dots \nu_2 \, \iota_1 \, dg(x) \\
&= \pi_1 \, \iota_1 \, dg(x) \quad = \quad 0
\end{aligned}
$$

__Corollary 2.13__

If $n > 0$ is an integer $g \in C^{\infty}(M;\mathbb{R})$ then for each $x \in M$ we can find $h_x \in C^{\infty}(M;\mathbb{R})$ such that

(1) $h_x \in Z_x^2$

(2) $(\partial^{(k)} h_x)(x) = (\partial^{(k)} g)(x)$, $2 \le k \le n+1$

(3) $x \to j_n(dh_x)(x)$ is a C^{∞} section of Ψ^n

<u>Proof</u>: Choose g_x as in lemma 2.12 and then define $h_x = g - g(x) - g_x$. Then (1) and (3) are clear and (2) follows from the fact that $\partial^{(k)}$ annihilates constants if $k \geq 1$.

<u>Corollary 2.14</u>

If $n > 0$ is an integer and $x \in M$ we can find a chart (U,ϕ) on M with $x \in U$ such that $\phi(x) = 0$ and

$$(\partial^{(k)} \phi_j)(x) = 0 \quad , \quad 2 \leq k \leq n+1 \quad , \quad 1 \leq j \leq m.$$

Now let (U,ϕ) be a chart on M. If α is a multi-index we define

$$(d\phi)^{\alpha} = d\phi_1 \odot \cdots \odot d\phi_1 \odot \cdots \odot d\phi_m \odot \cdots \odot d\phi_m$$

where $d\phi_j$ appears α_j times, $j = 1,.,m$. Then $(d\phi)^{\alpha}$, $|\alpha| = n$, is a local frame for $S^n T^*$ over U. The dual frame of $S^n T$ over U is denoted by $(\frac{\partial}{\partial\phi})^{\alpha}$, $|\alpha| = n$, and these sections may be interpreted in the usual way as differential operators.

Having fixed the notation we can now describe $\partial^{(n)}$ and $D^{(n)}$ in coordinates.

<u>Theorem 2.15</u>

Let $N > 0$ be an integer and let $x \in M$. Let (U,ϕ) be a chart on M with $x \in U$ such that $\phi(x) = 0$ and

$$(\partial^{(k)} \phi_j)(x) = 0 \quad , \quad 2 \leq k \leq N+1 \quad , \quad 1 \leq j \leq m$$

Suppose E is trivial over U and we have a frame $s_1 \ldots s_p$ for E over U such that

$$(D^{(k)} s_j)(x) = 0 \quad , \quad 1 \leq k \leq N \quad , \quad 1 \leq j \leq p$$

If $g \in C^{\infty}(M;\mathbb{R})$ then we have

$$(1) \quad (\partial^{(n)} g)(x) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \left(\frac{\partial}{\partial \phi} \right)^{\alpha} g(x) \; (d\phi)^{\alpha}(x)$$

for $0 \leq n \leq N+1$, and

$$(2) \quad (D^{(n)} g s_j)(x) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \left(\frac{\partial}{\partial \phi} \right)^{\alpha} g(x) \; (d\phi)^{\alpha}(x) \otimes s_j(x)$$

for $0 \leq n \leq N$, $1 \leq j \leq p$

<u>Proof</u>: By theorem 2.7 and by hypothesis, if $n \leq N$

$$D^{(n)}(g s_j)(x) = \sum_{k=0}^{n} \binom{n}{k} (\partial^{(k)} g)(x) \odot (D^{(n-k)} s_j)(x)$$

$$= (\partial^{(n)} g)(x) \otimes s_j(x)$$

and so it is sufficient to prove (1).

By Taylor's theorem, since $\phi(x) = 0$

$$g - \sum_{|\alpha| \leq n} \frac{1}{\alpha!} \left(\frac{\partial}{\partial \phi} \right)^{\alpha} g(x) \; \phi^{\alpha} \in Z_x^{n+1}$$

and hence

$$(\partial^{(n)} g)(x) = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} \left(\frac{\partial}{\partial \phi} \right)^{\alpha} g(x) \; (\partial^{(n)} \phi^{\alpha})(x) \quad .$$

By corollary 2.8

$$(*) \qquad (\partial^{(n)} \phi^{\alpha})(x) = \sum_{|\beta|=n} \frac{n!}{\beta!} (\partial^{(\beta_1)} \phi_1^{\alpha_1})(x) \odot \cdots \odot (\partial^{(\beta_m)} \phi_m^{\alpha_m})(x)$$

Since $\phi_j^{\alpha_j} \in Z_x^{\alpha_j}$, if $\beta_j < \alpha_j$ then $(\partial^{(\beta_j)} \phi_j^{\alpha_j})(x) = 0$. Hence in (*) it suffices to sum over $\beta \geq \alpha$. In particular if $|\alpha| = n$ we have

$$(\partial^{(n)} \phi^{\alpha})(x) = \frac{n!}{\alpha!} (\partial^{(\alpha_1)} \phi_1^{\alpha_1})(x) \odot \cdots \odot (\partial^{(\alpha_m)} \phi_m^{\alpha_m})(x)$$

But by corollary 2.8 and since $\phi(x) = 0$ we have

$$(\partial^{(\alpha_j)} \phi_j^{\alpha_j})(x) = \alpha_j! \ (\partial^{(1)}\phi_j)(x) \odot \cdots \odot (\partial^{(1)}\phi_j)(x)$$

and so if $|\alpha| = n$ then

$$(\partial^{(n)} \phi^{\alpha})(x) = n! \ (d\phi)^{\alpha}(x) .$$

It now suffices to prove that (*) vanishes if $|\alpha| < n$. If $|\alpha| < n$, $|\beta| = n$ and $\alpha \leq \beta$, then for some j, $\beta_j > \alpha_j$ and by corollary 2.8

$$(\partial^{(\beta_j)} \phi_j^{\alpha_j})(x) = \sum \frac{\beta_j!}{i_1! \cdots i_q!} (\partial^{(i_1)} \phi_j)(x) \odot \cdots \odot (\partial^{(i_q)} \phi_j)(x)$$

where we sum over $i_1 + \cdots + i_q = \beta_j$ and where $q = \alpha_j$. Since $\beta_j > \alpha_j$ each term in this sum contains an i_p with $2 \leq i_p \leq n \leq N+1$ and by hypothesis each such term vanishes.

We now return to the consideration of lower order symbols of differential operators. The following theorem is well-known.

<u>Theorem 2.16</u>

If $P \in \mathrm{Diff}_n(E,F)$ then there exist unique morphisms

$$\tau_{n-k}(P) : S^k T^* \otimes E \to F \qquad k = 0,1,2,\dots,n$$

such that

$$P = \sum_{k=0}^{n} i^{-k} \tau_{n-k}(P) \circ D^{(k)}$$

<u>Proof</u>: If $P = \theta j_n$ then we define

$$\tau_{n-k}(P) = i^k \theta \nu_n \dots \nu_{k+1} \iota_k$$

and check that this works. Uniqueness is clear from the direct sum representation of j_n.

The factor of i^k in theorem 2.16 was chosen to make the connection with the symbols a bit neater. Relative to the pair (D,∇) we have the injections

$$\nu_k \dots, \nu_1 : E \to J^k(E)$$

$$\dot{\nu}_k \dots, \dot{\nu}_1 : T^* \to \Psi^k$$

which are right inverses for the obvious projections. These injections induce a morphism

$$\kappa_k : S^{n-k} T^* \otimes E \to S^{n-k} \Psi^k \otimes J^k(E) \quad , \quad k = 0,1,\dots n$$

<u>Theorem 2.17</u>

If $P \in \mathrm{Diff}_n(E,F)$ then

$$\tau_k(P) = \sigma_k(P) \circ \kappa_k \qquad k = 0,1,\dots,n.$$

and in particular

$$\tau_0(P) = \sigma_0(P) .$$

<u>Proof</u>: If $x \in M$, $e \in E_x$, $v \in T_x^*$ choose $s \in \Gamma^\infty(E)$ such that $s(x) = e$ and $(D^{(h)} s)(x) = 0$, $1 \leq h \leq n$, and choose $g \in C^\infty(M:\mathbb{R})$ such that $dg(x) = v$ and $(\partial^{(h)} g)(x) = 0$ for $2 \leq h \leq n+1$ and $g(x) = 0$. (These choices are possible by lemmas 2.10 and 2.12). By definition of κ_k and by the direct sum decompositions of j_k and $j_k d$ we have

$$\kappa_k(\delta^{n-k} v \otimes e) = \delta^{n-k} j_k(dg)(x) \otimes j_k(s)(x)$$

and hence by theorem 2.16

$$\sigma_k(P) \cdot \kappa_k(\delta^{n-k} v \otimes e) = \frac{i^{n-k}}{(n-k)!} P(g^{n-k} s)(x)$$

$$= \tau_k(P) \cdot (\delta^{n-k} v \otimes e)$$

$$+ \sum_{h=0}^{k-1} \frac{i^{h-k}}{(n-k)!} \tau_h(P) \cdot D^{(n-h)}(g^{n-k} s)(x)$$

The computation of the first term here follows from the fact that $D^{(n-k)} = \mu_{n-k} j_{n-k}$ where μ_{n-k} splits the exact jet bundle sequence, and from the definition of the injection in that sequence, or else from computations similar to those in theorem 2.15.

To show that the terms in the sum vanish it is sufficient to show if $p < h \leq n$ then

$$D^{(h)}(g^p s)(x) = 0$$

We have

$$D^{(h)}(g^p s)(x) = \sum_{q=0}^{h} \binom{h}{q} (\partial^{(q)} g^p)(x) \odot (D^{(h-q)} s)(x)$$
$$= (\partial^{(h)} g^p)(x) \otimes v$$

by hypothesis on s.

Hence it suffices to show $(\partial^{(h)} g^p)(x) = 0$ for $p < h \leq n$. By corollary 2.8 we have

$$(\partial^{(h)} g^p)(x) = \sum \frac{h!}{i_1! \ldots i_p!} (\partial^{(i_1)} g)(x) \odot \cdots \odot (\partial^{(i_p)} g)(x)$$

where we sum over $i_1 + \ldots + i_p = h$.

Since $g(x) = 0$ we need only consider terms for which each $i_j \geq 1$. Since $p + 1 \leq h$, in any such term we have each $i_j \leq h-p+1 \leq n-p+1 \leq n+1$ and some $i_j \geq 2$. Thus in each of the terms we consider there is an i_j with $2 \leq i_j \leq n+1$, and hence by hypothesis on g, the sum vanishes.

It is also possible to give a formula which expresses $\sigma_k(P)$ in terms of $\tau_j(P)$, $0 \leq j \leq k$. However we will derive such an expression more generally for pseudo-differential operators, so we omit it for now. (see theorem 9.4). The motivation for presenting theorem 2.17 is that it shows how to define the τ_k for pseudo-differential operators, even though the decomposition given by theorem 2.16 no longer exists.

§3 Generalized Symbols

The projection $\omega_1 \ldots \omega_n : \Psi^n \to T^*$ is given by $j_n(dg) \to dg$. Let A^n be its kernel. Then A^n is a C^∞ vector subbundle and hence by local triviality $\Psi^n - A^n$ is an open submanifold of Ψ^n. Let $p_n : \Psi^n - A^n \to M$ be the restriction of the natural projection. Then if $n \geq 0$ is an integer and $z \in \mathbb{C}$ we define

$$\mathrm{Smbl}_z^n (E,F)$$

$$= \{ \sigma \in \mathrm{HOM}(p_n^* J^n(E),\ p_n^* F) : \sigma \cdot \lambda w = \lambda^z \sigma \cdot w\ ,\ \lambda > 0,\ w \in \Psi^n - A^n \}$$

where $\mathrm{HOM} = \Gamma^\infty \mathrm{Hom}$.

We may also describe $\mathrm{Smbl}_z^n(E,F)$ as the set of functions

$$\sigma : \Psi^n - A^n \to \mathrm{Hom}(J^n(E),F)$$

with the following properties

(1) σ is fibre preserving

(2) σ is positively homogeneous of degree z

(3) if $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M:\mathbb{R})$ and $dg \neq 0$ on supp s then $x \to \sigma \cdot j_n(dg)(x) \cdot j_n(s)(x)$ is a C^∞ section of F. (This section is well-defined on all of M if we make the convention that it is zero if $x \notin$ supp s).

Clearly $\mathrm{Smbl}_z^n(E,F)$ has a natural structure as a left $C^\infty(M:\mathbb{C})$-module. We let $\mathrm{Smbl}_z(E,F) = \mathrm{Smbl}_z^0(E,F)$. Then since $\Psi^0 = T^*$ and $A^0 = (0)$ the module $\mathrm{Smbl}_z(E,F)$ is the same as the one considered in

Palais [13,pg. 54] for real z. The natural projections $J^{n+1}(E) \to J^n(E)$ and $\Psi^{n+1} \to \Psi^n$ induce an inclusion $\text{Smbl}_z^n(E,F) \subseteq \text{Smbl}_z^{n+1}(E,F)$ so that in particular $\text{Smbl}_z(E,F) \subseteq \text{Smbl}_z^n(E,F)$ and for this reason we refer to the elements of $\text{Smbl}_z^n(E,F)$ as generalized symbols.

<u>Theorem 3.1</u>

Let $z \in \mathbb{C}$ and let $n \geq 0$ be an integer. Suppose for each $s \in \Gamma_c^\infty(E)$ and each $g \in C^\infty(M:\mathbb{R})$ such that $dg \neq 0$ on supp s we have

$$P(s,g) \in \Gamma^\infty(F)$$

such that

(a) if $\lambda > 0$, $dg \neq 0$ on supp s, then $P(s,\lambda g) = \lambda^z P(s,g)$

(b) if $dg \neq 0$ on supp $s \cup$ supp s', $a,b \in \mathbb{C}$ then

$$P(as + bs',g) = aP(s,g) + bP(s',g).$$

(c) if $dg \neq 0$ on supp s, $j_n(s)(x) = 0$, then $P(s,g)(x) = 0$

(d) if $dg_i \neq 0$ on supp s, $i = 1,2$, and $j_n(dg_1)(x) = j_n(dg_2)(x)$

then $P(x,g_1)(x) = P(x,g_2)(x)$

Then there exists a unique $\sigma \in \text{Smbl}_z^n(E,F)$ such that if $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M:\mathbb{R})$, $dg \neq 0$ on supp s then

$$(*) \quad \sigma \cdot j_n(dg)(x) \cdot j_n(s)(x) = P(s,g)(x) \quad , \quad x \in M$$

<u>**Proof:**</u> Let $x \in M$. Given $w \in \Psi_x^n - A_x^n$ and $v \in J^n(E)_x$ choose $g \in C^\infty(M;\mathbb{R})$ such that $j_n(dg)(x) = w$. Since $w \notin A_x^n$ it follows that $dg(x) \neq 0$. Thus there is an open neighborhood U of x such that $dg \neq 0$ on U. Now choose $s \in \Gamma_c^\infty(E|_U)$ such that $j_n(s)(x) = v$, and then define

$$\sigma \cdot w \cdot v = P(s,g)(x)$$

Suppose $s' \in \Gamma_c^\infty(E)$, $g' \in C^\infty(M;\mathbb{R})$, $dg' \neq 0$ on supp s' and $j_n(dg')(x) = w$, $j_n(s')(x) = v$. As before there is a neighborhood W of x such that $dg' \neq 0$ on W. Choose $s'' \in \Gamma_c^\infty(E|_{W \cap U})$ such that $j_n(s'')(x) = v$. Then by (d) we have

$$P(s'',g)(x) = P(s'',g')(x)$$

Now $dg \neq 0$ on supp $s'' \cup$ supp $s \supseteq \text{supp}(s - s')$ and hence by (c) and (b) we have

$$0 = P(s'' - s,g)(x) = P(s'',g)(x) - P(s,g)(x)$$

Similarly we have

$$0 = P(s'' - s',g')(x) = P(s'',g')(x) - P(s',g')(x)$$

It follows that

$$P(s',g')(x) = P(s,g)(x)$$

and hence $\sigma \cdot w \cdot v$ is well-defined. In particular it follows by (b) that $\sigma \cdot w \cdot v$ is $\mathbb{C}$-linear in v and so

$$\sigma : \Psi_x^n - A_x^n \to \text{Hom}(J^n(E),F)_x$$

Carrying out this construction for each $x \in M$ yields a function

$$\sigma : \Psi^n - A^n \to \mathrm{Hom}(J^n(E),F)$$

which clearly satisfies condition (1) for a generalized symbol. Condition (2) is satisfied because of (a) and condition (3) is satisfied by hypothesis. Hence it follows that $\sigma \in \mathrm{Smbl}_z^n(E,F)$. Clearly σ satisfies (*), and by construction is unique.

<u>Theorem 3.2</u>

Let $\sigma_1 \in \mathrm{Smbl}_z^n(E,F)$ and $\sigma_2 \in \mathrm{Smbl}_w^k(F,G)$. Then there exists a unique generalized symbol $\sigma_2 \circ \sigma_1 \in \mathrm{Smbl}_{z+w}^{n+k}(E,G)$ such that if $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp s then

$$(\sigma_2 \circ \sigma_1) \cdot j_{n+k}(dg) \cdot j_{n+k}(s)$$

$$= \sigma_2 \cdot j_k(dg) \cdot j_k(\sigma_1 \cdot j_n(dg) \cdot j_n(s))$$

<u>Proof:</u> If $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp s, define

$$P(s,g) = \sigma_2 \cdot j_k(dg) \cdot j_k(\sigma_1 \cdot j_n(dg) \cdot j_n(s)) .$$

Then it suffices to verify hypotheses (c) and (d) of theorem 3.1

(c): Suppose $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp s, and that $j_{n+k}(s)(x) = 0$. Then $j_n(s) \in Z_x^{k+1} \cdot \Gamma^\infty(J^n(E))$ and so since $\sigma_1 \cdot j_n(dg) : J^n(E) \to F$ is a morphism it follows that $\sigma_1 \cdot j_n(dg) \cdot j_n(s) \in Z_x^{k+1} \cdot \Gamma^\infty(F)$ whence $P(s,g)(x) = 0$.

(d): Suppose $s \in \Gamma_c^\infty(E)$, $g_i \in C^\infty(M;\mathbb{R})$, $dg_i \neq 0$ on supp s, $i = 1,2$, and that $j_{n+k}(dg_1)(x) = j_{n+k}(dg_2)(x)$. Then

$$j_n(dg_1) - j_n(dg_2) \in Z_x^{k+1} \cdot \Gamma^\infty(\Psi^n) \quad \text{and so by the chain rule}$$

$$\sigma_1 \cdot j_n(dg_1) \cdot j_n(s) - \sigma_1 \cdot j_n(dg_2) \cdot j_n(s) \in Z_x^{k+1}\, \Gamma^\infty(F)$$

from which it follows that

$$j_k(\sigma_1 \cdot j_n(dg_1) \cdot j_n(s)\,)(x) = j_k(\sigma_1 \cdot j_n(dg_2) \cdot j_n(s)\,)(x) .$$

Since $k \leq n+k$ we also have $j_k(dg_1)(x) = j_k(dg_2)(x)$ and hence $P(s,g_1)(x) = P(s,g_2)(x)$.

Remark 3.3

Since $\mathrm{Diff}_n(E,F) \subseteq \mathrm{Smbl}_0^n(E,F)$ theorem 3.2 in particular applies to composition of generalized symbols with differential operators.

Given a C^∞ map

$$\rho : T_x^* - (0) \to \mathrm{Hom}(E_x,F_x)$$

we can define the (Fréchet) derivatives of ρ

$$\rho^{(k)} : T_x^* - (0) \to \mathcal{L}_s^k(T_x^*\,,\ \mathrm{Hom}(E_x,F_x)\,) = \mathrm{Hom}(S^k T_\otimes^* E,F)_x$$

(J. Dieudonné [4, ch.8]) and we have the following result.

Theorem 3.4

If $\tau \in \mathrm{Smbl}_z(E,F)$ then for each integer $n \geq 0$ we have

$$\tau^{(n)} \in \mathrm{Smbl}_{z-n}(S^n\, T_\otimes^* E,\ F)$$

<u>Proof</u>: It is sufficient to observe that the degree of homogenuity is correct. Since the derivatives $\tau^{(n)}$ are defined inductively it suffices to consider $n = 1$.

If $v \in T^*_x - (0)$, $u \in T^*_x$, $e \in E_x$, $\lambda > 0$ then

$$\tau^{(1)} \cdot \lambda v \cdot u_{\otimes} e = \lim_{t\to 0} t^{-1}(\tau \cdot (\lambda v + tu) - \tau \cdot \lambda v) \cdot e$$

$$= \lim_{t\to 0} (\lambda t)^{-1}(\tau \cdot \lambda(v + tu) - \tau \cdot \lambda v) \cdot e$$

$$= \lambda^{z-1}\, \tau^{(1)} \cdot v \cdot u_{\otimes} e$$

We now compute the derivatives $\tau^{(n)}$ in coordinates. If N is a C^∞ manifold then $E \times N$ will denote the pullback of E by the projection of $M \times N$ onto M. Let $\tau \in \mathrm{Smbl}_z(E,F)$ and let (U,ϕ) be a chart on M. Then for each $\xi \in \mathbb{R}^m$ we define $g_\xi \in C^\infty(U;\mathbb{R})$ by

$$g_\xi = \sum_{j=1}^{m} \xi_j\, \phi_j$$

so if $\xi \neq 0$ then $dg_\xi \neq 0$ on U.

Now define $p \in \Gamma^\infty(\mathrm{Hom}(E,F)_{|U} \times \mathbb{R}^m - (0))$

by

$$p(\xi) \cdot s = \tau \cdot dg_\xi \cdot s \quad , \quad s \in \Gamma^\infty(E_{|U}), \ \ \xi \in \mathbb{R}^m - (0)$$

Then clearly $\tau^{(n)} \in \mathrm{Smbl}_{z-n}(S^n\, T^*_{\otimes} E, F)$ is given over U by

$$\tau^{(n)} \cdot dg_\xi \cdot (d\phi)^\alpha_{\otimes}\, s = p^{(\alpha)}(\xi) \cdot s \quad , \ |\alpha| = n$$

where $p^{(\alpha)} = \left(\frac{\partial}{\partial \xi}\right)^\alpha p$.

Now let D be a covariant derivative on E and let ∇ be a covariant derivative on T^*. Let $D^{(k)}$ be the total differentials induced by the pair (D,∇) and let $\partial^{(k)}$ be the total differentials induced by the pair

(d,∇).

The following theorem will be useful for describing the symbols of pseudo-differential operators. (see theorem 9.4)

Theorem 3.5

For each pair of integers k,ℓ with $0 \leq 2\ell \leq k$ there exists a unique generalized symbol

$$\chi_{k,\ell}(D,\nabla) \in \mathrm{Smbl}_{\ell}^{k-\ell}(E, S^k\, T^*_{\otimes}E)$$

such that if $s \in \Gamma_c^{\infty}(E)$, $g \in C^{\infty}(M;\mathcal{R})$, $dg \neq 0$ on supp s and if for each $x \in M$ we choose $h_x \in C^{\infty}(M;\mathcal{R})$ such that

(1) $h_x \in Z_x^2$

(2) $(\partial^{(q)} h_x)(x) = (\partial^{(q)} g)(x)$, $2 \leq q \leq k+1$

(3) $x \to j_k(dh_x)(x)$ is a C^{∞} section of Ψ^k

then

$$\chi_{k,\ell}(D,\nabla) \cdot j_{k-\ell}(dg)(x) \cdot j_{k-\ell}(s)(x) = D^{(k)}(h_x^{\ell}\, s)(x)$$

Proof: First we observe that h_x exists by corollary 2.13. If h'_x is another candidate for h_x , then by (1) and (2) and the direct sum decomposition of $j_k d$ we have $h_x - h'_x \in Z_x^{k+2}$. Then since $h_x \in Z_x^2$ and $h'_x \in Z_x^2$ it is clear that $(h_x)^{\ell} - (h'_x)^{\ell} \in Z_x^{k+\ell+1} \subseteq Z_x^{k+1}$. Hence $\chi_{k,\ell}(D,\nabla) \cdot j_{k-\ell}(dg)(x) \cdot j_{k-\ell}(s)(x)$ is well-defined.

To show that $\chi_{k,\ell}(D,\nabla) \in \mathrm{Smbl}_{\ell}^{k-\ell}(E, S^k\, T^*_{\otimes}E)$ it suffices to show that the hypotheses of theorem 3.1 are satisfied.

First we must show that $x \to D^{(k)}(h_x^\ell s)(x)$ is a C^∞ section of $S^k T^*_\otimes E$. By lemma 2.7 it suffices to show that $x \to \partial^{(q)}(h_x^\ell s)(x)$ is a C^∞ section of $S^q T^*$ for $q \leq k$, but that is clear by condition (3) on h_x.

Now conditions (a) and (b) of theorem 3.1 clearly hold, and (c) is also clear since $h_x^\ell \in Z_x^{2\ell} \subseteq Z_x^\ell$. For (d), assume $g' \in C^\infty(M;\mathbb{R})$, $dg' \neq 0$ on supp s and $j_{k-\ell}(dg')(x) = j_{k-\ell}(dg)(x)$. Let h_x' be an "h_x" corresponding to g'. Then by conditions (1) and (2) we have $h_x - h_x' \in Z_x^{k-\ell+2}$, and so since $h_x \in Z_x^2$ and $h_x' \in Z_x^2$ we have $(h_x)^\ell - (h_x')^\ell \in Z_x^{k+1}$ whence (d) follows.

The definition of $\chi_{k,\ell}$ also works for $2\ell > k$, but then we just get zero. Note that

$$\chi_{k,0}(D,\nabla) \cdot j_k(dg) \cdot j_k(s) = D^{(k)}s.$$

<u>Remark 3.6</u>

We recall that the pair (D,∇) induces injections

$$\nu_n \ldots \nu_1 : E \to J^n(E)$$

and that the pair (d,∇) induces injections

$$\dot{\nu}_n \ldots \dot{\nu}_1 : T^* \to \Psi^n$$

which are right inverses for the obvious projections. Moreover, we have the restriction

$$\dot{\vartheta}_n \cdots \dot{\vartheta}_1 : T^* - (0) \to \Psi^n - A^n$$

Now given $\sigma \in \mathrm{Smbl}_z^n(E,F)$ we define the <u>singular part of σ relative to (D,∇)</u> as the composition

$$T^* - (0) \longrightarrow \Psi^n - A^n \xrightarrow{\sigma} \mathrm{Hom}(J^n(E),F) \longrightarrow \mathrm{Hom}(E,F)$$

(See remark 9.10 for motivation for the name "singular part"). This definition is obviously motivated by theorem 2.17. The map

$$\mathrm{Smbl}_z^n(E,F) \to \mathrm{Smbl}_z(E,F)$$

which carries a generalized symbol to its singular part relative to (D,∇) is surjective and is clearly a left inverse for the obvious inclusion.

Let $\sigma \in \mathrm{Smbl}_z^n(E,F)$ and let $\tau \in \mathrm{Smbl}_z(E,F)$ be the singular part of σ relative to (D,∇). In view of the direct sum decompositions of Ψ^n (i.e. of $j_n d$) and of $J^n(E)$ (i.e. of j_n) given in §2 we have: if $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp s, $x \in M$, $(D^{(k)}s)(x) = 0$, $1 \leq k \leq n$, $(\partial^{(k)}g)(x) = 0$, $2 \leq k \leq n+1$, then

$$\tau \cdot dg(x) \cdot s(x) = \sigma \cdot j_n(dg)(x) \cdot j_n(s)(x)$$

and by lemmas 2.10 and 2.12 we may compute τ in this fashion.

<u>Remark 3.7</u>

Let $\tau \in \mathrm{Smbl}_z(E,F)$ and let $P \in \mathrm{Diff}_n(F,G)$. In remark 3.3 we defined the composition $P \circ \tau \in \mathrm{Smbl}^n_z(E,G)$. We now define $P_* \tau \in \mathrm{Smbl}_z(E,G)$ as the singular part of $P \circ \tau$ relative to (D,∇). (It will always be clear from context which covariant derivatives are employed to define $P_* \tau$).

§4 Formal Symbols

A sequence of distinct complex numbers $z_0, z_1, z_2, \ldots$ will be called a <u>sequence of exponents</u> if

(a) $Re\, z_0 \geq Re\, z_1 \geq Re\, z_2 \geq \ldots$.

(b) $\limsup_{n\to\infty} Re\, z_n = -\infty$

We do not distinguish sequences that differ only in order of arrangement.

Now we define $\mathcal{GF}(E,F)$ to be the left $C^\infty(M;\mathbb{C})$-module of all formal sums

$$\sum_{j=0}^{\infty} \sigma_j$$

where $\sigma_j \in \mathrm{Smbl}_{z_j}^{n_j}(E,F)$, $z_0, z_1, \ldots$ is a sequence of exponents and $0 \leq n_j \leq Re(z_0 - z_j)$. The elements of $\mathcal{GF}(E,F)$ are called <u>generalized formal symbols</u>.

We define $\mathcal{F}(E,F)$ to be the left $C^\infty(M;\mathbb{C})$-module of all formal sums

$$\sum_{j=0}^{\infty} \tau_j$$

where $\tau_j \in \mathrm{Smbl}_{z_j}(E,F)$ and $z_0, z_1, \ldots$ is a sequence of exponents. The elements of $\mathcal{F}(E,F)$ are called <u>formal symbols</u>. Clearly we have an inclusion

$$\mathcal{F}(E,F) \subseteq \mathcal{GF}(E,F)$$

If $\sum_{j=0}^{\infty} \sigma_j \in \mathcal{GF}(E,F)$, $\sum_{k=0}^{\infty} \tilde{\sigma}_k \in \mathcal{GF}(F,G)$ we define

$$(\sum_{k=0}^{\infty} \tilde{\sigma}_k) \circ (\sum_{k=0}^{\infty} \sigma_j) = \sum_{j,k=0}^{\infty} \tilde{\sigma}_k \circ \sigma_j$$

(see theorem 3,2). If $\sum_{j=0}^{\infty} \sigma_j$ has exponents $z_0, z_1, \ldots$ and $\sum_{k=0}^{\infty} \tilde{\sigma}_k$ has exponents $w_0, w_1, \ldots$ then clearly

$$\sum_{j,k=0}^{\infty} \tilde{\sigma}_k \circ \sigma_j \in \mathcal{GF}(E,G)$$

has exponents $z_j + w_k$, $j,k=0,1,2,\ldots$. With this definition of composition $\mathcal{GF}(E,E)$ becomes an algebra over $\mathbb{C}$. We do not consider this algebra in any detail.

Let D_E be a covariant derivative on E, D_F a covariant derivative on F, ∇ a covariant derivative on T^*, and let

$$\sum_{j=0}^{\infty} \tau_j \in \mathcal{F}(E,F)$$

$$\sum_{k=0}^{\infty} \tilde{\tau}_k \in \mathcal{F}(F,G)$$

Then we define

$$(\sum_{k=0}^{\infty} \tilde{\tau}_k) \cdot (\sum_{j=0}^{\infty} \tau_j) \in \mathcal{F}(E,G)$$

to be

$$\sum_{j,k,\ell=0}^{\infty} \frac{i^{-\ell}}{\ell!} \tilde{\tau}_k^{(\ell)} \circ (D_F^{(\ell)} * \tau_j)$$

Since $*$ depends on ∇ and D_E, and $D_F^{(\ell)}$ depends on ∇ and D_F it follows that this composition depends on all the covariant derivatives.

In particular the pair (D_E, ∇) induces the structure of an algebra on $\mathcal{F}(E) = \mathcal{F}(E,E)$. The resulting algebra will be denoted by $\mathcal{F}_{(D_E,\nabla)}(E)$. Later we will see that these algebras are all isomorphic to an algebra invariantly associated to E, and in particular are associative (see theorem 9,9).

§5 Pseudo-Differential Operators

In this section we define the pseudo-differential operators of L. Hörmander [8] except that we include operators whose symbols allow complex degrees of homogenuity. The "asymptotic series" that occur are then no longer true asymptotic series since distinct terms may have the same order of growth, but this fact does not introduce any serious complications. The introduction of complex degrees is necessary in order to consider complex powers of suitable operators (Seeley [16 and 17]).

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$

be a continuous $\mathbb{C}$-linear map. Then P is called a pseudo-differential operator if there is a sequence of exponents $z_0, z_1, \ldots$ (see §4) such that whenever $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$, on supp s, then there is an asymptotic expansion

$$e^{-i\lambda g} P(e^{i\lambda g} s) \sim \sum_{j=0}^{\infty} P_j(s,g)\lambda^{z_j}$$

in the following sense:

For each $s \in \Gamma_c^\infty(E)$, for each integer $N > 0$, and for each compact subset K of $C^\infty(M;\mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp s, the set

$$\{ \lambda^{-z_N} (e^{-i\lambda g} P(e^{i\lambda g} s) - \sum_{j<N} P_j(s,g)\lambda^{z_j}) : \lambda \geq 1, \ g \in K \}$$

is a bounded subset of $\Gamma^\infty(F)$.

Let $\mathcal{P}(E,F)$ be the set of all pseudo-differential operators from E to F. Then it is clear that $\mathcal{P}(E,F)$ is a left $C^\infty(M:\mathbb{C})$-module (and a right $C^\infty(M:\mathbb{C})$-module as well).

<u>lemma 5.1</u>

The asymptotic expansion associated to a pseudo-differential operator is unique.

(By uniqueness here we mean modulo zero terms and modulo order of arrangement of the terms of the series. As the proof will show the lemma is true under a much weaker definition of asymptoticity than used here.)

<u>Proof</u>: Suppose the lemma is false so we can find a sequence of exponents $z_0, z_1, \ldots$ and complex numbers $a_k \in \mathbb{C}$. $k = 0,1,2..$ such that

$$0 \sim \sum_{k=0}^{\infty} a_k \lambda^{z_k} \qquad (\lambda \to +\infty)$$

i.e. for each integer $n \geq 0$ if we define

$$f_n(\lambda) = \sum_{k=0}^{n-1} a_k \lambda^{z_k - z_n}$$

then f_n is bounded for $\lambda \geq 1$. Now note we have a recursive relation

$$(*) \quad f_{n+1}(\lambda) = (f_n(\lambda) + a_n)\lambda^{z_n - z_{n+1}}$$

We assume that $a_0 \neq 0$. Since $z_0, z_1, \ldots$ is a sequence of exponents we can find an integer $N \geq 0$ such that

$$Re\ z_0 = \ldots\ldots\ldots = Re\ z_N > Re\ z_{N+1}$$

and hence by (*) since $f_N(\lambda)$ is bounded for $\lambda \geq 1$ and $Re(z_N - z_{N+1}) > 0$ we have

$$\lim_{\lambda\to\infty} f_N(\lambda) + a_N = 0$$

On the other hand since $z_k - z_N$ is pure imaginary and nonzero for $0 \leq k \leq N-1$ it follows that

$$\lim_{t\to\infty} t^{-1} \int_1^{e^t} \lambda^{-1}(f_N(\lambda) + a_N)d\lambda = a_N$$

Let $\varepsilon > 0$. There is a $T \geq 0$ such that $\lambda > e^T$ implies $|f_N(\lambda) + a_N| < \varepsilon$ and hence

$$|a_N| \leq \lim_{t\to\infty} t^{-1} \left| \int_{e^T}^{e^t} \lambda^{-1}(f_N(\lambda) + a_N)d\lambda \right|$$

$$\leq \lim_{t\to\infty} t^{-1} \varepsilon(t - T) = \varepsilon$$

Thus $a_N = 0$. Hence we may discard this term from the asymptotic expansion. Then it follows by the same argument that $a_{N-1} = 0$, and hence by finite induction $a_0 = 0$, which contradicts our assumption that $a_0 \neq 0$.

<u>Corollary 5.2</u>

If $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$ we have

(i) If $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$, on supp s then

$$P_n(s,\lambda g) = \lambda^{z_n} P_n(s,g) , \quad \lambda > 0$$

(ii) If $s, s' \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp $s \cup$ supp s' and and $a, b \in \mathbb{C}$ then

$$P_n(as + bs', g) = a\,P_n(s, g) + b\,P_n(s',g)$$

Comparing Corollary 5.2 with theorem 3.1 indicates the direction in which we plan to go. (see theorem 9.2).

It follows from corollary 5.2 if $P_0 \neq 0$ then $\mathcal{R}e\ z_0$ is uniquely determined by P. This real number is called the order of P. P is said to have order $-\infty$ if $P_j = 0$ for all j. If $r \in \mathbb{R}$ or $r = -\infty$ we define

$$\mathcal{P}_r(E,F)$$

to be the submodule of $\mathcal{P}(E,F)$ consisting of pseudo-differential operators of order $\leq r$.

lemma 5.3

If $P \in \mathcal{P}(E,F)$ the corresponding P_j are local i.e. if $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp s, then

$$\text{supp } P_j(s,g) \subseteq \text{supp } s$$

Moreover, if $g' \in C^\infty(M;\mathbb{R})$ and $g - g' = 0$ on supp s then

$$P_j(s,g) = P_j(s,g')$$

Proof: We prove the last part first. Since $g - g' = 0$ on supp s we have

$$e^{i\lambda(g-g')}\, e^{-i\lambda g}\, P(e^{i\lambda g}\, s) = e^{-i\lambda g'}\, P(e^{i\lambda g'}\, s)$$

Now for each integer $n > 0$

$$\{ \lambda^{-z_n} (e^{-\lambda g'} P(e^{i\lambda g'} s) - \sum_{j<n} P_j(s,g')\lambda^{z_j}) : \lambda \geq 1 \}$$

is a bounded subset of $\Gamma^\infty(F)$ and hence since $e^{i\lambda(g-g')}$ is bounded in $C^0(M:\mathbb{C})$ it follows that

$$\{ \lambda^{-z_n} (e^{-i\lambda g} P(e^{i\lambda g} s) - \sum_{j<n} P_j(s,g')e^{i\lambda(g-g')} \lambda^{z_j}) : \lambda \geq 1 \}$$

is a bounded subset of $\Gamma^0(F)$. But

$$\{ \lambda^{-z_n} (e^{-i\lambda g} P(e^{i\lambda g} s) - \sum_{j<n} P_j(s,g)\lambda^{z_j}) : \lambda \geq 1 \}$$

is a bounded subset of $\Gamma^\infty(F)$. Thus it follows for each integer $n > 0$

$$\{ \sum_{j<n} (P_j(s,g) - P_j(s,g')e^{i\lambda(g-g')})\lambda^{z_j-z_n} : \lambda \geq 1 \}$$

is a bounded subset of $\Gamma^0(F)$.

Now for each $x \in M$, $e^{i\lambda(g-g')(x)}$ is periodic in λ and hence there exists a real number $r_x > 0$ such that

$$\{ \sum_{j<n} (P_j(s,g)(x) - P_j(s,g')(x)) (kr_x)^{z_j-z_n} : k = 1,2,\ldots \}$$

is a bounded subset of F_x, and hence

$$\{ \sum_{j<n} (P_j(s,g)(x) - P_j(s,g')(x))\lambda^{z_j-z_n} : \lambda \geq 1 \}$$

is a bounded subset of F_x. Hence by the proof of lemma 5.1 we conclude

$$P_j(s,g) = P_j(s,g')$$

and moreover by considering $kr_x + t$ above we can conclude that if

$g(x) - g'(x) \neq 0$ then for any complex number w such that $|w| = 1$ we have

$$P_j(s,g)(x) = P_j(s,g')(x)w$$

whence $P_j(s,g)(x) = 0$. But for any $x \in M\text{-supp } s$ we may choose g' satisfying the hypotheses and such that $g(x) - g'(x) \neq 0$. Thus $\text{supp } P_j(s,g) \subseteq \text{supp } s$.

Lemma 5.3 implies that we can localize pseudo-differential operators. Let $P \in \mathcal{P}(E,F)$ and let U be an open subset of M. Then we define $P_{|U}$ to be the composition

$$\Gamma_c^\infty(E_{|U}) \xrightarrow{i} \Gamma_c^\infty(E) \xrightarrow{P} \Gamma^\infty(F) \xrightarrow{r} \Gamma^\infty(F_{|U})$$

where i is the inclusion map and r is the restriction map. $P_{|U}$ is clearly continuous.

Corollary 5.4

$P_{|U} \in \mathcal{P}(E_{|U},F_{|U})$. In fact if $s \in \Gamma_c^\infty(E_{|U})$, K is a compact subset of $C^\infty(U;\mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp s then

$$e^{-i\lambda g} P_{|U} (e^{i\lambda g} s) \sim \sum_{j=0}^{\infty} P_j(s,ug)\lambda^{z_j}$$

uniformly for $g \in K$, where $u \in C_c^\infty(U;\mathbb{R})$ is any function such that $u = 1$ on supp s.

Proof:

If we choose u as indicated then

$$e^{-i\lambda(u-1)g}(e^{-i\lambda g}\, P_{|U}\, (e^{i\lambda g}\, s)\,)$$

$$= \; e^{-i\lambda ug}\, P(e^{i\lambda ug}\, s) \sim \sum_{j=0}^{\infty} P_j(s,ug)\lambda^{z_j}$$

in $\Gamma^\infty(F_{|U})$, uniformly for $g \in K$ (since uK is a compact subset of $C^\infty(M;\mathbb{R})$ and the restriction map is continuous). But by lemma 5.3 $P_j(s,ug)$ is independent of the choice of u, so we may define $P_j'(s,g) \in \Gamma^\infty(F_{|U})$ by

$$P_j'(s,g) \;=\; P_j(s,ug)$$

for any u satisfying the hypotheses. But now boundedness of sets in $\Gamma^\infty(F_{|U})$ is checked only over compact subsets of U, and given any compact subset A of U we can choose u so that $u = 1$ in a neighborhood of A as well as on supp s. Hence

$$e^{-i\lambda g}\, P_{|U}\, (e^{i\lambda g}\, s) \;\sim\; \sum P_j'(s,g)\lambda^{z_j}$$

as required.

<u>Corollary 5.5</u>

If $P \in \mathcal{P}(E,F)$, $u,v \in C_c^\infty(M;\mathbb{C})$ and supp u $\cap$ supp v $= \varnothing$ then $vPu \in \mathcal{P}_{-\infty}(E,F)$.

<u>Proof</u>: Immediate by lemma 5.3

Remark 5.6

From the fact that the symbols are local and a finite partition of unity argument we see that if $P \in \mathcal{P}(E,F)$ and if $\mathcal{U}$ is any open cover of M then $P \in \mathcal{P}_r(E,F)$ if and only if $P_{|U} \in \mathcal{P}_r(E_{|U},F_{|U})$ for each $u \in \mathcal{U}$. Thus the order of P is entirely determined by the behaviour of its distribution-kernel near the diagonal in M×M. One can show that these distribution-kernels are C^∞ sections in the complement of the diagonal, and are actually C^∞ sections everywhere when P has order $-\infty$. (see Appendix).

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a linear map. Then for any open subset U of M we can define $P_{|U}$ as we did for pseudo-differential operators. We will say that P is pseudo-local if for each $u,v \in C_c^\infty(M:\mathbb{C})$ such that $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$ we have $uPv \in \mathcal{P}_{-\infty}(E,F)$ (see Corollary A.5) By Corollary 5.5 pseudo-differential operators are pseudo-local.

lemma 5.7

Let $P : \Gamma^\infty_c(E) \to \Gamma^\infty(F)$ be a linear map and let $\mathcal{U}$ be an open cover of M. Suppose one of the following hypotheses holds:

(a) P is pseudo-local

(b) $\{U \times U : U \in \mathcal{U}\}$ is an open cover of $M \times M$

It follows if $P_{|U} \in \mathcal{P}(E_{|U},F_{|U})$ for each $U \in \mathcal{U}$ then for any $v,u \in C_c^\infty(M:\mathbb{C})$ we have $vPu \in \mathcal{P}(E,F)$.

<u>Proof</u>: One proves part (a) first, and then shows that if (b) holds then P is pseudo-local. The technique is the same as used in [19], so we do not reproduce it here.

If A is a subset of M we define $\Gamma^{\infty}(E;A)$ to be the set of all $s \in \Gamma^{\infty}(E)$ with supp $s \subseteq A$.

<u>lemma 5.8</u>

If $P : \Gamma_c^{\infty}(E) \to \Gamma^{\infty}(F)$ is a linear map and if for each $u,v \in C_c^{\infty}(M;\mathbb{R})$ vPu is continuous, then P is continuous.

<u>Proof</u>: $\Gamma_c^{\infty}(E)$ carries the strongest locally convex topology for which all the inclusions $\Gamma^{\infty}(E;A) \to \Gamma_c^{\infty}(E)$ are continuous for compact $A \subseteq M$. Hence it suffices to show for each compact $A \subseteq M$, $P : \Gamma^{\infty}(E;A) \to \Gamma^{\infty}(F)$ is continuous. Choosing $u \in C_c^{\infty}(M;\mathbb{R})$ so that $u = 1$ in a neighborhood of A we see that $vP : \Gamma^{\infty}(E;A) \to \Gamma^{\infty}(F)$ is continuous. But now a continuous semi-norm on $\Gamma^{\infty}(F)$ only depends on what happens over a compact subset of M, and hence $P : \Gamma^{\infty}(E;A) \to \Gamma^{\infty}(F)$ is continuous.

<u>lemma 5.9</u>

Let $P : \Gamma_c^{\infty}(E) \to \Gamma^{\infty}(F)$ be a linear map and let $\mathcal{U}$ be an open cover of M. Suppose <u>one</u> of the following hypotheses holds:

(a) P is pseudo-local

(b) $\{ U \times U ; U \in \mathcal{U} \}$ is an open cover of $M \times M$

Then $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$ if and only if for each $U \in \mathcal{U}$, $P_{|U} \in \mathcal{P}(E_{|U}, F_{|U})$ with exponents $z_0, z_1, \ldots$.

<u>Proof:</u> Necessity follows by corollary 5.4. For sufficiency, by lemma 5.7, $vPu \in \mathcal{P}(E,F)$ with exponents $z_0, z_1 \ldots$ for each $v,u \in C_c^\infty(M:\mathbb{C})$ and hence by lemma 5.8 P is continuous. Now given $s \in \Gamma_c^\infty(E)$ we can choose $u \in C_c^\infty(M:\mathbb{C})$ so that $us = s$ and hence it follows that $vP \in \mathcal{P}(E,F)$ with exponents $z_0, z_1 \ldots$ for each $v \in C_c^\infty(M:\mathbb{C})$. But since boundedness of subsets of $\Gamma^\infty(F)$ is only checked over compact subsets of M, it then follows that $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$.

$P \in \mathcal{P}(E,F)$ is said to be <u>almost local</u> if for each compact subset A of M there is a compact subset A' of M such that

(a) $s \in \Gamma^\infty(E:A)$ implies $Ps \in \Gamma^\infty(E:A')$

(b) $s \in \Gamma_c^\infty(E)$ and $s_{|A'} = 0$ implies $Ps_{|A} = 0$

Since we have assumed M is paracompact it follows by a theorem of Smirnov [7, pg. 81] that M is metrizable. Choose a metric on M. Then we have

<u>lemma 5.10</u>

If $P \in \mathcal{P}(E,F)$ then there exists $P' \in \mathcal{P}(E,F)$ such that P' is almost local and $P - P' \in \mathcal{P}_{-\infty}(E,F)$.

In fact given any $\varepsilon > 0$ we can choose P' so that if $s \in \Gamma_c^\infty(E)$ then supp P's lies in the ε-neighborhood of supp s. In this case we say that P' is ε-local.

<u>Proof</u>: Let $(U_i)_{i\in I}$ be a locally finite cover of M by open sets of diameter less than ε and with compact closures. By the shrinking lemma there is a locally finite open cover $(V_i)_{i\in I}$ of M such that $\bar{V}_i \subseteq U_i$ for each $i \in I$. Choose $u_i \in C_c(U_i:\mathbb{R})$ such that $u_i = 1$ in a neighborhood of $\bar{V}_i$. Choose $v_i \in C_c(V_i:\mathbb{R})$ such that $0 \leq v_i \leq 1$ and $\sum_{i\in I} v_i = 1$. Then define

$$P' = \sum_{i\in I} v_i P u_i$$

Then $P' : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ is a well-defined linear map since for each $s \in \Gamma_c^\infty(E)$, $P's$ involves only a finite sum. If $v,u \in C_c^\infty(M:\mathbb{C})$ then $vP'u$ is a finite sum of pseudo-differential operators and hence $vP'u \in \mathcal{P}(E,F)$ and has the same exponents as P. By the proof of lemma 5.9 it follows that $P' \in \mathcal{P}(E,F)$ and P' has the same exponents as P. By local finiteness and since $\operatorname{supp} u_i s \subseteq \operatorname{supp} s$ it is clear that we have an asymptotic expansion

$$e^{-i\lambda g} P'(e^{i\lambda g} s) \sim \sum_{j=0}^{\infty} \sum_{i\in I} v_i P_j(u_i s,g)\lambda^{z_j}$$

Since $s-u_i s$ vanishes in a neighborhood of $\operatorname{supp} v_i$, $\operatorname{supp}(s-u_i s) \subseteq \operatorname{supp} s$ and P_j is local it follows that $v_i P_j(s-u_i s,g) = 0$. Then since $\operatorname{supp} u_i s \subseteq \operatorname{supp} s$ and $\operatorname{supp}(s-u_i s) \subseteq \operatorname{supp} s$ it follows by "linearity" that

$$v_i P_j(s,g) = v_i P_j(u_i s,g)$$

and hence $P - P' \in \mathcal{P}_{-\infty}(E,F)$. That P' is ε-local is clear.

Essentially the same proof of lemma 5.10 was given in [19]. We give a proof here since we refer to the proof later.

lemma 5.11

If $P \in \mathcal{P}(E,F)$ is almost local then $P : \Gamma_c^\infty(E) \to \Gamma_c^\infty(F)$ is continuous and moreover the asymptotic expansion for P holds in $\Gamma_c^\infty(F)$.

Proof: As in lemma 5.8 it suffices to show $P : \Gamma^\infty(E:A) \to \Gamma_c^\infty(F)$ is continuous for each compact $A \subseteq M$. But by definition it is clear that this map preserves bounded sets. Since $\Gamma^\infty(E:A)$ is metrizable and so in particular is first countable it follows that $P : \Gamma^\infty(E:A) \to \Gamma_c^\infty(F)$ is continuous (A. Friedman [6, pg. 18]). The last statement is obvious, since the P_j are local.

Remark 5.12

If $\mathcal{U}$ is the set of all strict charts on M then $\{U \times U : U \in \mathcal{U}\}$ is an open cover of $M \times M$. Thus lemma 5.9 reduces the study of pseudo-differential operators on manifolds to pseudo-differential operators over open submanifolds of Euclidean space.

Now we consider how the study of pseudo-differential operators on vector bundles may be reduced to the consideration of pseudo-differential operators on functions. The following simple lemma is quite useful.

<u>lemma 5.13</u>

Given $u \in C_c^\infty(M:\mathbb{C})$ there exists $s_1,\dots,s_N \in \Gamma_c^\infty(E)$ and $t_1,\dots,t_N \in \Gamma_c^\infty(E^*)$ such that for any $s \in \Gamma^\infty(E)$ and any $t \in \Gamma^\infty(E^*)$

$$u\,s = \sum_{j=1}^{N} < t_j,\ s > s_j$$

$$u\,t = \sum_{j=1}^{N} < t,\ s_j > t_j$$

<u>Proof</u>: By local triviality and a finite partition of unity argument over supp u.

If $s \in \Gamma^\infty(E)$ then multiplication by s induces a continuous linear map

$$\Gamma_c^\infty(1) \to \Gamma^\infty(E)$$

If $w \in \Gamma^\infty(F^*)$ then contraction by w induces a continuous linear map

$$\Gamma^\infty(F) \to \Gamma^\infty(1)$$

<u>lemma 5.14</u>

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a linear map. Then $wPs \in \mathcal{P}(1,1)$ for each $w \in \Gamma_c^\infty(F^*)$, $s \in \Gamma_c^\infty(E)$ if and only if for each $u,v \in C_c^\infty(M:\mathbb{C})$ we have $vPu \in \mathcal{P}(E,F)$.

<u>Proof</u>: Sufficiency: Given $w \in \Gamma_c^\infty(F^*)$, $s \in \Gamma_c^\infty(E)$ choose u,v

in $C_c(M;\mathbb{C})$ such that $vw = w$ and $us = s$. Then $wPs = w(vPu)s \in P(1,1)$ is clear by continuity of the maps induc ' by w and s.

Necessity: Given $v \in C_c^\infty(M;\mathbb{C})$ by lemma 5.13 we can find $w_1,\ldots,w_q \in \Gamma_c^\infty(F^*)$ and $h_1,\ldots,h_q \in \Gamma_c^\infty(F)$ such that for any $h \in \Gamma^\infty(F)$

$$vh = \sum_{k=1}^{q} < w_k , h > h_k$$

Given any $u \in C_c^\infty(M;\mathbb{C})$ by lemma 5.13 we can find $t_1,\ldots,t_p \in \Gamma_c^\infty(E^*)$ and $s_1,\ldots,s_p \in \Gamma_c^\infty(E)$ such that for any . ε .c(E)

$$us = \sum_{\ell=1}^{p} < t_\ell , s > s_\ell$$

Then $$vPu = \sum_{k=1}^{q} \sum_{\ell=1}^{p} \{ (w_k P s_\ell) \cdot < t_\ell , \cdot > \} \cdot h_k$$

Now by hypothesis $w_k Ps_\ell \in P(1,1)$ and hence $(w_k Ps_\ell) \cdot < t_\ell , \cdot >$ is in $P(E,1)$ which implies that $(w_k Ps_\ell) \cdot < t_\ell, \cdot> h_k$ is in $P(E,F)$.

<u>Corollary 5.15</u>

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a linear map and let $\mathcal{U}$ be an open cover of M. Suppose <u>one</u> of the following hypotheses holds:

(a) P is pseudo-local

(b) $\{ U \times U : U \in \mathcal{U} \}$ is an open cover of $M \times M$

It follows if for each $U \in \mathcal{U}$, $w \in \Gamma_c^\infty(F^*|_U)$, $s \in \Gamma_c^\infty(E|_U)$ we have $wPs \in P(1_{|U}, 1_{|U})$ with exponents $z_0,z_1\ldots$ then $P \in P(E,F)$ with exponents $z_0,z_1\ldots\ldots$

<u>Proof</u>: By lemma 5.14 if $u,v \in C_c^\infty(U:\mathbb{C})$ then $vPu \in P(E_{|U},F_{|U})$ with exponents $z_0,z_1,\ldots$ Hence by the proof of lemma 5.9, $P_{|U} \in P(E_{|U},F_{|U})$ with exponents $z_0,z_1\ldots$ and so by lemma 5.9 $P \in P(E,F)$ with exponents $z_0,z_1,\ldots$.

Remark 5.16

If E and F have constant dimension then the set V of all open subsets V of M such that $\bar{V}$ is compact and E and F are trivial over a neighborhood of $\bar{V}$ is an open cover of M and moreover $\{ V \times V : V \in V \}$ is an open cover of $M \times M$. Hence Corollary 5.14 implies that a linear map $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ is an element of $P(E,F)$ if and only if for each $V \in V$, $P_{|V}$ is a matrix (relative to local frames) of pseudo-differential operators acting on functions, with some given sequence of exponents $z_0, z_1,\ldots$ independent of V.

Remark 5.17

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a continuous linear map and let $z_0,z_1\ldots$ be a sequence of exponents. Then the definition of a pseudo-differential operator may be expressed in the form:

Given $s \in \Gamma_c^\infty(E)$, K a compact subset of $C^\infty(M;\mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp s, and a continuous semi-norm ρ on $\Gamma^\infty(F)$ then is a sequence of reals $t_0 \geq t_1 \geq t_2 \ldots \to -\infty$ (possibly depending on ρ) such that

$$\rho(e^{-i\lambda g} P(e^{i\lambda g} s) - \sum_{j<N} P_j(s,g)\lambda^{z_j}) = O(\lambda^{t_N})$$

uniformly for $g \in K$, as $\lambda \to +\infty$.

This formulation has the advantage that the t_N's are quite arbitrary, so that when proving a given operator is a pseudo-differential operator we do not require explicit information regarding the order of growth of the 'remainder terms'. This formulation of the definition is used implicitly by Hörmander [8].

§6 Local Analysis of Pseudo-differential Operators

This section is a review of some of the results of L. Hörmander [8]. Only a slight modification of Hörmander's proofs is necessary to allow for complex degrees of homogenuity. We assume in this section that M is an open submanifold of $\mathbb{R}^m$.

Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, z_2, \ldots\ldots$

If $s \in \Gamma_c^\infty(E)$ we define

$$p(s,\xi) = e^{-i<\cdot,\xi>} P(e^{i<\cdot,\xi>} s) \quad , \quad \xi \in \mathbb{R}^m$$

Then $p(s,\cdot)(\cdot) \in \Gamma^\infty(\mathbb{R}^m \times F)$ by continuity of P. We also define

$$p_j(s,\xi) = P_j(s, <\cdot,\xi>) \quad , \quad \xi \in \mathbb{R}^m - (0)$$

Then p_j is positively homogeneous of degree z_j in ξ and we have an asymptotic expansion (in $\Gamma^\infty(F)$)

$$p(s,\lambda\xi) \sim \sum_{j=0}^{\infty} p_j(s,\xi)\lambda^{z_j} \quad , \quad \lambda \to \infty$$

uniformly for ξ in compact subsets of $\mathbb{R}^m - (0)$. Then according to [8, lemma 2.3] we have

lemma 6.1

$p_j(s,\cdot) \in \Gamma^\infty((\mathbb{R}^m - (0)) \times F)$ and if $N > 0$ is an integer, α is an m-multi-index, and ρ is any continuous semi-norm on $\Gamma^\infty(F)$ then

$$\rho(p^{(\alpha)}(s,\xi) - \sum_{j<N} p_j^{(\alpha)}(s,\xi)) = O(|\xi|^{Re\, z_N - |\alpha|})$$

for $|\xi| \geq 1$, where $p^{(\alpha)} = (\frac{\partial}{\partial\xi})^\alpha p$

Corollary 6.2

For each continuous semi-norm ρ on $\Gamma^{\infty}(F)$ and each multi-index α there is a constant C such that

$$\rho(p^{(\alpha)}(s,\xi)) \leq C(1+|\xi|)^{Re\, z_0 - |\alpha|}$$

Corollary 6.2 is the key property in the definition of a larger class of pseudo-differential operators in Hörmander [9].

Now let $v \in C_c^{\infty}(M:\mathbb{C})$. If $\hat{v}$ is the Fourier transform of v, then by the inversion formula, for any $s \in \Gamma_c^{\infty}(E)$ we have

$$vs = (2\pi)^{-m} \int e^{i<\cdot,\xi>} s\, \hat{v}(\xi)d\xi$$

where the integral converges absolutely in $\Gamma_c^{\infty}(E)$. By the continuity of P it follows that

$$P(vs) = (2\pi)^{-m} \int e^{i<\cdot,\xi>} p(s,\xi)\, \hat{v}(\xi)d\xi$$

Thus it is natural to consider integral operators of this form where the kernel has an asymptotic expansion of the form given in lemma 6.1. Concerning the existence of such kernels we have [8, proposition 3.1].

lemma 6.3

Let $z_0, z_1,$ be a sequence of exponents. Suppose $K_j \in \Gamma^{\infty}(F \times \mathbb{R}^m - (0))$ is positively homogeneous of degree z_j in $\xi \in \mathbb{R}^m - (0)$. Then there exists $K \in \Gamma^{\infty}(F \times \mathbb{R}^m)$ such that for each continuous semi-norm ρ on $\Gamma^{\infty}(F)$, each integer $N > 0$, and each multi-index β we have

$$\rho(K^{(\beta)}(\cdot,\xi) - \sum_{j<N} K_j^{(\beta)}(\cdot,\xi)) = O(|\xi|^{Re\, z_N - |\beta|})$$

for $|\xi| \geq 1$.

Let K_j and K be as in lemma 6.3. Then we have an estimate of the form given in corollary 6.2 and hence we may define a continuous linear map

$$Q : \Gamma_c^\infty(1) \to \Gamma^\infty(F)$$

by

$$Qu = (2\pi)^{-m} \int e^{i < \cdot,\xi >} K(\cdot,\xi)\hat{u}(\xi)d\xi$$

Then Hörmander [8, lemma 3.2] proves

lemma 6.4

For each $u \in \Gamma_c^\infty(1)$

$$e^{-i < x,\xi >} Q(e^{i < \cdot,\xi >} u)(x) \sim \sum_{\alpha,j} \frac{i^{-|\alpha|}}{\alpha!} K_j^{(\alpha)}(x,\xi)\left(\frac{\partial}{\partial x}\right)^\alpha u(x)$$

as $|\xi| \to \infty$. More precisely, if $N,J > 0$ are integers and ν is a continuous semi-norm on $\Gamma^0(F)$ then

$$\nu(e^{-i < \cdot,\xi >} Q(e^{i < \cdot,\xi >} u) - \sum_{\substack{|\alpha|<N \\ j<J}} \frac{i^{-|\alpha|}}{\alpha!} K_j^{(\alpha)}(\cdot,\xi)\left(\frac{\partial}{\partial x}\right)^\alpha u)$$

$$= O(|\xi|^{Re\, z_0 - N} + |\xi|^{Re\, z_J})$$

for $|\xi| \geq 1$, uniformly for u in bounded subsets of $\Gamma_c^\infty(1)$.

This lemma together with some careful Fourier transform estimates [8, lemmas 3.4 and 3.5] is then used to prove the main existence theorem [8, theorem 3.3] which may be stated as follows.

Theorem 6.5

$Q \in \mathcal{P}(1,F)$ with exponents $z_j - k$, $j,k = 0, 1, 2.....$
If $f \in \Gamma_c^\infty(1)$, $g \in C^\infty(M;\mathbb{R})$ and $dg \neq 0$ on supp f then

$$\sum_{\ell=0}^{\infty} Q_\ell(f,g)(x) = \sum_{\alpha,j} \frac{i^{-|\alpha|}}{\alpha!} K_j^{(\alpha)}(x,\xi_x)\left(\frac{\partial}{\partial x}\right)^\alpha (f\, e^{ih_x})(x)$$

where ξ_x = grad g(x) , $h_x(y) = g(y) - g(x) - \langle y - x, \xi_x \rangle$

Moreover if B_0 is a bounded subset of $\Gamma_c^\infty(1)$ and B is a bounded subset of $C^\infty(M;\mathbb{R})$ and there is a constant $c > 0$ such that $f \in B_0$ and $g \in B$ implies $|\text{grad } g(x)| \geq c$ on supp f, then the asymptotic expansion

$$e^{-i\lambda g}\, Q(e^{i\lambda g} f) \sim \sum_{\ell=0}^{\infty} Q_\ell(f,\lambda g)$$

is uniform for $f \in B_0$ and $g \in B$.

§7 Composition of Pseudo-Differential Operators

If B_0 is a bounded subset of $\Gamma_c^\infty(E)$ we define supp B_0 to be the closure of the union of the supports of the elements of B_0. Thus supp B_0 is a compact set.

Theorem 7.1

Choose a Riemannian metric on the cotangent bundle T^*. Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$

Let B_0 be a bounded subset of $\Gamma_c^\infty(E)$ and let B be a bounded subset of $C^\infty(M;\mathbb{R})$ such that there is a constant $c > 0$ such that if $s \in B_0$, $g \in B$ then $|dg(x)| \geq c$ for $x \in$ supp s.

Then the asymptotic expansion associated to P

$$e^{-i\lambda g}\, P\,(e^{i\lambda g}\, s) \sim \sum_{k=0}^{\infty} P_k(s,g)\lambda^{z_k}$$

is uniform for $s \in B_0$ and $g \in B$. Moreover for each k

$$\{\, P_k(s,g) : s \in B_0\,,\ g \in B \,\}$$

is a bounded subset of $\Gamma_c^\infty(F)$.

Note if B_0 is a bounded subset of $\Gamma_c^\infty(E)$ and K is a compact subset of $C^\infty(M;\mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp B_0, then there is a constant $c > 0$ such that the hypotheses above are satisfied with $B = K$.

<u>**Proof**</u>: The last statement follows from the fact that the map

$$C^{\infty}(M \times \mathbb{R}) \times M \to \mathbb{R} \; : \; (g,x) \to |dg(x)|$$

is continuous and positive on the compact set $K \times \operatorname{supp} B_0$.

To prove the first part we observe since $\operatorname{supp} B_0$ is compact, by lemma 5.13 there exist $s_1,\ldots,s_N \in \Gamma_c^{\infty}(E)$ and $t_1,\ldots,t_N \in \Gamma_c^{\infty}(E^*)$ such that for any $s \in B_0$ we have

$$s = \sum_{k=1}^{N} < t_k, s > s_k$$

Then $B_{0,k} = \{ < t_k, s > : s \in B_0 \}$ is a bounded subset of $\Gamma^{\infty}{}_c(1)$, $\operatorname{supp} B_{0,k} \subseteq \operatorname{supp} B_0$ and

$$Ps = \sum_{k=1}^{N} (Ps_k)(<t_k, s >) \quad , \; s \in B_0$$

Hence it suffices to consider the operators Ps_k , i.e. we may assume $E = 1$.

By a finite partition of unity argument, since $\operatorname{supp} B_0$ is compact, it suffices to prove the theorem for the operators Pu, for any $u \in C_c^{\infty}(U:\mathbb{C})$, for any chart (U,ϕ).

By a finite partition of unity argument and the definition of a bounded subset of $\Gamma^{\infty}(F)$ it now suffices to show that the asymptotic expansion associated to vPu is uniform over $\operatorname{supp} v$ for any $v \in C_c^{\infty}(V;\mathbb{C})$, for any chart (V,ψ) .

Given u,v as above we may write $u = u_1 + u_2$, $v = v_1 + v_2$ where $u_i \in C_c^{\infty}(U:\mathbb{C})$, $v_i \in C_c^{\infty}(V:\mathbb{C})$ and

$$\text{supp } v_1 \cup \text{supp } u \subseteq U$$
$$\text{supp } v_2 \cup \text{supp } u_1 \subseteq V$$
$$\text{supp } v_2 \cap \text{supp } u_2 = \emptyset$$

Then cutting down and translating the charts associated to U and V we can produce a chart (W,θ) such that

$$\text{supp } v_2 \cup \text{supp } u_2 \subseteq W$$

Since $vPu = v_1Pu + v_2Pu + v_2Pu_2$ it follows that it suffices to prove if (U,ϕ) is a chart on M, $u,v \in C_c^\infty(U:\mathbb{C})$ then the asymptotic expansion associated to the operator vPu is uniform on the sense required over supp v. Thus we may assume that M is an open submanifold of $\mathbb{R}^m$.

If we pick $u \in C_c^\infty(M:\mathbb{C})$ such that $u = 1$ in a neighborhood of supp B_0 then the operator $Pu \in P(1,F)$ is an integral operator of the type considered in theorem 6.5 and hence the conclusion follows since the existence of the constant c in the hypotheses here for one Riemannian metric on T^*, implies the existence of such a constant for any Riemannian metric (since supp B_0 is compact) and hence in particular for the natural Riemannian metric on T^* when M is an open submanifold of $\mathbb{R}^m$.

Thus we have proved the first part. Then for any integer $N \geq 0$ we have

$$P_N(s,g) = (e^{-ig}P(e^{ig}\, s) - \sum_{j<N} P_j(s,g))$$
$$- (e^{-ig}P(e^{ig}\, s) - \sum_{j\leq N} P_j(s,g))$$

If $N > 0$ the right hand side is bounded in $\Gamma^\infty(F)$ uniformly for $s \in B_0$, $g \in B$ by what we have just proved, and if $N = 0$, the right hand side is again bounded uniformly for $s \in B_0$, $g \in B$ by continuity of P and by what we have just proved. Since $\operatorname{supp} P_N(s,g) \subseteq \operatorname{supp} B_0$ it follows that $\{ P_N(s,g) : s \in B_0, g \in B \}$ is a bounded subset of $\Gamma_c^\infty(F)$.

<u>lemma 7.2</u>

If $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ is a continuous linear map, B_0 is a bounded subset of $\Gamma^\infty_c(E)$, and B is a bounded subset of $C^\infty(M:\mathbb{R})$ then for each continuous semi-norm ρ on $\Gamma^\infty(F)$ there is an integer $n \geq 0$ such that

$$\rho(e^{-i\lambda g} P(e^{i\lambda g} s)) = \mathcal{O}(\lambda^n)$$

for $\lambda \geq 1$, uniformly for $g \in B$, $s \in B_0$. The integer n depends only on P, ρ and supp B_0.

<u>Proof</u>: The conclusion is clear since P is continuous and since there exists a differential operator $D \in \mathrm{Diff}(F,1)$ and a compact subset A of M such that for any $w \in \Gamma^\infty(F)$, $\rho(w) \leq \sup_A |D(w)|$.

<u>Theorem 7.3</u>

Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$ and let $Q \in \mathcal{P}(F,G)$ with exponents $w_0, w_1, \ldots$

(a) If $f \in C_c^\infty(M:\mathbb{C})$ then $QfP \in \mathcal{P}(E,G)$ with exponents $z_j + w_k$ and

$$\sum_{\ell=0}^{\infty} (QfP)_\ell(s,g) = \sum_{j,k=0}^{\infty} Q_k(fP_j(s,g),g)$$

(b) If P is almost local then $QP \in P(E,G)$ with exponents z_j+w_k and

$$\sum_{\ell=0}^{\infty} (QP)_\ell(s,g) = \sum_{j,k=0}^{\infty} Q_k(P_j(s,g),g)$$

Proof: Continuity of QfP in (a) is clear and continuity of QP in (b) follows by lemma 5.11. Then (b) follows from (a) by taking $f=1$ in a neighborhood of supp $s \cup$ supp Ps to obtain the appropriate asymptotic expansion. Hence it suffices to prove (a).

Let $s \in \Gamma_c^\infty(E)$, let K be a compact subset of $C^\infty(M;\mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp s. Then for each integer $N > 0$ there is a bounded subset B_N of $\Gamma^\infty(F)$ such that

$$e^{-i\lambda g}\, fP(e^{i\lambda g}\, s) - \sum_{j=0}^{N-1} f\, P_j(s,g)\lambda^{z_j} = \lambda^{z_N} f\, b(\lambda)$$

where $b(\lambda) \in B_N$, $\lambda \geq 1$.

Thus we obtain

$$e^{-i\lambda g}\, QfP(e^{i\lambda g}\, s) - \sum_{j=0}^{N-1} e^{-i\lambda g}\, Q(e^{i\lambda g}\, f\, P_j(s,g)\,)\lambda^{z_j}$$

$$= e^{-i\lambda g}\, Q(e^{i\lambda g}\, f\, b(\lambda)\,)\lambda^{z_N}$$

Since supp $fP_j(s,g) \subseteq$ supp s it follows by theorem 7.1 for each integer $J > 0$ there is a bounded subset $B_{J,j}$ of $\Gamma^\infty(G)$ such that

$$e^{-i\lambda g}\, Q(e^{i\lambda g}\, fP_j(s,g)\,) - \sum_{k=0}^{J-1} Q_k(fP_j(s,g),g)\lambda^{w_k}$$

$$= \lambda^{w_J}\, c_j(\lambda)$$

when $C_j(\lambda) \in B_{J,j}$, $\lambda \geq 1$, $g \in K$. It follows that

$$e^{-i\lambda g}\, QfP(e^{i\lambda g}\, s) - \sum_{j=0}^{N-1} \sum_{k=0}^{J-1} Q_k(fP_j(s,g),g)\lambda^{z_j + w_k}$$

$$= - \sum_{j=0}^{N-1} C_j(\lambda)\lambda^{w_J + z_j} + e^{-i\lambda g}\, Q(e^{i\lambda g}\, fb(\lambda))\lambda^{z_N}$$

Hence if ρ is any continuous semi-norm on $\Gamma^{\infty}(G)$ then by lemma 7.2 there is an integer $n \geq 0$ depending only on Q, ρ and supp f such that

$$\rho(e^{-i\lambda g}\, QfP(e^{i\lambda g}\, s) - \sum_{j=0}^{N-1} \sum_{k=0}^{J-1} Q_k(fP_j(s,g),g)\lambda^{z_j + w_k})$$

$$= O(\lambda^{Re(w_J + z_0)} + \lambda^{Re\, z_N + n})$$

as $\lambda \to \infty$, and hence by remark 5.17 we are done.

<u>Remark 7.4</u>

Let $P_r(E) = P_r(E,E)$

$P(E) = P(E,E)$

Suppose $P \in P(E,F)$, $Q \in P(F,G)$. Choose $P' \in P(E,F)$ such that $P - P' \in P_{-\infty}(E,F)$ and P' is almost local (lemma 5.10). Then $QP' \in P(E,G)$. Now if $P'' \in P(E,F)$, P" is almost local and $P - P'' \in P_{-\infty}(E,F)$ then $P' - P'' = (P - P'') - (P - P')$ is in $P_{-\infty}(E,F)$ and hence $QP' - QP'' \in P_{-\infty}(E,G)$, i.e. $QP' = QP''$ modulo $P_{-\infty}(E,G)$.

Then the composition of pseudo-differential operators induces the structure of an associative algebra on $P(E) / P_{-\infty}(E)$ and we denote this algebra by $P(E, \text{mod}-\infty)$.

If M is compact then every element of $P(E)$ is almost local and hence $P(E)$ itself is an associative algebra. In this case $P_{-\infty}(E)$ is an ideal in $P(E)$ and the quotient algebra $P(E) / P_{-\infty}(E)$ is of course $P(E, \text{mod}-\infty)$.

§8 Asymptotic Sums of Pseudo-Differential Operators

If $Q^0, Q^1 \ldots \in \mathcal{P}(E,F)$ then $P \in \mathcal{P}(E,F)$ is called an asymptotic sum of $Q^0, Q^1, \ldots$

$$P \sim \sum_{j=0}^{\infty} Q^j$$

if

$$\limsup_{n\to\infty} \text{order} \left(P - \sum_{j=0}^{n} Q^j \right) = -\infty$$

Note if we change the Q^j's by adding operators of order $-\infty$ then P will be an asymptotic sum of the new operators also.

Theorem 8.1

If $P \sim \sum_{j=0}^{\infty} Q^j$ then P is unique modulo operators of order $-\infty$, and if $s \in \Gamma_c^{\infty}(E)$, $g \in C^{\infty}(M;\mathbb{R})$ and $dg \neq 0$ on supp s then

$$(1) \quad \sum_{k=0}^{\infty} P_k(s,g) = \sum_{j,\ell=0}^{\infty} Q_\ell^j(s,g)$$

A necessary and sufficient condition for a sequence Q^j, $j=0,1,2\ldots$ in $\mathcal{P}(E,F)$ to admit an asymptotic sum is

$$(2) \quad \limsup_{n\to\infty} \text{order } Q^n = -\infty$$

Proof: Suppose P and P' are both asymptotic sums of the Q^j's. Then

$$P - P' = \left(P - \sum_{j=0}^{n} Q^j\right) - \left(P' - \sum_{j=0}^{n} Q^j\right)$$

implies $P - P' \in P_{-\infty}(E,F)$ since it follows from the definition of pseudo-differential operators that

$$\bigcap_{r \in \mathbb{R}} P_r(E,F) = P_{-\infty}(E,F)$$

Since

$$Q^n = (P - \sum_{j=0}^{n-1} Q^j) - (P - \sum_{j=0}^{n} Q^j)$$

the necessity of (2) is clear, and in addition (1) makes sense. (The equality of such formal sums means of course equality of the parts homogeneous of the same degree). Now for each $r \in \mathbb{R}$ there exists an integer $N \geq 0$ such that $n \geq N$ implies

$$\text{order } (P - \sum_{j=0}^{n} Q^j) \leq r$$

$$\text{order } Q^n \leq r$$

and hence (1) clearly follows.

It remains to prove that (2) is sufficient. When (2) holds then (1) makes sense, and we observe that it suffices to find $P \in P(E,F)$ such that (1) holds. By the proof of lemma 5.10 it suffices to consider the case where M is an open submanifold of $\mathbb{R}^m$. Then by lemma 6.1 and theorem 6.5 for each $s \in \Gamma_c^\infty(E)$ there exists $P^s \in P(1,F)$ such that

$$P^s \sim \sum_{j=0}^{\infty} Q^j s$$

Now let V be an open subset of M such that $\bar{V}$ is compact and there exists $s_1 .. s_p \in \Gamma_c^\infty(E)$ such that $s_1 .. s_p$ is a local frame for E in a neighborhood of $\bar{V}$. Let $t_1 . . t_p \in \Gamma_c^\infty(E^*)$ be such that $< t_i, s_j > = \delta_{ij}$ on V and then define $P' \in P(E,F)$ by

$$P's = \sum_{i=1}^{p} P^{s_i} < t_i, s >$$

Clearly $P'|_V \sim \sum_{j=0}^{\infty} Q^j|_V$. Then by the proof of lemma 5.10 again, we can construct $P \in P(E,F)$ such that $P \sim \sum_{j=0}^{\infty} Q^j$.

If $Q^0, Q^1 \ldots \in P(E,F)$ we say $Q \in P(E,F)$ is an <u>asymptotic limit</u> of the sequence $Q^0, Q^1 \ldots .$

$$Q = \lim_{n \to \infty} \text{asymp } Q^n$$

provided we have

$$\limsup_{n \to \infty} \text{ order } (Q - Q^n) = -\infty$$

Clearly this condition is equivalent to requiring that for some (and so far any) integer $k \geq 0$

$$Q - Q^k \sim \sum_{j=k}^{\infty} (Q^{j+1} - Q^j)$$

By theorem 8.1 a necessary and sufficient condition for the asymptotic limit to exist is

$$\limsup_{n \to \infty} \text{ order } (Q^{n+1} - Q^n) = -\infty$$

and hence it follows that

$$\text{order } Q \leq \limsup_{n \to \infty} \text{ order } Q^n \quad .$$

lemma 8.2

If $Q^0, Q^1 \ldots$ is a sequence in $P(E,F)$ of almost local operators and if

$$\limsup_{n\to\infty} \text{ order } (Q^{n+1} - Q^n) = -\infty$$

then there exists $Q \in P(E,F)$ such that Q is almost local and

$$Q = \operatorname*{lim\,asymp}_{n\to\infty} Q^n$$

If $P \in P(F,G)$ and $S \in P(E,G)$ and

$$\limsup_{n\to\infty} \text{ order } (PQ^n + S) = r \geq -\infty$$

then $PQ + S \in P_r(E,G)$.

Proof: Q exists by the above remarks and by lemma 5.10. By definition it follows that

$$PQ + S = \operatorname*{lim\,asymp}_{n\to\infty} (PQ^n + S)$$

and hence

$$\text{order } (PQ + S) \leq \limsup_{n} (PQ^n + S) = r.$$

§9 The Symbols of Pseudo-Differential Opetators

Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1 \ldots$ and let (U,ϕ) be a chart on M. If $s \in \Gamma_c^\infty(E_{|U})$ then the operator $(Ps)_{|U}$ is an integral operator of the form considered in theorem 6.5 and hence we have if $u \in C_c^\infty(U:\mathbb{C})$, $g \in C^\infty(M:\mathbb{R})$, $dg \neq 0$ on supp u, then

$$(1) \quad \sum_j P_j(us,g)(x) = \sum_{\alpha,j} \frac{i^{-|\alpha|}}{\alpha!} p_j^{(\alpha)}(s,\xi_x)(x)\left(\frac{\partial}{\partial\phi}\right)^\alpha (ue^{ih_x})(x)$$

where if $g_\xi = \sum \xi_k \phi_k$ $(\xi \in \mathbb{R}^m)$ then ξ_x is determined by $dg(x) = dg_{\xi_x}(x)$, and where $h_x = g - g(x) - g_{\xi_x} + g_{\xi_x}(x)$.

Theorem 9.1

Let n_ℓ be the largest integer such that $0 \leq n_\ell \leq Re(z_0 - z_\ell)$. Then we have:

(a) If $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M:\mathbb{R})$, $dg \neq 0$ on supp s, and $j_{n_\ell}(s)(x) = 0$ then $P_\ell(s,g)(x) = 0$.

(b) If $s \in \Gamma_c^\infty(E)$, $g,g' \in C^\infty(M:\mathbb{R})$, $dg \neq 0$ on supp s, $dg' \neq 0$ on supp s, and $j_{n_\ell}(dg)(x) = j_{n_\ell}(dg')(x)$ then

$$P_\ell(s,g)(x) = P_\ell(s,g')(x)$$

Proof: By lemma 5.3 it suffices to consider $x \in$ supp s, and we may cut down supp s. In particular we may assume that there is a chart

(U,ϕ) on M, an open neighborhood V of x, such that $\bar{V}$ is compact, $\bar{V} \subseteq U$, supp $s \subseteq V$, and E is trivial in a neighborhood of $\bar{V}$. Then there exist $s_1,\ldots,s_p \in \Gamma_c^\infty(E|_U)$ such that $s_1,\ldots,s_p$ is a local frame for E over V. Thus we have unique functions $u_1,\ldots,u_p \in C_c(V:\mathbb{R})$ such that

$$s = \sum u_k s_k$$

Then by (1) we have

$$(2) \quad P_\ell(s,g)(x) = \sum_{k=1}^{p} \sum_{\alpha,j,n} \frac{i^{-|\alpha|+n}}{\alpha!\, n!} p_j^{(\alpha)}(s_k,\xi_x)(x)\left(\frac{\partial}{\partial\phi}\right)^{\alpha}(u_k h_x^n)(x)$$

where we sum over $z_\ell - z_j = n-|\alpha|$ and $|\alpha| \geq 2n$. Moreover since $n-|\alpha| \leq n-2n \leq -n \leq 0$ it follows that in (2) we are summing only over $j \leq \ell$, for if $j > \ell$ then $Re(z_\ell - z_j) \geq 0$ and in the case where $Re(z_\ell - z_j) = 0$ we have $z_\ell - z_j$ is pure imaginary and (since $j > \ell$) is nonzero, and so cannot be equal to $n-|\alpha|$ which is an integer.

(a): By hypothesis u_k vanishes of order $n_\ell + 1$ at x, and hence $u_k h_x^n$ vanishes of order $n_\ell + 2n + 1$ at x. In (2) we differentiate $u_k h_x^n$, $|\alpha|$ times where $|\alpha|-n = z_j - z_\ell$. Since $j \leq \ell$ we have $|\alpha|-n = Re(z_j - z_\ell) \leq Re(z_0 - z_\ell)$ and so since $|\alpha|-n$ is an integer, $|\alpha|-n \leq n_\ell$. Thus $u_k h_x^n$ is differentiated at most $n_\ell + n$ times and hence $P_\ell(s,g)(x) = 0$.

(b): Let h_x' be the "h_x" corresponding to g'. Since $j_{n_\ell}(dg)(x) = j_{n_\ell}(dg')(x)$ it follows that $dg(x) = dg'(x)$ and so ξ_x is the same for both g and g'. Since $u_k h_x^n$ is differentiated at most $n_\ell + n$ times in (2) (same argument as in (a)) it follows by Leibnitz rule that to prove (b) it suffices to show that $(h_x)^n - (h_x')^n$ vanishes of order $n_\ell + n + 1$ at x. Since h_x and g (h_x' and g') have the same derivatives of order two and higher at x and since h_x (h_x') vanishes of order two at x if follows from $j_{n_\ell}(dg)(x) = j_{n_\ell}(dg')(x)$ that $h_x - h_x'$ vanishes of order $n_\ell + 2$ at x. Since h_x and h_x' vanish of order two at x it now follows that $(h_x)^n - (h_x')^n$ vanishes of order $n_\ell + n + 1$ at x which proves (b).

Theorem 9.2

Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$ and let n_k be the largest integer such that

$$0 \le n_k \le \mathcal{R}e(z_0 - z_k) \qquad k = 0,1,2,\ldots$$

Then for each integer $k \ge 0$ there exists a unique generalized symbol

$$\sigma_k(P) \quad \varepsilon \quad \mathrm{Smbl}_{z_k}^{n_k} \; (E,F)$$

such that if $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp s, then

$$P_k(s,g) = \sigma_k(P) \cdot j_{n_k}(dg) \cdot j_{n_k}(s)$$

Proof: By theorem 9.1, corollary 5.2 and theorem 3.1.

Thus the coefficients of the asymptotic series associated to P define a generalized formal symbol and so we have a linear map

$$\sigma : \mathcal{P}(E,F) \to \mathcal{GF}(E,F)$$

with kernel $\mathcal{P}_{-\infty}(E,F)$.

Theorem 9.3

Let $P \in \mathcal{P}(E,F)$ and let $Q \in \mathcal{P}(F,G)$.

(a) If $f \in C_c^\infty(M;\mathbb{C})$ then

$$\sigma(QfP) = \sigma(Q) \circ f\,\sigma(P)$$

(b) If P is almost local then

$$\sigma(QP) = \sigma(Q) \circ \sigma(P)$$

Proof: By theorem 7.3 and theorem 3.2

It follows that $\sigma : \mathcal{P}(E) \to \mathcal{GF}(E)$ induces an injective homomorphism of algebras

$$\sigma : \mathcal{P}(E, \text{mod-}\infty) \to \mathcal{GF}(E).$$

Now let ∇ be a covariant derivative on T^* and let D_E be a covariant derivative on E. Let $\partial^{(k)}$ be the total differentials induced by the pair (d,∇) and let $D_E^{(k)}$ be the total differentials induced by the pair (D_E, ∇). If $P \in \mathcal{P}(E,F)$ then for each integer $k \geq 0$ let $\tau_k(P)$ be the singular part of $\sigma_k(P)$ relative to (D_E,∇). Then

$$\tau_{(D_E,\nabla)}(P) = \sum_{k=0}^{\infty} \tau_k(P)$$

defines a linear map

$$\tau_{(D_E,\nabla)} : P(E,F) \to F(E,F)$$

and we have:

Theorem 9.4

$$(3) \quad \sigma(P) = \sum_{\substack{j,n,\ell=0 \\ 2\ell\leq n}}^{\infty} \frac{i^{\ell-n}}{\ell!n!} \tau_j(P)^{(n)} \circ \chi_{n,\ell}(D_E,\nabla)$$

Thus $\sigma(P)$ is uniquely determined by $\tau_{(D_E,\nabla)}(P)$ and the kernel of $\tau_{(D_E,\nabla)}$ is $P_{-\infty}(E,F)$.

Proof: Let $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;\mathbb{R})$, $dg\neq 0$ on supp s. Let $x \in M$, and let $N > 0$ be an integer.

Let (U,ϕ) be a strict chart on M with $x \in U$ such that $\phi(x) = 0$ and

$$(\partial^{(q)}\phi_j)(x) = 0, \quad 2 \leq q \leq N+1, \quad 1 \leq j \leq m$$

(exists by corollary 2.14).

Let $s_1,\ldots,s_p \in \Gamma_c^\infty(E_{|U})$ be a local frame for E over an open neighborhood $V \subseteq U$ of x such that

$$(D_E^{(q)} s_j)(x) = 0, \quad 1 \leq q \leq N, \quad 1 \leq j \leq p$$

(exists by corollary 2.11)

To prove equality in (3) at x, it suffices to prove equality up to a finite number of terms, increasing with N. For each choice of N (and corresponding U and V, etc.) since everything in (3) is local we may cut down the support of s, and in particular we may assume supp $s \subseteq V$. Then there exist unique functions $u_1,\dots,u_p \in C_c^\infty(V;\mathbb{R})$ such that

$$s = \sum_{k=1}^{p} u_k s_k$$

and by (1) we have

$$(4) \quad \sum_{j=0}^{\infty} P_j(s,g)(x) = \sum_{k=1}^{p} \sum_{\substack{\alpha,j,\ell \\ |\alpha| \geq 2\ell}} \frac{i^{-|\alpha|+\ell}}{\alpha!\ \ell!} p_j^{(\alpha)}(s_k,\xi_x)(x)\left(\frac{\partial}{\partial\phi}\right)^\alpha (u_k h_x^\ell)(x)$$

where ξ_x is determined by $dg(x) = dg_{\xi_x}(x)$ and where $h_x = g-g(x)-g_{\xi_x}$, where $g_\xi = \sum_{k=1}^{m} \xi_k \phi_k$.

Now

$$p_q(s_k,\xi) = P_q(s_k,g_\xi)$$

$$= \sigma_q(P) \cdot j_{n_q}(dq_\xi) \cdot j_{n_q}(s_k)$$

and so if $Re(z_0-z_j) \leq N$ then by remark 3.6

$$p_j(s_k,\xi)(x) = \tau_j(P) \cdot dg_\xi(x) \cdot s_k(x)$$

Then by the computation following theorem 3.4 (pg. 35) if $|\alpha| = n$ and $Re(z_0-z_j) \leq N$ we have

$$\tau_j(P)^{(n)} \cdot dg_\xi(x) \cdot (d\phi)^\alpha(x) \otimes s_k(x) = p_j^{(\alpha)}(s_k,\xi)(x)$$

Thus by theorem 2.15 (1) if $n \leq N$, $Re(z_0-z_j) \leq N$ we have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} p_j^{(\alpha)}(s_k,\xi)(x)\left(\frac{\partial}{\partial\phi}\right)^\alpha (u_k h_x^\ell)(x)$$

$$= \frac{1}{n!} \tau_j(P)^{(n)} \cdot dg_\xi(x) \cdot D_E^{(n)}(h_x^\ell u_k s_k)(x)$$

Since $dg_{\xi_x}(x) = dg(x)$ it follows that

$$(5)\quad \sum_{k=1}^{p} \sum_{\alpha,j,\ell} \frac{i^{-|\alpha|+\ell}}{\alpha!\ \ell!}\ p_j^{(\alpha)}(s_k,\xi_x)(x)\left(\frac{\partial}{\partial\phi}\right)^{\alpha}(u_k\ h_x^{\ell})(x)$$
$$(\text{sum over } 2\ell \le |\alpha| \le N,\ Re(z_0-z_j) \le N)$$

$$= \sum_{n,j,\ell} \frac{i^{-n+\ell}}{n!\ \ell!}\ \tau_j(P)^{(n)} \cdot dg(x) \cdot D_E^{(n)}(h_x^{\ell}s)(x)$$
$$(\text{sum over } 2\ell \le n \le N,\ Re(z_0-z_j) \le N)$$

Now by the choice of the chart (U,ϕ) and by definition of h_x we have

$$(\partial^{(q)}\ h_x)(x) = (\partial^{(q)}\ g)(x) \qquad 2 \le q \le N+1$$

and hence by theorem 3.5 if $0 \le 2\ell \le n \le N$ then

$$\chi_{n,\ell}(D_E,\nabla) \cdot j_{n-\ell}(dg)(x) \cdot j_{n-\ell}(s)(x) = D_E^{(n)}(h_x^{\ell}\ s)(x)$$

Then by the definition of the composition of generalized symbols (theorem 3.2), (5) is equal to

$$\sum_{n,j,\ell} \frac{i^{-n+\ell}}{\alpha!\ell!}\ (\tau_j(P)^{(n)} \circ \chi_{n,\ell}(D_E,\nabla)\ ) \cdot j_{n-\ell}(dg)(x) \cdot j_{n-\ell}(s)(x)$$
$$(\text{sum over } 2\ell \le n \le N\ ,\ Re(z_0-z_j) \le N)$$

Now the $(\alpha,j,\ell)^{th}$ term in (5) is positively homogeneous of degree $z_j-|\alpha| + \ell$ and $Re(z_j-|\alpha| + \ell) \le Re\ z_j$. Hence it now follows that

$$\sum_{Re(z_0-z_j) \le N} \sigma_j(P) = \sum_{n,j,\ell} \frac{i^{-n+\ell}}{n!\ \ell!}\ \tau_j(P)^{(n)} \circ \chi_{n,\ell}(D_E,\nabla)$$
$$(\text{sum over } 2\ell \le n \le N\ ,\ Re(z_0-z_j) \le N,$$
$$Re\ z_j-n+\ell \ge Re z_k)$$

where $k \geq 0$ is the largest integer such that $Re(z_0 - z_k) \leq N$. Then the theorem follows since N is arbitrary.

Theorem 9.4 is the invariant form of the formula obtained by Hörmander for the generalized symbols in coordinates [8, theorem 4.2].

Now consider in addition to the covariant derivatives ∇ and D_E, a covariant derivative D_F on F. Let $D_F^{(k)}$ be the total differentials induced by the pair (D_F, ∇). Let $P \in \mathcal{P}(E,F)$, $Q \in \mathcal{P}(F,G)$, let $f \in C_c^\infty(M:\mathbb{C})$ and let $R = QfP$. Now let

$$\tau_{(D_E,\nabla)}(P) = \sum_{j=0}^{\infty} \tau_j(P)$$

$$\tau_{(D_F,\nabla)}(Q) = \sum_{k=0}^{\infty} \tau_k(Q)$$

$$\tau_{(D_E,\nabla)}(R) = \sum_{\ell=0}^{\infty} \tau_\ell(R)$$

Then we have the following coordinate free version of Hormander's composition formula [8, theorem 4.3].

<u>Theorem 9.5</u>

$$\tau_{(D_E,\nabla)}(R) = \tau_{(D_F,\nabla)}(Q) \cdot f\, \tau_{(D_E,\nabla)}(P)$$

i.e.

$$\sum_{\ell=0}^{\infty} \tau_\ell(R) = \sum_{n,j,k=0}^{\infty} \frac{i^{-n}}{n!} \tau_k(Q)^{(n)} \circ \left(D_F^{(n)} * f\, \tau_j(P)\right)$$

(The * operation is defined in Remark 3.7)

Proof: By theorem 9.4 and theorem 9.3 we have

$$\sum_{\ell=0}^{\infty} \sigma_\ell(R)$$

$$= \sum_{\substack{n,k,h=0 \\ 2h\le n}}^{\infty} \sum_{\substack{q,j,p=0 \\ 2p\le q}}^{\infty} \frac{i^{h-n+p-q}}{n!h!q!p!} \tau_k(Q)^{(n)} \circ \chi_{n,h}(D_F,\nabla) \circ \tau_j(fP)^{(q)} \circ \chi_{q,p}(D_E,\nabla)$$

Let the exponents of P be $z_0, z_1, \ldots$ and let the exponents of Q be $w_0, w_1, \ldots$ Then the $(n,k,h,q,j,p)^{th}$ term in the above expression is positively homogeneous of degree $w_k - n + h + z_j - q + p$ where $2h \le n$ and $2p \le q$. The exponents of R are $z_j + w_k$, $j,k = 0,1,2,\ldots$ which we order appropriately and write as r_ℓ, $\ell = 0,1,2,\ldots$ Then if $N > 0$ is an integer we have

$$(6) \qquad \sum_{\substack{\ell \\ Re(r_0 - r_\ell) \le N}} \sigma_\ell(R)$$

$$= \sum_{n,k,h} \sum_{q,j,p} \frac{i^{h-n+p-q}}{n!h!q!p!} \text{ (same as above)}$$

(sum over $2h \le n$, $2p \le q$, $Re(w_0 - w_k) + Re(z_0 - z_j) \le N$,

$Re(w_k - n + h + z_j - q + p) \ge Re\, r_{\ell'}$.)

where ℓ' is the largest integer such that $Re(r_0 - r_{\ell'}) \le N$. In order to prove the theorem we take the singular part of both sides of (5) relative to (D_E,∇). So let $x \in M$, $s \in \Gamma_c^\infty(E)$, $g \in C^\infty(M;R)$, $dg \ne 0$ on supp s and suppose that

$$g(x) = 0$$

$$(\partial^{(\ell)} g)(x) = 0 \quad , \quad 2 \le \ell \le N+1$$

$$(D^{(\ell)} g)(x) = 0 \quad , \quad 1 \le \ell \le N$$

Then we have

$$(7)\qquad \sum_{\substack{\ell \\ Re(r_0-r_\ell)\le N}} \sigma_\ell(R)\cdot j_{n_\ell}(dg)(x)\cdot j_{n_\ell}(s)(x)$$

$$= \sum_{\substack{\ell \\ Re(r_0-r_\ell)\le N}} \tau_\ell(R)\cdot dg(x)\cdot s(x)$$

where n_ℓ is the largest integer such that $0 \le n_\ell \le Re(r_0-r_\ell)$. The right hand side of (6) becomes

$$(8)\quad \sum\sum \frac{i^{h-n+p-q}}{n!h!p!q!}\ \tau_k(Q)^{(n)}\cdot dg(x)\cdot D_F^{(n)}$$

$$\left(h_x^h\ \tau_j(fP)^{(q)}\cdot dg\cdot(\chi_{q;p}(D_E,\nabla)\cdot j_{q-p}(dg)\cdot j_{q-p}(s)\ \right)(x)$$

where $h_x \in Z_x^2$ satisfies

$$(\partial^{(\ell)}h_x)(x) = (\partial^{(\ell)}g)(x) \qquad 2\le \ell\le N+1$$

and so by the condition on g we have

$$h_x \in Z_x^{N+2}$$

whence $h_x^h \in Z_x^{Nh+2h}$, $h_x^p \in Z_x^{Np+2p}$.

Then $\chi_{q;p}(D_E,\nabla)\cdot j_{q-p}(dg)\cdot j_{q-p}(s)$ vanishes of order $Np+2p-q$ at x, since it involves at most q derivatives of h_x^p. It follows that the $(n,k,h,q,j,p)^{th}$ term in (8) vanishes of order

$$N(h + p) + 2(h + p) - q - n$$

Now since $Re(w_k - n + h + z_j - q + p) \geq Re\, r_{\ell'}$, in (8) and $r_0 = w_0+z_0$ it follows that

$$-n+h-q+p \geq Re(w_0-w_k) + Re(z_0-z_j) - Re(r_0-r_{\ell'})$$

$$\geq Re(w_0-w_k) + Re(z_0-z_j) - N$$

$$\geq -N$$

and so

$$h + p \geq n + q - N$$

Thus $\quad N(h + p) + 2(h + p) - q - n \geq N(h + p) + n + q - 2N$

and so if $h \geq 1$ or $p \geq 1$ we have

$$N(h + p) + 2(h + p) - q - n$$

$$\geq n + q - N$$

$$\geq h + p$$

$$\geq 1$$

Thus all terms in (8) in which $h \geq 1$ or $p \geq 1$ vanish. Hence we are left only with terms in which $h = 0$ and $p = 0$, which gives

$$(9) \quad \sum \sum \frac{i^{-n-q}}{n!\, q!} \tau_k(Q)^{(n)} \cdot dg(x) \cdot D_F^{(n)} (\tau_j(fP)^{(q)} \cdot dg \cdot D_E^{(q)} s)(x)$$

and since we proved above that $h + p \geq n + q - N$ it follows that in (9) we have

$$n + q \leq N$$

Now by theorem 2.15 since

$$(D_E^{(\ell)} s)(x) = 0 \quad , \quad 1 \le \ell \le N$$

it follows that

$$D_E^{(q)} s$$

vanishes of order $N - q + 1 \ge n + 1$ at x if $1 \le q \le N$ and hence all terms in (9) for which $q \ge 1$ vanish, and so we are left only with the terms for $q = 0$. Thus

$$(10) \qquad \sum_{\substack{\ell \\ Re(r_0 - r_\ell) \le N}} \tau_\ell(R) \cdot dg(x) \cdot s(x)$$

$$= \sum_{n,j,k} \frac{i^{-n}}{n!} \tau_k(Q)^{(\;\;)} \cdot dg(x) \cdot D_F^{(n)}(\tau_j(fP) \cdot dg \cdot s)(x)$$

(sum over $Re(w_0 - w_k) + Re(z_0 - z_j) \le N$, $Re(w_k - n + z_j) \ge Re\, r_\ell$.)

Since in (9) we had $n + q \le N$ it follows that in (10) we have $n \le N$ and hence by the definition of $*$ (remark 3.7) the theorem follows.

Corollary 9.6

In addition to the hypotheses of theorem 9.5 assume that P is almost local. Then

$$\tau_{(D_E,\nabla)}(QP) = \tau_{(D_F,\nabla)}(Q) \cdot \tau_{(D_E,\nabla)}(P)$$

i.e.

$$\sum_{\ell=0}^{\infty} \tau_\ell(QP) = \sum_{n,j,k=0}^{\infty} \frac{i^{-n}}{n!} \tau_k(Q)^{(n)} \circ (D_F^{(n)} * \tau_j(P))$$

<u>Proof:</u> We simply omit f in the proof of theorem 9.5.

<u>lemma 9.7</u>

Let $\tau \in \mathrm{Smbl}_z(E,F)$. Then there exists $P \in \mathcal{P}(E,F)$ with exponents $z, z-1, z-2, \ldots$ such that

$$\sigma_0(P) = \tau$$

<u>Proof:</u> If (U,ϕ) is a chart on M such that E admits a local frame $s_1,\ldots,s_p \in \Gamma^\infty(E_{|U})$ define

$$K_j(x,\xi) = \tau \cdot dg_\xi(x) \cdot s_j(x)$$

Then by theorem 6.5 there exists $P_j \in \mathcal{P}(1_{|U}, F_{|U})$ such that

$$\sigma_0(P_j) \cdot dg \cdot f = \tau \cdot dg \cdot fs_j$$

and P_j has exponents $z, z-1, z-2, \ldots$.

Now let $t_1,\ldots,t_p \in \Gamma^\infty(E^*_{|U})$ be the dual frame to $s_1,\ldots,s_p$ and define

$$Ps = \sum_{j=1}^{p} P_j < t_j, s > , \quad s \in \Gamma_c^\infty(E_{|U}) .$$

Then $P \in \mathcal{P}(E_{|U}, F_{|U})$, $\sigma_0(P) = \tau_{|U}$ and P has exponents $z, z-1, z-2, \ldots$

Now globalize by a partition of unity argument as in lemma 5.10.

lemma 9.8

Let $\tau \in \mathrm{Smbl}_z(E,F)$. Then there exists $Q \in P(E,F)$ with exponents $z, z-1, z-2 \ldots$ such that

$$\tau_{(D_E,\nabla)}(Q) = \tau$$

i.e. the formal symbol of Q relative to (D_E,∇) consists of the single term τ.

It follows that the generalized formal symbol of Q is given by by

$$\sigma(Q) = \sum_{\substack{n,\ell=0 \\ 2\ell\leq n}}^{\infty} \frac{i^{\ell-n}}{\ell!n!} \tau^{(n)} \circ \chi_{n,\ell}(D_E,\nabla)$$

Proof: The last statement follows by theorem 9.4. By lemma 9.7 there is $Q_0 \in P(E,F)$ with exponents $z, z-1, z-2, \ldots$ such that

$$\sigma_0(Q_0) = \tau$$

Now define $\tau^1 \in F(E,F)$ by

$$\tau^1 = \tau - \tau_{(D_E,\nabla)}(Q_0)$$

so τ^1 has exponents $z-1,\ z-2,\ z-3, \ldots.$

Suppose we have defined $\tau^n \in F(E,F)$ $(n \geq 1)$

$$\tau^n = \sum_{k=0}^{\infty} \tau_k^n$$

with exponents $z-n,\ z-n-1,\ z-n-2, \ldots.$ Then by lemma 9.7 there is $Q_n \in P(E,F)$ with exponents $z-n, z-n-1, z-n-2, \ldots$ such that

$$\sigma_0(Q_n) = \tau_0^{\,n}$$

Now define

$$\tau^{n+1} = \tau^n - \tau_{(D_E,\nabla)}(Q_n)$$

Then τ^{n+1} has exponents $z-n-1$, $z-n-2$, $z-n-3$, .. and so we may proceed inductively. Then the order of Q_n is at most $\mathcal{R}e\, z - n$ and hence by theorem 8.1 an asymptotic sum

$$Q \sim \sum_{n=0}^{\infty} Q_n$$

exists, and moreover

$$\sigma(Q) = \sum_{n=0}^{\infty} \sigma(Q_n)$$

(this sum makes sense since the orders are decreasing to $-\infty$).

It follows that

$$\tau_{(D_E,\nabla)}(Q) = \sum_{n=0}^{\infty} \tau_{(D_E,\nabla)}(Q_n)$$

$$= \sum_{n=0}^{\infty} (\tau^n - \tau^{n+1})$$

$$= \tau^0 = \tau.$$

Theorem 9.9

The sequence

$$0 \longrightarrow \mathcal{P}_{-\infty}(E,F) \longrightarrow \mathcal{P}(E,F) \xrightarrow{\tau_{(D_E,\nabla)}} \mathcal{F}(E,F) \longrightarrow 0$$

is exact, and moreover

$$\tau_{(D_E,\nabla)} : \mathcal{P}(E) \to \mathcal{F}(E)$$

induces an isomorphism of algebras

$$\tau_{(D_E,\nabla)} : \mathcal{P}(E,\text{mod-}\infty) \to \mathcal{F}_{(D_E,\nabla)}(E)$$

In particular it follows that $\mathcal{F}_{(D_E,\nabla)}(E)$ is an associative algebra.

Proof: By corollary 9.6 and theorem 9.4 it suffices to prove that

$$\tau_{(D_E,\nabla)} : \mathcal{P}(E,F) \to \mathcal{F}(E,F)$$

is surjective.

Let $\tau = \sum_{j=0}^{\infty} \tau_j \in \mathcal{F}(E,F)$ have exponents $z_0, z_1, z_2, \ldots$ By lemma 9.8 for each integer $k \geq 0$ there exists $Q_k \in \mathcal{P}(E,F)$ with exponents z_k, z_k-1, $z_k-2, \ldots$ such that

$$\tau_{(D_E,\nabla)}(Q_k) = \tau_k .$$

Then

$$\limsup_{k\to\infty} \text{order } Q_k \leq \limsup_{k\to\infty} \mathcal{R}e\, z_k = -\infty$$

and hence by theorem 8.1 an asymptotic sum

$$Q \sim \sum_{k=0}^{\infty} Q_k$$

exists, and moreover

$$\sigma(Q) = \sum_{k=0}^{\infty} \sigma(Q_k) .$$

It follows that

$$\tau_{(D_E,\nabla)}(Q) = \sum_{k=0}^{\infty} \tau_{(D_E,\nabla)}(Q_k) = \sum_{k=0}^{\infty} \tau_k = \tau$$

Thus $\tau_{(D_E,\nabla)}$ is surjective.

Remark 9.10

Suppose we say that a generalized formal symbol $\sigma \in GF(E,F)$ is integrable if there exists $P \in P(E,F)$ such that $\sigma(P) = \sigma$. Then theorem 9.9 and theorem 9.4 imply that $\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell \in GF(E,F)$ is integrable if and only if for some (and hence for any) pair (D,∇) (where D is a covariant derivative on E, and ∇ is a covariant derivative on T^*) we have

$$\sum_{\ell=0}^{\infty} \sigma_\ell = \sum_{\substack{j,n,k=0 \\ 2k \leq n}}^{\infty} \frac{i^{k-n}}{k!n!} \tau_j^{(n)} \circ \chi_{n,k}(D,\nabla)$$

where τ_j is the singular part of σ_j relative to (D,∇).

Notice that the generalized symbols $\chi_{n,k}(D,\nabla)$ extend naturally to C^∞ sections over all of Ψ^{n-k}. Thus the τ_j contain all the singularities of the σ_ℓ. This fact was the motivation for the name 'singular part'.

§10 The Geometric Transpose of Generalized Symbols

The orientation bundle Θ of M is the real C^∞ line bundle associated to the frame bundle $\mathcal{J}$ of M by the one-dimensional representation of $GL(m:\mathbb{R})$ which sends a matrix into the sign of its determinant. (see M. F. Atiyah and R. Bott [1 and 2]). If (U,ϕ) is a chart on M let $[\phi]$ be the C^∞ section of $\Theta_{|U}$ which corresponds to the unique $GL(m:\mathbb{R})$-map $\mathcal{J}_{|U} \to \mathbb{R}$ which sends the frame $d\phi_1,\dots,d\phi_m$ onto 1. (Here the left action of $GL(m:\mathbb{R})$ on $\mathbb{R}$ is multiplication by the sign of the determinant.) Then $[\phi]$ is a nonvanishing C^∞ section of Θ over U, i.e., a local frame for Θ over U.

We define the volume bundle or density bundle of M by

$$\Omega = \Theta \otimes \Lambda^m T^*$$

If $\rho \in \Gamma^k(\Omega)$ $(0 \leq k \leq \infty)$, then for each chart (U,ϕ) on M we have a unique function $\rho_\phi \in C^k(U:\mathbb{R})$ such that

$$\rho_{|U} = \rho_\phi [\phi] \otimes d\phi_1 \wedge \dots \wedge d\phi_m$$

The functions ρ_ϕ constitute a C^k density on M. A C^∞ density ρ is called a smooth positive measure on M if for each chart (U,ϕ) we have $\rho_\phi > 0$. Since M is paracompact there exists a Riemannian metric on T^*. Such a metric induces a smooth positive measure on M, which is clearly a frame for Ω. Thus Ω is trivial.

By S. Sternberg [20, ch. III §3] there exists a unique continuous

$\mathbb{R}$-linear map

$$\int_M : \Gamma_c^\infty(\Omega) \to \mathbb{R}$$

such that if (U,ϕ) is a chart on M and $\rho \in \Gamma_c^\infty(\Omega_{|U})$ then

$$\int_M \rho = \int_{\mathbb{R}^m} (\rho_\phi \circ \phi^{-1})(x)dx$$

If E is a complex C^∞ vector bundle on M we define the <u>geometric dual</u> E' of E by

$$E' = \mathrm{Hom}(E, \Omega \otimes \mathbb{C}) = E^* \otimes \Omega$$

Since Ω is a trivial line bundle, $\Omega^* \otimes \Omega$ is canonically isomorphic to 1 and hence we have a natural isomorphism $E'' = E$. In particular

$$(S^k T^* \otimes E)' = S^k T \otimes E'.$$

The existence of adjoints of differential operators [13] implies that we have a natural $\mathbb{C}$-linear isomorphism

$$\mathrm{HOM}(J^n(E),F) \simeq \mathrm{HOM}(J^n(F'),E') : \mu \to \mu'$$

such that if $s \in \Gamma_c^\infty(E)$, $w \in \Gamma_c^\infty(F')$ then

$$\int_M < w,\mu \cdot j_n(s) > = \int_M < \mu' \cdot j_n(w), s >$$

<u>lemma 10,1</u>

Let $\sigma \in \mathrm{Smbl}_z^n(E,F)$. Then there exists a unique $\sigma' \in \mathrm{Smbl}_z^{2n}(F',E')$

such that if $s \in \Gamma_c^\infty(E)$, $w \in \Gamma_c^\infty(F')$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on $\operatorname{supp} s \cup \operatorname{supp} w$ then

$$\int_M \langle w, \sigma \cdot j_n(-dg) \cdot j_n(s) \rangle = \int_M \langle \sigma' \cdot j_{2n}(dg) \cdot j_{2n}(w), s \rangle$$

(note the minus sign)

<u>Proof</u>: Let $g \in C^\infty(M;\mathbb{R})$ and let U be an open subset of M such that $dg \neq 0$ on U. Then

$$\sigma \cdot j_n(-dg) \in \operatorname{Hom}(J^n(E)_{|U}, F_{|U})$$

admits a unique transpose

$$(\sigma \cdot j_n(-dg))' \in \operatorname{Hom}(J^n(F')_{|U}, E'_{|U}) \subseteq \operatorname{Hom}(J^{2n}(F')_{|U}, E'_{|U})$$

If $w \in \Gamma_c^\infty(F'_{|U})$ define

$$P(w,g) = (\sigma \cdot j_n(-dg))' \cdot j_{2n}(w)$$

By uniqueness this definition is independent of U, and hence for any $w \in \Gamma_c^\infty(F')$, $g \in C^\infty(M;\mathbb{R})$ such that $dg \neq 0$ on $\operatorname{supp} w$

$$P(w,g) \in \Gamma^\infty(E')$$

is well-defined.

It remains to verify the hypotheses of theorem 3.1. (a), (b) and (c) are clear, so it suffices to prove (d). Suppose $w \in \Gamma_c^\infty(F')$, $g_1, g_2 \in C^\infty(M;\mathbb{R})$, $dg_1 \neq 0$ on supp w, $dg_2 \neq 0$ on supp w and suppose $x \in M$ and $j_{2n}(dg_1)(x) = j_{2n}(dg_2)(x)$. Clearly $\operatorname{supp} P(s,g_i) \subseteq \operatorname{supp} w$ $i=1,2$, so without loss of generality we may assume $x \in \operatorname{supp} w$. Let U be an open subset of M such that

$$\text{supp } w \subseteq U \subseteq \{ y \in M : dg_1(y) \neq 0 \quad \text{and} \quad dg_2(y) \neq 0 \}$$

and consider the differential operators $\sigma \cdot j_n(-dg_1)$, $\sigma \cdot j_n(-dg_2)$ on U (they are well-defined by choice of U). If we look at the coordinate representations, the coefficients of the transpose involve derivatives of order at most n of the coefficients of the given operators, and hence by the chain rule, derivatives of the coefficients of dg_1 and dg_2 of at most order 2n. Thus since

$$j_{2n}(dg_1)(x) = j_{2n}(dg_2)(x)$$

we have that

$$(\sigma \cdot j_n(-dg_1))' \cdot j_{2n}(w)(x) = (\sigma \cdot j_n(-dg_2))' \cdot j_{2n}(w)(x)$$

and so

$$P(w,g_1)(x) = P(w,g_2)(x)$$

which verifies hypothesis (d) of theorem 3.1.

Remark 10.2

Lemma 10.1 is rather unsatisfactory since we get 2n where we would prefer to have n. If $\sigma \in \text{Smbl}_z^n(E,F)$ occurs as a generalized symbol of a pseudo-differential operator we will see that we actually have $\sigma' \in \text{Smbl}_z^n(F',E')$. This anomaly suggests that the modules $\text{Smbl}_z^n(E,F)$ are really too big, and that certain submodules may be more natural in our context.

Let D_E be a covariant derivative on E and let ∇ be a covariant derivative on T^*. Let ℓ, k be integers with $0 \leq 2\ell \leq k$ and consider

$$\chi_{k,\ell}(D_E,\nabla) \in \mathrm{Smbl}_{\ell}^{k-\ell}(E, S^k T^* \otimes E)$$

(see theorem 3,5)

<u>lemma 10,3</u>

$$(\chi_{k,\ell}(D_E,\nabla))' \in \mathrm{Smbl}_{\ell}^{k-\ell}(S^k T \otimes E', E')$$

and if $u \in \Gamma^{\infty}(T)$, $v \in \Gamma_c^{\infty}(E')$, $g \in C^{\infty}(M;\mathbb{R})$ $dg \neq 0$ on supp v, then

$$(\chi_{k,\ell}(D_E,\nabla))' \cdot j_{k-\ell}(dg) \cdot j_{k-\ell}(s)$$

$$= (-1)^{\ell} \sum_{q=0}^{k} \binom{k}{q} (D^{(k-q)})' (H_q\, s^{k-q}\, u \otimes v)$$

where if for each $x \in M$ we choose $h_x \in C^{\infty}(M;\mathbb{R})$ such that

(1) $h_x \in Z_x^2$

(2) $(\partial^{(n)} h_x)(x) = (\partial^{(n)} g)(x)$, $2 \leq n \leq k+1$

(3) $x \to j_k(dh_x)(x)$ is a C^{∞} section of Ψ^k .

then

$$H_q(x) = \langle s^{n} u, \partial^{(q)} (h_x^{\ell}) \rangle (x)$$

<u>**Proof:**</u> First note if g , $g' \in C^{\infty}(M;\mathbb{R})$ and

$$j_{k-\ell}(dg)(x) = j_{k-\ell}(dg')(x)$$

then $(h_x)^{\ell} - (h'_x)^{\ell} \in Z_x^{k+\ell+1} \subseteq Z_x^{k+1}$ and hence $H_q - H'_q \in Z_x^{k-q+1}$

whence

$$(D^{(k-q)})'(H_q \, s^{k-q} \, u \otimes v)(x) = (D^{(k-q)})'(H'_q \, s^{k-q} \, u \otimes v)(x)$$

Hence the first statement follows once we prove the formula for $(\chi_{k,\ell}(D_E,\nabla))'$.

Let $s \in \Gamma_c^{\infty}(E)$. Then

$$\chi_{k,\ell}(D_E,\nabla) \cdot j_{k-\ell}(-dg)(x) \cdot j_{k-\ell}(s)(x)$$

$$= (-1)^{\ell} \, D_E^{(k)} \, (h_x^{\ell} \, s)(x)$$

$$= (-1)^{\ell} \sum_{q=0}^{k} \binom{k}{q} \, (\partial^{(q)} \, h_x^{\ell}) \odot (D_E^{(k-q)} \, s)(x)$$

(by theorem 2,7) and hence

$$< s^k \, u \otimes v \, , \; \chi_{k,\ell}(D_E,\nabla) \cdot j_{k-\ell}(-dg) \cdot j_{k-\ell}(s) > (x)$$

$$= (-1)^{\ell} \sum_{q=0}^{k} \binom{k}{q} < s^q u, \partial^{(q)}(h_x^{\ell}) > (x) < s^{k-q} u \otimes v, D_E^{(k-q)} s > (x)$$

$$= (-1)^{\ell} \sum_{q=0}^{k} \binom{k}{q} < H_q \, s^{k-q} \, u \otimes v, \; D_E^{(k-q)} \, s > (x)$$

and hence

$$\int_M < (\chi_{k,\ell}(D_E,\nabla))' \cdot j_{k-\ell}(dg) \cdot j_{k-\ell}(s^k u \otimes v) \, , \, s >$$

$$= \int_M < s^k u \otimes v, \; \chi_{k,\ell}(D_E,\nabla) \cdot j_{k-\ell}(-dg) \cdot j_{k-\ell}(s) >$$

$$= (-1)^{\ell} \sum_{q=0}^{k} \binom{k}{q} \int_M < H_q s^{k-q} \, u \otimes v, \; D_E^{(k-q)} \, s >$$

$$= \int_M < (-1)^{\ell} \sum_{q=0}^{k} \binom{k}{q} \, (D_E^{(k-q)})' \, (H_q s^{k-q} \, u \otimes v) \, , \; s >$$

Corollary 10.4

If $w \in \Gamma_c^\infty(S^k T \otimes E')$, $g \in C^\infty(M,\mathbb{R})$ $dg \neq 0$ on supp w, $x \in M$ and

$$(\partial^{(q)} g)(x) = 0 , \quad 2 \leq q \leq k-\ell+1$$

then

$$(\chi_{k,\ell}(D_E,\nabla))' \cdot j_{k-\ell}(dg)(x) \cdot j_{k-\ell}(w)(x)$$

$$= \begin{cases} (D_E^{(k)})' \quad (w)(x) & \text{if } \ell=0 \\ 0 & \text{if } \ell>0. \end{cases}$$

Proof: By polarization it suffices to consider $w = \delta^k u \otimes v$ where $u \in \Gamma^\infty(T)$, $v \in \Gamma_c^\infty(E')$. By hypothesis on g we have $h_x \in Z_x^{k-\ell+2}$ and hence $h_x^\ell \in Z_x^{k\ell-\ell^2+2}$ and so

$$H_q \in Z_x^{k\ell-\ell^2+2\ell-q}$$

If $\ell \geq 1$ then since $2\ell \leq k$ we have

$$\begin{aligned} k\ell - \ell^2 + 2\ell - q &\geq k + k(\ell-1) - \ell^2 + 2\ell - q \\ &\geq k + 2\ell(\ell-1) - \ell^2 + 2\ell - q \\ &\geq k + \ell^2 - q \\ &\geq k - q + 1 \end{aligned}$$

and so

$$(D_E^{(k-q)})' \; (H_q \, \delta^{k-q} u \otimes v)(x) = 0.$$

If $\ell = 0$ then $h_x^\ell = 1$ and $\partial^{(q)}(1) = 0$ for $q \geq 1$. Thus only the $q = 0$ term remains and $H_0 = 1$.

§11 The Geometric Transpose of a Pseudo-Differential Operator

lemma 11.1

Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1, \ldots$ and let n_j be the largest integer such that $0 \le n_j \le Re(z_0 - z_j)$. Suppose that there exists $P' \in \mathcal{P}(F',E')$ such that

$$\int_M < w, Ps > = \int_M < P'w, s >$$

for each $w \in \Gamma_c^\infty(F')$, $s \in \Gamma_c^\infty(E)$.

Then P' is unique, has exponents $z_0, z_1, \ldots$ and

$$\sigma_j(P') = \sigma_j(P)'$$

so in particular $\sigma_j(P)' \in \mathrm{Smbl}_{z_j}^{n_j}(F',E')$

Proof: Uniqueness of P' is clear. Let $s \in \Gamma_c^\infty(E)$, $w \in \Gamma_c^\infty(F')$, $g \in C^\infty(M;\mathbb{R})$ $dg \ne 0$ on supp $s \cup$ supp w. Then

$$e^{i\lambda g} P(e^{-i\lambda g} s) \sim \sum_{k=0}^\infty \lambda^{z_k} \sigma_k(P) \cdot j_{n_k}(-dg) \cdot j_{n_k}(s)$$

$$e^{-i\lambda g} P'(e^{i\lambda g} w) \sim \sum_{k=0}^\infty \lambda^{z_k'} \sigma_k(P') \cdot j_{n_k'}(dg) \cdot j_{n_k'}(w)$$

and

$$(1) \quad \int_M < e^{-i\lambda g} P'(e^{i\lambda g} w), s > = \int_M < w, e^{i\lambda g} P(e^{-i\lambda g} s) >$$

Then by the continuity of

$$\int_M < w, \cdot > : \Gamma^{\infty}(F) \to \mathbb{C}$$

$$\int_M < \cdot, s > : \Gamma^{\infty}(E') \to \mathbb{C}$$

we obtain two asymptotic expansions for (1);

$$\sum_{k=0}^{\infty} \lambda^{z_k} \int_M < w, \sigma_k(P) \cdot j_{n_k}(-dg) \cdot j_{n_k}(s) >$$

and

$$\sum_{k=0}^{\infty} \lambda^{z'_k} \int_M < \sigma_k(P') \cdot j_{n'_k}(dg) \cdot j_{n'_k}(w) , s >$$

By the proof of lemma 5.1 it follows that we may assume $z'_k = z_k$, $k=0,1,2...$ (since any z'_ℓ (resp. z_j) for which such an equality does not hold corresponds to a zero term in the asymptotic expansion of P' (resp. P)). Then $n'_k = n_k$ and

$$\int_M < w, \sigma_k(P) \cdot j_{n_k}(-dg) \cdot j_{n_k}(s) >$$

$$= \int_M < \sigma_k(P') \cdot j_{n_k}(dg) \cdot j_k(w) , s >$$

whence

(2) $\sigma_k(P') = \sigma_k(P)'$

By lemma 10.1 we know $\sigma_k(P)' \in \mathrm{Smbl}_{z_n}^{2n_k}(F',E')$ but (2) implies that $\sigma_k(P)'$ is actually in $\mathrm{Smbl}_{z_k}^{n_k}(F',E')$.

We denote the complexified volume bundle $\Omega \otimes \mathbb{C}$ simply by Ω.

lemma 11.2

Suppose M is an open submanifold of $\mathbb{R}^m$, and $P \in \mathcal{P}(1,1)$ with exponents $z_0, z_1, \ldots$. Then there exists a unique $P' \in \mathcal{P}(\Omega,\Omega)$ such that if $f \in C_c^\infty(M:\mathbb{C})$, $\omega \in \Gamma_c^\infty(\Omega)$ then

$$\int_M (Pf)\omega = \int_M f(P'\omega)$$

Moreover P' has exponents $z_0, z_1, z_2, \ldots$.

Proof: By theorem 4.4 in L. Hörmander [8] there exists a unique $Q \in \mathcal{P}(1,1)$ such that if $u, v \in C_c^\infty(M:\mathbb{C})$ then

$$\int_M v(Pu)dx = \int_M u(Qv)dx$$

Now given any $\omega \in \Gamma_c^\infty(\Omega)$ there exists a unique $v \in C_c^\infty(M:\mathbb{C})$ such that $\omega = vdx$. Thus the Lebesgue measure dx induces an isomorphism

$$\psi : \Omega \cong 1$$

Now define $P' = \psi^{-1} Q \psi$. Then $P' \in \mathcal{P}(\Omega,\Omega)$ and if $u \in C_c^\infty(M:\mathbb{C})$ $\omega \in \Gamma_c^\infty(\Omega)$, $\omega = vdx$, then

$$\begin{aligned} \int_M (Pu)\,\omega &= \int_M (Pu)\,vdx \\ &= \int_M u(Qv)dx \\ &= \int_M u(\psi^{-1}Q\,\psi)(\omega) \\ &= \int_M u(P'\omega) \end{aligned}$$

That P' has exponents $z_0, z_1, \ldots$ follows by lemma 11.1.

<u>lemma 11.3</u>

Let $P \in \mathcal{P}(1,1)$ with exponents $z_0, z_1 \ldots$ Then there exists a unique $P' \in \mathcal{P}(\Omega,\Omega)$ such that if $f \in C_c^\infty(M:\mathbb{C})$, $\omega \in \Gamma_c^\infty(\Omega)$ then

$$\int_M (Pf)\,\omega = \int_M f\,(P'\,\omega)$$

Moreover P' has exponents $z_0, z_1, \ldots\ldots$

<u>Proof:</u> Let (U,ϕ) be a chart on M. By lemma 11.2 $(P_{|U})' \in \mathcal{P}(\Omega_{|U},\Omega_{|U})$ exists. Now if $u,v \in C_c^\infty(U:\mathbb{C})$ then

$$u\,(P_{|U})'\,v \in \mathcal{P}(\Omega,\Omega)$$

and clearly $u(P_{|U})'\,v = (vPu)'$. In the proof of theorem 7.1 we showed that if $u,v \in C_c^\infty(M:\mathbb{C})$ have supports in coordinate neighborhoods then $u = u_1+u_2$ and $v = v_1+v_2$ where

$$\operatorname{supp} v_1 \cup \operatorname{supp} u$$
$$\operatorname{supp} v_2 \cup \operatorname{supp} u_1$$
$$\operatorname{supp} v_2 \cup \operatorname{supp} u_2$$

are each contained in some coordinate neighborhoods. It follows that $(vPu)'$ exists and

$$(vPu)' = (v_1Pu)' + (v_2Pu_1)' + (v_2Pu_2)'$$

Then by a partition of unity argument $(vPu)'$ exists for each $u,v \in C_c^\infty(M:\mathbb{C})$. Then there is a unique linear map

$$P' : \Gamma_c^\infty(\Omega) \to \Gamma^\infty(\Omega)$$

such that

$$uP'v = (vPu)'$$

for each $u,v \in C_c^\infty(M:\mathbb{C})$ and P' has the desired property (see argument in theorem 11.5). By lemma 5.7, remark 5.12, and construction we have $uP'v \in P(\Omega,\Omega)$. By lemma 11.1, $vP'u$ has exponents $z_0, z_1 \ldots$ and so by the proof of lemma 5.9, $P' \in P(\Omega,\Omega)$ and P' has exponents $z_0, z_1 \ldots$

Theorem 11.4

Let $P \in P(E,F)$ with exponents $z_0, z_1 \ldots$ Then there exists a unique $P' \in P(F',E')$ such that if $s \in \Gamma_c^\infty(E)$, $w \in \Gamma_c^\infty(F')$ then

$$\int_M < w, Ps > = \int_M < P'w, s > .$$

Moreover P' has exponents $z_0, z_1 \ldots$. P' is called the geometric transpose of P.

Proof: If $u \in \Gamma_c^\infty(F^*)$, $s \in \Gamma_c^\infty(E)$ then $uPs \in P(1,1)$ admits a (unique) geometric transpose $(uPs)' \in P(\Omega,\Omega)$ by lemma 11.3.

If $f \in C_c^\infty(M:\mathbb{C})$, $\omega \in \Gamma_c^\infty(\Omega)$ then

$$(3) \quad \int_M < u \otimes \omega, (Ps)(f) > = \int_M < u, P(fs) > \omega$$

$$= \int_M (uPs)(f)\, \omega$$

$$= \int_M f (uPs)'(\omega)$$

Since Ω is a trivial line bundle any $w \in \Gamma_c^\infty(F')$ may be written as $w = u \otimes \omega$ where $u \in \Gamma_c^\infty(F^*)$, $\omega \in \Gamma_c^\infty(\Omega)$. Thus (3) implies that $(Ps)'$

exists and

$$(Ps)' (u \otimes \omega) = (uPs)'(\omega)$$

i.e.

$$(Ps)' u = (uPs)'$$

But then by lemma 5.14 and the proof of lemma 5.9 $(Ps)' \in P(F',\Omega)$ and $(Ps)'$ has exponents $z_0, z_1, \ldots$.

Now (3) also implies

$$(Pfs)' u = f (uPs)'$$

whence $s \to (Ps)'$ is $C_c^\infty(M:\mathbb{C})$-linear. Thus we can define a linear map

$$Q : \Gamma_c^\infty(F') \to \Gamma^\infty(E')$$

by

$$< Q(w), s > = (Ps)'(w)$$

for $w \in \Gamma_c^\infty(F')$, $s \in \Gamma_c^\infty(E)$.

Now choose $f \in C_c^\infty(M:\mathbb{C})$ such that $f = 1$ on supp s. Then

$$\begin{aligned} \int_M < Q(w), s > &= \int_M (Ps)'(w) \\ &= \int_M f (Ps)'(w) \\ &= \int_M < w, (Ps)(f) > \\ &= \int_M < w, Ps > \end{aligned}$$

Thus $Q = P'$, i.e. P' exists.

If $s \in \Gamma_c^\infty(E)$, $u \in \Gamma_c^\infty(F^*)$, $\omega \in \Gamma_c^\infty(\Omega)$ then

$$\begin{aligned} sP'u\ (\omega) &= \langle P'(u \otimes \omega),\ s \rangle \\ &= (Ps)'(u \otimes \omega) \\ &= (uPs)'(\omega) \end{aligned}$$

so $sP'u = (uPs)' \in \mathcal{P}(\Omega,\Omega)$ with exponents $z_0, z_1 \ldots$ Since Ω is a trivial line bundle the proof of lemma 5.9 implies $P' \in \mathcal{P}(F',E')$ with exponents $z_0, z_1 \ldots$

Let D_E be a covariant derivative on E, $D_{F'}$ a covariant derivative on F', and ∇ a covariant derivative on T^*.

Let $P \in \mathcal{P}(E,F)$ with exponents $z_0, z_1 \ldots$ and let $P' \in \mathcal{P}(F',E')$ be the geometric transpose of P.

Let $\tau_j(P)$ be the singular part of $\sigma_j(P)$ relative to the pair (D_E,∇) and let $\tau_j(P')$ be the singular part of $\sigma_j(P')$ relative to the pair $(D_{F'},\nabla)$.

<u>Theorem 11.5</u>

$$\sum_{k=0}^{\infty} \sigma_k(P') = \sum_{j,n=0}^{\infty} \frac{i^{-n}}{n!} (D_E^{(n)})' * (\tau_j(P)^{(n)})'$$

<u>Proof:</u> By theorem 9.4 we have

$$\sum_{k=0}^{\infty} \sigma_k(P) = \sum_{\substack{j,n,\ell=0 \\ 2\ell \le n}}^{\infty} \frac{i^{\ell-n}}{n!\,\ell!} \tau_j(P)^{(n)} \circ \chi_{n,\ell}(D_E,\nabla)$$

and hence by lemma 11.1

$$\sum_{k=0}^{\infty} \sigma_k(P') = \sum_{\substack{j,n,\ell=0 \\ 2\ell \le n}}^{\infty} \frac{i^{\ell-n}}{n!\,\ell!} (\chi_{n,\ell}(D_E,\nabla))' \circ (\tau_j(P)^{(n)})'$$

By lemma 10.3, lemma 10.1 and theorem 3.2

$$(\chi_{n,\ell}(D_E,\nabla))' \circ (\tau_j(P)^{(n)})'$$

is in $\mathrm{Smbl}^{n-\ell}_{z_j-n+\ell}(F',E')$. Hence for any integer $N > 0$ we have

$$\sum_{\substack{k \\ Re(z_0-z_k) \leq N}} \sigma_k(P')$$

$$= \sum_{j,n,\ell} \frac{i^{\ell-n}}{n!\ell!} (\chi_{n,\ell}(D_E,\nabla))' \circ (\tau_j(P)^{(n)})'$$

(sum over $2\ell \leq n$, $n-\ell \leq N$, $Re\, z_j - n+\ell \geq Re\, z_{k'}$)

when k' is the largest integer such that $Re(z_0 - z_{k'}) \leq N$.

Now let $x \in M$, $w \in \Gamma_c^\infty(F')$, $g \in C^\infty(M;\mathbb{R})$, $dg \neq 0$ on supp w, and suppose

$$(\partial^{(q)} g)(x) = 0 , 2 \leq q \leq N+1$$

$$(D_{F'}^{(q)} w)(x) = 0, 1 \leq q \leq N$$

Then by (4) we have

$$(5) \qquad \sum_{\substack{k \\ Re(z_0-z_k) \leq N}} \tau_k(P') \cdot dg(x) \cdot w(x)$$

$$= \sum_{j,n,\ell} \frac{i^{\ell-n}}{n!\ell!} (\chi_{n\ell}(D_E,\nabla))' \cdot j_{n-\ell}(dg)(x) \cdot ((\tau_j(P)^{(n)})' \cdot dg \cdot w)(x)$$

(sum over $2\ell \leq n$, $n-\ell \leq N$, $Re\, z_j - n+\ell \geq Re\, z_{k'}$)

Since $n-\ell \leq N$ by the condition on g and by corollary 10.4 all terms in the right hand side of (5) in which $\ell > 0$ vanish, and by the same corollary the remaining terms yield

$$(6)\quad \sum_{j,n} \frac{i^{-n}}{n!} \; (D_E{}^{(n)})' \cdot \left((\tau_j(P)^{(n)})' \cdot dg \cdot w \right) (x)$$

(sum over $n \leq N$, $Re\ z_j - n \geq Re\ z_{k'}$)

By the conditions on g and w (6) is equal to

$$\sum_{j,n} \frac{i^{-n}}{n!} \left((D_E{}^{(n)})' * (\tau_j(P)^{(n)})' \right) \cdot dg(x) \cdot w(x)$$

(sum over $n \leq N$, $Re\ z_j - n \geq Re\ z_{k'}$)

which proves the theorem.

<u>Remark 11.6</u>

In theorem 11.5

$$\tau_j(P)^{(n)} \in Smbl_{z_j-n}(S^n T^* \otimes E, F)$$

i.e. $\tau_j(P)^{(n)}$ is a map

$$T^* - (0) \rightarrow Hom(S^n T^* \otimes E, F)$$

Then $(\tau_j(P)^{(n)})'$ is given by the composition

$$T^* - (0) \xrightarrow{-1} T^* - (0) \xrightarrow{\tau_j(P)^{(n)}} Hom(S^n T^* \otimes E, F) \xrightarrow{a} Hom(F', S^n T \otimes E')$$

where -1 denotes multiplication by -1, and a maps a homomorphism onto its adjoint.

§12 Elliptic Operators

Let $P \in \mathcal{P}(E,F)$ have exponents $z_0, z_1, \ldots$ We say that P is *elliptic* if $\mathrm{Re}\, z_1 < \mathrm{Re}\, z_0$ and if for each $x \in M$, $\omega \in T_x^* - (0)$

$$\sigma_0(P) \cdot \omega : E_x \to F_x$$

is an isomorphism.

Theorem 12.1

Let $P \in \mathcal{P}(E,F)$ be elliptic and have exponents $z_0, z_1 \ldots$ Then there exists an almost local operator $Q \in \mathcal{P}(F,E)$ of order $-\mathrm{Re}\, z_0$ such that

$$PQ - 1 \in \mathcal{P}_{-\infty}(F)$$

and for each $f \in C_c^\infty(M:\mathbb{C})$, if U is an open set on which $f = 1$, then

$$(QfP - 1)_{|U} \in \mathcal{P}_{-\infty}(E_{|U})$$

Moreover Q is unique modulo $\mathcal{P}_{-\infty}(F,E)$.

Proof: It suffices to consider the case where P is almost local and to show we can construct an almost local $Q \in \mathcal{P}(F,E)$ such that

$$PQ - 1 \in \mathcal{P}_{-\infty}(F)$$
$$QP - 1 \in \mathcal{P}_{-\infty}(E)$$

By hypothesis there exists a unique $\tau \in \mathrm{Smbl}_{-z_0}(F,E)$ such that $\sigma_0(P) \circ \tau = 1$ and $\tau \circ \sigma_0(P) = 1$.

Then by lemma 9.7 there exists $Q_0 \in P(F,E)$ such that

$$\sigma_0(Q_0) = \tau$$

and Q_0 has exponents $-z_0, -z_0-1, -z_0-2, \ldots$ Thus

$$\text{order } (PQ_0 - 1) \leq \max \{-1, Re(z_1 - z_0)\} = c < 0$$
$$\text{order } (Q_0P - 1) \leq \max \{-1, Re(z_1 - z_0)\} = c < 0$$

Now define $Q_n \in P(F,E)$ by

$$Q_n = \sum_{k=0}^{n} (1 - Q_0P)^k Q_0 = \sum_{k=0}^{n} Q_0(1 - PQ_0)^k$$

Then

$$PQ_n - 1 = -(1 - PQ_0)^{n+1}$$
$$Q_nP - 1 = -(1 - Q_0P)^{n+1}$$
$$Q_{n+1} - Q_n = (1 - Q_0P)^{n+1} Q_0 = Q_0(1 - PQ_0)^{n+1}$$

Thus

$$\text{order } (PQ_n - 1) \leq (n+1)c$$
$$\text{order } (Q_nP - 1) \leq (n+1)c$$
$$\text{order } (Q_{n+1} - Q_n) \leq (n+1)c - Re\, z_0$$

Since $c < 0$ it follows by lemma 8.2 that there exists an almost local $Q \in P_{-Re\, z_0}(F,E)$ such that $Q = \operatorname{lim\,asymp}_{n\to\infty} Q_n$

$$PQ - 1 \in P_{-\infty}(F)$$
$$QP - 1 \in P_{-\infty}(E).$$

If $\tilde{Q}$ is another candidate for Q then $\tilde{Q} - Q = (\tilde{Q}P - 1)Q + \tilde{Q}(1 - PQ) \in P_{-\infty}(F,E)$, which completes the proof.

Remark 12.2

Let D_E be a covariant derivative on E, D_F a covariant derivative on F, and ∇ a covariant derivative on T^*. Let P and Q be as in theorem 12.1 and let

$$\sum_{k=0}^{\infty} \tau_k(P)$$

be the formal symbol of P relative to (D_E, ∇) and let

$$\sum_{j=0}^{\infty} \tau_j(Q)$$

be the formal symbol of Q relative to (D_F, ∇). Then by theorem 9.5 we have

$$\sum_{n,j,k=0}^{\infty} \frac{i^{-n}}{n!} \tau_k(P)^{(n)} \circ \left(D_E^{(n)} * \tau_j(Q)\right) = 1$$

$$\sum_{n,j,k=0}^{\infty} \frac{i^{-n}}{n!} \tau_j(Q)^{(n)} \circ \left(D_F^{(n)} * \tau_k(P)\right) = 1$$

and theorem 12.1 implies that these equations may be solved uniquely for $\tau_j(Q)$.

APPENDIX

Operators of Order $-\infty$

In this appendix we prove that the pseudo-differential operators of order $-\infty$ are precisely those linear maps given by integrals with smooth kernels. In particular we characterize pseudo-local operators without reference to pseudo-differential operators. This characterization makes hypothesis (a) of lemma 5.7, lemma 5.9 and corollary 5.15 perhaps easier to check in practice.

lemma A.1

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a linear map. Then $P \in \mathcal{P}_{-\infty}(E,F)$ if and only if for each chart (U,ϕ) on M, each $s \in \Gamma_c^\infty(E_{|U})$ we have $P_{|U}\, s \in \mathcal{P}_{-\infty}(1_{|U}, F_{|U})$.

<u>Proof</u>: If $P \in \mathcal{P}_{-\infty}(E,F)$ then by corollary 5.4 $P_{|U}s \in \mathcal{P}_{-\infty}(1_{|U}, F_{|U})$. The converse follows by corollary 5.15.

lemma A.2

Suppose M is an open submanifold of $\mathbb{R}^m$ and let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a continuous linear map. If $s \in \Gamma_c^\infty(E)$ we define

$$p(s,\xi) = e^{-i\langle\cdot,\xi\rangle}\, P(e^{i\langle\cdot,\xi\rangle}\, s)$$

Then the following conditions are equivalent,

(i) For each $s \in \Gamma_c^\infty(E)$, $Ps \in P_{-\infty}(1,F)$

(ii) For each $s \in \Gamma_c^\infty(E)$, each continuous semi-norm ρ on $\Gamma^\infty(F)$ and each pair of multi-indexes α,β

$$\sup \{ \rho(\xi^\alpha p^{(\beta)}(s,\xi)) : \xi \in \mathbb{R}^m \} < \infty$$

(iii) For each $s \in \Gamma_c^\infty(E)$, each continuous semi-norm ρ on $\Gamma^\infty(F)$ and each multi-index α

$$\sup \{ \rho(\xi^\alpha p(s,\xi)) : \xi \in \mathbb{R}^m \} < \infty$$

<u>Proof:</u> (i) $\Rightarrow$ (ii) by corollary 6.2

(ii) $\Rightarrow$(iii) is trivial

(iii) $\Rightarrow$(ii) is clear since

$$\frac{\partial}{\partial \xi_j} p(s,\xi) = - i\, x_j\, p(s,\xi) + i\, p(x_j s,\xi)$$

(ii) $\Rightarrow$(i) by theorem 6.5

Let $\check{\otimes}$ denote the "exterior" tensor product.

<u>Theorem A.3</u>

Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a continuous linear map. Then $P \in P_{-\infty}(E,F)$ if and only if there exists

$$K \in \Gamma^\infty(F \check{\otimes} E')$$

such that if $s \in \Gamma_c^\infty(E)$ then

$$Ps(x) = \int_M < K(x,\cdot), s >$$ Moreover K is unique.

<u>Proof:</u> By considering operators of the form wP where $w \in \Gamma_c^\infty(F^*)$, by the uniqueness of K we see there is no loss of generality in assuming $F = 1$.

If K exists it is unique. Hence it suffices to prove existence of K locally in $M \times M$ and hence by remark 5.12 it suffices to construct a C^∞ integral kernel for $P_{|U}$ for each chart (U,ϕ) on M. Conversely by lemma A.1 if K exists to prove $P \in P_{-\infty}(E,F)$ it suffices to consider $P_{|U}$ and hence $K_{|U \times U}$ for each chart (U,ϕ) on M. Thus we may assume that M is an open submanifold of $\mathbb{R}^m$.

Then $P \in P_{-\infty}(E,1)$ if and only if $Ps \in P_{-\infty}(1,1)$ for each $s \in \Gamma_c^\infty(E)$ is clear and hence by lemma A.2 $P \in P_{-\infty}(E,1)$ if and only if for each $s \in \Gamma_c^\infty(E)$ each pair of multi-indexes α,β we have

$$\left(\frac{\partial}{\partial x}\right)^\alpha \xi^\beta p(s,\xi)(x)$$

is uniformly bounded for all $\xi \in \mathbb{R}^m$ and for x in any compact set.

(i) Suppose we have $K \in \Gamma^\infty(1 \check{\otimes} E')$ which is an integral kernel for P. Then $K(x,y) = \bar{K}(x,y)dy$ where $\bar{K} \in \Gamma^\infty(1 \check{\otimes} E^*)$ and we have

$$p(s,\xi)(x) = \int e^{i<y-x,\xi>} \bar{K}(x,y)s(y)dy$$

and hence integrating by parts

$$\left(\frac{\partial}{\partial x}\right)^\alpha \xi^\beta p(s,\xi)(x)$$

$$= \sum_{0\le\gamma\le\alpha} \binom{\alpha}{\gamma} i^{|\beta|} \int e^{i<y-x,\xi>} \left(\frac{\partial}{\partial y}\right)^{\beta+\gamma} \left(\frac{\partial}{\partial x}\right)^{\alpha-\gamma} (\bar{K}(x,y)s(y))dy$$

which is uniformly bounded for x in a compact set. Thus $P \in P_{-\infty}(E,1)$.

(ii) Conversely suppose $P \in \mathcal{P}_{-\infty}(E,1)$. Then for each $s \in \Gamma_c^\infty(E)$

$$K_s(x,y) = (2\pi)^{-m} \int e^{i<x-y,\xi>} p(s,\xi)(x)d\xi$$

defines $K_s \in \Gamma^\infty(1 \check{\otimes} 1)$ and if $u \in C_c^\infty(M:\mathbb{C})$ then

$$P(us)(x) = \int K_s(x,y)u(y)dy$$

Now a local triviality and a partition of unity argument yields $K \in \Gamma^\infty(1 \check{\otimes} E^*)$ such that

$$(Ps)(x) = \int K(x,y)s(y)dy.$$

Corollary A,4

If $Q : \Gamma^\infty(F) \to \Gamma^\infty(G)$ is a continuous linear map and $P \in \mathcal{P}_{-\infty}(E,F)$ then $QP \in \mathcal{P}_{-\infty}(E,G)$.

Corollary A.5

If $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ is a linear map then P is pseudo-local if and only if for each pair of disjoint compact subsets A_1, A_2, of M there exists $K \in \Gamma^\infty(F \check{\otimes} E')$ such that

$$(Ps)(x) = \int_M < K(x,\cdot), s >$$

if $s \in \Gamma_c^\infty(E)$, supp $s \subseteq A_1$ and $x \in A_2$.

Remark A,6

Suppose M is compact and $P \in \mathcal{P}(E,F)$ is a bijective elliptic operator. Since $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$ are Fréchet spaces it follows that

P^{-1} is continuous. Now let $Q \in \mathcal{P}(F,E)$ be a parametrix for P, i.e.

$$QP - 1 \in \mathcal{P}_{-\infty}(E)$$

$$PQ - 1 \in \mathcal{P}_{-\infty}(F)$$

Then $P^{-1} = Q + P^{-1}(1 - PQ)$ implies by corollary A.4 that $P^{-1} \in \mathcal{P}(F,E)$ and that $P^{-1} - Q \in \mathcal{P}_{-\infty}(F,E)$.

BIOGRAPHY

The author was born in Copenhagen, Denmark on July 31, 1942. He attended Ryesgade (København) elementary school from September 1949 to September 1951 at which time his family moved to Canada. He attended half a dozen elementary schools in Ontario until March 1957 when his family moved to British Columbia. In Vancouver, B.C. he attended Vancouver Technical High School (1957 - 1960), Magee High School (1960 - 1961) and the University of British Columbia (1961 - 1964) where he received his B.Sc. in Honours Mathematics in May 1964. He came to Massachusetts Institute of Technology in September 1964 on a Woodrow Wilson Fellowship (1964 - 1965) and was supported during 1965 - 1968 by National Research Council of Canada Scholarships and in addition held an M.I.T. Teaching Assistantship during 1967 - 1968. He returned temporarily to Vancouver during the summer of 1965 to marry Marguerite K. A. McCrindle, who returned to Cambridge with him and has since been employed in various secretarial positions at M.I.T.

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