Essays in Applied Theory

by

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ABSTRACT

In "Incomplete Contracting with Endogenous Competition," I consider a variant of the incomplete contracting with renegotiation model introduced by Hart and Moore (1988). I study a trading relation between a buyer and seller, where some ex-ante relationship specific investment on the part of the seller is needed to generate value for the buyer. Uncertainty is revealed ex post, in that prior to the investment stage, the buyer does not know which type of service she may need, and it is impossible to describe under what precise circumstances she needs a particular service. The contract can take only two broad forms: (1) a specification of the nature of a service to be provided under all circumstances, or (2) a general option contract, namely, a menu of services that the seller agrees to provide at predetermined terms.

In addition to uncertainty regarding state realizations which was the focus of the literature thus far, I consider a different source of contracting friction, namely, uncertainty about downstream profitability. I show that depending on the specific assumptions, the competitor may invest in, produce and sell an imperfect substitute or free-ride on the incumbent supplier's investment and replicate a perfect substitute with positive probability. However, in a subgame-perfect equilibrium, the incumbent supplier will correctly anticipate potential entrants and change its ex-ante investment to account for downstream competition. It may also distort the level of its ex-ante investment to deter future entry.

In this paper, information affects the outcome of economic transactions, but the presence and absence of information is exogenously given by assumptions about the form of competitions. In my paper "Selling Information: Multidimensional Oligopolistic Competition," I model the information structure as a variable endogenously chosen to optimally manage competition. Specifically, I consider a model of an economic transaction between an upstream monopolist and several downstream oligopolists.

The downstream parties may be E-commerce retailers who compete over a heterogeneous customer base. Each party may have some prior assessment over the distribution of customer types, but would benefit from incremental knowledge on customer information. The upstream party, in this scenario, is an information vendor, who has access to technology required to develop a targeting device. Since information is valuable, to extract surplus the upstream party would like to improve the quality of information. Such motive is counterbalanced by the incentive to manage competition.

The upstream monopolist supplies a menu of multi-dimensional intermediate goods from which the downstream oligopolists select. The oligopolists then use the previously purchased intermediate goods to produce the final products and compete with each other.

The model enriches Bonatti (2015)'s multi-dimensional information product model by considering what the information product is used for (competing for heterogenous customers by personalized pricing) and how downstream competition affects the value of each dimension of the

information product. The key feature of the model is that the information good is intermediate, whose value is affected by the extent of ex post competition among the

The model captures the indirect externalities conferred in the market for information. Specifically, the value of customer information to a given firm is no longer determined solely by the characteristics of that firm and the those customers. Instead, the value now also depends on the market competition structure among all downstream firms. For example, a model of competition of customer information has features similar to an arms race: having better information over the opponent allows one to better engage in better price discrimination, but it also increases the value of information to the opponent and induces more aggressive demand for information on the part of the opponents.

In addition to economic transactions, in my third paper I study the role of information in managing and orienting actions beyond the market. In "Information Theory Foundation of Propaganda," I develop a model of strategic information signaling with an informed sender and a continuum of imperfectly informed receivers. The sender sends a costly signal to disrupt receivers' coordination action and to bias their aggregate action away from the true state towards the sender's desired state. The receivers want to match their actions to the true state and also seek to coordinate with each other. The leading application of the model is an authoritarian regime sending propaganda to its citizens to prevent them from learning the true strength of the regime and taking collective actions.

In equilibrium the sender's manipulation does not succeed in changing the mean of the receivers' beliefs, but manipulation makes their interpretation of the signal noisier. This model helps resolve an empirical puzzle: since we observe propaganda, regimes apparently think it works, in some way, but can propaganda work, even if the citizens who see it know it is biased information? In the model, propaganda works not through changing beliefs per se, but through adding noise and confusion into the communication structure, so that citizens, who value coordination, are more likely to redirect their attention across various sources of information.

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Chapter 1: Incomplete Contracting with Endogenous Competition

1 Introduction

When drafting a contract, the parties involved frequently cannot describe all relevant contingencies that may arise during the course of the relationship. The state space may be so complicated that it is impossible to anticipate or prohibitively costly to specify all scenarios. The details of the transactions may be observable only to the parties involved and unverifiable in the court. In practice, the parties are likely to end up with a highly incomplete contract.

We take as the departure point the incomplete contracting approach to institution design pioneered by Simon (1951). Specifically, we consider a model where a seller can undertake an unobservable investment today to raise the value of the service to be provided to the buyer in the future. This investment is sunk by the time the buyer and the seller negotiate over service provision in a spot contract, so that the seller is "held up" by the buyer, in the sense of Klein, Crawford and Alchian (1978). When holdup problems are present, employment contract emerges as a potential way to protect the seller's ex-ante investment against ex post opportunisim by the buyer.

Following the incomplete contract literature, we assume that prior to the investment stage, the buyer does not know which type of service she may need, and it is impossible to describe under what precise circumstances she needs a particular service. The contract can specify only the nature of a service to be provided under all circumstances, or a general option contract, namely, a menu of services that the seller agrees to provide at predetermined terms.

In addition to uncertainty regarding state realizations, we consider a new source of dispute that has been ignored in the prior literature, namely, changes in the competitive environment. Specifically, we consider the possibility of entry by a competitive fringe. The competitive supplier's entry may change the market dynamics in the input market and distorts input costs in a way that is ex ante unpredictable to the parties involved. Depending on the specific assumptions, the competitor may invest in, produce and sell an imperfect substitute service, or free-ride on the incumbent supplier's investment and replicate a perfect substitute with positive probability. In a sub-game equilibrium, the incumbent supplier will correctly anticipate the possibility of a competitive entry and change its ex-ante investment to account for potential downstream competition. The incumbent may also distort its investment in the direction so as to deter future entry.

Starting with the timeline of the game as in Hart and Moore (1988), we consider the effect of introducing competition to each of the three intermediate stages of the game: the ex-ante investment stage, the realization of state uncertainty stage, and the renegotiation stage. Our baseline assumes symmetric information regarding state distribution, cost functions and value configurations. Depending on the mode of competition, we show that competition (1) affects the level of ex-ante investment, but the direction of such effect is ambiguous, (2) alters the relative desirability of sales versus employment contract, and (3) interacts with the other primitives of the model (e.g., complexity of state distribution, marginal returns of value and cost configurations.)

We believe the addition of competition into the classic Hart and Moore (1988) model is important in four ways.

First, it substantially relaxes the restrictive assumption of "bilateral monopoly" common in the literature. The assumption that the two parties locked in by specific investment are immediately lifted from perfect competition, transformed into a pair of bilateral monopoly and shielded from outside competition is unrealistic. In practice, contracting partners face alternative opportunities and reneging temptations throughout their relationship. The fact that we sometimes see two parties engaged solely with each other should be viewed as an equilibrium outcome of the model, not it primitives. In fact, true bilateral monopoly is rare, but oligopolistic and monopolistic competition are common, which is especially prominent when the industries in question have developed broad capabilities and the parties involved serve multiple markets. The empirical observations suggest that we need to go beyond the two extreme cases (perfect competition and monopoly) and explore the intermediate cases.

Second, it provides a method to systematically incorporate different forms of regulatory practices into the model of incomplete contracting. The classic Hart and Moore (1988) framework exists in vacuum and is effectively silent on regulatory environment, which in practice, almost all countries have some antitrust laws and regulations in place. In this paper, we will explore various assumptions regarding the competitive environment between S and T, specifically, at what stage of the game T enters the market, to what extent can T free-ride on S's investment, and how successfully can T imitate the good/service to be provided. Although this paper is aimed to study regulations about supply side competitions, the methodology we develop is general enough to be applied to study of other institutions. Our analysis also yields a few general lessons: assessment of the competitive environment is central to the issue of institutional design. Decision-making procedures must take into account the competitor's potential actions. The choice of discretion versus rules depends on the availability of information regarding the potential competitor. Accountability and reneging temptation depends on the details of the competitive environment. It is also important to note that our framework treats the regulatory environment regarding competitive entry as exogenous. In practice, the choice of regulations is likely to be the second-best solution to some constrained optimization problem, where the constraints involved are financial, political and cultural. Understanding the objective as well as the constraints in such optimization problem is an important question for future empirical works.

Third, it sheds light on the Alchian and Demsetz(1972) critique that there is no difference between an employer ordering an employee around and a customer ordering a grocer to deliver a basket of goods. The conventional response is that the employment contract is a commitment to serve in the future as opposed to a spot contract to sell goods to a customer. Our analysis highlights another source of difference between two forms of relationships. A customer is not restricted to order delivery from any single grocer (she may go to a different grocer, or not use any delivery service at all) and a grocer caters to more than one customer's need. The two parties, although engaged in a contractual relationship, are embedded in competition over which they have little effect. On the other hand, the employer-employee relationship is much better insulated from the outside environment. An employee may moonlight for another boss, but her ability to do so is limited by time constraint, monitoring technologies. An employer may lay off one employee in favor of another, but his discretion is severed limited by the terms of labor contract and the hassles involved in hiring another person. In other words, the two relations differ in the degree to which they are exposed to outside competition.

Fourth, it offers a link between Hart and Moore (1988) and Grossman and Hart (1986). The Hart and Moore (1988) framework assumes the ex-ante investment is exclusively used within the bilateral relationship, while Grossman and Hart (1986) allows for alternative use of investment (outside of the relationship), through its effect on the renegotiation surplus. Our analysis generalizes the idea of Grossman and Hart (1986) and the subsequent work by Baker, Gibbons and Murply (2002). We treat "alternative use of investment" as an integral part of the model by viewing the alternative value of investment as equilibrium outcome of a strategic player (the third-party competitor).

Though not the focus of this paper, we should mention that competition also has implication on ex-post inefficiencies. Williamson (1985) views institutional arrangements such as authority relations as integral to market economies. However, the ensuing literature has abstracted from ex post inefficiencies from contractual disputes and focuses entirely on ex ante investment inefficiencies. Even when ex post inefficiencies are explicitly considered, they are modeled as arisen from frictions during the bargaining procedure or from behavioral distortions. We offer a richer view of this interaction. The incompleteness of contracts induces the contracting parties to attempt to interpret the competitive environment to their own advantage ex post, which in turn can lead to both ex ante and ex post inefficiencies. For example, if the value of the imperfect substitute produced by the competitor is private information of the buyer, during the renegotiation stage, the buyer will attempt to exaggerate its value. If the marginal cost of production changes due to new entrants (potentially because of the changes in the input supply market), and if such changes are unobservable to the buyer, the seller will have incentives to misrepresent the true production cost. Hence, when facing competition, the contracting parties may get involved in inefficient contractual disputes ex post. Anticipation of such ex-post opportunism will further distort the level of ex-ante investment. In our model, institutions can be understood as responses designed to overcome the potential inefficiencies in incomplete contracting environment. While an employment relation can be valuable in reducing the scope for ex post haggling, it may also backfire through encouraging competition or allowing the employer to inefficiently appropriate the rents created by the other party's ex ante investments. Whether the details of the transaction should be left to the discretion of the employer depends on the specifics of the competitive environment.

Admittedly, more empirical evidence is needed to establish the importance of competition and non-exclusivity relationship-specific investment. Before we turn to the formal analysis of the model, we offer a real world observation that illustrates the importance of competitive environment. Consider a grocery delivery service provider, who provide delivery service to its customers while charging a fixed amount of membership fee plus a variable amount per usage of its delivery service. To make its services valuable, the provider will need to make significant ex-ante investment: developing a web application, hiring a shopping team, establishing relationship with local grocers, etc. Customers who subscribe to grocery delivering service is not excluded from other grocers. For example, the buyer may walk to the nearby grocery store and purchase on her own. The value of the seller's investment depends on some underlying states that is not feasible to contract upon. For example, on rainy days and busy weekends, such delivery services are clearly desirable, while during other times, the customer may simply use the web application to get price comparison information for free and then call in the the grocery store to pick up her orders. This is a case where the incumbent (the grocery delivery service customer)'s ex-ante investment is not exclusive, and can potentially benefit its competitor (the in-store grocery pickup service).

The two questions facing the grocery delivery company and the customer body are: How should the contract between the delivery company and its customers be structured, so that the surplus generated in the relationship will not eroded by the competitor? Anticipating potential competition during some states, how should ex-ante investment be adjusted?

The rest of the paper attempts answer to these questions. Section 2 and 3 introduce and analyze the baseline model. Section 4 and 5 discuss the implications of the model and offer several applications. Section 6 introduces a few potential ways to extend the model. Section 7 concludes.

2 Setup

We consider a buyer B (the "market") which can be served by either an incumbent supplier (S) or a competitor (T). Throughout our analysis, we will assume that the seller does not have alternative contractual opportunities. In this section, we will first introduce the setup of the model in the absence of the third-party supplier (T).

Suppose there are n types of service, denoted by a_i (i = 1,...,n), that the seller can provide to the the buyer who has unit demand. Each service yields some (non-positive) value to the buyer, but which of the n types of services is the most desirable is unknown ex-ante and dependent on some state of nature l. We assume provision of the service is observable and verifiable, i.e., a contract can be written between B and S to specify a payment contingent on delivery of the service.

The timing of the game is as follows.

In date 1, S and B negotiate terms of trade $(a,p) \subset (A,P) \equiv \{(a_i,p_i)|i = 1,2,...,n\}$, Note that this notation allows for a menu of potential services, together with corresponding prices. If (a,p) is singleton, then we say the contract is a fixed service contract: a given type of service is to be provided regardless of the state realizations. If (a,p) contains at least two elements, then we say the contract is a flexible service contract: more than one type of services is allowed and the choice of provision depends on future realization of states. For the most part of our analysis, we assume symmetry across states and services, so we will without loss of generality focus on the two polar cases: the case where (a, p) is singleton and the case where $(a, p) = (A, P)$.

After the terms of trades are agreed upon, in date 2, S makes a relationship-specific [1] investment I at some private cost $\kappa(I)$. In our baseline analysis, we assume the level of investment is observable to both S and B.

In date 3, the state of the world becomes public. The specific realization of the state, together with the level of ex-ante investment, yields some value-cost configuration for the terms of trade agreed upon in date 1.

In date 4, given the knowledge of the state realization, the two parties get together and renegotiate the terms of trade to some other (a', p') via Nash bargaining. If the bargaining fails, the status quo terms of trade (a, p) will be implemented. We assume the bargaining procedure is efficient. In the extension section of the paper, we will discuss the implication of inefficient bargaining (due to information asymmetry) and in the event of competitive entry, the effect of coalition bargaining.

Finally, in date 5, the renegotiated production plan a' is carried out. The outcome of trade (e.g., demand, equilibrium price) will be determined based on market conditions, e..g, whether a competitor is present and whether the competitor produces a substitute service. We will return to the specifics in the next section.

Figure 1, 2, 3, and 4 summarize the timing of the model.

In the remaining part of this section, we will abstract away from specific rules and regulations that determine entry environment and will rather discuss the cases of the model in terms of the timeline illustrated in Figure 1, 2, 3, and 4, which cover all possible ways in which T and S can interact which each other. We summarize the three cases as below.

(1) T is a business stealer. In this scenario, T is symmetric to S (i.e., they have the same potential production technology, same access to information, etc.) with the exception that S has the opportunity to renegotiate the terms of trade with B after the realization of θ, while T only has access to the spot market. This scenario naturally arises when S is an existing company considering offering a new good/service to its customer body, while T is startup offering the same new good/service (without access to existing customer body). T is constrained relative to S in that T is limited to engage in spot contracting with B while S has other options as part of the available set of contractual relations. This scenario arises naturally when the incumbent has multiple capacities and is considering extending its service to a new business area, while the third-party competitor is new comer in the same area of business. For example, S could be Amazon.com extending its range of products to grocery delivery services (Amazon Fresh) and T could be Instacart.com, which is a startup offering grocery delivery services via smartphone applications.

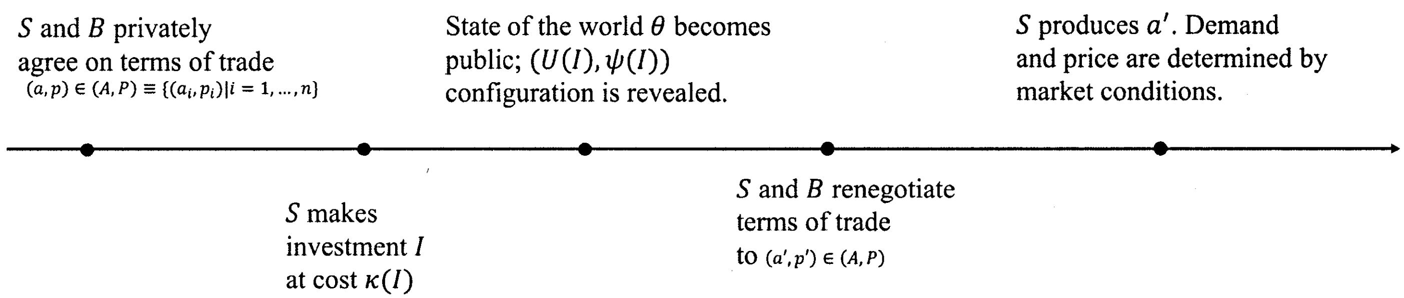
(2) T is a market raider. In this scenario, T can potentially copy the fruit of S's investment and develop a competitive product which is similar but not identical to S's. The probability of T's success is decreasing in S's ex-ante investment: the more S invests in developing the product, the more complicated it becomes which makes it harder for another supplier to copy. In this case, T's impact on S is twofold. First, the possibility of T copying

[1] Following the convention in the literature (c.f. Segal and Whinston (2013)), we say investments are relationship-specific if the renegotiation surplus in non-decreasing in the level of investment for all state realizations. We should note that the notion of "renegotiation surplus" in our model depends endogenously on the competitor's entry decision, which in turn is determined in equilibrium as a function of the ex-ante investment level of the incumbent.

S's investment lowers S's investment incentives. Second, in the event of successful copying, the alternative product T develops changes the bargaining surplus at stake at the renegotiation stage between S and B, which in turns has implications on S's investment incentives. The key difference between this case and the previous one is that the competitor can free ride on (part of) the investment expenditure of the incumbent. This case is best suited to study competition when the kind of ex-ante investment is partially non-exclusive, either due to the nature of the investment (e.g., investment in basic science and discovery), or due to insufficient intellectual property protections. For example, we could imagine S being Celera is who invests in genome sequencing (which is considered knowledge in the public domain and excluded from IP laws) and T being a biological technology firm specializing in genome-based medical technology.

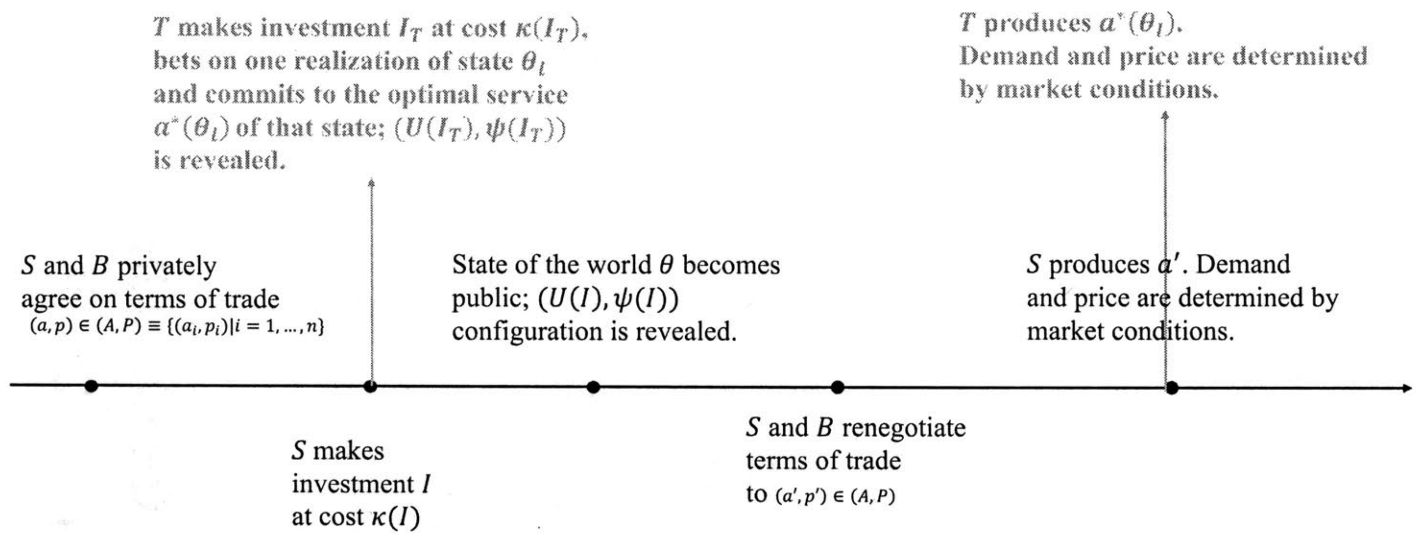
(3) T is a knockoff producer. In this scenario, T can potentially copy S's product and develop a competitive product which is identical (i.e., perfectly substitute) to S's product. As is the case in the previous scenario, T's impact on S's investment incentives is ambiguous. On one hand, S may over-invest to make its product harder for S to copy. On the other hand, T being a potential competitor limits the potential profit to be gained from developing the product, which induces S to under-invest. For example, one may wonder whether the horse race between iPhone and its imitator makes iPhone upgrade its product too frequently (to outrun its imitators) or not frequent enough (since new products will be copied by and potential profits shared with imitators). The key distinction between the last two cases of competition is whether the products by S and T are perfect or imperfect substitutes. In our baseline analysis with unit demand and a representative customer, the difference matters only through changing the renegotiation surplus. We should note that the distinction between the two will be much richer in a model with differentiated preferences and multi-dimensional valuation (e.g., when quality enters consumer preferences). The welfare implications of the two modes of competition are also likely to be different. We offer a brief discussion in the discussion section of the paper.

Figure 1: Baseline: Without T



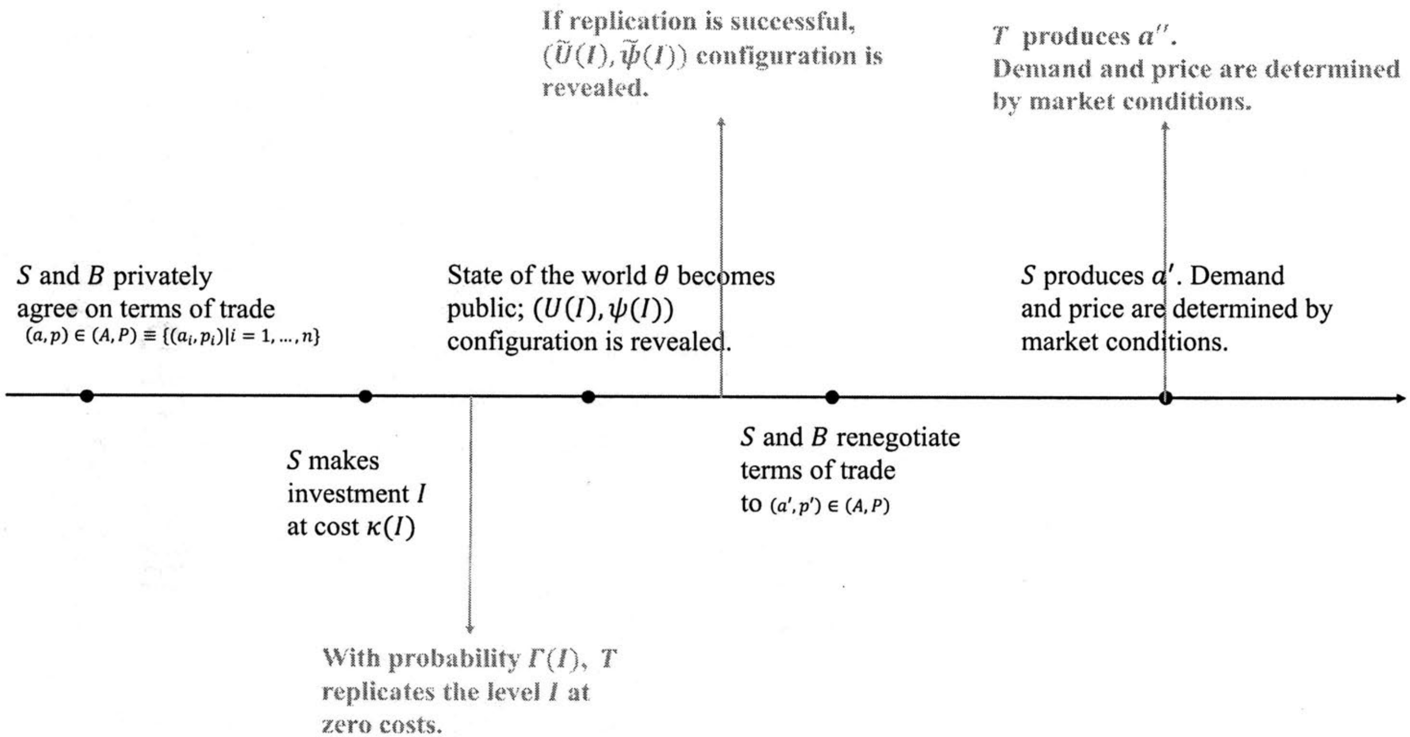
Note: The figure above shows the timing of the game in the absence of the third party competitor T. The timeline is identical to the one introduced in Hart and Moore (1988), with the exception that we allow the terms of trade to exclude some types of services a_j from the set of all possible services A. In practice, however, we assume all states are symmetric. Hence the set of admissible services will be either singleton or the set of all possible services A.

Figure 2: T is a business stealer



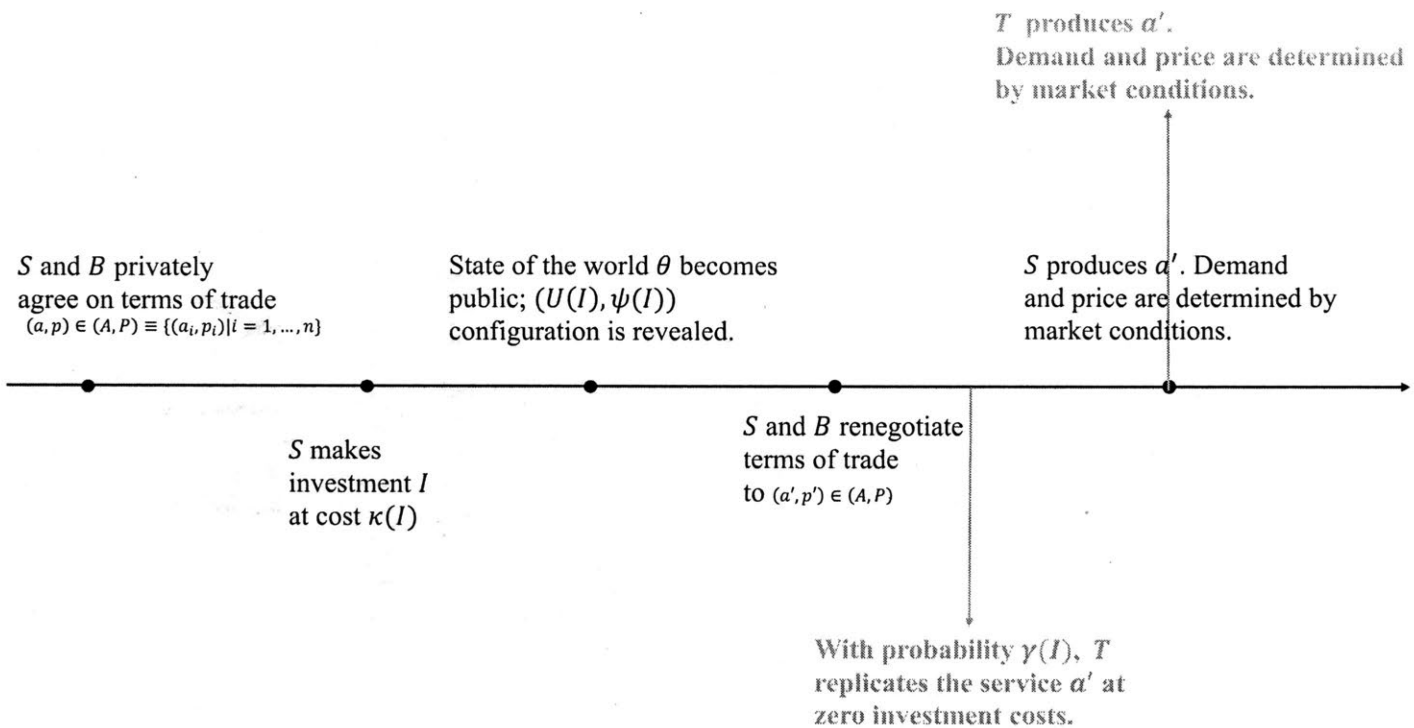
Note: The figure above shows the timing of the game when the third-party competitor T is a business stealer. In this scenario, T is symmetric to S (i.e., they have the same potential production technology, same access to information, etc.) with the exception that S has the opportunity to renegotiate the terms of trade with B after the realization of θ, while T only has access to the spot market. This scenario naturally arises when S is an existing company considering offering a new good/service to its customer body, while T is startup offering the same new good/service (without access to existing customer body). T is constrained relative to S in that T is limited to engage in spot contracting with B while S has other options as part of the available set of contractual relations.

Figure 3: T is a market raider



Note: The figure above shows the timing of the game when the third-party competitor T is a market raider. In this scenario, T can potentially copy the fruit of S's investment and develop a competitive product which is similar but not identical to S's. The probability of T's success is decreasing in S's ex-ante investment: the more S invests in developing the product, the more complicated it becomes which makes it harder for another supplier to copy. In this case, T's impact on S is twofold. First, the possibility of T copying S's investment lowers S's investment incentives. Second, in the event of successful copying, the alternative product T develops changes the bargaining surplus at stake at the renegotiation stage between S and B, which in turns has implications on S's investment incentives.

Figure 4: T is a knockoff producer



Note: The figure above shows the timing of the game when the third-party competitor T is a knockoff producer. In this scenario, T can potentially copy S's product and develop a competitive product which is identical (i.e., perfectly substitute) to S's product. As is the case in the previous scenario, T's impact on S's investment incentives is ambiguous. On one hand, S may over-invest to make its product harder for S to copy. On the other hand, T being a potential competitor limits the potential profit to be gained from developing the product, which induces S to under-invest.

3 Analysis

Consider a risk-neutral buyer (B) and a risk-neutral seller (S). Throughout our analysis, we will assume that the seller does not have alternative contractual opportunities . In the first subsection, we will assume that S is the only seller available. Later, we will relax this assumption and allow the buyer to potentially contract with a third-party (T) instead of S.

In all cases below, we will assume there are n types of service that the seller can provide, denoted by a_i (i = 1,...,n). The buyer will need only of these services ex post. Provision of the service is observable and verifiable, i.e., a contract can be written between B and S specifying a payment contingent on delivery of the service.

Which service is the most desirable will depends on the realization of some underlying state. Suppose there are $N \geq n$ states of nature (l), which affects the buyer's and the seller's payoffs. Assume the state distribution is common knowledge among all parties.

Suppose the seller S can make an ex-ante investment I, which will affect the buyer's and the seller's payoff. We allow the effect of ex-ante investment on payoffs to vary across states. Specifically, let $U = U(a_i, \theta_l, I)$ with $\frac{\partial U}{\partial I} > 0$ and $\frac{\partial^2 U}{\partial I^2} < 0$. Let $\psi = \psi(a_i, \theta_l, I)$ with $\frac{\partial \psi}{\partial I} < 0$ and $\frac{\partial^2 \psi}{\partial I^2} > 0$. To simplify our analysis, we will assume the cost of investment is $\kappa(I) = I$.

If they trade service a_i at price P_i, and if the realized state is θ_l, then B and S get

$$U(a_i, \theta_l, I) - P_i$$

and

$$P_i - \psi(a_i, \theta_l, I) - I$$

respectively.

Given the level of ex-ante investment I and the state realization θ_l, we define the ex-post optimal choice of service as $a^*(\theta_l, I) \equiv \underset{a_i}{argmax}\ U(a_i, \theta_l, I) - \psi(a_i, \theta_l, I)$

The efficient level of ex ante investment solves

$$\underset{I \geq 0}{max} \{\mathbb{E}\left[U(a^*(\theta_l, I), \theta_l, I) - \psi(a^*(\theta_l, I), \theta_l, I)\right] - I\}$$

where the expectation is over the distribution of θ_l.

To highlight the role of ex-ante investment and make our analysis more tractable, for the remaining part of the paper, we will make several simplifying assumption regarding the state of the world.

Specifically, we will assume that there are three symmetric states ($N = 3$) and three potential services available. We will assume that in any given state, the value and cost configurations are positively correlated. This assumption generates the interesting scenario where the interests of S and B are in conflict: while S would like to choose

	θ_1	θ_2	θ_3
a_1	$(\varphi(I), c(I))$	$(\Phi(I), C(I))$	(0,0)
a_2	(0,0)	$(\phi(I), c(I))$	$(\Phi(I), C(I))$
a_3	$(\Phi(I), C(I))$	(0,0)	$(\phi(I), c(I))$

Table 1: State-service match

the service that reduces cost, B would like to choose the serve that yields the highest value. To further simplify the functional forms of the payoff functions, we assume in any state θ_l $(l = 1, 2, 3)$, $(U(a_i, \theta_l, I),\ \psi(a_i, \theta_l, I) \in \{(0,0),\ (\phi(I), c(I)),\ (\Phi(I), C(I)\}$. In other words, for each investment level I, there are only three possible (U, ψ) configurations, i.e., $(0,0)$, (ϕ, c) and (Φ, C), and which one occurs depends on the realization of θ.

For each investment level I, we assume the magnitudes of ϕ, c, Φ, C satisfy

$$0 < \Phi(I) - C(I) < \phi(I) - c(I) < \Phi(I) - c(I) < \Phi(I)$$

which implies (ϕ, c) is the **efficient (U, ψ) configuration**. We will refer to (Φ, C) as the high-cost configuration and $(0,0)$ the worthless configuration.

We will order the labels of a_i in the following way:

The above assumptions can be relaxed without changing the main predictions of the model. In the extension section, we will discuss the implication of more complex state space. Loosely speaking, N is a measure of overall **complexity** of the environment, and for each a_i, the ratio $\frac{Pr(U(a_i,\theta_l,I)=\psi(I))}{Pr(U(a_i,\theta_l,I)<\psi(I))}$, i.e., the odds of successfully "guessing" the state of the world measures the predictability of the environment. As we will see, these two measures affect the relative desirability of sales v.s employment contract via their effects on the equilibrium investment level.

3.1 Baseline case: without T

We will first discuss the model where the third-party T is excluded.

Recall that the two parties S and B trade service a_i at price P_i, then B and S get

$$U(a_i, \theta_i, I) - P_i$$

and

$$P_i - \psi(a_i, \theta_i, I) - I$$

respectively.

If the contract can be made state-contingent, the two parties can solve the joint surplus maximization problem state by state.

The first best investment level I^*maximizes $\phi(I) - c(I) - I$, i.e., $\phi'(I^*) - c'(I^*) = 1$.

If T does not exist and the contracting environment is incomplete (i.e., the contract cannot be made state-contingent), then the contract between B and S can be one of the following three cases.

3.1.1 Spot Contract

We say B and S sign a spot contract if the they choose to wait until state is realized and write a contract. If the realized state is θ_l, the joint surplus is maximized by choosing a_l such that $U(a_l, \theta_l, I) = \phi(I)$ and $\psi(a_l, \theta_l) = c(I)$.

We will assume that the two parties divide equally the gains from trade. Under this assumption, the buyer and the seller each gets $\frac{\phi(I)-c(I)}{2}$ ex-post.

The ex-ante level of investment I, denoted I^{Spot}, is then chosen to maximize $\frac{\phi(I)-c(I)}{2} - I$. i.e., I^{Spot} satisfies

$$\phi'(I^{Spot}) - c'(I^{Spot}) = 2$$

Comparing I^{Spot} and I^*, we obtain the familiar observation that contractual incompleteness leads to under-investment relative to the first-best.

3.1.2 Sales Contract (Fixed service delivery)

An alternative solution to the problem of incompleteness of the contracting environment is to write a fixed-service menu of the form $\{a_i, p_i\}$, which specifies a single service to be provided ex-post at pre-specified terms $\{a_i, p_i\}$, regardless of the actual realization of the state.

Following the assumptions of Hart and Moore (1988), we allow for efficient renegotiation after the realization of the state. Suppose the contract $\{a_i, p_i\}$ is signed and the actual state turns out to be θ_i, the existing term of service is already efficient and no renegotiation is necessary. On the other hand, suppose θ_j $(j \neq i)$ occurs where the surplus at at the pre-specified term is $(\Phi(I), C(I))$, the two parties can renegotiate to the efficient configuration $(\phi(I), c(I))$ by changing the service to be delivered to a_i. This process of renegotiation yields extra surplus of the amount $[\phi(I) - c(I) - (\Phi(I) - C(I))]$, which, under the assumption of equal bargaining power, the renegotiated price will reflect equal division of surplus between the two parties. Similarly, if θ_l is such that $(0, 0)$ occurs, the two parties can renegotiate to another a_i to gain the extra surplus $[\phi(I) - c(I)]$ to be dived between the seller and buyer.

When states are symmetric, the seller's ex-ante payoff is thus

$$\begin{aligned}
\pi^S(I,p) \equiv & \frac{1}{3}[p - c(I)] + \frac{1}{3}\left\{p - C(I) + \frac{1}{2}[\phi(I) - c(I) - (\Phi(I) - C(I))]\right\} \\
& + \frac{1}{3}\left\{p + \frac{1}{2}(\phi(I) - c(I))\right\} - I \\
& = p - \frac{1}{3}c(I) - \frac{1}{3}C(I) + \frac{1}{3}[\phi(I) - c(I)] - \frac{1}{6}[\Phi(I) - C(I)] - I
\end{aligned}$$

Under fixed service delivery contract, the optimal I, denoted as I^{Sales} satisfies

$$\frac{1}{3}[\phi'(I^{Sales}) - c'(I^{Sales})] - \frac{1}{6}[\Phi'(I^{Sales}) - C'(I^{Sales})] - \frac{1}{3}c'(I^{Sales}) - \frac{1}{3}C'(I^{Sales}) = 1$$

Unlike in the spot contract case, where the equilibrium I^{Spot} only depends on the efficient configuration (ϕ, c), here, I^{Sales} is also affected by the form of the inefficient configurations. From the expression $\pi^S(I,p)$, we see that the inefficient configurations matter through their effect on the renegotiation surplus, which is the central lesson from Hart and Moore (1988).

Consider the case when $\Phi = \phi$ and $C = c$. The above expression simplifies to

$$\phi'(I^{Sales}) - 5c'(I^{Sales}) = 6$$

Since ϕ' is decreasing and c' is increasing in I, we have $I^{Spot} < I^{Sales} < I^{FB}$.

Intuitively, when there's no dispute regarding which state is more favorable ($\Phi = \phi$, $C = c$), there is no need to wait for the state realization, since the ex-ante renegotiation surplus ($\frac{1}{2}(\phi(I) - c(I))$) can be used to induce S to put in more investment, which improves on the second best investment level under spot contract. This equilibrium level of investment I^{Sales} is less than the first-best level I^{FB}, since S only internalizes half of the total surplus at stake $(\phi(I) - c(I))$.

3.1.3 Employment Contract (Flexible service delivery)

Finally, we consider the contract with flexible service delivery, i.e. a menu of services from which the buyer can choose ex-post, which must be delivered at pre-specified terms $\{A, p(a)|a \in A\}$. For this reason, we will refer to the buyer as the employer and the seller the employee. Since all a_i's are symmetric ex-ante, in equilibrium, A will contains all a's and $p(a)$ has the same value for all a, denoted by $p(a) = p$. To make this case meaningful in practice, we will assume that the seller's amount of investment outlay I is dictated by the buyer-employer ex-ante and compensated by the buyer-employer ex post. [2]

[2]We ignore the case of seller-employment contract, where the seller chooses the type of service from A to implement, since the equilibrium investment under seller-employment is identical to that under spot contracting mode. To see this, note that under seller-employment contract, S solves

$$\max_I \{p - \psi(a, \theta_i)\}$$

and will always select the useless configuration $(0, 0)$, which will then be changed to the efficient configuration (ϕ, c) during the renegotiation stage, yielding a total renegotiation surplus of the amount $\phi(I) - c(I)$. The buyer's payoff is then

$$-p + \frac{1}{2}(\phi(I) - c(I))$$

and the seller's payoff is then

$$p + \frac{1}{2}(\phi(I) - c(I)) - I$$

The seller's choice of ex-ante investment is solution to

$$\max_I \left\{p + \frac{1}{2}(\phi(I) - c(I)) - I\right\}$$

which yields the same FOC as the spot contracting problem.

The strength of such a contract is to allow the buyer some flexibility in choosing the most desirable service after the state is realized. The weakness is that the service favored by the buyer (the one that yields $\Phi(I)$ ex post) is also inefficiently highly costly at $C(I)$. In equilibrium, this inefficient outcome will be renegotiated to the efficient configuration $(\phi(I), c(I))$, and the total surplus from renegotiation $[\phi(I) - c - \Phi(I) + C(I)]$ will be equally divided between the two parties, yielding

$$(\Phi(I) - p) + \frac{1}{2}[\phi(I) - c(I) - \Phi(I) + C(I)]$$

for the buyer and

$$(p - C(I)) + \frac{1}{2}[\phi(I) - c(I) - \Phi(I) + C(I)]$$

for the seller.

Consider the choice of I by the buyer.

The level of ex-ante investment I^{BE} dictated by the buyer solves

$$\max_{I \geq 0} \left\{ \Phi(I) - p + \frac{1}{2}[\phi(I) - c(I) - \Phi(I) + C(I)] - I \right\}$$

i.e.,

$$2\Phi'(I^{BE}) + [\phi'(I^{BE}) - c'(I^{BE})] - [\Phi'(I^{BE}) - C'(I^{BE})] = 2$$

Consider the case when $\Phi = \phi$ and $C = c$. The above expression simplifies to $\Phi'(I^{BE}) = 1$. [3]

The intuition is simple: when $\Phi = \phi$ and $C = c$, there is no surplus generated in the renegotiation stage. Since investment does not modify B's benefit sharing in the renegotiation stage and serves only to increase the status quo payoff $\Phi(I) - I$, B will dictate the level of investment to maximize the level of status quo payoff. As the marginal effect of I on the net surplus in the efficient configuration $(\phi'(I) - c'(I))$ becomes larger relative to the marginal effect of I on the net surplus in the high-cost configuration $(\Phi'(I) - C'(I))$, B will be induced to dictate a different level of investment, taking into consideration the additional incentives at the renegotiation stage.

3.2 Illustrative Examples

3.2.1 Example with co-linear value and cost configurations

As discussed above, even without third-party competition, which form of contract is better depends crucially on the shape of $\phi(.)$, $\Phi(.)$, $c(.)$, and $C(.)$.

Consider the simple case where the functional forms of ϕ' and Φ' are co-linear and the functional forms of C'

Note that this holds because the least-cost service also has zero value to the buyer. If the least-cost service gives strictly positive value to the buyer, then spot contracting and seller employment contracting will give different level of ex-ante investment.

[3] Note that by assumption, $\Phi'(I) < \phi'(I)$, so the buyer may still under-rinvesst under the employment contract relative to the first-best leve.

and c' are co-linear, i.e., suppose $\phi'(I) = m\Phi'(I)$, where $m > 1$, $\phi'(I) \geq 0$ and $c'(I) = c$ and $C'(I) = kc$, where $c > 0$. Note that we also $k < 0$ to account for the scenario where C includes large amount of fixed cost but small amount of variable cost.

The optimality conditions for the spot contract, the sales contract and the employment contract are, respectively:

$$\Phi'(I^{Spot}) = \frac{2+c}{m}$$

$$\Phi'(I^{Sales}) = \frac{3 + (2 - \frac{k}{2})c}{(m - \frac{1}{2})}$$

$$\Phi'(I^{BE}) = 1$$

Suppose $m = 3/2$, $c = 1$

$$\Phi'(I^{Spot}) = 2$$

$$\Phi'(I^{Sales}) = 5 - \frac{1}{2}k$$

$$\Phi'(I^{BE}) = 1$$

The second-best contract is buyer employment when $0 < k \leq 8$ and sales contract when $6 < k \leq 10$.

Suppose $m = 6$, $c = 1$

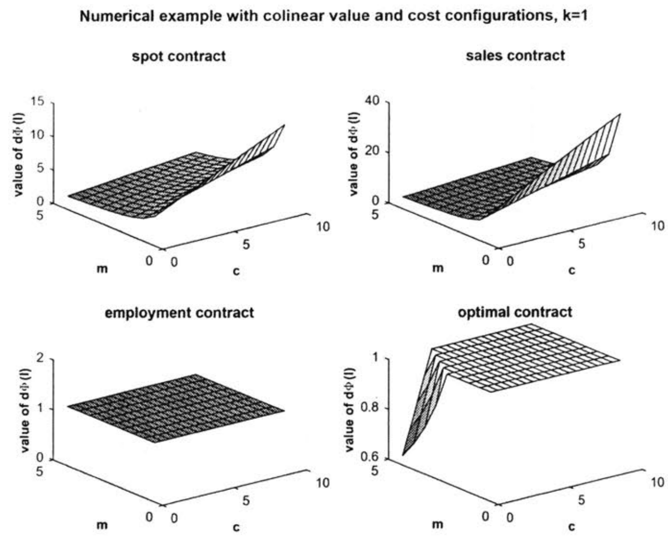
$$\Phi'(I^{Spot}) = \frac{1}{2}$$

$$\Phi'(I^{Sales}) = \frac{10}{11} - \frac{1}{11}k$$

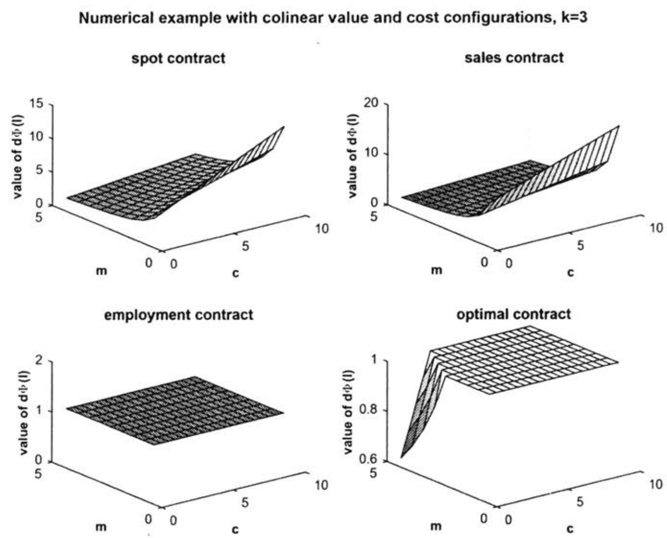
$$\Phi'(I^{BE}) = 1$$

The second-best contract is spot contracting when $0 < k \leq \frac{9}{2}$ and sales contract when $\frac{9}{2} < k \leq 10$.

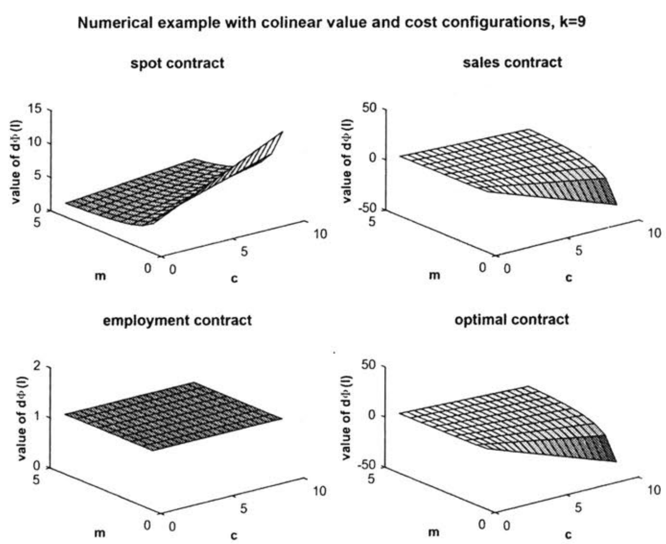
The figures below show the value of $\Phi'(I)$ for the three types of contracts when $k = 1$, 3 and 9.



(a) $k = 1$



(b) $k = 3$



(c) $k = 9$

Figure 5: Value of $\Phi'(I)$ for numerical example with co-linear value and cost configurations

3.2.2 Example with non-colinear value and cost configurations

To illustrate the forces at work in this model, we consider the following numerical example. Suppose an upstream biotechnological lab (S) invests I dollars to develop the treatment for a certain type of cancer for a downstream firm (B). There are three types of treatment that can be potentially developed: chemotherapy (A), hormone therapy (B), and stem cell transplant (C).

Because of the nature of research development, the terms of trade cannot be specified at the time of development. Further, the value of each type of treatment depends on the regulatory environment (insurance coverage, government funding in medical research, etc.), which we will abstractly refer to as state a, b, and c. The gross value and cost of each type of treatment in various states is summarized in the following table.

	a	b	c
A	$V(I), C(I)$	$v(I), c(I)$	0
B	0	$V(I), C(I)$	$v(I), c(I)$
C	$v(I), c(I)$	0	$V(I), C(I)$

Table 2: Payoff Matrix

$\Pi(I) \equiv V(I) - C(I)$ and $\pi(I) \equiv v(I) - c(I)$ are strictly increasing, strictly concave, bounded above, and satisfy $V'(0) - C'(0) > 1$, $v'(0) - c'(0) > 1$ and $V(I) - C(I) < v(I) - c(I)$ $\forall I > 0$. We assume the three states occur with equal probability.

For concreteness, let $v(I) = \sqrt{I}$, $c(I) = \frac{1}{2}I^2$, $V(I) = log(I+1)+1$, and $C(I) = I^2 + log(2) - 1$, where $0 \leq I \leq 1$. The value of these functions are plotted in the figure below.

Consider the case where there is no third party that can potentially develop a competing product. In this case, the upstream and downstream firms are locked in a bilateral bargaining situation, resulting in an ex-post efficient trade and in an equal split of the surplus.

If S and B write a spot contract, i.e., they contract on delivery of a specific kind of treatment after the state is realized and becomes known. Under ex-post spot contracting, the payoff to the upstream party is $\frac{1}{2}\pi(I) - I$, so the optimal investment satisfies $\frac{1}{2}\pi'(I) = 1$, which is lower than the first-best investment level $x^{FB} = 0.18$ (solution to $\pi'(I) = 1$). We have $x^{spot} = 0.06$.

If S and B write a fixed service delivery contract, the solution to S's optimization problem yields

$$\frac{4}{3}\pi'(I) + \frac{5}{3}\Pi'(I) = 2.$$

We have $x^{fixed} = 0.19$.

If S and B write a flexible service delivery contract, the value of x is given by

$$\pi'(I) + \Pi'(I) = 2.$$

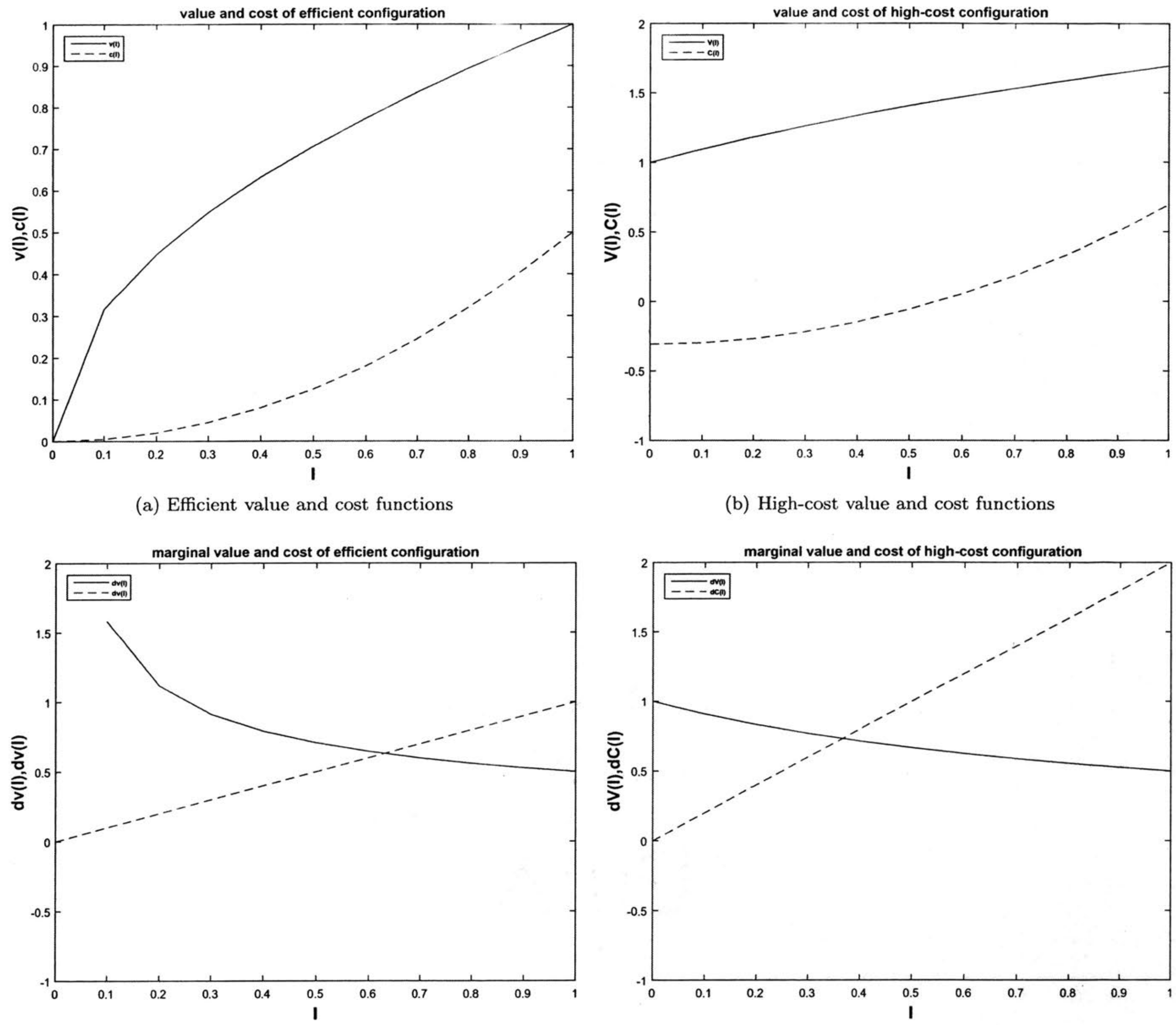
We have $x^{flexible} = 0.12$.

In this case, the fixed service delivery contract is the second-best. This is so because $\Pi' < \pi'$ by our functional form assumption.

To anticipate our discussion in the next section, consider a potential third-party competitor when S and B have a fixed service delivery contract in place.

It is intuitive to that the optimal choice of x is higher in this case. To see this, consider a marginal increase in x from x^{fixed}. Since $\pi'(x^{fixed}) > 0$ and $\Pi'(x^{fixed}) > 0$, this change has two first-order effects: it improves S's bargaining position if T correctly bets the state, and it increases the total surplus at stake if T incorrectly bets the state. In our scenario, this change constitutes a first-order improvement.

In this example, we see that the potential entrant improves on the second-best effort choice. We will discuss the role of potential entry in full details in the remaining part of this section.



(a) Efficient value and cost functions

(b) High-cost value and cost functions

(c) Efficient marginal value and cost functions

(d) High-cost marginal value and cost functions

Figure 6: Efficient v.s high-cost configurations for numerical example with no

3.3 T is a business stealer

In this scenario, T is symmetric to S (i.e., they have the same potential production technology, same access to information, etc.) with the exception that S has the opportunity to renegotiate the terms of trade with B after the realization of θ, while T only has access to the spot market. This scenario naturally arises when S is an existing company considering offering a new good/service to its customer body, while T is startup offering the same new good/service (without access to existing customer body). T is constrained relative to S in that T is limited to engage in spot contracting with B while S has other options as part of the available set of contractual relations.

As illustrated in 2, when the third party T is a business stealer. T has the same access to production technology as S and will bet on a state of nature θ_i and provides the optimal service a_i for that state. Let I_S and I_T denote S and T's investment level. Suppose that if B is indifferent between S and T, then she chooses S.

As discussed in the baseline analysis, fixing the level of I_T and I_S, which supplier's service will be accepted depends on the form of existing contract between S and B. As shown in the previous section, spot contracting is dominated by either sales or employment contracting. Hence we will focus on the case of sales contract and employment contract below.

3.3.1 The Case of Sales Contract

Suppose the existing contract between S and B is sales contract.

If T develops $(\phi(I_T), c(I_T))$ (which happens with probability $\frac{1}{3}$), then we will say T successfully "guesses" the state.

Suppose B trades with S, she gets

$$\pi^B(I_S,p) \equiv \frac{1}{3}[\phi(I_S)-p] + \frac{1}{3}\left\{\Phi(I_S)-p+\frac{1}{2}\left[\phi(I_S)-c(I_S)-\Phi(I_S)+C(I_S)\right]\right\} + \frac{1}{3}\left\{-p+\frac{1}{2}[\phi(I_S)-c(I_S)]\right\}$$

Suppose B defects and engages in spot contracting with T, she gets

$$\frac{\phi(I_T)-c(I_T)}{2}$$

To prevent B from defecting, p must be set at a level such that

$$\pi^B(I_S,p) \geq \frac{\phi(I_T)-c(I_T)}{2}$$

The lowest value of p that satisfies this expression is given by

$$p_1^*(I_T) = \frac{1}{3}\phi(I_S) + \frac{1}{3}\left\{\Phi(I_S)+\frac{1}{2}\left[\phi(I_S)-c(I_S)-\Phi(I_S)+C(I_S)\right]\right\} + \frac{1}{6}[\phi(I_S)-c(I_S)] - \frac{\phi(I_T)-c(I_T)}{2}$$

Now, consider the case where T develops $(\Phi(I_T), C(I_T))$ (which happens with probability $\frac{1}{3}$).

Suppose B defects and engages in spot contracting with T, she gets

$$\frac{\Phi(I_T)-C(I_T)}{2}$$

To prevent B from defecting, p must be set so that

$$\pi^B(I_S,p) \geq \frac{\Phi(I_T)-C(I_T)}{2}$$

The lowest value of p that satisfies this expression is given by

$$p_2^*(I_T) = \frac{1}{3}\phi(I_S) + \frac{1}{3}\left\{\Phi(I_S) + \frac{1}{2}\left[\phi(I_S) - c(I_S) - \Phi(I_S) + C(I_S)\right]\right\} + \frac{1}{6}[\phi(I_S) - c(I_S)] - \frac{\Phi(I_T) - C(I_T)}{2}$$

Finally, if T fails (which happens with probability $\frac{1}{3}$), the renegotiation stage between S and B proceeds as in the baseline case where T is absent.

Recall that $\phi(I_T) - c(I_T) > \Phi(I_T) - C(I_T)$. Hence, we have $p_1^* < p_2^*$. Intuitively, competition hurts S the hardest in the case where T successfully "guessed" the state.

We assume T is a profit maximizer. Given the (common) knowledge about the distribution of states and the corresponding payoff configurations, the level of I_T will be determined in equilibrium from T 's maximization problem:

$$\max_{I_T \geq 0} \frac{1}{3} \cdot \frac{\phi(I_T) - c(I_T)}{2} + \frac{1}{3}\frac{\Phi(I_T) - C(I_T)}{2} - I_T$$

Let I_T^*denotes the solution to the above problem.

Under our assumption that I_T is observable to both S and T, the renegotiated price p can be conditioned on the observed (and agreed) level of I_T. Hence, S's payoff will be $\pi^S(I_S, p_1^*(I_T^*))$ with probability $\frac{1}{3}$, $\pi^S(I_S, p_2^*(I_T^*))$ with probability $\frac{1}{3}$, $\pi^S(I_S, p)$ with probability $\frac{1}{3}$, where p denotes the trading price in the absence of T and

$$\begin{aligned}\pi^S(I,p) &\equiv \frac{1}{3}\left[p - c(I)\right] + \frac{1}{3}\left\{p - C(I) + \frac{1}{2}\left[\phi(I) - c(I) - (\Phi(I) - C(I))\right]\right\} \\ &+ \frac{1}{3}\left\{p + \frac{1}{2}(\phi(I) - c(I))\right\} - I \\ &= p - \frac{1}{3}c(I) - \frac{1}{3}C(I) + \frac{1}{3}[\phi(I) - c(I)] - \frac{1}{6}[\Phi(I) - C(I)] - I\end{aligned}$$

The equilibrium investment level I_S^* is determined as the maximizer of the ex-ante expected producer surplus

$$\frac{1}{3}\pi^S(I_S, p_1^*(I_T^*)) + \frac{1}{3}\pi^S(I_S, p_2^*(I_T^*)) + \frac{1}{3}\pi^S(I_S, p)$$

where $p = \frac{\phi(I_S) - c(I_S)}{2}$. (Recall that in the absence of T, the equilibrium price p is obtained from equal surplus sharing.)

It's easy to see the effect of competition on I_S^*: as in Grossman and Hart (1986), competition gives B a better outside option (relative to no-trade), and thus strengthens B's bargaining power.

3.3.2 The Case of Employment Contract

Now, assume that S and B are in a buyer employment relation. In this case, the two parties specify a menu of services and the corresponding prices, from which the buyer can choose ex-post, which must be delivered at prespecified terms $\{A, p(a)|a \in A\}$. As discussed in the previous section, given that all a_i's are symmetric ex-ante, in equilibrium, A will contains all a's and $p(a)$ has the same value for all a, denoted by $p(a) = p$.

Without T, optimal I dictated by the buyer-employer solves

$$\underset{I}{max}\bar{\pi}^B(I,p) \equiv \underset{I}{max}\left\{\Phi(I) - p + \frac{1}{2}\left[\phi(I) - c(I) - \Phi(I) + C(I)\right] - I\right\}$$

Let $\bar{\pi}^S(I,p) = p - C(I) - \frac{1}{2}\left[\phi(I) - c(I) - \Phi(I) + C(I)\right]$ denote the seller's surplus under buyer-emloyment contract.

Now, suppose T is a business stealer, who bets on a state of nature θ_i and provides the optimal service a_i for that state.

The analysis of the previous subsection carries over, with $\pi^S(I,p)$ replaced by $\bar{\pi}^S(I,p)$:

If T develops $(\phi(I_T), c(I_T))$ (which happens with probability $\frac{1}{3}$), then we say T successfully "guesses" the state.

Suppose B trades with S, she gets

$$\bar{\pi}^B(I_S,p) = \Phi(I_S) - p + \frac{1}{2}\left[\phi(I_S) - c(I_S) - \Phi(I_S) + C(I_S)\right] - I_S$$

Suppose B defects and engages in spot contracting with T, she gets

$$\frac{\phi(I_T) - c(I_T)}{2}$$

To prevent B from defecting, p must be set at a level such that

$$\bar{\pi}^B(I_S,p) \geq \frac{\phi(I_T) - c(I_T)}{2}$$

The lowest value of p that satisfies this expression is given by

$$p_1^*(I_T) = \Phi(I_S) + \frac{1}{2}\left[\phi(I_S) - c(I_S) - \Phi(I_S) + C(I_S)\right] - I_S - \frac{\phi(I_T) - c(I_T)}{2}$$

Now, consider the case where T develops $(\Phi(I_T), C(I_T))$ (which happens with probability $\frac{1}{3}$).

Suppose B defects and engages in spot contracting with T, she gets

$$\frac{\Phi(I_T) - C(I_T)}{2}$$

To prevent B from defecting, p must be set so that

$$\bar{\pi}^B(I_S,p) \geq \frac{\Phi(I_T) - C(I_T)}{2}$$

The lowest value of p that satisfies this expression is given by

$$p_2^*(I_T) = \Phi(I_S) + \frac{1}{2}\left[\phi(I_S) - c(I_S) - \Phi(I_S) + C(I_S)\right] - I_S - \frac{\Phi(I_T) - C(I_T)}{2}$$

Finally, if T fails (which happens with probability $\frac{1}{3}$), the renegotiation stage between S and B proceeds as in the baseline case where T is absent.

Recall that $\phi(I_T) - c(I_T) > \Phi(I_T) - C(I_T)$. Hence, we have $p_1^* < p_2^*$. Intuitively, competition hurts S the hardest in the case where T successfully "guessed" the state.

As with the previous section, we assume T is a profit maximizer. Given the (common) knowledge about the distribution of states and the corresponding payoff configurations, the level of I_T will be determined in equilibrium from T 's maximization problem:

$$\max_{I_T \geq 0} \frac{1}{3} \cdot \frac{\phi(I_T) - c(I_T)}{2} + \frac{1}{3}\frac{\Phi(I_T) - C(I_T)}{2} - I_T$$

which yields the same solution I_T^* as in the case of sales contract.

Under our assumption that I_T is observable to both S and T, the renegotiated price p can be conditioned on the observed (and agreed) level of I_T. Hence, B's payoff will be $\bar{\pi}^B(I_S, p_1^*(I_T^*))$ with probability $\frac{1}{3}$, $\bar{\pi}^B(I_S, p_2^*(I_T^*))$ with probability $\frac{1}{3}$, $\bar{\pi}^B(I_S, p)$ with probability $\frac{1}{3}$, where p denotes the trading price in the absence of T and

$$\bar{\pi}^S(I, p) = p - C(I) - \frac{1}{2}\left[\phi(I) - c(I) - \Phi(I) + C(I)\right]$$

The equilibrium investment level I_S^* is determined as the maximizer of the ex-ante expected producer surplus

$$\frac{1}{3}\bar{\pi}^B(I_S, p_1^*(I_T^*)) + \frac{1}{3}\bar{\pi}^B(I_S, p_2^*(I_T^*)) + \frac{1}{3}\bar{\pi}^B(I_S, p)$$

One important observation is that, while competition might change the level of ex-ante investment I_S, it does not change the **relative** desirability of sales contract v.s. employment contract. This conclusion depends crucially on our assumption that T is a business stealer, meaning that the optimal choice of I_T^* is independent of I_S and of the specific form of contracting between S and B.

3.4 T is a market raider

Now we turn to the case where T is a market raider. With some probability $\Gamma(I_S)$, she can replicate the exact investment I_S that S puts in and produce a service **after** the realization of θ but **before** the renegotiation state between B and S, without paying the ex-ante investment layout $\kappa(I_S) = I_S$. Note that we assume T still needs to pay the variable cost $\psi(I_S) \in \{0,\ c(I_S),\ C(I_S)\}$

As we will see, the equilibrium outcome of the game depends crucially on the functional form of $\Gamma(I_S)$. Throughout this section, we will assume the discovery probability Γ is decreasing in I_S: the more the seller investment, the harder it is for T to replicate its investment (potentially due to financial constraint).

For a given level of I_S, whose service will be accepted depends on the form of existing contract between S and B.

We consider the case of sales contract and the case of buyer employment contract in turn.

3.4.1 The Case of Sales Contract

Suppose the existing contract between S and B is sales contract.

Note that if T successfully replicate I_S, after the realization of θ, she will always choose a_i which yields the efficient configuration $(\phi(I_S), c(I_S))$, and sell to B at price $\frac{\phi(I_S)-c(I_S)}{2}$.

Suppose B trades with S, she gets

$$\pi^B(I_S,p) \equiv \frac{1}{3}[\phi(I_S)-p] + \frac{1}{3}\left\{\Phi(I_S) - p + \frac{1}{2}[\phi(I_S) - c(I_S) - \Phi(I_S) + C(I_S)]\right\} + \frac{1}{3}\left\{-p + \frac{1}{2}[\phi(I_S) - c(I_S)]\right\}$$

If T's replication is successful and suppose B defects and trades with T, she gets

$$\frac{\phi(I_S) - c(I_S)}{2}$$

B will choose T over S if $\pi^B(I_S,p) \leq \frac{\phi(I_S)-c(I_S)}{2}$. To prevent this from happening, p will be set at p^* so that B is indifferent, i.e.,

$$p^* = \frac{1}{3}\phi(I_S) + \frac{1}{3}\left\{\Phi(I_S) + \frac{1}{2}[\phi(I_S) - c(I_S) - \Phi(I_S) + C(I_S)]\right\} + \frac{1}{6}[\phi(I_S) - c(I_S)] - \frac{\phi(I_S) - c(I_S)}{2}$$

If T's replication is unsuccessful, we are back in the case where T is absent.

Given a specific functional form Γ, S's ex-ante surplus can be written as

$$\hat{\pi}^S(I,p) \equiv (1-\Gamma(I)) \cdot \pi^S(I_S,p) + \Gamma(I) \cdot \pi^S(I_S,p^*)$$

where

$$\begin{aligned}
\pi^S(I,p) \equiv{}& \frac{1}{3}[p - c(I)] + \frac{1}{3}\left\{p - C(I) + \frac{1}{2}[\phi(I) - c(I) - (\Phi(I) - C(I))]\right\} \\
&+ \frac{1}{3}\left\{p + \frac{1}{2}(\phi(I) - c(I))\right\} - I \\
={}& p - \frac{1}{3}c(I) - \frac{1}{3}C(I) + \frac{1}{3}[\phi(I) - c(I)] - \frac{1}{6}[\Phi(I) - C(I)] - I
\end{aligned}$$

and p is the equilibrium price in the absence of T, i.e., $p = \frac{\phi(I_S)-c(I_S)}{2}$.

To see the intuition of this result, consider the two polar cases: $\Gamma(I) = 0$ and $\Gamma(I) = 1$.

The first case occurs if T is effectively banned from entry. The analysis then reduces to the baseline no-competition scenario.

The second case occurs if T can perfectly free-ride on S's investment and compete with the incumbent in the product market by offering the spot contract. The market competition effectively reduces the trading price from p to p^*.

Since $\hat{\pi}^S(I,p)$ is linear in $\Gamma(.)$, we immediately obtain the comparative statistics result: for all$\Gamma(.) \in (0,1]$ that is strictly increasing, **fixing** the level of ex-ante investment, S is worse off with competition than without.

Note that this statement does not hold if we allow ex-ante investment to adjust freely. On the one hand, higher investment hurts S. If T can successfully copies the investment, the potential gain from trade downstream decreases, which induces S to lower I_S. On the other hand, higher investment also protects S's monopoly position, as it decreases the probability of T's successful entry. In this sense, ex-ante investment serves as de facto monopoly protection against T's invasion. The effect of ex-ante investment on S's surplus is thus ambiguous.

3.4.2 The Case of Employment Contract

Now suppose the existing contract between S and B is buyer employment contract. The timing of the game is illustrated in 3.

Recall that when T is a market raider, T can potentially copy the fruit of S's investment and develop a competitive product which is similar but not identical to S's. The probability of T's success is decreasing in S's ex-ante investment: the more S invests in developing the product, the more complicated it becomes which makes it harder for another supplier to copy.

If T successfully replicates I_S, after the realization of θ, she chooses a_i which yields $(\phi(I_S), c(I_S))$ and splits the surplus with B, provided that B accepts this offer.

Before renegotiation, the buyer solves

$$\max_a \{U(a, \theta_l, I) - p\}$$

Hence, choose a_j that yields $\Phi(I_S)$.

In the absence of T, S and B will renegotiate to efficient service a_l, where the optimal I_S^* solves

$$\max_I \left\{ \Phi(I) - p + \frac{1}{2} [\phi(I) - c(I) + \Phi(I) - C(I)] - I \right\}$$

and gives B a surplus of the amount

$$\Phi(I_S^*) - p + \frac{1}{2} [\phi(I_S^*) - c(I_S^*) + \Phi(I_S^*) - C(I_S^*)] - I_S^*$$

B will accept S instead of T if this is greater than $\frac{\phi(I_S^*)-c(I_S^*)}{2}$, which holds if

$$\Phi(I_S^*) - p + \frac{1}{2}\left[\phi(I_S^*) - c(I_S^*) + \Phi(I_S^*) - C(I_S^*)\right] \geq \frac{\phi(I_S^*) - c(I_S^*)}{2}$$

i.e.,

$$p \leq \Phi(I_S^*) + \frac{1}{2}\left[\Phi(I_S^*) - C(I_S^*)\right]$$

Note that if $\Phi(I_S^*) + \frac{1}{2}\left[\Phi(I_S^*) - C(I_S^*)\right] \geq \left[\phi(I_S^*) - c(I_S^*)\right]$, then this constraint is not binding: by assumption, the trading p must satisfy $p \leq \phi(I_S) - c(I_S)$ if trade is to take place. In this case, the employment contract serves as buffer against competitive pressure. Compared with the scenario without T, equilibrium investment level will be the same.

If on the other hand, $\Phi(I_S^*) + \frac{1}{2}\left[\Phi(I_S^*) - C(I_S^*)\right] < \phi(I_S^*) - c(I_S^*)$, then the constraint is binding. Define

$$\tilde{p}^* = \Phi(I_S^*) + \frac{1}{2}\left[\Phi(I_S^*) - C(I_S^*)\right]$$

which is the relevant (constrained) trading price when T's replication is successful.

Given a specific functional form Γ, S's ex-ante surplus can be written as

$$\tilde{\pi}^S(I_S^*, p) \equiv (1 - \Gamma(I_S^*)) \cdot \bar{\pi}^S(I_S^*, p) + \Gamma(I_S^*) \cdot \bar{\pi}^S(I_S^*, \tilde{p}^*)$$

where

$$\bar{\pi}^S(I, p) = p - C(I) - \frac{1}{2}\left[\phi(I) - c(I) - \Phi(I) + C(I)\right]$$

and p is the equilibrium price in the absence of T, i.e., $p = \frac{\phi(I_S^*)-c(I_S^*)}{2}$.

3.5 T is a knockoff producer

Finally, we turn to the case where T is a knockoff producer.

With some probability $\gamma(I_S)$, she can replicate at zero cost the exact service a_i that S was to deliver to B in her absence and sell it to B **after** the renegotiation state between B and S, without the ex-ante investment layout $\kappa(I_S) = I_S$. As in the previous case, we assume the discovery probability γ is decreasing in I_S: the more the seller investment, the harder it is for T to replicate its service.

Suppose T successfully copies S's investment. Bertrand competition will reduce the supplier's surplus to 0.

Hence, regardless of the form of contract between S and T, S's payoff will be reduced by a factor of $(1 - \gamma(I_S))$.

As with the previous case, the role of I_S is two-fold: One one hand, S will tend to under-invest due to the decrease in the amount of surplus (only a fraction, $\gamma(I_S)$ of the previous amount). On the other hand, S will tend to over-invest to deter T from entry. On net, the effect of competition on I_S is ambiguous.

We should remark that the simplicity of this final case is partly due to our assumption that of perfect price (Bertrand) competition.

In an enriched version of the model with continuous (as opposed to unit) demand, quantity competition emerges as an alternative mode of interaction between T and S. In the discussion section, we briefly allude to this possibility.

4 Discussion

4.1 The role of complexity of states

In our previous analysis, we assume that there are only three symmetric states, and three symmetric types of services.

In this section, we briefly discuss how these assumptions are not crucial to the analysis, and show that how the formulas are modified when there are L states and N ($N \leq L$) types of potential services.

Let f denotes the probability distribution of states, with $\sum_l f(\theta_l) = 1$ and $f(\theta_l) \geq 0 \, \forall l$. For simplicity, assume there are three configurations as before. Consider the set of efficient actions in state θ_i, denoted as A_i, i.e., if $a_l \in A_i$, $U(a_l, \theta_i, I) - \psi(a_l, \theta_i, I) = \phi(I) - c(I) > U(a_j, \theta_i, I) - \psi(a_j, \theta_i, I)$ for all j,. Let S_i denote the set of subscripts of actions in A_i. Similarly, let the set of high cost actions in state θ_i be denoted as B_i, i.e., if $a_l \in B_i$, $U(a_l, \theta_i, I) - \psi(a_l, \theta_i, I) = \Phi(I) - C(I)$. Let $S_i^{'}$ denote the set of subscripts of actions in B_i. The buyer's ex-ante payoff under the sales contract is given by

$$\sum_{k \in S_i} f(\theta_k)[\phi(I) - p_i] + \sum_{k \in S_i'} f(\theta_k) \left\{ \Phi(I) - p_i + \frac{1}{2}(\phi(I) - c(I) - \Phi(I) + C(I)) \right\} + \sum_{k \in S_i^C \cap S_i'^C} f(\theta_k) \left\{ -p_i + \frac{1}{2}(\phi(I) - c(I)) \right\}$$

The seller's ex-ante payoff under the sales contract is given by

$$\sum_{k \in S_i} f(\theta_k)[p_i - c(I)] + \sum_{k \in S_i'} f(\theta_k) \left\{ p_i - C(I) + \frac{1}{2}(\phi(I) - c(I) - \Phi(I) + C(I)) \right\} + \sum_{k \in S_i^C \cap S_i'^C} f(\theta_k) \left\{ p_i + \frac{1}{2}(\phi(I) - c(I)) \right\} - I$$

To simplify the expression, assume all states are equiprobable, and that service a_l is the uniquely efficient action in state θ_l. Then the above expressions simplifies to

$$\frac{1}{n}[\phi(I) - p_i] + \frac{n-1}{2n} \left\{ \Phi(I) - p_i + \frac{1}{2}(\phi(I) - c(I) - \Phi(I) + C(I)) \right\} + \frac{n-1}{2n} \left\{ -p_i + \frac{1}{2}(\phi(I) - c(I)) \right\}$$

and

$$\frac{1}{n}[p_i - c(I)] + \frac{n-1}{2n}\left\{p_i - C(I) + \frac{1}{2}(\phi(I) - c(I) - \Phi(I) + C(I))\right\} + \frac{n-1}{2n}\left\{p_i + \frac{1}{2}(\phi(I) - c(I))\right\} - I$$

It is easy to see that the FOC is modified, with the weights reflecting the relative probability of the three configurations.

It is also important to note that increasing complexity of states does not favor one particular form of contract over another; it is the interaction of the state distribution and S's payoff distribution that determines the incentives for ex-ante investment I.

4.2 General Nash Bargaining and Coalition Bargaining

In our baseline model, we assume bilateral bargaining procedure, i.e., bargaining takes place between B and S and potentially between B and T, and the two parties involved have equal bargaining power.

This assumption is made for convenience. It is easy to see how the analysis in the previous sections can be enriched to include general bargaining environment.

Following the model of Hart and Moore (1990), we define a coalition as $T \subset \{B, S, T\}$ and the joint surplus it can achieve through efficient negotiation within the coalition as $S_T(I, \theta)$. In this notation, $S_{\{S,B\}}(I, \theta)$ is the amount of renegotiation surplus within the trading relation, while $S_{\{T,B\}}(I, \theta)$ is the amount of bargaining surplus between B and the competitive supplier T. Using this notation, a party i's marginal contribution to a pre-existing coalition T is

$$M_T^i(I, \theta) \equiv S_{T\cup\{i\}}(I, \theta) - S_T(I, \theta)$$

Let Π denote the set of orderings of the set $\{B, S, T\}$ and let f^{-i} denote the set of agents that appears before i in the ordering f. For an arbitrary probability distribution on Π, denoted as $\alpha \in \Delta(\Pi)$, we can derive the expression for the Shapley value

$$\hat{\pi}^i(I, \theta) = \sum_{f\in\Pi} \alpha(f) M_{f^{-i}}^i(I, \theta)$$

The analysis of the previous section is unchanged with the renegotiation surplus π^S and π^B replaced by $\hat{\pi}^S$ and $\hat{\pi}^B$.

4.3 Joint surplus v.s social welfare

Our previous analysis focuses on the surplus generated within the relationship between S and B, as is custom in the literature of incomplete contracting with bilateral monopoly. With competition, the notion of "surplus" is more delicate. The distinction is moot in the baseline framework discussed in this paper: as T earns zero profit on the equilibrium path, joint surplus and social welfare are essentially the same. This lesson is not general, however.

Once we relax the assumption of unit demand and representative buyer and allow for monopolistic competition between S and T, T can potentially make positive profit in equilibrium, and the notion of consumer surplus needs to be enriched. Although a full-fledged discussion of the distinction between joint surplus and social welfare is beyond the scope of this paper, we offer some thoughts in Section 4.5 of the paper, where we discuss the alternative interpretation of the model in the light of entry deterrence.

4.4 Asset ownership v.s employment contract

An employment contract gives the employer the discretion to order what the employee should do. Similarly, ownership of a productive asset gives the owner the right to asset usage and revenue generated by the asset as she sees fit. The main difference between the two strands of theory is that the employer in the sense of Hart and Moore (1988) is the residual claimant on the cash flow, while the asset owner in the sense of Grossman and and Hart (1986) and Hart and Moore (1990) also has the right to exclude others from using the asset.

A robust prediction of the theory of Grossman and and Hart (1986) and Hart and Moore (1990) is that property rights serve as protection against ex post opportunism, and that the ownership of productive assets should be allocated to the party whose investment is most sensitive to the degree of protection against ex post opportunism.

We believe that the theory of Grossman and and Hart (1986) and Hart and Moore (1990) is well-suited to study entrepreneurial firms run by owner-managers, where the boundaries of assets are (1) well-defined, e.g., when assets are machinery or a bundle of human and organizational capitals developed by the firm, and (2) excludible, e.g., parties having no ownership can be excluded from using the asset. The second assumption fails when the asset in question provides information that's freely available. For example, a grocery shopping company develops a website that allows users to order grocery online from local stores at a certain service fee. By posting the catalogs online, consumers can potentially use it to compare prices without actually using the software for delivery services. In this example, the "excludibility" assumption fails because the owner of the asset (software) cannot prevent the other party from benefiting from the asset.

4.5 Alternative interpretation of the model

In this section, we offer an alternative interpretation of the model in light of Tirole's theory of entry accommodation. As we will show, the "fixed cost of entry" in Tirole's model, is analogous to the asymmetry between T and S arisen from the difference in timing of entry.

This analogy helps us connect the IO literature, where the focus of analysis is on how competitive environment affects firms' choice of instruments (price, quantity, or irreversible investment) and the OE literature, where the focus of analysis is on how the internal organization and contractual relationships between two parties can be modified to account for the external environment (e.g., uncertainty of states, incompleteness of contract).

From our analysis in the previous section, we see that in both sales contract and employment contract. S's profit function can be written in the form of $\pi^S(I^S, I^T)$ with $\partial\pi^S/\partial I^S > 0$, $\partial^2\pi^S/(\partial I^S)^2 < 0$.

For simplicity, consider the numerical example where we posit the profits of the two firms are

$$\pi^S(I^S, I^T) = I^S(1 - I^S - I^T)$$

and

$$\pi^T(I^S, I^T) = I^T(1 - I^S - I^T)$$

Note that these are reduced-form profit functions arising from short-run product-market competition with given capacities. The important thing is that I's are "strategic substitutes" since$\pi^i_j < 0$ and $\pi^i_{ij} < 0$ for $i \neq j$. In words, each firm dislikes investment by the other firm, and each firm's marginal value of investment decreases with the other firm's capital investment. If there's no fixed cost of entry and T and S choose investment simultaneously, then the solution to the problem is $I^S = I^T = \frac{1}{3}$ and $\pi^S = \pi^T = \frac{1}{9}$, which is the standard Cournot competition outcome.

If we introduce asymmetry in timing while assuming there's no fixed cost of entry, we obtain the familiar form of Stackelberg game: In period 1, S chooses I^S. In period 2, T observes I^S and chooses I^T. The perfect Nash equilibrium outcome in this sequential game is $I^S = \frac{1}{2}$, $I^T = \frac{1}{4}$, $\pi^S = \frac{1}{8}$ and $\pi^T = \frac{1}{16}$.

Now, suppose T has some fixed cost of entry:

$$\pi^T(I^S, I^T) = \begin{cases} I^T(1 - I^S - I^T) - F & if\ I^T > 0 \\ 0 & otherwise \end{cases}$$

where $F < \frac{1}{16}$. It is easy to see that the solution to the problem features potential entry deterrence, depending on the value of F, which features prominently in Tirole's theory of entry deterrence:

If F is large enough (sufficiently close to $\frac{1}{16}$), S will optimally deter the entry of T by choosing $I^S = 1 - 2\sqrt{F}$. This level of investment is greater than the monopoly level of investment $\frac{1}{2}$. If F is small (sufficiently close to 0), S will accommodate T's entry.

Now, we turn to a slightly more abstract version of the model and interpret the connection between this framework and that introduced in the previous sections.

Consider the following three-period model. In period 1, the incumbent S makes some governance agreement with B, which we will refer to as G^S. In period 2, S makes some ex-ante investment choices K^S. Note that these ex-ante choices may take place sequentially and have arbitrary temporal interdependence, i.e., $K^S = (K^S_1, K^S_2(K^S_1), K^S_3(K^S_1, K^S_2(K^S_1), ...)$ which is then observed by the third-party firm T who then decides whether to enter. In the final period, the two firms make simultaneous ex-post choices x^T and x^S (e.g., price, quantity, or quality). Their profits are given by

$$\pi^S(G^S, K^S, x^S, x^T)$$

and

$$\pi^T(G^S, K^S, x^S, x^T)$$

respectively.

Let $\{x_S^*(K^S), x_T^*(K^S)\}$ denote the third-period equilibrium outcome. Let $K_S^*(G)$ denote the optimal choice of ex-ante investment given governance structure G. Suppose it is optimal for the incumbent to deter entry. She will chooses G^S so that

$$\pi^T(G^S, K_S^*(G^S), x_S^*(K_S^*(G^S)), x_T^*(K_S^*(G^S))) = 0$$

By the Envelope Theorem, $\frac{\partial \pi^T}{\partial x_T} = 0$, so the optimal choice of G^S is determined by

$$\begin{aligned}\frac{d\pi^S}{dG^S} &= \frac{\partial \pi^T}{\partial G^S} + \frac{\partial \pi^T}{\partial K^S} \cdot \frac{dK_S^*}{dG} + \frac{\partial \pi^T}{\partial x^S} \cdot \frac{dx^S}{dK^S} \cdot \frac{dK_S^*}{dG} \\ &= \frac{\partial \pi^T}{\partial G^S} + (\frac{\partial \pi^T}{\partial K^S} + \frac{\partial \pi^T}{\partial x^S} \cdot \frac{dx^S}{dK^S}) \cdot \frac{dK_S^*}{dG}\end{aligned}$$

The firm term on the right, $\frac{\partial \pi^T}{\partial G^S}$ measures the effect of governance choice on the third-party's profit. (e.g., in the Baker, Gibbons and Murphy (2002) work, this is the effect of contractual relations (vertical integration v.s non-integration) between the upstream biotech and the downstream pharmaceutical company (Pfizer) on the profit of the alternative downstream party (Merck). This is the "BGM effect". (One difference is K^S is binary in the Baker, Gibbons and Murphy (2002) framework, but this distinction is inconsequential.)

The second term is the effect that motivates the literature on property rights theory: the choice of governance structure changes S's ex-ante investment behavior, which is measured by the term $\frac{dK_S^*}{dG^S}$ (the "Hart and Moore effect"), which in turn affects the third-party's profitability (through $\frac{\partial \pi^T}{\partial K^S} + \frac{\partial \pi^T}{\partial x^S} \cdot \frac{dx^S}{dK^S}$).

The term $\frac{\partial \pi^T}{\partial K^S}$ is the direct effect of competition. For example, if K^S is ex-ante relationship-specific investment in customer loyalty made by S to B, we might reasonably think a greater clientele reduces the size of the market available to T and hence $\frac{\partial \pi^T}{\partial K^S}$<0 .

The term $\frac{\partial \pi^T}{\partial x^S} \cdot \frac{dx^S}{dK^S}$ is the strategic interaction effect. Continuing our example of K^S being investment in customer loyalty, suppose x^S measures effort in new-product development. It's reasonably to think that $\frac{\partial \pi^T}{\partial x^S} < 0$ and $\frac{dx^S}{dK^S} < 0$, so the combined sign of the interaction effect is positive.

The term $(\frac{\partial \pi^T}{\partial K^S} + \frac{\partial \pi^T}{\partial x^S} \cdot \frac{dx^S}{dK^S})$ is the focus of Tirole's work on taxonomy of business strategy. (The "Tirole effect")

The above illustrates how the current paper enriches the class incomplete contracting literature with strategic interaction, via the Tirole effect.

5 Applications

In our previous discussion, we maintain that the one-dimensional ex-ante investment I^S is the only tool available for S to deter entry by T. In this section, we provide a more general interpretation of I^S, and show that many real world mechanisms essentially play the role of I^S which serves to deter potential entrants.

5.1 Termination Fee

Aghion and Bolton (1987) considers a bilateral trade model with termination fee. In their model, gains from trade for the buyer and incumbent are obtained at the expense of future (potentially more efficient) sellers in signing "exclusive contracts". In short, contracts that specify penalties for early termination used to extract efficiency rents from future entrants. As we will shortly see, termination fee d is equivalent to a particular form of the Γ function,

$$\Gamma(I) = \begin{cases} 1 & if\ I < d \\ 0 & if\ I \geq d \end{cases}$$

which is increasing and takes value between $[0, 1]$.

First consider the scenario studied by Aghion and Bolton (1987). Suppose there are two transaction periods $t = 0$ and $t = 1$. In period 0 an incumbent firm can offer a service at cost $c_t > 0$ to a representative buyer B, who is willing to pay at most $v = 1$. Suppose $c_l \leq \frac{1}{2}$. In period 1 a new entrant may be able to provide the service at cost c_E.

As there is only one firm in period 0, the equilibrium price will satisfy $p_0 = 1$. In period 1, entry occurs whenever the entrant's cost is lower than the ongoing cost of the incumbent, i.e., $c_E \leq c_l$. Whenever entry occurs, Bertrand competition between the incumbent and the entrant ensures $p_1 = c_l$. In the absence of entry, the period 1 game is identical to the period 0 game, and the incumbent continues to charge $p_1 = 1$.

As an illustration, we will make a simplifying assumption that the prior distribution of c_E is uniform on $[0, 1]$. Under this distribution, B's ex-ante expected payoff under spot contracting is $(1-c_l)c_l$ and the incumbent's expected payoff is $1 - c_l + (1 - c_l)^2$.

Now, suppose the incumbent and the buyer sign a long-term employment contract in period 0, specifying both p_0 and p_1 as well as a penalty for early termination, $d > 0$. Under such a contract the buyer would switch to the entrant in period 1 only if p_E satisfies $1 - p_E \geq 1 - p_1 + d$.

Hence, the ex-ante probability of entry given a value of d satisfies $Pr(entry) = Pr(c_E \leq p_1 - d) = p_1 - d$. In a Bertrand equilibrium, the entrant would offer price $p_E = p_1 - d$ to attract the buyer, so that B's expected payoff under the long-term contract is $(1 - p_0) + (1 - p_1)$. The incumbent's ex-ante expected payoff is

$$p_0 - c_l + (p_1 - c_l)(1 - p_1 + d) + d(p_1 - d)$$

Incentive compatibility of the long-term contract over the short-term contract requires $(1-p_0)+(1-p_1) \geq (1-c_l)c_l$.

Combining the above conditions, the incumbent solves for the optimization problem

$$\max_{(p_0,p_1,d)} \{p_0 - c_l + (p_1 - c_l)(1 - p_1 + d) + d(p_1 - d)\}$$

subject to the incentive compatibility constraint.

Without loss of generality, we can normalize the price in period 0 by assuming $p_0 = 1$. Then the solution to the program is

$$d^* = \frac{1 + (1 - c_l)(1 - 2c_l)}{2} > 0$$

and the equilibrium price of entry is $Pr(Entry) = p_1 - d^* = \frac{c_l}{2}$.

Note that the employment contract serves to extract part of the efficiency rent of the new entrant. It is easy to see that the employment contract give rise to inefficiency even if it can be renegotiated ex-post, as long as the entrant's cost c_E remains private information.

Our model can be viewed at one of endogenizing d. Instead of treating termination fee as a choice variable of the model, we might equivalently specify the copying probability function $\Gamma(c_l) = 1 - Pr(Entry) = 1 - p_1 + d^*$, where $d^* = \frac{1+(1-c_l)(1-2c_l)}{2}$.

One crucial distinction between this framework and our model is that entry occurs on the equilibrium path in this framework. This is due to the randomness of the cost variable c_l. But the gist of model is essentially identical. We remark that this model can be extended in one important way: suppose learning takes place over time in the sense that $\mathbb{E}\left[c_E(1 - p_0)\right]$ is increasing in p_0 (in words, more demand in period 0 gives the incumbent an opportunity to learn from their experience and makes it more formidable (on average) for the potential entrant to compete with. We will discuss the implication of allowing for dynamic interaction briefly in the extension section of the paper.

5.2 Vertical Integration as Instrument for Price-discrimination

Our previous analysis assumes competition comes from the supply side, and the margin of adjustment is the ex-ante investment which can be used to deter or accommodate entry. The essence of the analysis remains unchanged if the role of B and S are reversed, as illustrated in the following example.

Consider the following example from Joskow (2008). A monopolist produces an intermediate good, which is used as the sole input by two competitive industries producing different final good, 1 and 2. We assume the goods cater to two entirely different customer groups who have independent demands, with elasticity such that $\epsilon_2 > \epsilon_1$. For example, one could imagine good 1 being solar-powered vehicles and good 2 being solar-powered residential lighting system. While both use the same intermediate good, i.e., solar-powered batteries, they are subject to very different business law and regulations with distinct potential customers. The reason for good 1's relatively inelastic demand could be due to tax subsidy, the lack of substitutes, capital lock-in, among other things.

Because of the technology and the fact that down-stream industries are competitive, each final good's price is equal to the intermediate price charged to the industry manufacturing it. If arbitrage can be prevented between the two industries, we obtain the optimal pricing formula for the intermediate good from the inverse elasticity rule, i.e., the optimal prices p_1^* and p_2^* for the intermediate good 1 and 2, respectively, satisfy

$$p_2^* = \frac{c}{1 - \frac{1}{\epsilon_2}} < p_1^* = \frac{c}{1 - \frac{1}{\epsilon_1}}$$

One caveat to this analysis, however, is that the monopolist might not be able to prevent arbitrage between the two down-stream industries. Industry 2 might well buy the intermediate good at the low price p_2^* and resell it to industry 1. One text-book solution to this problem is to exclude market 2 altogether and sell only to industry 1, provided that industry 1 is sufficiently large and profitable.

Another less obvious solution illustrated in Joskow (2008) is for the monopolist intermediate supplier (S) to internalize industry 2, set the internal transaction price at p_2^*, and sell its intermediate good at price p_1^* to industry 1. Viewed a little differently, the two buyers may collude (through arbitrage) to undermine the monopolist's ability to price discriminate. Collusion is essentially the flip-side of competition. Competition shrinks the profit margin of the suppliers while increases the buyer's surplus. Collusion undermines price-discrimination which benefits the buyers (by decreasing the price faced by the buyer with lower elasticity) while hurts the seller. As in our baseline analysis, changing the contractual or ownership structure between two business parties can potentially lead to gains from trade that were otherwise impossible.

6 General informational structure: the role of asymmetric information

There are two periods. S, the incumbent, is monopoly at date 1 and chooses price p_1 for $t = 1$. T the entrant, then decides to enter or stay out in the second period. As opposed to our baseline analysis, we assume S's production technology is private information. For simplicity, suppose S's cost is low with probability x and high with probability $(1 - x)$. Let $M_1^t(p_1)$ denote the monopoly profit where $t = L$ or H denotes the cost type. Let p_m^L and p_m^H denote the monopoly prices charged by the incumbent when $t = L$ and H respectively.

S knows its cost from the start while T does not know firm 1's cost. Assume T learns S's cost immediately after entering if it decides to enter. The second period duopoly competition is independent of the price in period 1. Let D_1^t and D_2^t denote the duopoly profits of S and T, respectively.

Assume costs are such that $D_2^H > 0 > D_2^L$. That is, under symmetric information, T would enter only if S's cost were high. Let δ be the common discount factor.

S would like to signal the information that it has low cost (so as to deter entry), but it may only do so via charging low price p_1^L. We will solve for the Perfect Bayesian Nash Equilibrium of the game.

There are two kinds of potential equilibria: the pool equilibrium, where the first-period price is independent of the cost level, and firm 2's posterior belief is identical to its prior belief; separating equilibrium, where the first-period price fully reveals the cost type to the entrant.

Focusing on separating equilibrium, we have the necessary and sufficient conditions:

$$M_1^H - M_1^H(p_1^L) \geq \delta(M_1^H - D_1^H)$$

namely, the high-cost type prefers inducing entry and earning the duopoly profits in the second than than charging the low-cost type's price p_1^L.

Similarly,

$$M_1^L - M_1^L(p_1^L) \leq \delta(M_1^L - D_1^L)$$

(note that the low-cost type could charge the monopoly price and gets at worst $M_1^L + \delta D_1^L$ (at worst, p_m^L induces entry).

Assume the high-cost type would wish to pool if p_1^L were equal to p_m^L:

$$M_1^L - M_1^H(p_m^L) < \delta(M_1^H - D_1^H)$$

i.e., there is no separating equilibrium in which each type behaves as in a full-information context.

Assume the monopoly demand is equal to the market size. We can obtain the interval of prices $[\tilde{\tilde{p}}_1, \tilde{p}_1]$ where $\tilde{p}_1$ is such that

$$M_1^H - M_1^H(p_1^L) = \delta(M_1^H - D_1^H)$$

We will focus on the least-cost separating price $\tilde{p}_1$ since charging any other price in $[\tilde{\tilde{p}}_1, \tilde{p}_1)$ is dominated for the high-cost type (he is better off charging the monopoly price regardless of his expectations about the effect of price on entry). .

From the above analysis, we get the familiar lesson from game theory: despite the fact that the incumbent manipulates his price, the entrant is not fooled. Entry occurs exactly when it would have occurred under symmetric information.

Now, turning to pooling equilibrium, which exists if the condition

$$xD_2^L + (1-x)D_2^H < 0$$

is satisfied. It's easy to see that if the pool equilibrium exists, the low-cost type charges its monopoly price and the high-cost type imitates the low-cost type by charging the same price. Note that the effect on welfare is ambiguous. On the one hand, the high-cost type lowers its price which in general increases welfare. On the other hand, the second-period entry is deterred which in general decreases welfare.

7 Conclusion

This paper considers a variant of the incomplete contracting with renegotiation model introduced by Hart and Moore (1988). We consider a trading relation between a buyer and seller, where some ex-ante relationship specific investment on the part of the buyer is needed to generate value for the buyer. Uncertainty is revealed ex post, in that prior to the investment stage, the buyer does not know which type of service she may need, and it is impossible to describe under what precise circumstances she needs a particular service. The contract can specify only the nature of a service to be provided under all circumstances, or a general option contract, namely, a menu of services that the seller agrees to provide at predetermined terms.

In addition to uncertainty regarding state realizations which was the focus of the literature thus far, we consider a different source of dispute, namely, changes in the competitive environment. We show that depending on the specific assumptions, the competitor may invest in, produce and sell an imperfect substitute or free-ride on the incumbent supplier's investment and replicate a perfect substitute with positive probability. However, in a sub-game equilibrium, the incumbent supplier will correctly anticipate potential entrants and change its ex-ante investment to account for downstream competition. It may also distort the level ex-ante investment to deter future entry.

Before concluding, we briefly summarize three key features of the model: the first two are consistent with Hart and Moore (1988) are the departure point of the analysis, while the third one is the main innovation of this paper.

Complexity and uncertainty of the economic environment. As is standard in the incomplete contracting literature, we assume contracts are incomplete. In other words, optimization is limited to the institutional constraints, including limitation in contractual language, i.e., the inability to describe in full details certain events before their realizations, even if such events can be verified ex post, among others. Specifically, we will assume there is a menu of potential goods and services that can be produced by S (or T) and sold to B. We assume the state of the world θ which is ex ante uncertain affects which good or service is the most desirable. But the states are sufficiently complex such that no state-contingent contract can be specified ex-ante.

Noncontractibility and appropriability of ex-ante investment. As in Hart and Moore (1988), we assume some non-contractible ex-ante investment I by the supplier is required to create value for B. Such investment is only worthwhile to the supplier if the demand for the service is sufficient high. We will assume that under some cases, such investment is partially appropriable, thus creating free-riding possibility for the third-party supplier. Whether a certain investment can be excluded from a third-party depends on both the regulatory environment and the nature of the investment. Countries' IP protection laws differ in length of protection, revenue sharing and antitrust regulations and leniency in execution. Certain investment, for example, investment in basic science that leads to new knowledge, or investment in platform that makes information freely accessible, are by construction accessible to the public.

Endogenous third-party supplier. While neither S or T has alternative contractual opportunities, the buyer can freely choose which supplier, S or T, to obtain the good/service from. Unlike Grossman and Hart (1986) who treat T as essentially exogenous to the contractual environment between S and B , we assume S and T are **both** profit maximizers. Specifically, S takes into account the effect of his action (investment and production) on the behavior of T and T bases his entry and production decisions on the profitability of such actions.

We should emphasize that the present paper only considers a very specific form of competition, namely, price competition over unit demand represented by a single buyer's valuation. Other modes of competition (e.g., quantity or quality competition) and assumptions regarding the demand side (e.g., separate markets, multi-dimensional preferences) will undoubtedly enrich the analysis.

A main shortcoming of the present analysis is that the competitive environment is treated as exogenously given. This is justifiable if the timing of entry represents constraints in the regulatory environment, which is difficult to change in the short tun. However, in the general equilibrium version of the model, we might reasonably endogenize the regulatory environment and ask the admittedly more important question: how should regulatory policies be designed to maximize social surplus from the trading relationship? Theoretical research and empirical evidence on this front will be invaluable.

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Chapter 2: Selling Information: Multidimensional Oligopolistic Competition

1 Introduction

Thanks to the emergence of the internet and the development of information technology, the most basic of economics transactions - the buying and selling of goods and services - continues to undergo changes which have profoundly impacted the way economic activities are organized. In 1999, the first year in which Census data on E-commerce became available [1], sales realized through electronic retailing represented a meager 1 percent of total U.S. retail sales. By the first quarter of 2018, business-to-consumer E-commerce increased to 9.5 percent of total U.S. retail sales.

The rapid growth of e-commerce has sparked changes in both firms' and consumers' behaviors. Economic researches (e.g. Bakos (2001)) document that the growth of e-commerce has brought about reduction in consumers' search cost, downward pressure on retailer markups, and narrower price dispersion for a variety of consumer goods. More importantly, with internet as the leading platform for business, information has increasingly become an integral part of transactions, which has revived the attention on the study of the role of information in economics since Akerlof (1970) and Rothschild and Stigliz (1976).

Economists have long established that when customer preferences are heterogeneous, price discrimination can be strictly profit-enhancing (Machlup (1995)). When information is private, however, the seller has to rely on some customer recognition mechanisms to segment the market (Varian (1989)). Prior to the internet era, personalized pricing was already practiced by direct marketers. For example, Young (1997) reports that AT&T lures its competitors' customers by mailing out coupon offers whose values vary based on the demographic characteristics that predict one's purchasing power. With the advent of internet, such practice has become much more widespread. In early 2000s, an FTC report found that over 99% of online companies collect electronic fingerprints of individuals visiting their websites (Seligman and Taylor (2001)).

The traditional approach to modeling customer recognization is to build a dynamic framework in which an infinitely-lived monopolist firm faces either overlapping generations of forward-looking customers (Villas-Boas (2004)) or long-lived customers making repeated purchases (Farrell (1984), Farrell (1986), Milgrom and Roberts (1986)). The underlying assumption of this type of models is that a firm can enhance its understanding of the customers only by leveraging the information revealed through the purchasing history.

However, if a firm can obtain additional information asset about customer types directly, just as it can purchases physical capital directly from the market of capital, then it can potentially increase profits by making tailored offers

[1] https://www.census.gov/retail/mrts/www/data/pdf/ec_current.pdf

to the customers without relying on the history of previous transactions. The active market of personal information with prominent players like Double Click and I-Behavior (Taylor (2004)) illustrates the feasibility and profitability of the direct market for information.

In fact, outside of the realm of academic research (Shapiro and Varian(1998), Varian(1998)), information has long been recognized as an important business asset by market participants. When the E-commerce retailer Toysmart.com filed for bankruptcy in 2000, its creditors considered the firm's customer profile data as one of its key assets. The online business giant Amazon.com explicitly states in its privacy notice that the company view "customer information (as) one of the transferred business assets" if business continues to expand.

Although the economic literature on E-commerce business models is flourishing and the debates on customer privacy protection ongoing (Varian (1997), Shapiro (2000), Shapiro (2001), Calzolari and Pavan (2004),), there has been little effort in formally building a model of optimal design of information as business asset [2]. This paper represents our attempt to fill in the gap. To this end, we propose a model with a monopolistic upstream data vendor and several downstream firms engaging in oligopolistic competition. The upstream monopolist, having knowledge of downstream competitive structure, designs the information product taking into account these concerns. The product must capture the multi-dimensionality nature of information. The product must be incentive compatible: downstream parties with different types must find it beneficial to choose the product designed for it but not its opponents. The product must optimally manage downstream competition. The model we develop combines an multidimensional screening problem (the outer stage) and an oligoposlistic competition problem (the inner stage), the parameters of which depend on the outcomes of the outer stage. Although both stages are necessary in the market of information, which aspect is more important depends on the specific applications. We will discuss two main applications of the model in details.

The focuses of these two models are slightly different. For the first model, we will emphasize the effect of ex-post competition. Specifically, we will work with the simplest model that incorporates features of both horizontal and vertical differentiation. We consider two downstream duopolists competing over the same, heterogeneous customer base. Customers vary in their preferences and price sensitivity, which is privately known only to the firm in question. If such information is public knowledge, a firm has incentive to charge higher prices to those who have higher intrinsic preferences for its product and those who have low price sensitivity. When such information is not available to the firm, however, it must trade off the benefit from charging a higher price to the risk of misclassifying a switcher as a loyal customer. The downstream firms may improve their profitability prospect by purchasing a targeting device from a single upstream information vendor. Specifically, the upstream vendor can design a multi-dimensional targeting device which improves the precision of the downstream parties' prior assessment of customer types. Since the downstream parties preferences over the customer base are different and such preferences are private knowledge, the upstream vendor, when designing a menu of information product, must take into account

[2] A notable exception is Adamati and Pfleiderer (1986) who study the behaviors of speculative traders who purchase information from a monopolistic seller.

both profitability (how much surplus can one extract from each party) and feasibility (whether the party will find it incentive-compatible to purchase the product designed for it). This model is best suited to study industry structures where the downstream parties are direct competitors and the mode of competition is well-defined, e.g., where the upstream party is a data vendor (e.g., customer data platform (CDP)) and the downstream parties are two competing e-commerce retailers (e.g., Amazon.com and Walmart.com).

For the second variant of the model, we will enrich the dimension of ex-ante selection at the expense of simplifying the ex-post competition structure. We will keep the mode of ex-post competition in the abstract. Instead we will propose a "conflict matrix" which loosely captures the idea that the downstream parties operate in imperfectly overlapping markets and may compete with each other. As we will show later, the structure of conflicts will have implication on the optimal menu of information product provided by the upstream party. This model is closely related to the optimal commodity bundling literature (Armstrong (1999)). The key difference here is that the multi-dimensional good in question is an intermediate good, whose value is endogenous to the form of interactions among the downstream players. This version of the model is best suited to study complicated industry structures where the downstream parties have overlapping practices, e.g., the market for IT infrastructure where the upstream party is a cloud computing service provider (e.g., AWS) and the downstream parties are corporate customers of AWS whose scopes of operation overlap with each other.

The menu of information products as described above has several key features. First, unlike the classic case of adverse selection, an information product is intrinsically multidimensional and the monopolist's objective function cannot simply be collapsed into a one-dimensional screening problem. This feature arises naturally in B2C settings where customers vary in brand loyalty and purchasing power and it is conceivable for a retailer to target different segments of the customer base with different degrees of intensity. Empirically, Blake, Nosko and Tadelis (2014) report significant consumer heterogeneity in paid search effectiveness. For non-brand keywords, the authors find that new and infrequent users are positively influenced by advertisements but that frequent users' behaviors are not influenced by much. In the case of B2B transactions of IT infrastructures, the multi-dimensional nature of the product in question is even more salient. As the leading provider of IT infrastructure, AWS offers menu of services including click-stream analytics, data warehousing, recommendation engines, fraud detection, event-driven ETL, and internet-of-things (IoT) processing done through cloud-based computing. Its corporate customers can select from the menu of provisions with customized order in each category.

Second, information is an intermediate good, in the sense that it represents a business asset to the firm who possesses it, but its market value is only realized through improved matching between customers and the final products. Hence, the optimal design of information produce must take into consideration its effect on downstream competition. This is related to De Long (1999)'s point that the market for information good "is almost never a market for today's tangible goods, but rather for a bundle of present goods, future goods, and future services. The initial purchase is not a complete transaction in itself, but rather a down payment on the establishment of a relationship." [3]

Third, information is both non-excludable and non-rival. In traditional monopolistic screening models, the underlying assumption is that buyers do not find it profitable to trade the product in question in secondary, ex-post market: the no-arbitarge condition is implicitly taken care of by the incentive constraints. However, given the non-excludibility and non-rivalry nature of information, it is natural to think that the downstream firms might find it profitable to augment their existing information product with the products their rivals purchase, should such exchanges be mutually beneficial (potentially with transfers). The natural questions to ask are: (1) when will such downstream transactions be feasible and what conditions must be satisfied for such exchanges to be mutually beneficial? (2) when are the implications on downstream industry profits and on societal welfare (including all players, both upstream and downstream) should such transactions be allowed? In general, finer information benefits the firms but harms the customers (the intuition is similar to that found in the canonical price discrimination literature). The answers to these questions will have implications on the optimal design of data privacy protection and related regulations, which we will only briefly touch upon in this paper and leave the details for future researches.

Before we discuss the details of the model, we should briefly discuss a key assumption of the model, namely, that the upstream party is monopolist player. This assumption can be justified in a number of ways. Theoretically, the scalability nature of information means the market for it has features of natural monopoly which can sustain only one market player. Further, the high fixed-cost and the low variable-cost involved in the production of information services imply significant lock-in and high switching cost, which is conceivable since setting up the infrastructure for developing information product is costly and migrating one's current data warehouse to an alternative provider is costly. This assumption is also justified empirically as the market for information usually features a persistent, dominant market player. According to SEC filings, AWS is several times larger than its closest competitor, making it the de facto monopolist player in this area. Nevertheless we stress that the assumption of upstream monopolist is fundamentally a modeling choice. For example, although Amazon is the dominant player in customer tracking service, Google has recently emerged as a important challenger. The partnership of Google and Walmart can be understood in terms of a model involving four parties, with two upstream players and two downstream players. We will offer a brief illustration of duopolistic upstream parties in the extension section of the model. For the baseline model, we take the object of analysis as a market with a fixed number of players and ignore potential future market entrants.

Our work is closely related to several strands of literature.

Relation to nonlinear pricing models with private information

The paper extends the theory of second best since Meade (1955) and Lipsey and Lancaster (1956). Nonlinear

[3]Although we will not explore the dynamic version of the model, it is worth noting that intermediate goods, in their most extreme form, mean that the implication of the purchase is not restricted to current period final-good market transactions and may extend to all future market transactions.

pricing models in the presence of private information have been studied extensively in Mussa and Rosen (1978), Roberts (1979), Spence (1980), Maskin and Riley (1984), Wilson (1993).These models usually feature downward distortions relative to the full-information benchmark (except for the top type, "no distortion at the top"). The framework has many applications in optimal taxation, insurance contract design, etc. The underlying assumptions of this strand of models are (1) the monopolist provider is limited to a single-dimensional instrument (tax rate, insurance premium etc.); (2) the downstream parties or customers can be ordered consistently along a single, well-defined characteristics (e.g., in the taxation literature, skills or marginal cost of effort); (3) there's no further interactions among the downstream parties, after the service in question has been purchased.

When these assumptions are satisfied, the problem is referred to as standard screening problem. Several general results can be established with some generality. When the payoff functions satisfy certain regularity and sorting conditions (the most important of which is referred to in the literature as the single-crossing condition), only local incentive constraints are binding, in which case the optimal solution features no bunching (generically). In most real world applications, however, characteristics of downstream parties/players are frequently multi-dimensional, so are the instruments of the upstream monopolist. Seade (1979) studied this problem in the optimal taxation setting and develop the solution using calculus of variation techniques. Baron and Myerson (1982) study a two-dimensional variant of the problem in the regulation setting (the two instruments being fixed cost and marginal cost of pollution reduction) while allowing for stochastic mechanisms. Armstrong (1996) studied nonlinear bundle-pricing problem and shows that the participation constraint typically binds for a nontrivial set of customers. Rochet and Chone (1998) give a comprehensive characterization of the solution to the multi-dimensional screening model with quasi-linear utility functions and introduced the techniques of ironing and sweeping.

Relation to industrial organization model of oligopolistic competition

The inner stage of the model closely resembles a standard oligopolistic market structure, where a firm does not design its pricing strategy passively. Instead, each player must incorporate strategic interactions of other decision makers. Following the technique proposed by Bulow, Geanakopos and Klemperer (1985), we extends the standard Bertrand model where two identical firms produce a homogeneous good to heterogeneous firms facing customers with various preferences. The optimality of privacy protection crucially depends on the strategic substitutability and/or complementarity of the information products. The leading applications of this paper consider only simultaneous, non-cooperative games of short-run price competition, but other variants e.g., Cournot-style quantity competition, repeated-interaction paradigms can be incorporated into the baseline framework, as long as equilibria of the downstream competition games can be pinned down conditioning on the first stage choice variables.

Relation to recent development in information economics and the study of consumer privacy

The early literature on information economics proved that asymmetric information where one party has access to information while others do not present a distinct kind of problem where attempt to exploit informational advantage leads to allocative inefficiencies and market distortions.

Calzolari and Pavan (2004), Acquisti and Varian (2004), Dodds (2002), and Cathieu (2002) provide the theoretical foundation for studying consumer and firm behaviors in E-commerce. The main focus of this early literature is to derive conditions under which the firm(s) will benefit from committing keep customer information private, and when these conditions do not hold, what kind of regulatory interventions are necessary. The related works by Hann et al. (2002) and Deck and Wilson (2002) empirically explore consumers' preferences for privacy, which could alter the welfare implications of the theoretical models.

More recently, Bergemann, Bonatti and Smolin (2017) consider a model of the sale of supplemental information, where a monopolist data seller owns a database containing information about the underlying state. The data buyers with private and incomplete information buy from the upstream monopolist. The problem is to design a menu of information with various choices of data quality and price combinations and induce self-selection by the downstream parties. This model is closely related to the outer-stage of our paper. The key difference is their work takes the sale of information as the end in and of itself: ex post utility is pinned down by own-action and states, with no reference to the information and/or other downstream parties.

Relation to foreclosure literature

Models have been developed in which vertical integration may result in market foreclosure as a means to indirectly forestall competition in downstream markets. These models typically rely on assumptions about contractual environments between non integrated entities (e.g. linear pricing) and about the commitment ability of integrated entities (e.g., an integrated supplier's commitment to not undercut a rival). In contrast, Hart and Tirole (1991) uses the theory of ownership and residual control rights to formulate the decision of vertical integration and foreclosure. Their model assumes that the upstream and downstream firms do not know ex ante which intermediate good is the relevant one to trade and it is too costly to specify contingent forward contracts (due to the large number of potential states, for example). Hence the the only way to change ex post behavior is through changes of control rights over asset uses. Vertical integration matters through its effect on profit-sharing: non-integrated units have incentives to trade with other parties. In contrast, when the units are integrated, their owners have residual rights of control over the unit's assets and can effectively restrict access to the unit's production by other third parties. In these foreclosure style model, the upside of integration is to enable full- or partial- foreclosure to fend off competition. The downside reflects the standard incentives v.s coordination trade-off: lower incentives of asset users since investment is expropriated by the owner, loss of local information about the user's performance may lead to dulled or perverse incentives that benefit the asset user at the expense of the asset owner, etc.

A model similar in spirit was developed by Ordover, Saloner, and Salop (1990). Unlike the VI story, exclusion could be achieved via enforcing commitments, either through explicit contracts, e.g., exclusive-dealing contract, or implicit reputation effect. Another related model was introduced by Bolton and Whinston (1989) who study the motives for vertical integration and foreclosure from the perspective of incomplete contracts. They focus on the situation in which downstream firms operate in distinct product markets, each making specific investments to the upstream monopolist, who in term provides one unit of intermediate good to the downstream firms ex post, that is, after the investments are made and costs become sunk. They consider a model where the upstream firm is sometimes capacity-constrained: in some states. both downstream firms can be served while in others the upstream firm has only one unit of intermediate good available.

Their model involves Nash bargaining so that each downstream firm pays the input price determined by the other downstream firm's willingness to pay. Each downstream thus receives at the margin the incremental contribution of the marginal product of its investment, so non-integration is socially optimal (when the outside option is binding). On the other hand, integration (and exclusion) may be privately optimal. By integrating one of the downstream parties, U can achieve greater post-negotiation surplus at the expense of the the other, non integrated downstream party.

Our model offers a more complete explanation for observed market exclusion. In addition to the incentives to manage downstream competition (standard in the foreclosure literature), there's also an information incentive. Providing cheaper and inferior information product to the low-value types (loosely speaking) will induce high-value types to misrepresent themselves as low types, making it harder to differentiate among potential users of the product. Such exclusion is generic (to be made precise later) when types have multi-dimensional private information, which is a reasonable assumption to make.

Relation to contracting complexity literature

In theory, the complexity of contract grows in proportion to the number of dimensions of private information. In real world however, contract usually takes simple form, which is especially puzzling in the market of information service, where the dimension of private information is high and potentially evolving over time and there's little upfront cost in setting up complicated pricing mechanism.

Traditional approach to this problem is to assume that the contracting environment is sufficiently complex which renders it impossible to perfectly specify all relevant circumstance. This could be due to existing regulations. For example in the Hart and Moore (1988) model, specific performance contracts cannot be enforced by courts. In their model, the quality of the good is unverifiable and the optimal form of the contract takes on a single form. This conclusion relies on the use of equilibrium renegotiation and message communication mechanism. Relatedly, the Grossman-Hart-Moore incomplete-contract paradigm stresses the role of unverifiable trade which may arise due to excessive complexity. This notion of complexity is formalized in Segal (1999) which provides a characterization of second-best contract when the complexity of the trading environment grows without bound. Another approach is to show that when the dimensionality of the monopolist's problem is large, then the optimal menu can be approximated by a simpler menu. For example, Armstrong (1999) shows when the type distribution is sufficiently regular (loosely, if the ratio of expectation and variance σ/μ vanishes as $n \to \infty$, then the difference in profit between the optimal second-best contract and the optimal third-best contract (i.e., a two-part tariff) tends to zero. This assumption holds when for instance the values of different types are independently distributed.

Our model shows that complexity reduction may arise naturally in the optimal contract for two reasons. First, the multidimensional nature of private information makes it generally impossible to write a differentiating contract that screens all types perfectly. This is related to the notion of path-dependence in and conservativeness of the vector field representing type distributions (to be explained later). Second, the downstream interactions post the contracting stage introduces further ambiguity in the preference space, making it more likely that types will be bunched.

The rest of the paper is organized as follows. Section 2 and 3 discuss the two versions of the baseline model, with Section 2 focusing on the ex post competition (the inner stage) and Section 3 focusing on the ex ante selection (the outer stage). Section 4 provides the infinite-dimensional extension of the model. Section 5 concludes.

2 Model

2.1 Background

Information vendors specialize in collecting, storing, processing and analyzing personal information from customers as a means of determining customers preferences. These companies uses predictive analytics for targeted marketing to increase customer satisfaction and build company loyalty. Specifically, customer data can be leveraged to provide services including:

Personalized Recommendation System whereby a data vendor uses comprehensive collaborative filtering engine (CFE) to analyze previous purchases, shopping cart items, wish list, reviews and ratings, search entries, browsing patterns, returns and exchanges. This information can be used to recommend additional products to the same customer or other customers exhibiting similar behavior patterns. According to industry estimate, such method increases the company's revenue by up to 30% annually.

Anticipatory Shipping Model and Supply Chain Optimization whereby the data provider uses big data for predicting future purchases and sending these items to local distribution centers or warehouses so that they will be ready for shipping once a order is placed. Such predictive analytics can decrease delivery time and overall expenses, optimize shipping schedules and increase product sales and profit margins by providing more timely delivery and thus achieving higher customer satisfactions. According to industry research, optimization of delivery schedule, route and product groupings can reduce shipping expenses.such reduce shipping costs by 10 to 40%.

Price Optimization which can be employed to attract more customers and increase profits. Prices are updated according to real-time web activity, competitors' prices, product availability, order history etc. It's estimated that price optimization can increase annual profit by 25% .

In recent years, the marketing landscape has been disrupted by the emergence of customer data platform (CDP) which provides a high-tech alternative approach to traditional enterprise-level data warehousing. By definition (https://www.cdpinstitute.org/cdp-basics), CDP "puts marketing in direct control of the data unification project, helping to ensure it is focused directly on marketing requirements". By applying "specialized technologies and pre-built processes that are tailored precisely to meet marketing data needs", CDP allows "a faster, more efficient solution than general purpose technologies that try to solve many problems at once".

Different from the corporate IT department, CDP proposes a "marketer-managed system" where "marketing is in charge of deciding what goes into the system". Further, it "creates a persistent, unified customer database" by "capturing data from multiple systems, linking information related to the same customer, and storing the information to track behavior over time". The goal of CDP products is to "target marketing messages and track individual-level marketing results", and to analyze and manage customer interactions. Another key feature of CDP is its open access to multiple downstream customers. A leader in CDP development, Experian boasts "consumer data, cross-channel media partnerships, and marketing campaign measurement capabilities" which makes Experian "the connective marketing tissue for thousands of brands around the globe".

Another example that has received considerable public attention is the competition between E-commerce retailing giants Amazon.com and Walmart.com. In 2017, Walmart announced an alliance with Google. In addition to voice-assisted shopping lists, re-ordering on demand, the new partnership allows the traditional brick-and-mortar retailer a "killer app" to access customer information and infer buying preferences. Amazon, although much smaller in its scale of operation, was a trailblazer in customer-tracking. Its integrated recommendation system monitors buying behavior over time and gets smarter over time as browsing data accumulates and customers supply richer information through purchasing, returning and reviewing products. The recent purchase of Whole Foods further allows Amazon to launch Wi-Fi enabled in-store shopping and tag its customers' in-store activities.

2.2 Setup

We consider a model with two downstream parties D_1 and D_2 with private types. In our leading application, we imagine D_1 and D_2 to be e-commerce retailers facing customers with heterogeneous preferences. Customers are ex-ante identical, with preference type ω drawn from a set Ω. Note that this setup is general enough to capture both vertical price discrimination model (where $\Omega = \{H, L\}$) and horizontal price discrimination model (where $\Omega = [0, 1]$ and $\omega \in \Omega$ measures a given customer's relative preference for D_1 over D_2).

Faced with a customer of type ω, each retailer D_i can take an action $a_i \in A$. The payoff from the action will depend on the type of the customer ω and the action of the competitor a_{-i}. Denote D_i's ex post utility function by $u_i(a_i, a_{-i}, \omega)$.

If D_1 and D_2 engage in spatial competition on the unit interval as in the Hotelling model, then $a_i = p_i$ captures price, $\omega \in \Omega = [0, 1]$ posted to ω and we have

$$u_i(a_i, a_{-i}, \omega) = p_i \cdot I(p_i - p_{-i} \leq \omega) - c_i$$

where c_i is the marginal cost of production.

If demand for D_1 and D_2 are independent from each other, then u_i is independent from a_{-i} and we can write $u_i(a_i, a_{-i}, \omega) = u_i(a_i, \omega)$. Bergemann and Bonatti (2015) consider a model where the action a is taken by a decision maker to match with the state ω, in which case $|A| = |\Omega| = N$ where N is the total (finite) number of possible states and ex post utility of the decision maker is $u(a, \omega) = I(a = \omega)$.

Prior distribution of customer base We model the difference in D_1's and D_2's customer base in the following way. Suppose there is a common prior $\mu \in \Delta\Omega$ over customer types. Each party D_i has an interim belief about the distribution of customer types. Specifically, D_i privately observes a signal $r_i \in R$ drawn from an information structure $\lambda : \Omega \to \Delta R$ (common knowledge) and forms the interim belief $\theta_i \in \Delta\Omega$ via Bayes' rule:

$$\theta_i = \theta(\omega|r_i) = \frac{\lambda(r_i|\omega)\mu(\omega)}{\int_{\omega'} \lambda(r_i|\omega')\mu(\omega')}$$

We will equate θ_i with the interim belief function $\theta(\omega|r_i)$ and refer to it as D_i's type.

From an external observer's point of view, the common prior $\mu \in \Delta\Omega$ and the distribution of signals $\lambda : \Omega \to \Delta R$ induce a distribution of retailer type distributions $\lambda \circ \mu \equiv F \in \Delta\Theta$.

For the static version of the model, we will take θ as the primitive of our setup. In the dynamic version of the model, we could consider how outcomes of market interactions in the previous period provide further information about $\mu \in \Delta\Omega$ and shape D_i's interim belief in the current period.

This formulation is abstract. For the main application of the model, we will equate r_i as a binary signal where $r_i = 1$ if the customer visits D_i's store and $r_i = 0$ if the customer does not visit $D_i's$ store. Note that D_i only observes whether a customer visits her own store, but not the visiting decisions the customer made about other stores. We assume the information structure $\lambda : \Omega \to \Delta R$ arises from the customers' optimal visiting decisions. Specifically, we will assume that customer ω visits a store if and only if her valuation of the store's product is strictly positive. To fix ideas, suppose there are three equally weighted types of customers: type ω_1 only likes D_1's product, type ω_2 only likes D_2's product, and type ω_3 has strictly positive valuation for both D_1's and D_2's products. Given the assumptions, we have $\lambda(\omega_1) = (1, 0)$, $\lambda(\omega_2) = (0, 1)$ and $\lambda(\omega_3) = (1, 1)$ where the vector denotes (r_1, r_2). Hence if D_1 observes $r_1 = 1$, he correctly infers that ω must be either of type ω_1 or ω_3, so $\theta_1 = (\frac{1}{2}, 0, \frac{1}{2})$. The equal-weighted distribution of ω thus induces an equal-weighted distribution of θ.

Information technology and information vendor The downstream retailers may benefit from additional information technology. Such technology will help the retailers obtain a better understanding about customer types. The link between information quality and profitability may be through cost-reduction channel (by more efficient advertisement targeting) or through revenue-enhancing channel (by charging a price closer to the customer's reservation price or providing a tailored product/service according to the customer's specific taste).

We refer to an information product $I = \{\pi, S\}$ as consisting of a set of signals $s \in S$ and a likelihood function

$$\pi : \Omega \to \Delta S$$

Given I, we define π_{ij} as the conditional probability of obtaining signal s_j if the a given customer's true type is ω_i, i.e.,

$$\pi_{ij} \equiv Pr(s_j|\omega_i)$$

If both the state space and the signal space are finite, we can represent π as a stochastic matrix

$$\pi = (\pi_{ij})_{i\in\{1,\ldots,|\Omega|\},j\in\{1,\ldots,|S|\}}$$

with $\pi_{ij} \geq 0$ and $\sum_j \pi_{ij} = 1 \; \forall i$.

We assume such information devices are designed by a monopolist upstream information vendor U, who designs a menu $\mathscr{M} = \{\mathscr{I}, t\}$ of information products, where $\mathscr{I} = \{I\}$ is a collection of information technology and $t : \mathscr{I} \to \mathbb{R}^+$is the price for each device.

We assume that the information vendor commits to $\mathscr{M}$ before the realization of ω and θ, and that none of a, ω, s are contractible.

Timing of the game

The timing of the game is as follows.

Stage 1: the seller posts menu $\mathscr{M}$.

Stage 2: Customer type ω and retailer type θ are privately realized.

Stage 3: Retailer D_i chooses an information technology $I_i \in \mathscr{I}$ at the associated price $t(I_i)$. The choice of I_i is observed by the other parties D_j, $j \neq i$.

Stage 4: Customer ω visits all desirable stores (browzes all candidate websites).

Stage 5: Using the technology I_i, retailer D_i forms an assessment of ω's type (if ω visits D_i) and chooses an action $a_i \in A$ based on the assessment.

Stage 6: Customer ω makes purchasing decision based on the profile of actions a_i for all D_i she visited.

Stage 7: Payoffs to D_1 and D_2 are realized.

To fix ideas, we will focus on the case where Ω has finite dimension $|N|$. If neither D_1 or D_2 buys additional customer information, given type (prior assessment of customer profile based on V_i) actions will be chosen to maximize expected utility given the other type's action. Suppose the opponent's action profile is $G_{a_{-i}} : A \to \Delta_A$

where Δ_A denotes a probability distribution on the support of the action space, A. Then we have

$$
\begin{aligned}
u(\theta_i, G_{a_{-i}}) &\equiv \max_{\Delta_{a_i}} \left\{ \sum_k \theta_{ki} u(a_i, G_{a_{-i}}, \omega_k) \right\} \\
&= \max_{\Delta_{a_i}} \left\{ \sum_k \theta_{ki} \mathbb{E}_{G_{a_{-i}}} u(a_i, a_{-i}, \omega_k) \right\} \\
&= \max_{\Delta_{a_i}} \left\{ \mathbb{E}_\omega \left[\mathbb{E}_{G_{a_{-i}}} u(a_i, a_{-i}, \omega_k) \right] \right\}
\end{aligned}
$$

Abusing notation, define

$$
G^*_{a_i}(G_{a_{-i}}) \equiv G^*_a(\theta_i, G_{a_{-i}}) = \underset{\Delta_{a_i}}{argmax} \left\{ \sum_k \theta_{ki} u(a_i, G_{a_{-i}}, \omega_k) \right\}
$$

The equilibrium of the pricing game at stage 5 is given by the intersection of the best-response action distributions $G^*_{a_i}(G_{a_{-i}})$ and $G^*_{a_{-i}}(G_{a_i})$.

If retailer D_i has access to information technology $I = \{\pi, S\}$where $|S| = J$, then the expected gross utility after observing signal $s_j \in S$ is given by

$$
u(\theta_i, G_{a_{-i}}, s_j) \equiv \max_{\Delta_{a_i}} \left[\sum_{k=1}^{N} \frac{\theta_{ki}\pi_{kj}}{\sum_{m=1}^{J} \theta_{ki}\pi_{km}} u(a_i, G_{a_{-i}}, \omega_k) \right]
$$

with slight abuse of notation, we have

$$
G^*_{a_i}(G_{a_{-i}}, s_j) \equiv G^*_{a_i}(\theta_i, G_{a_{-i}}, s_j) = \underset{\Delta_{a_i}}{argmax} \left[\sum_{k=1}^{N} \frac{\theta_{ki}\pi_{kj}}{\sum_{m=1}^{J} \theta_{ki}\pi_{km}} u(a_i, G_{a_{-i}}, \omega_k) \right]
$$

We will focus on the case where the information technologies are uncorrelated, i.e.,

$$
Pr[s_j|\theta_i] = Pr[s_j|\theta_i, \theta_{-i}] = \sum_{k=1}^{N} \theta_{ik}\pi_{kj} \ \forall \theta_i, \theta_{-i} \in \Theta
$$

and

$$
Pr[s_{-j}|\theta_{-i}] = Pr[s_{-j}|\theta_{-i}, \theta] = \sum_{k=1}^{N} \theta_{-ik}\pi'_{kj} \ \forall \theta_i, \theta_{-i} \in \Theta
$$

Note that this assumption does not mean that the signal distributions are uncorrelated. It does mean, however, the signal distribution of one's own device is unchanged had the opponent chosen a different device from the menu.

Since

$$
Pr[\theta_i, \theta_{-i}|s_j, s_{-j}] \propto p(s_j, s_{-j}|\theta_i, \theta_{-i}) \cdot f_\Theta(\theta_i) \cdot f_\Theta(\theta_{-i})
$$

where $p(s_j, s_{-j}|\theta_i, \theta_{-i}) = p(s_j|\theta_i, \theta_{-i}) \cdot p(s_{-j}|\theta_i, \theta_{-i}) = p(s_j|\theta_i) \cdot p(s_{-j}|\theta_{-i})$ by the assumption.

Hence the assumption allows us to factor the posterior signal distribution into two independent components:

$$p(\theta_i|s_j)\cdot p(\theta_{-i}|s_{-j})$$

Consider the case where the ex post signal realization is (s_j, s_{-j}) and is known to both parties. Then the equilibrium of the pricing game at stage 5 is given by the following utility functions and the intersection of the best-response function distributions

$$u(G_{a_{-i}}, s_j, s_{-j}) = \left[\sum_{k=1}^{N} \frac{\theta_{ki}\pi_{kj}}{\sum_{m=1}^{J}\theta_{ki}\pi_{km}} u(a_i, G_{a_{-i}}(G_{a_i}, s_{-j}), \omega_k)\right]$$

$$u(G_{a_i}, s_{-j}, s_j) = \left[\sum_{k=1}^{N} \frac{\theta_{ki}\pi_{k(-j)}}{\sum_{m=1}^{J}\theta_{ki}\pi_{km}} u(a_{-i}, G_{a_i}(G_{a_{-i}}, s_j), \omega_k)\right]$$

$$G^*_{a_i}(G_{a_{-i}}, s_j, s_{-j}) = \underset{\Delta_{a_i}}{argmax}\left[\sum_{k=1}^{N} \frac{\theta_{ki}\pi_{kj}}{\sum_{m=1}^{J}\theta_{ki}\pi_{km}} u(a_i, G_{a_{-i}}(G_{a_i}, s_{-j}), \omega_k)\right]$$

$$G^*_{a_{-i}}(G_{a_i}, s_{-j}, s_j) = \underset{\Delta_{a_{-i}}}{argmax}\left[\sum_{k=1}^{N} \frac{\theta_{ki}\pi_{k(-j)}}{\sum_{m=1}^{J}\theta_{ki}\pi_{km}} u(a_{-i}, G_{a_i}(G_{a_{-i}}, s_j), \omega_k)\right]$$

In general, the signal realization of the opponent party $-i$ is unknown to party i, so the value from a given action profile Δ_{a_i}can only be assessed based on the distribution of s_{-j}, $\Delta_{s_{-j}}$

$$G^*_{a_i}(G_{a_{-i}}, s_j, \Delta_{s_{-j}}) = \underset{\Delta_{a_i}}{argmax}\left[\mathbb{E}_{\Delta_{s_{-j}}}\left[u(G_{a_{-i}}, s_j, s_{-j})\right]\right]$$

$$G^*_{a_{-i}}(G_{a_i}, s_{-j}, \Delta_{s_j}) = \underset{\Delta_{a_{-i}}}{argmax}\left[\mathbb{E}_{\Delta_{s_j}}\left[u(G_{a_i}, s_{-j}, s_j)\right]\right]$$

Let

$$u(G^*_{a_{-i}}(G_{a_i}, s_{-j}, \Delta_{s_j}), s_{-j}, s_j) \equiv \underset{\Delta_{a_{-i}}}{max}\mathbb{E}_{\Delta_{s_{-j}}}\left[u(G_{a_{-i}}, s_j, s_{-j})\right]$$

$$u(G^*_{a_i}(G_{a_{-i}}, s_j, \Delta_{s_{-j}}), s_j, s_{-j}) \equiv \underset{\Delta_{a_i}}{max}\mathbb{E}_{\Delta_{s_j}}\left[u(G_{a_i}, s_{-j}, s_j)\right]$$

denote the corresponding value functions

The expected payoff to type θ_i when she purchases I_i (which generates signal structure S_j) and the opponent purchases I_{-i} (which generates signal structure S_{-j}) is given by

$$\mathbb{E}_{\Delta_{s_j}}\left\{u(G^*_{a_i}(G_{a_{-i}}, s_j, \Delta_{s_{-j}}), s_j, s_{-j})\right\}$$

and

$$\mathbb{E}_{\Delta_{s_{-j}}}\left\{u(G^*_{a_{-i}}(G_{a_i}, s_{-j}, \Delta_{s_j}), s_{-j}, s_j)\right\}$$

Note that these pair of functions depend only on I_i, I_{-i} and private types θ_i and θ_{-i}. The value of information device I_i to type θ_i, when the opponent has type θ_{-i} and purchases I_{-i} is given by

$$V(I_i, I_{-i}; \theta_i\theta_{-i}) \equiv \mathbb{E}_{\Delta_{s_j}}\left\{u(G^*_{a_i}(G_{a_{-i}}, s_j, \Delta_{s_{-j}}), s_j, s_{-j})\right\} - u(\theta_i, a_{-i}(\theta_{-i}))$$

and

$$V(I_{-i}, I_i; \theta_{-i}\theta_i) \equiv \mathbb{E}_{\Delta_{s_{-j}}}\left\{u(G^*_{a_{-i}}(G_{a_i}, s_{-j}, \Delta_{s_j}), s_{-j}, s_j)\right\} - u(\theta_{-i}, a_i(\theta_i))$$

This expression highlights a few complications of the model.

In our model, although the information technologies are not directly correlated, they interact by affecting the optimal actions chosen by the downstream parties. This is a main departure from Bergemann and Bonatti (2015) who assume the value of information only depends on the own party's type and choice of information product.

This aspect of the model is realistic in the setting where the downstream firms produce (partially) substitutable products and compete over the same customer base.

If instead the downstream parties produce tailor-made products with little or no scope for substitution, it may be reasonable to drop the dependence of V on I_{-i} and θ_{-i}. For example, if utility derived from a given action a_i is given by

$$u(a_i, \omega) = I(a_i = \omega)$$

i.e., payoff is realized only when the action is matched to the state, then one party's utility is completely independent from the other party's action and hence unaffected by the information technology purchased by the other player. This is the case considered in Bergemann and Bonatti (2015).

We will enrich the Bergemann and Bonatti (2015) framework in several steps.

First, we will introduce a simple form of price competition with heterogeneous customers, while both of the duopolists have access to an imperfect technology. We will show that our framework subsumes this model as a special case.

2.3 Baseline: one-dimensional information with interdependent downstream actions

Assume there is a unit mass of customers with heterogeneous preferences. some are loyal to D_1 and some to D_2. Some are indifferent between D_1's and D_2's product per se and will purchase from the party that offers a better price. The loyal customers are price-insensitive: they will purchase from the preferred downstream retailer as long as the price they face is below their reservation price, which we normalize to 1. Switchers, on the other hand, will buy from the firm with the lower price and will mix with 50-50 probability when offered the same price by D_1 and D_2.

Assume a fraction γ_1 of customers is loyal to D_1 ($\omega = L_1$), a fraction γ_2 of customers is loyal to D_2 ($\omega = L_2$) and the remaining is indifferent ($\omega = S$). γ_1, γ_2 and χ satisfy $\gamma_1 + \gamma_2 + \chi = 1$.

Since customers loyal to D_i will never show up at D_{-i}, we define type θ_i by the prior distribution over (L_1, L_2, S) as

$$\theta_1 = [\frac{\gamma_1}{\gamma_1 + \chi}, 0, \frac{\chi}{\gamma_1 + \chi}]$$

$$\theta_2 = [0, \frac{\gamma_2}{\gamma_2 + \chi}, \frac{\chi}{\gamma_2 + \chi}]$$

Without loss of generality, we assume the signal space of any information device has the same dimension as Ω, i.e., $|S| = |\Omega| = 3$. The information product I_i will help firm D_i form perception about customer types. A given customer can be classified by I_i as loyal to D_i or indifferent between D_i and D_{-i}. Note that a customer loyal to D_{-i} will never show up at D_i.

The information product for D_1 can be equivalently expressed in terms of transition matrix $\pi : \Omega \to \Delta S$ where the rows denote the state space and the columns denote the signal space:

$$\pi^1 = \begin{bmatrix} \pi_{11}^1 & 0 & 1 - \pi_{11}^1 \\ 0 & 1 & 0 \\ 1 - \pi_{33}^1 & 0 & \pi_{33}^1 \end{bmatrix}$$

$$\pi^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pi_{22}^2 & 1 - \pi_{22}^2 \\ 0 & 1 - \pi_{33}^2 & \pi_{33}^2 \end{bmatrix}$$

If $\pi_{11}^1 = 1$ and $\pi_{33}^1 = 1$, then the information product I_1 has perfect precision, in the sense that $Pr(s = L_1|\omega = L_1) = 1$, $Pr(s = S|\omega = S) = 1$.

If $\pi_{11}^1 = \frac{\gamma_1}{\gamma_1+\chi}$ and $\pi_{33}^1 = \frac{\chi}{\gamma_1+\chi}$, then the information product I_1 has no value, since the posterior assessments represent no improvement over the prior assessments:

$$Pr(s = L_1|\omega = L_1) = \frac{\gamma_1}{\gamma_1 + \chi} = Pr(\omega = L_1)$$

$$Pr(s = S|\omega = S) = \frac{\chi}{\gamma_1 + \chi} = Pr(\omega = S)$$

Similarly, if $\pi_{22}^2 = 1$ and $\pi_{33}^2 = 1$, then the information product I_2 has perfect precision, in the sense that $Pr(s = L_2|\omega = L_2) = 1$, $Pr(s = S|\omega = S) = 1$.

If $\pi_{22}^2 = \frac{\chi}{\gamma_2+\chi}$ and $\pi_{33}^2 = \frac{\gamma_2}{\gamma_2+\chi}$, then the information product I_2 has no value, since the posterior assessments

True type	D_1 perception	D_2 perception	Size	Purchasing price
L_1	l_1	-	$\gamma_1 P_1(l_1\|L_1)$	p_{1L}
	s	-	$\gamma_1 P_1(s\|L_1)$	p_{1S}
S	l_1	l_2	$\chi P_1(l_1\|S)P_2(l_2\|S)$	$min\{p_{1L}, p_{2L}\}$
	l_1	s	$\chi P_1(l_1\|S)P_2(s\|S)$	$min\{p_{1L}, p_{2S}\}$
	s	l_2	$\chi P_1(s\|S)P_2(l_2\|S)$	$min\{p_{1S}, p_{2L}\}$
	s	s	$\chi P_1(s\|S)P_2(s\|S)$	$min\{p_{1S}, p_{2S}\}$
L_2	-	l_2	$\gamma_2 P_2(l_2\|L_2)$	p_{2L}
	-	s	$\gamma_2 P_2(s\|L_2)$	p_{2S}

Table 1: True type v.s perceived type

represent no improvement over the prior assessments:

$$Pr(s = L_2|\omega = L_2) = \frac{\gamma_2}{\gamma_2 + \chi} = Pr(\omega = L_1)$$

$$Pr(s = S|\omega = S) = \frac{\chi}{\gamma_1 + \chi} = Pr(\omega = S)$$

Note that in the configuration of each I_i, we have two degrees of freedom: (π^1_{11}, π^1_{33}) for I_1 and (π^2_{22}, π^2_{33}) for I_2. For the sake of illustration, we will first impose an extra restriction that $\pi^1_{11} = \pi^1_{33}$ and $\pi^2_{22} = \pi^2_{33}$. In words, we allow each information device to differ in precision, but we do not allow the degree of precision to vary across different customer types (loyal customers or switchers).

Abusing notation, we will use $I_1 \in [0,1]$ and $I_2 \in [0,1]$ to measure the precision of D_1 's and D_2's information devices. Assume

$$Pr_i(s = L_i|\omega = L_i) = I_i + (1 - I_i)\frac{\gamma_i}{\gamma_i + \chi}$$

$$Pr_i(s = S|\omega = L_i) = (1 - I_i)\frac{\chi}{\gamma_i + \chi}$$

$$Pr_i(s = L_i|\omega = S) = (1 - I_i)\frac{\gamma_i}{\gamma_i + \chi}$$

$$Pr_i(s = S|\omega = S) = I_i + (1 - I_i)\frac{\chi}{\gamma_i + \chi}$$

The information device gives a type recommendation (with noise) to the retailer. The retailer then chooses a price, which is the "action" in the general setup introduced above. The action (posted price) can vary depending on the signal received: retailer i may choose one price p_{iS} for perceived switchers and another price p_{iL} for perceived loyal customers. For rotational simplicity, we will abbreviate $s = L_i$ to l_i and $s = S$ to s. The table below summarizes the notation we have introduced so far.

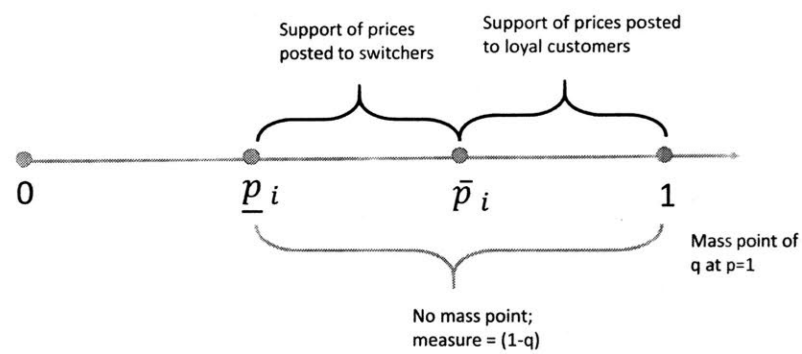


Figure 1: Illustration of pricing strategy

Next, we turn to the customers' choice.

First consider two extreme cases. If targeting technology is null, i.e., firms can only post uniform price to all consumers, then in equilibrium, firms will randomize prices (c.f., Varian (1980)). If targeting technology is perfect, i.e., firms can perfectly distinguish the loyal customers from the switchers, then each firm will charge the reservation price which equals 1 by assumption to the loyal customers and charge the marginal cost of production which equals 0 by assumption to the switchers. In other words, firms have full monopoly power among their own loyalty base, and compete a la Bertrand for the switchers.

More generally, suppose the targeting technology is imperfect. In this setting, it is easy to show that the equilibrium features mixing pricing strategy (c.f. Narasimhan (1988)): the retailer charges higher (random) price to its perceived loyal customers than to its perceived switchers. Further, supports for p_S and p_L are continuous and do not overlap. There will be no mass point in the support for p_S and the only potential mass point of measure $q \in (0,1)$ in the support for $p_L$is the monopoly price 1. Graphically, the equilibrium pricing strategy can be expressed as follows.

Given a pricing strategy, let $G_{iL}(p) = Pr(p_{iL} < p)$ and $G_{iS}(p) = Pr(p_{iS} < p)$ denote the CDF.

D_1's expected revenue from perceived loyal customers, when the price charged is p is

$$R_{1L}(p) = p \cdot \{\gamma_1 P_1(l_1|L_1) + \chi P_1(l_1|S)P_2(l_2|S) \cdot (1 - G_{2L}(p)) + \chi P_1(l_1|S)P_2(s|S) \cdot (1 - G_{2S}(p))\}$$

D_1's expected revenue from perceived switchers, when the price charged is p is

$$R_{1S}(p) = p \cdot \{\gamma_1 P_1(s|L_1) + \chi P_1(s|S)P_2(l_2|S) \cdot (1 - G_{2L}(p)) + \chi P_1(s|S)P_2(s|S) \cdot (1 - G_{2S}(p))\}$$

The expressions for D_2 are similar.

Given the form of equilibrium strategy, solving the problem boils down to solving for the cutoff values of the support, $\underline{p}$, $\bar{p}$, and the size of the mass point at $p = 1$, namely q.

The boundary conditions are: $G_{iL}(\bar{p}) = 0$, $G_{iL}(1) = 1$ or $(1 - q)$, $G_{iS}(\underline{p}) = 0$, and $G_{is}(\bar{p}) = 1$.

By construction, any price in the support yields the same expected revenue. So we have the indifference conditions: $R_{1L}(\bar{p}) = R_{1S}(\bar{p})$, $R_{1L}(\bar{p}) = R_{1L}(1)$, and $R_{1S}(\bar{p}) = R_{1S}(\underline{p})$.

For notational simplicity, write $\phi_i = \gamma_i Pr_i(l_i|L_i)$, $\rho = \chi Pr_1(s|S)Pr_2(s|S)$, and $\psi = \chi Pr_1(l_1|S)Pr_2(l_2|S)$. The solutions to the problem are $\bar{p} = \frac{\phi_i}{\phi_i+\psi}$, $\underline{p} = \frac{\chi-\rho}{\chi}\bar{p}$ where $q = \frac{\phi_i-\phi_j}{\phi_i+\psi}$, $G_{1L}(p) = 1+\frac{\phi_2}{\psi} - \frac{\phi_2+\psi q}{\psi p}$, $G_{2L}(p) = 1+\frac{\phi_1}{\psi} - \frac{\phi_1}{\psi}\frac{1}{p}$, and $G_{1S}(p) = G_{2S}(p) = 1 - \frac{\chi-\rho}{\rho}(\frac{\phi_i}{\phi_i+\psi}\frac{1}{p} - 1)$.

Using the above ingredients, we can derive expressions for D_1's and D_2's revenue, as functions of I_1 and I_2:

$$R_1(I_1, I_2) = \frac{\phi_i}{\phi_i + \psi}(\phi_i + \psi + \chi - \rho)$$

$$R_2(I_1, I_2) = \frac{\phi_i}{\phi_i + \psi}(\phi_j + \psi + \chi - \rho)$$

To find the value of information, we consider the solution to the problem in the absence of information technology. In this case, we have

$$Pr_i(s = L_i|\omega = L_i) = \frac{\gamma_i}{\gamma_i + \chi}$$

$$Pr_i(s = S|\omega = L_i) = \frac{\chi}{\gamma_i + \chi}$$

$$Pr_i(s = L_i|\omega = S) = \frac{\gamma_i}{\gamma_i + \chi}$$

$$Pr_i(s = S|\omega = S) = \frac{\chi}{\gamma_i + \chi}$$

And by construction: $\phi_i = \frac{\gamma_i^2}{\gamma_i+\chi}$, $\rho = \frac{\chi^3}{(\gamma_1+\chi)(\gamma_2+\chi)}$, and $\psi = \frac{\gamma_1\gamma_2\chi}{(\gamma_1+\chi)(\gamma_2+\chi)}$.

Suppose D_1 is the market leader in the sense that $\gamma_1 > \gamma_2$. The equilibrium payoffs are

$$R_1^0 \equiv R_1(0,0) = \frac{\gamma_1}{\gamma_1(\gamma_2 + \chi) + \gamma_2\chi}\left\{\frac{\gamma_1[\gamma_1(\gamma_2 + \chi) + \gamma_2\chi] + \chi(\gamma_1 + \chi)(\gamma_2 + \chi) - \chi^3}{(\gamma_1 + \chi)}\right\} = \gamma_1$$

$$R_2(I_1, I_2) = \frac{\phi_i}{\phi_i + \psi}(\phi_j + \psi + \chi - \rho)$$

$$R_1^0 - R_2^0 = \frac{\gamma_1}{\gamma_1(\gamma_2+\chi)+\gamma_2\chi} \cdot [\frac{\gamma_1^2}{\gamma_1+\chi} - \frac{\gamma_2^2}{\gamma_2+\chi}]$$

$$R_2^0 = \gamma_1 - \frac{\gamma_1}{\gamma_1(\gamma_2+\chi)+\gamma_2\chi} \cdot [\frac{\gamma_1^2}{\gamma_1+\chi} - \frac{\gamma_2^2}{\gamma_2+\chi}]$$

Consider the case where $\chi = 0$, then

$$R_2^0 = R_2^0 = max\{\gamma_1 - \frac{\gamma_1 - \gamma_2}{\gamma_2}, 0\}$$

So R_2 is positive when $\frac{1}{\gamma_2} - \frac{1}{\gamma_1} > 1$ holds.

$$\gamma_1^2 + \gamma_1 - 1 > 0$$

or

$$\gamma_1 \geq \frac{-1+\sqrt{5}}{2} \approx 0.618$$

Let $V_1(I_1, I_2) \equiv R_1(I_1, I_2) - R_1^0$ and $V_2(I_1, I_2) \equiv R_2(I_1, I_2) - R_2^0$ denote the value of information to D_1 and D_2 respectively.

Note that the value of I_1 to V_1 in general depends on I_2 through its dependence on $\rho = \chi Pr_1(s|S)Pr_2(s|S) = \chi\pi_{33}^1\pi_{33}^2$ and $\psi = \chi Pr_1(l_1|S)Pr_2(l_2|S) = \chi(1-\pi_{33}^1)(1-\pi_{33}^2)$. The former measures competition over correctly identified switchers. The latter measures competition over mis-identified switchers. The sizes of both segments depend on both own information technology and information technology of the opposing party.

The value of I_2 to V_2 also depends on I_1. In addition to the channels mentioned above, it also depends on I_1 through the multiplicative term $\bar{p} = \frac{\phi_1}{\phi_1+\psi}$, where $\phi_1 = \gamma_1 Pr_1(l_1|L_1) = \gamma_1\pi_{11}^1$ is a measure of how well D_1 can correctly capture its own loyalty base. This term $\bar{p}$ is the cutoff between the price support for P_{1L} and P_{1S}.

Note that we ignore the marginal cost of production. If we introduce a positive marginal cost $c > 0$, we may ask how many downstream firms can be sustained in the equilibrium. This is the theme of Varian (1980)'s seminal work. We will return to the possibility of entry and exit later.

Next we consider the upstream firm U (the information vendor)'s problem.

Let θ^1 and θ^2 denote the types of D_1 and D_2, which we identify with the prior distribution over relevant states. The information device $I(\theta^k)$ can be equivalently expressed as a matrix $\pi_{ij}(\theta^k)$ where $\pi_{ij} = Pr(s = s_j|\omega = \omega_i)$ measures the probability of sending signal s_j when the true state is ω_i.

Let $V_1(I_1, I_2)$ and $V_2(I_1, I_2)$ denote the value of information which we derived above. The upstream data seller's problem is

$$\max_{\{I_1, I_2\}} t(I_1) + t(I_2)$$

s.t.

$$V_1(I_1, I_2) - t(I_1) \geq V_1(I_2, I_2) - t(I_2)$$

$$V_2(I_2, I_1) - t(I_2) \geq V_2(I_1, I_1) - t(I_1)$$

$$V_1(I_1, I_2) - t(I_1) \geq 0$$

$$V_2(I_2, I_1) - t(I_2) \geq 0$$

We prove the following preliminary results:

Responsive information structure. Every incentive-compatible menu $\mathscr{M}$ can be represented as a collection of $I(\theta)$ in which θ takes a different action for each signal $s \in S(\theta)$.

The proof of this result is similar to Myerson (1986): suppose $\mathscr{M}$ contains an experiment $I(\theta)$ with more signals than actions, then the seller may combine all signals that lead to the same action. The value of $I(\theta)$ is unchanged for type θ, and $V(I(\theta), \theta')$ decreases for all $\theta' \neq \theta$, a relaxation of the incentive constraints.

This result allows us to represent each information device $I(\theta)$ as a square matrix with states ω on the rows and $s \in S$ on the columns.

We also order the signals so that the ordering of signals and the ordering of actions are aligned (i.e., if the signal is in the order: $L_i L_{-i}$,S, then the corresponding action action is p_{iL}, p_{null}, p_{iS}, where p_{null} is a place holder for the null action, as such signals never appear in equilibrium.)

"Possible distortion at the top". A common result in the screening literature is that "any optimal$\mathscr{M}$ contains the fully-informative I^*". In general this result no longer holds in the current setting. To see the intuition, we first discuss the sufficient conditions under which the result does hold.

We say the value of information is **positively sorted** if, $V_j(I, I') - V_j(\hat{I}, I') \leq V_i(I, I') - V_i(\hat{I}, I')$ for all I strictly dominating $\hat{I}$. In other words, a strict improvement in information quality benefits D_i more than D_j. It is easy to see that optimal menu must have $t(I_1) \geq t(I_2)$. i.e., I_1 gets the more expensive item.

If the condition of positive-sortedness holds, then the fully-informative I^* must be on the optimal$\mathscr{M}$.

To see this, consider replacing I_1 with I^* and charge

$$t(I^*) \in [max\{t(I_1), V_2(I_1^*, I_1^*) - V_2(I_2, I_1^*) + t(I_2)\}, \; V_1(I^*, I_2) - V_1(I_2, I_2) + t(I_2)]$$

Note that

$$V_2(I_1^*, I_1^*) - V_2(I_2, I_1^*) \leq V_2(I_1^*, I_1) - V_2(I_2, I_1) \leq V_1(I_1^*, I_1) - V_1(I_2, I_1) \leq V_1(I_1^*, I_2) - V_1(I_2, I_2)$$

and $V_1(I^*, I_2) - V_1(I_2, I_2) + t(I_2) \geq V_1(I_1, I_1) - V_1(I_1, I_2) + t(I_2) \geq t(I_1)$

So the interval

$$t(I^*) \in [max\,\{t(I_1), V_2(I_1^*, I_1^*) - V_2(I_2, I_1^*) + t(I_2)\}\,,\; V_1(I^*, I_2) - V_1(I_2, I_2) + t(I_2)]$$

is nonempty.

By construction, both incentive compatibility constraints

$$V_2(I_2, I_1^*) - t(I_2) - V_2(I_1^*, I_1^*) + t(I_1^*) \geq 0$$

and

$$V_1(I_1^*, I_2) - t(I_1^*) - V_2(I_2, I_2) + t(I_2) \geq 0$$

are satisfied.

In general, because V_i depends on both I_i and I_j and their interactions, full-information might not be part of the equilibrium.

To see this, consider the symmetric case where D_1 and D_2 have the same size of loyal bases: $\gamma_1 = \gamma_2 = \gamma$. We restrict attention to $I_1 = I_2 = I$.

To see why this is the case, note that D_i's equilibrium profit is

$$R_i(I, I) \equiv \gamma_i Pr_i(s = L_i | \omega = L_i) \cdot \left[1 + \frac{\chi - \chi Pr_i(s = S|\omega = S) Pr_j(s = S|\omega = S)}{\gamma_i - \chi Pr_i(s = S|\omega = S) Pr_j(s = L_j|\omega = S)}\right]$$

The multiplicative term $\gamma_i Pr_i(s = L_i|\omega = L_i)$ captures the firm's return to increasing prices to perceived loyal customers, which increases with I.

On the other hand, $\chi Pr_i(s = S|\omega = S) Pr_j(s = S|\omega = S)$ captures intensity of competition, and is increasing in I.

Finally, the term $\chi Pr_i(s = S|\omega = S) Pr_j(s = L_j|\omega = S)$ arising from correct targeting by D_i and mistargeting by D_j softens price competition.

For the loyal customers, higher I means higher (expected price). For the switchers, the relationship is more complicated: expected profit first increases with I (the mistargeting effect dominates) and then decreases (the price competition effect dominates).

Formally,

$$\frac{\partial^2 R}{\partial I^2} = \left(\frac{\partial^2}{\partial I_1^2} R_1 + 2\frac{\partial^2}{\partial I_1 \partial I_2} R_1 + \frac{\partial^2}{\partial I_2^2} R_2\right) |_{I_1 = I_2 = I} < 0$$

From the FOC

$$\frac{\partial R}{\partial I}|_{I=I^*} = \left(\frac{\partial R}{\partial I_1} + \frac{\partial R}{\partial I_2}\right) |_{I_1 = I_2 = I} = 0$$

we have

$$I^{opt} = \frac{(1-\chi)(1+3\chi)}{4\chi^2}\left[\frac{1+\chi}{\sqrt{(1-\chi)(1+3\chi)}} - 1\right] < 1$$

2.4 Generalizations: Two-dimensional information with interdependent downstream actions

2.4.1 Variant 1: Inducing participation v.s exclusion

Next, we relax the restriction that $\pi_{11}^1 = \pi_{33}^1$ and $\pi_{22}^2 = \pi_{33}^2$.

Abusing notation, we will use $I_1^L, I_1^S \in [0,1]$ and $I_2^L, I_s^S \in [0,1]$ to measure the precision of D_1 's and D_2's information devices. Note that $I_i^L \neq I_i^S$ in general, allowing for differential precision in the loyalty base dimension v.s the shopper dimension. With this notation, the posterior signal distribution is

$$Pr_i(s = L_i|\omega = L_i) = I_i^L + (1-I_i^L)\frac{\gamma_i}{\gamma_i+\chi}$$

$$Pr_i(s = S|\omega = S) = I_i^S + (1-I_i^S)\frac{\chi}{\gamma_i+\chi}$$

$$Pr_i(s = S|\omega = L_i) = (1-I_i^L)\frac{\chi}{\gamma_i+\chi}$$

$$Pr_i(s = L_i|\omega = S) = (1-I_i^S)\frac{\gamma_i}{\gamma_i+\chi}$$

Using the same notation in the simplified case, we derive the following solutions to the problem: $\bar{p} = \frac{\phi_i}{\phi_i+\psi}$, $\underline{p} = \frac{\chi-\rho}{\chi}\bar{p}$, $q = \frac{\phi_i-\phi_j}{\phi_i+\psi}$, $G_{1L}(p) = 1 + \frac{\phi_2}{\psi} - \frac{\phi_2+\psi q}{\psi p}$, $G_{2L}(p) = 1 + \frac{\phi_1}{\psi} - \frac{\phi_1}{\psi}\frac{1}{p}$, and $G_{1S}(p) = G_{2S}(p) = 1 - \frac{\chi-\rho}{\rho}(\frac{\phi_i}{\phi_i+\psi}\frac{1}{p} - 1)$.

Using the above ingredients, we can derive expressions for D_1's and D_2's revenue, as functions of I_1 and I_2:

$$R_1(I_1^L, I_1^S, I_2^L, I_2^S) = \frac{\phi_i}{\phi_i+\psi}(\phi_i+\psi+\chi-\rho)$$

$$R_2(I_1^L, I_1^S, I_2^L, I_2^S) = \frac{\phi_i}{\phi_i+\psi}(\phi_j+\psi+\chi-\rho)$$

where we write $\phi_i = \gamma_i Pr_i(l_i|L_i)$, $\rho = \chi Pr_1(s|S)Pr_2(s|S)$, and $\psi = \chi Pr_1(l_1|S)Pr_2(l_2|S)$ for notational simplicity.

Note that

$$\phi_i = \phi_i\left(I_1^L, I_1^S, I_2^L, I_2^S\right) = \gamma_i Pr_i(l_i|L_i) = \gamma_i I_i^L + (1-I_i^L)\frac{\gamma_i^2}{\gamma_i+\chi}$$

$$\rho = \rho\left(I_1^L, I_1^S, I_2^L, I_2^S\right) = \chi Pr_1(s|S)Pr_2(s|S) = \chi\left[\left(I_1^S + (1-I_1^S)\frac{\chi}{\gamma_1+\chi}\right)\left(I_2^S + (1-I_2^S)\frac{\chi}{\gamma_2+\chi}\right)\right]$$

$$\psi = \psi\left(I_1^L, I_1^S, I_2^L, I_2^S\right) = \chi Pr_1(l_1|S)Pr_2(l_2|S) = \chi\left[\left(1 - I_1^S\right)\left(1 - I_2^S\right)\left(\frac{\gamma_1}{\gamma_1 + \chi}\right)\left(\frac{\gamma_2}{\gamma_2 + \chi}\right)\right]$$

are functions of $\left(I_1^L, I_1^S, I_2^L, I_2^S\right)$.

Next, we turn to the upstream monopolist's menu design problem. The monopolist will post a menu (I_1, I_2) and the corresponding price $t(I_1)$ and $t(I_2)$ where $I_1 = (I_1^L, I_1^S)$ and $I_2 = (I_2^L, I_2^S)$ to induce D_1 to choose I_1 and D_2 to choose I_2. The problem is

$$\max_{\{I_1, I_2\}} t(I_1) + t(I_2)$$

s.t.

$$V_1(I_1, I_2) - t(I_1) \geq V_1(I_2, I_2) - t(I_2)$$

$$V_2(I_2, I_1) - t(I_2) \geq V_2(I_1, I_1) - t(I_1)$$

$$V_1(I_1, I_2) - t(I_1) \geq V_1(0, I_2)$$

$$V_2(I_2, I_1) - t(I_2) \geq V_2(I_1, 0)$$

There are two different ways to interpret the screening problem.

First, suppose the information product is 3. Note that the first row representing loyalty base to D_1 and the second row representing loyalty base to D_2 are distinguishable to D_1 and D_2.

$$\pi^1 = \begin{bmatrix} \pi_{11}^1 & 0 & 1 - \pi_{11}^1 \\ 0 & 1 & 0 \\ 1 - \pi_{33}^1 & 0 & \pi_{33}^1 \end{bmatrix}$$

$$\pi^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pi_{22}^2 & 1 - \pi_{22}^2 \\ 0 & 1 - \pi_{33}^2 & \pi_{33}^2 \end{bmatrix}$$

This interpretation is helpful for screening purposes, because the extra precision (π_{22}^2) in I_2 is useless to D_1 and the extra precision (π_{11}^1) in I_1 is useless to D_2. Specifically, define

$$\delta_1 \equiv \frac{\gamma_1^2}{\gamma_1 + \chi}$$

(the prior version of ϕ_1) and

$$\delta_2 \equiv \frac{\gamma_2^2}{\gamma_2 + \chi}$$

(the prior version of ϕ_2).

The incentive constraint of D_1 is

$$\frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) \geq \frac{\delta_1}{\delta_1+\psi}(\delta_1+\psi+\chi-\rho)$$

or

$$1 \geq \frac{-\psi}{(\delta_1+\psi)(\phi_1+\psi)}(\chi-\rho)$$

Note that if $\chi \geq \rho = \chi\left[\left(I_1^S+(1-I_1^S)\frac{\chi}{\gamma_1+\chi}\right)\left(I_2^S+(1-I_2^S)\frac{\chi}{\gamma_2+\chi}\right)\right]$, so this inequality holds trivially.

The incentive constraint of D_2 is

$$\frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) \geq \frac{\phi_1}{\phi_1+\psi}(\delta_2+\psi+\chi-\rho)$$

Note that $\phi_2 \geq \delta_2$, so this inequality holds trivially as well.

The problem then reduces to one of participation elicitation, i.e.,

$$\max_{\{I_1,I_2\}} t(I_1)+t(I_2)$$

s.t.

$$V_1(I_1,I_2)-t(I_1) \geq V_1(0,I_2)$$

$$V_2(I_2,I_1)-t(I_2) \geq V_2(I_1,0)$$

To solve this problem, we first derive an expression for $V_1(0,I_2)$ and $V_2(I_1,0)$.

Given a pricing strategy, let $G_{iL}(p)=Pr(p_{iL}<p)$ and $G_{iS}(p)=Pr(p_{iS}<p)$ denote the CDF.

D_1's expected revenue from perceived loyal customers, when the price charged is p is

$$R_{1L}(p)=p\cdot\{\gamma_1P_1(l_1|L_1)+\chi P_1(l_1|S)P_2(l_2|S)\cdot(1-G_{2L}(p))+\chi P_1(l_1|S)P_2(s|S)\cdot(1-G_{2S}(p))\}$$

D_1's expected revenue from perceived switchers, when the price charged is p is

$$R_{1S}(p)=p\cdot\{\gamma_1P_1(s|L_1)+\chi P_1(s|S)P_2(l_2|S)\cdot(1-G_{2L}(p))+\chi P_1(s|S)P_2(s|S)\cdot(1-G_{2S}(p))\}$$

If $I_2=0$, we have

$$P_2(l_2|S)=\frac{\gamma_2}{\gamma_2+\chi}$$

$$P_2(s|S)=\frac{\chi}{\gamma_2+\chi}$$

For (I_1,I_2), write $\phi_i=\gamma_iPr_i(l_i|L_i)$, $\rho=\chi Pr_1(s|S)Pr_2(s|S)$, and $\psi=\chi Pr_1(l_1|S)Pr_2(l_2|S)$.

For $(I_1, 0)$, write $\hat{\phi}_i = \gamma_i Pr_i(l_i|L_i)$, $\hat{\rho} = \chi Pr_1(s|S)Pr_2(s|S)$, and $\hat{\psi} = \chi Pr_1(l_1|S)Pr_2(l_2|S)$.

For $(0, I_2)$, write $\tilde{\phi}_i = \gamma_i Pr_i(l_i|L_i)$, $\tilde{\rho} = \chi Pr_1(s|S)Pr_2(s|S)$, and $\tilde{\psi} = \chi Pr_1(l_1|S)Pr_2(l_2|S)$.

Note that

$$\phi_1 = \hat{\phi}_1,\ \phi_2 > \hat{\phi}_2,\ \rho > \hat{\rho},\ and\ \psi < \hat{\psi}$$

$$\phi_1 > \tilde{\phi}_1,\ \phi_2 = \tilde{\phi}_2,\ \rho > \tilde{\rho},\ and\ \psi < \tilde{\psi}$$

Note that $\phi_1 > \phi_2$,

$$V_1(I_1, I_2) = \frac{\phi_1}{\phi_1 + \psi}(\phi_1 + \psi + \chi - \rho)$$

$$V_2(I_1, I_2) = \frac{\phi_1}{\phi_1 + \psi}(\phi_2 + \psi + \chi - \rho)$$

So $\hat{\phi}_1 > \hat{\phi}_2$ and

$$V_2(I_1, 0) = \frac{\hat{\phi}_1}{\hat{\phi}_1 + \hat{\psi}}(\hat{\phi}_2 + \hat{\psi} + \chi - \hat{\rho})$$

On the other hand, the sign of $\tilde{\phi}_1 - \tilde{\phi}_2$ is unclear.

If $\tilde{\phi}_1 > \tilde{\phi}_2$, then

$$V_1(0, I_2) = \frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$$

If $\tilde{\phi}_1 < \tilde{\phi}_2$, then

$$V_1(0, I_2) = \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$$

The constraint

$$t_2 \leq V_2(I_1, I_2) - V_2(I_1, 0)$$

is equivalent to

$$t_2 \leq \frac{\phi_1}{\phi_1 + \psi}(\phi_2 + \psi + \chi - \rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1 + \hat{\psi}}(\hat{\phi}_2 + \hat{\psi} + \chi - \hat{\rho})$$

and the constraint

$$t_1 \leq V_1(I_1, I_2) - V_1(0, I_2)$$

is equivalent to

$$t_1 \leq \frac{\phi_1}{\phi_1 + \psi}(\phi_1 + \psi + \chi - \rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}\right\}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$$

The table below summarizes the effect of changing $(I_1^L, I_1^S, I_2^L, I_2^S)$ on the relevant incentive constraints.

A few remarks are in order.

First note that an increase in I_1^L is desirable in general: such a change softens D_1's participation constraint and leaves D_2's incentive to participate unchanged. This will be the case as long as type 1's loyalty base γ_1 is sufficiently small (so perfect information on γ_1 does not reduce competition on χ to the extent that D_2 is unwilling to purchase

	ϕ_1	ϕ_2	ρ	ψ	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{\rho}$	$\hat{\psi}$	$\tilde{\phi}_1$	$\tilde{\phi}_2$	$\tilde{\rho}$	$\tilde{\psi}$		$V_1(I_1, I_2)$	$V_1(0, I_2)$	$V_2(I_1, I_2)$	$V_2(I_1, 0)$
I_1^L	↑	-	-	-	↑	-	-	-	-	-	-	-		↑↑	-	↑	↑
I_1^S	-	-	↑↑	↓↓	-	-	↑	↓	-	-	-	-		↓	-	↓	↓
I_2^L	-	↑	-	-	-	-	-	-	-	↑	-	-		-	- or↑	↑	-
I_2^S	-	-	↑↑	↓↓	-	-	-	-	-	-	↑	↓		↓	- or ↑ or ↓	↓	-

Notes: Across the same row-block, double arrows indicate larger changes in magnitude than single arrows.

Table 2: Effect of I_i^K, $K \in \{L, S\}$ on participation constraints

information on γ_2) or type 1's loyalty base γ_1 is sufficiently large (so the gain from selling perfect information to D_1 more than compensates for the loss in not being able to sell I_2^L to D_2). This is best seen from the first panel in the table below, where we set the parameters of the model to $f = \frac{1}{5}$ and $\chi = \frac{1}{3}$ and vary the value of γ from the lower bound (just above $\frac{1}{2}(1-\chi)$) to the upper bound (just below $(1-\chi)$).

Counter-intuitively, an increase in I_1^S hurts screening in this type of model. Mathematically, $V_1(I_1, I_2) = \frac{\phi_1}{\phi_1+\psi}(\phi_1 + \psi + \chi - \rho)$ is decreasing in I_1^S through its positive effect on the term $\rho = \chi Pr_1(s|S) Pr_2(s|S) = \chi\left[\left(I_1^S + (1-I_1^S)\frac{\chi}{\gamma_1+\chi}\right)\left(I_2^S + (1-I_2^S)\frac{\chi}{\gamma_2+\chi}\right)\right]$ (which enters $V_1(I_1, I_2)$ negatively) and its negative effect on $\psi = \chi Pr_1(l_1|S) Pr_2(l_2|S) = \chi\left[\left(1-I_1^S\right)\left(1-I_2^S\right)\left(\frac{\gamma_1}{\gamma_1+\chi}\right)\left(\frac{\gamma_2}{\gamma_2+\chi}\right)\right]$ (which enters $V_1(I_1, I_2)$ positively). This reflects the strategic complementarity nature of downstream competition: a better informed D_1 on the switcher region will be better able to price competitively, and hence intensify the competition on this part of the customer base, which in turn reduces the value of additional information.

The trade-offs involved in choosing (I_2^L, I_2^S) are broadly similar, with one additional concern. When the information provided to D_2 is sufficiently sophisticated, the industry ordering reflected in the relative magnitude of ϕ_1 versus ϕ_2 will be switched. When this is the case, D_2 overtakes D_1 as the industry leader, which roughly means the information advantage I_2^L over I_1^L is great enough to counterbalance the innate disadvantage in prior customer base γ_2 v.s γ_1. Mathematically, this happens when the sign of $\tilde{\phi}_1 - \tilde{\phi}_2$ changes. If $\tilde{\phi}_1 > \tilde{\phi}_2$, then $V_1(0, I_2) = \frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$. On the other hand, if $\tilde{\phi}_1 < \tilde{\phi}_2$, then $V_1(0, I_2) = \frac{\tilde{\phi}_2}{\tilde{\phi}_2+\tilde{\psi}}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$. Note that the multiplier term outside the parentheses switches subscripts. When γ_1 is small, the regular effect dominates and the increase in I_2^L helps to soften D_2's participation constraint, which is desirable from the upstream monopolist's point of view. When γ_1 is large, the countervailing effect through increasing $\tilde{\phi}_2$ dominates and the increase in I_2^L worsens D_1's participation constraint (through increasing $V_1(0, I_2)$), in which case it is desirable for the monopolist to reduce I_2^L and supply I_2^S instead. This can be seen from the bottom region of the table below. Similarly, when γ_1 is small, the regular incentive effect tends to push for a lower value of I_2^S since I_2^S tightens D_2's participation constraint, but as γ_1 increases, such incentive is muted relative to the industry ordering takeover effect through $V_1(0, I_2)$.

γ	f=1/5, χ=1/3					f=3/4, χ=1/3					f=1/5, χ=1/2					f=3/4, χ=1/2				
	l_{1L}	l_{1S}	l_{2L}	l_{2S}	value	l_{1L}	l_{1S}	l_{2L}	l_{2S}	value	l_{1L}	l_{1S}	l_{2L}	l_{2S}	value	l_{1L}	l_{1S}	l_{2L}	l_{2S}	value
lower bound	1.00	0.00	1.00	0.00	0.06	1.00	0.00	1.00	0.00	0.22	1.00	0.00	1.00	0.00	0.13	1.00	1.00	1.00	0.00	0.16
	1.00	0.00	1.00	0.00	0.06	1.00	0.00	1.00	0.00	0.24	1.00	0.00	1.00	0.00	0.13	1.00	1.00	1.00	0.00	0.17
	1.00	0.00	1.00	0.00	0.07	1.00	0.00	1.00	0.00	0.25	1.00	0.00	1.00	0.00	0.12	1.00	1.00	1.00	0.00	0.18
	1.00	0.00	1.00	0.00	0.07	1.00	0.00	1.00	0.00	0.28	1.00	0.00	1.00	0.00	0.12	1.00	0.00	1.00	0.00	0.19
	1.00	0.00	0.46	0.00	0.08	1.00	0.00	0.46	0.00	0.29	1.00	0.00	1.00	0.00	0.11	1.00	0.00	1.00	0.00	0.20
	1.00	0.00	0.44	0.00	0.08	1.00	0.00	0.44	0.00	0.30	1.00	0.00	1.00	0.00	0.11	1.00	0.00	1.00	0.00	0.21
	1.00	0.00	0.42	0.00	0.09	1.00	0.00	0.42	0.00	0.32	1.00	0.00	1.00	0.00	0.11	1.00	0.00	1.00	0.00	0.22
	1.00	0.00	0.41	0.00	0.09	1.00	0.00	0.41	0.00	0.34	1.00	0.00	1.00	0.00	0.10	1.00	0.00	1.00	0.00	0.23
	1.00	0.00	0.40	0.00	0.09	1.00	0.00	0.40	0.00	0.35	1.00	0.00	1.00	0.00	0.09	1.00	0.00	1.00	0.00	0.23
	1.00	0.00	0.40	0.00	0.10	1.00	0.00	0.40	0.00	0.37	1.00	0.00	1.00	0.00	0.08	1.00	0.00	1.00	0.00	0.24
	0.56	0.00	0.00	0.97	0.08	0.84	0.00	0.00	0.97	0.36	1.00	0.00	1.00	0.00	0.07	1.00	0.00	1.00	0.00	0.25
	0.48	0.00	0.00	0.92	0.08	0.72	0.00	0.00	0.92	0.35	1.00	0.00	1.00	0.00	0.07	1.00	0.00	1.00	0.00	0.26
	0.47	0.00	0.00	0.82	0.09	0.61	0.00	0.00	0.82	0.36	1.00	0.00	1.00	0.00	0.07	1.00	0.00	1.00	0.00	0.27
	0.47	0.07	0.07	0.89	0.09	0.60	0.07	0.07	0.89	0.37	1.00	0.00	1.00	0.00	0.07	1.00	0.00	1.00	0.00	0.28
	1.00	0.00	0.00	1.00	0.12	1.00	0.00	0.00	1.00	0.45	1.00	0.00	1.00	0.00	0.08	1.00	0.00	1.00	0.00	0.29
	1.00	0.00	0.00	1.00	0.13	1.00	0.00	0.00	1.00	0.48	1.00	0.00	1.00	0.00	0.08	1.00	0.00	1.00	0.00	0.30
	1.00	0.00	0.00	1.00	0.13	1.00	0.00	0.00	1.00	0.49	1.00	0.00	0.66	0.00	0.08	1.00	0.00	0.66	0.00	0.31
	1.00	0.00	0.00	1.00	0.13	1.00	0.00	0.00	1.00	0.50	1.00	0.00	0.61	0.00	0.09	1.00	0.00	0.61	0.00	0.32
	1.00	0.00	0.00	1.00	0.14	1.00	0.00	0.00	1.00	0.53	1.00	0.00	0.58	0.00	0.09	1.00	0.00	0.58	0.00	0.34
upper bound	1.00	0.00	0.00	1.00	0.15	1.00	0.00	0.00	1.00	0.55	1.00	0.00	0.55	0.00	0.09	1.00	0.00	0.55	0.00	0.35

Figure 2: Results of participation-inducing problem

2.4.2 Variant 2: Specialization and stratification

A different way to interpret the problem is that the upstream party's action consists of two steps. In step one, the upstream party can tell whether a customer belongs to D_i's turf or not. Assume the classification at step one is perfect. In step two, the upstream party sells a device that can further classify whether the customer is a loyal one or a switcher.

First, suppose the information product is 3. Note that the first row representing loyalty base to D_1 and the second row representing loyalty base to D_2 are distinguishable to D_1 and D_2. With this interpretation, the information devices take the following forms.

$$\pi^1 = \begin{bmatrix} \pi_H^1 & 1-\pi_H^1 \\ 1-\pi_L^1 & \pi_L^1 \end{bmatrix}$$

$$\pi^2 = \begin{bmatrix} \pi_H^2 & 1-\pi_H^2 \\ 1-\pi_L^2 & \pi_L^2 \end{bmatrix}$$

This interpretation makes it harder for the upstream party to screen.

Write $\phi_{12} = \gamma_1 Pr_2(l_2|L_2)$, $\phi_{21} = \gamma_2 Pr_1(l_1|L_1)$, $\rho_2 = \chi Pr_2(s|S)^2$, $\rho_1 = \chi Pr_1(s|S)^2$,and $\psi_2 = \chi Pr_2(l_2|S)^2$ and $\psi_1 = \chi Pr_1(l_1|S)^2$.

The incentive constraint of D_1 is

$$\frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho)-t_1 \geq \frac{\phi_{12}}{\phi_{12}+\psi_2}(\phi_{12}+\psi_2+\chi-\rho_2)-t_2 \; [IC_1]$$

Or equivalently,

$$\phi_1 + \frac{\phi_1}{\phi_1+\psi}(\chi-\rho) - t_1 \geq \phi_{12} + \frac{\phi_{12}}{\phi_{12}+\psi_2}(\chi-\rho_2) - t_2$$

The incentive constraint of D_2 is

$$\frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) - t_2 \geq \frac{\phi_1}{\phi_1+\psi_1}(\phi_2+\psi_1+\chi-\rho_1) - t_1 \; [IC_2]$$

The participation constraints of D_1 and D_2 are as in the previous model. Call them IR_1 and IR_2, respectively. We have

$$t_1 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2+\tilde{\psi}}\right\}(\tilde{\phi}_1+\tilde{\psi}+\chi-\tilde{\rho})$$

$$t_2 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1+\hat{\psi}}(\hat{\phi}_2+\hat{\psi}+\chi-\hat{\rho})$$

The incentive constraints can be written as

$$t_1 - t_2 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - \frac{\phi_{12}}{\phi_{12}+\psi_2}(\phi_{12}+\psi_2+\chi-\rho_2)$$

and

$$t_1 - t_2 \geq \frac{\phi_1}{\phi_1+\psi_1}(\phi_2+\psi_1+\chi-\rho_1) - \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho)$$

The graph below plots the four components as functions of I_1^L, I_1^S, I_2^L and I_2^S, when the ordering of D_1 and D_2 is unchanged (i.e., $\phi_1 > \phi_2$ throughout.)

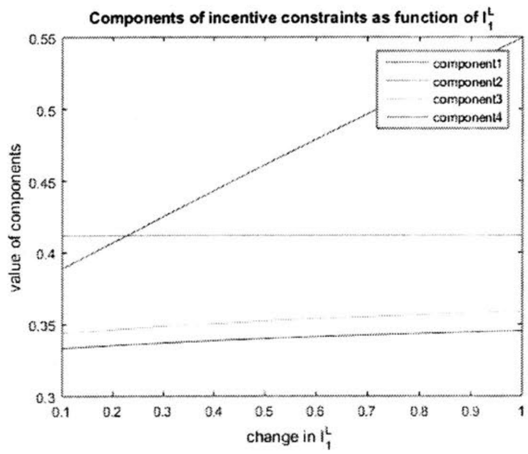
Note that an increase in I_1^L relaxes the first constraint and leaves the second one unchanged, so we have $I_1^L = 1$ in any optimal solution. On the other hand, an increase in I_1^S tightens the first constraint unchanged but relaxes the second one. Further, an increase in I_2^L tightens the first constraint and leaves the second one unchanged, so we have $I_2^L = 0$ in any optimal solution. Similarly, an increase in I_2^S relaxes the first constraint but tightens the second constraint. Note also that the above are only valid under the assumption that IC_i instead of IR_i binds.

Define $Pr_i(s|S) = s_i$. Substituting, we have $\phi_1 = \gamma_1$, $\phi_2 = \gamma_2$, $\phi_{12} = \gamma_1 \frac{\gamma_2}{\gamma_2+\chi}$, $\phi_{21} = \gamma_2$, $\rho = \chi s_1 s_2$, $\rho_2 = \chi s_2^2$, $\rho_1 = \chi s_1^2$,and $\psi_2 = \chi(1-s_2)^2$ and $\psi = \chi(1-s_1)(1-s_2)$.

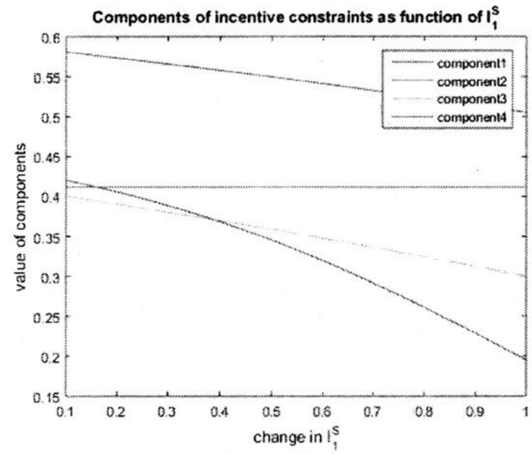
IC_1 and IC_2 becomes

$$t_1 - t_2 \leq F_1(s_1, s_2) \equiv \gamma_1 - \gamma_1\frac{\gamma_2}{\gamma_2+\chi} + \frac{\gamma_1}{\gamma_1+\chi(1-s_2)^2}(\chi-\chi s_1 s_2) - \frac{\gamma_1\frac{\gamma_2}{\gamma_2+\chi}}{\gamma_1\frac{\gamma_2}{\gamma_2+\chi}+\chi(1-s_2)^2}(\chi-\chi s_2^2)$$

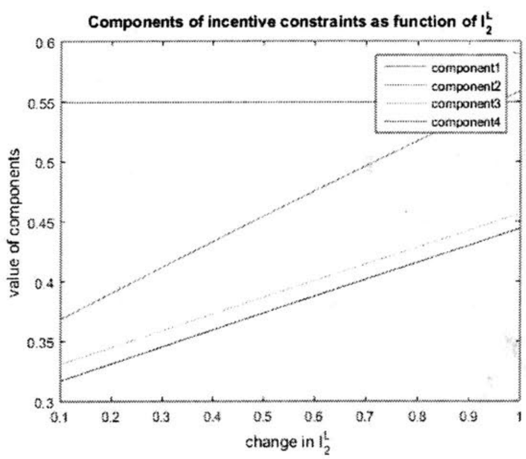
$$t_1 - t_2 \geq F_2(s_1, s_2) \equiv \frac{\gamma_1}{\gamma_1+\chi(1-s_1)^2}(\gamma_2+\chi(3-2s_1)) - \frac{\gamma_1}{\gamma_1+\chi(1-s_1)(1-s_2)}(\gamma_2+\chi(2-s_1-s_2))$$



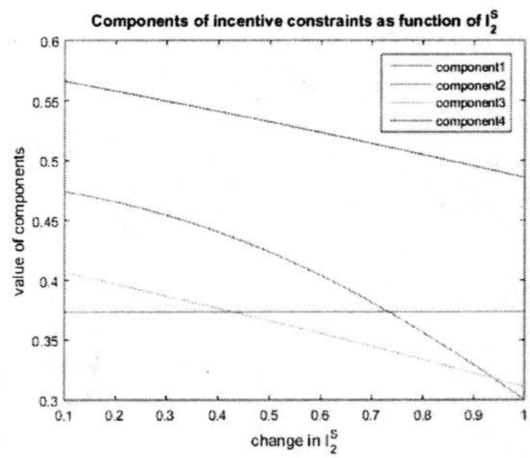
(a) I_1^L



(b) I_1^S

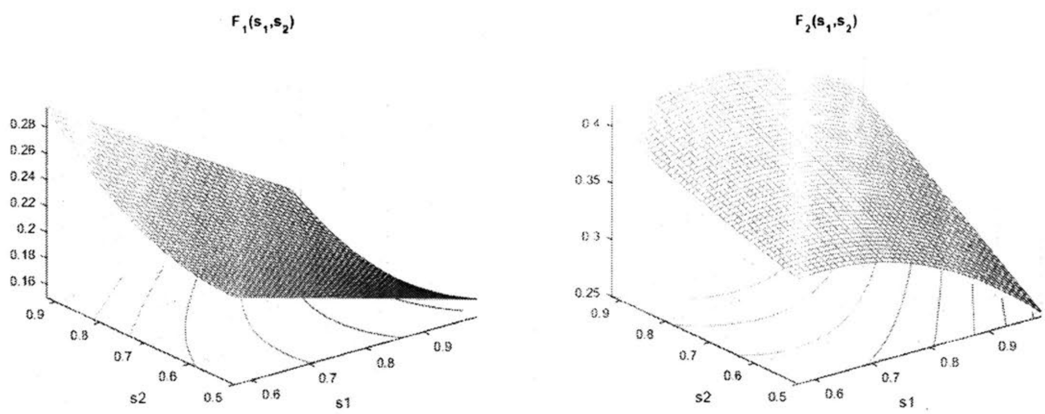


(c) I_2^L



(d) I_2^S

Figure 3: Components of incentive constraints and functions of I_1^L,I_1^S,I_2^L and I_2^S.



(a) $F_1(s_1, s_2)$ (b) $F_2(s_1, s_2)$

Figure 4: Value of $F_1(s_1, s_2)$ and $F_2(s_1, s_2)$

The figure below plots F_1 and F_2 as functions of (s_1, s_2).

When F_1 binds, the optimum is achieved at $s_2 = 1$ and $s_1 = \frac{\chi}{\chi+\gamma_1}$. This corresponds to the case of specialization: D_1 can perfectly identify its loyalty base, but has no further information on the switchers (and has to base its decisions on the prior assessment.) D_2 can perfectly identify the switchers using the information device, which does not provide any further information on the loyalty base.

When F_2 binds, the optimum is achieved at $s_2 = 1$ and $s_1 \in (\frac{\chi}{\chi+\gamma_1}, 1)$, where the interior value of s_1 reflects the trade-off between tightening D_1's incentive constraint and relaxing D_2's incentive constraint. Importantly, with downstream price competition, it is never optimal for D_1 to have perfect information $(I_1^L, I_1^S) = (1, 1)$ as long as D_2 obtains some additional information, i.e., $(I_2^L, I_2^S) \geq (\frac{\gamma_2}{\chi+\gamma_2}, \frac{\chi}{\chi+\gamma_2})$ with strict inequality in at least one coordinate.

Formally, at most 2 of the following 4 constraints are binding

$$t_1 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2+\tilde{\psi}}\right\}(\tilde{\phi}_1+\tilde{\psi}+\chi-\tilde{\rho})\ [IR1]$$

$$t_2 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1+\hat{\psi}}(\hat{\phi}_2+\hat{\psi}+\chi-\hat{\rho})\ [IR2]$$

$$t_1 - t_2 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - \frac{\phi_{12}}{\phi_{12}+\psi_2}(\phi_{12}+\psi_2+\chi-\rho_2)\ [IC1]$$

$$t_1 - t_2 \geq \frac{\phi_1}{\phi_1+\psi_1}(\phi_2+\psi_1+\chi-\rho_1) - \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho)\ \ [IC2]$$

and only one of $[IC1]$ and $[IC2]$ can bind.

Before we proceed to discuss the 4 possible cases of binding constraints, note that

$$\phi_1(I_1^L) = \gamma_1(\frac{\gamma_1}{\gamma_1+\chi}(1-I_1^L)+I_1^L)$$

$$\phi_1'(I_1^L) = \frac{\gamma_1 \chi}{\gamma_1 + \chi}$$

$$\psi(I_1^S, I_2^S) = \chi(\frac{\gamma_1}{\gamma_1 + \chi}(1 - I_1^S))(\frac{\gamma_2}{\gamma_2 + \chi}(1 - I_2^S))$$

$$\frac{\partial}{\partial I_1^S}\psi(I_1^S, I_2^S) = -\frac{\chi\gamma_1}{\gamma_1 + \chi}(\frac{\gamma_2}{\gamma_2 + \chi}(1 - I_2^S))$$

$$\frac{\partial}{\partial I_2^S}\psi(I_1^S, I_2^S) = -\frac{\chi\gamma_2}{\gamma_2 + \chi}(\frac{\gamma_1}{\gamma_1 + \chi}(1 - I_1^S))$$

$$\rho(I_1^S, I_2^S) = \chi(\frac{\chi}{\gamma_1 + \chi}(1 - I_1^S) + I_1^S)(\frac{\chi}{\gamma_2 + \chi}(1 - I_2^S) + I_2^S)$$

$$\frac{\partial}{\partial I_1^S}\rho(I_1^S, I_2^S) = \frac{\chi\gamma_1}{\gamma_1 + \chi}(\frac{\chi}{\gamma_2 + \chi}(1 - I_2^S) + I_2^S))$$

$$\frac{\partial}{\partial I_2^S}\rho(I_1^S, I_2^S) = \frac{\chi\gamma_2}{\gamma_2 + \chi}(\frac{\chi}{\gamma_1 + \chi}(1 - I_1^S) + I_1^S)$$

$$\tilde{\psi}(I_2^S) = \chi(\frac{\gamma_1}{\gamma_1 + \chi})(\frac{\gamma_2}{\gamma_2 + \chi}(1 - I_2^S))$$

$$\tilde{\psi}'(I_2^S) = -\chi(\frac{\gamma_1}{\gamma_1 + \chi})(\frac{\gamma_2}{\gamma_2 + \chi})$$

$$\tilde{\phi}_2(I_2^L) = \gamma_2(\frac{\gamma_2}{\gamma_2 + \chi}(1 - I_2^L) + I_2^L)$$

$$\tilde{\phi}_2'(I_2^L) = \gamma_2(\frac{\chi}{\gamma_2 + \chi})$$

$$\tilde{\rho}(I_2^S) = \chi\frac{\chi}{\gamma_1 + \chi}(\frac{\chi}{\gamma_2 + \chi}(1 - I_2^S) + I_2^S)$$

$$\tilde{\rho}'(I_2^S) = \chi\frac{\chi}{\gamma_1 + \chi}(\frac{\gamma_2}{\gamma_2 + \chi})$$

$$\rho_2(I_2^S) = \chi(\frac{\chi}{\gamma_2 + \chi}(1 - I_2^S) + I_2^S)^2$$

$$\rho_2'(I_2^S) = 2\chi(\frac{\chi}{\gamma_2 + \chi}(1 - I_2^S) + I_2^S)\frac{\gamma_2}{\gamma_2 + \chi}$$

$$\psi_2(I_2^S) = \chi(\frac{\gamma_2}{\gamma_2 + \chi}(1 - I_2^S))^2$$

$$\psi_2'(I_2^S) = 2\chi(\frac{\gamma_2}{\gamma_2 + \chi}(1 - I_2^S))(-\frac{\gamma_2}{\gamma_2 + \chi})$$

$$\phi_{12}'(I_2^L) = \gamma_1(\frac{\chi}{\gamma_2 + \chi})$$

These expressions will be used repeatedly in the discussions to follow. In what follows, we will offer a full-fledged solution to the symmetric case where $\gamma_1 = \gamma_2 = \chi = \frac{1}{3}$ and $f = \frac{1}{2}$. The general case follows similarly. We will discuss the effect of other parameter value choices on the optimal solution. To save space, we leave the detailed derivation to the appendix.

A full characterization of the solution to the model is lengthy. In general, there are several potential forms of optima.

Exclusion: D_1 has perfect information $I_1^L = I_1^S = 1$ and D_2 is excluded from the market $I_2^L = I_2^S = 0$.

Stratification/ Market leader and follower: D_1 has premium information on both L and S segments: $I_1^L > I_2^L$ and $I_1^S > I_2^S$.

Specializaition: D_1 has premium information on the L segment while D_2 specializes on the S segment: $I_1^L > I_2^L$ and $I_2^S > I_1^S$.

Reversal/overtaking: The above three cases, with the order of D_1 and D_2 reversed.

Which form of equilibrium arises as the optimal structure depends on

(1) **The relative sizes of loyalty bases γ_1 and γ_2**, which affects the return to I_i^L and the incentive constraint.

(2) **The distribution of loyalty base v.s switchers region**, which affects equilibrium payoff when the mistargeting effect is nontrivial.

(3) **The (endogenous) ordering of D_1 and D_2**, captured by the relative magnitudes of ϕ_1 v.s ϕ_2, where $\phi_i = \gamma_i Pr_i(l_i|L_i)$. The equilibrium support of the pricing function is determined by $max\{\phi_1, \phi_2\}$. Note that the ex-ante weaker downstream player (the one with lower value of γ) can have a higher value of ϕ (provided that the $Pr(l|L)$ is large enough) and overtake the opponent in the prescence of additional targeting technology.

(4) **The (endogenous) complementarity of I_1 and I_2**, which is relevant when targeting is imperfect ($I < 1$). When the prior difference between $D_1$and D_2 is small (i.e., the difference between γ_1 and γ_2 is small), and the level of I is low, an increase in I_1 will **benefit** D_2 via the mistargeting effect: an increase in I_1 will induce D_1 to raise its price to the loyal customers, but when targeting is imperfect, the higher price will be mistakenly posted to switchers, which softens the competition on the switchers region and allow D_2 to benefit (by enlarging its reach on the switchers region).

Market share From the previous discussion, it is easy to derive the market shares of D_1 and D_2 are, respectively

$$s_1(I_1, I_2) = \frac{1}{2} + \frac{1}{2} \cdot \frac{\phi_1}{\phi_1 + \psi}(\phi_1 - \phi_2)$$

and

$$s_2(I_1, I_2) = \frac{1}{2} - \frac{1}{2} \cdot \frac{\phi_1}{\phi_1 + \psi}(\phi_1 - \phi_2)$$

γ	I_{1L}	I_{1S}	I_{2L}	I_{2S}	value
	f=0.6, χ=0.5				
lower bound	1.00	0.00	1.00	0.00	0.15
	1.00	0.00	1.00	0.00	0.16
	1.00	0.00	1.00	0.00	0.16
	1.00	0.00	1.00	0.00	0.17
	1.00	0.00	1.00	0.00	0.17
	1.00	0.00	1.00	0.00	0.18
	1.00	0.38	0.50	0.00	0.16
	1.00	0.42	0.50	0.00	0.17
	1.00	0.46	0.50	0.00	0.18
	1.00	0.50	0.50	0.00	0.19
	1.00	0.53	0.50	0.00	0.20
	1.00	0.57	0.50	0.00	0.21
	1.00	0.61	0.50	0.00	0.22
	1.00	0.64	0.50	0.00	0.23
	1.00	0.68	0.50	0.00	0.24
	1.00	0.72	0.50	0.00	0.24
	1.00	0.75	0.50	0.00	0.25
	1.00	0.79	0.50	0.00	0.26
	1.00	0.83	0.50	0.00	0.27
	1.00	0.86	0.50	0.00	0.28
upper bound	1.00	0.90	0.50	0.00	0.28

γ	I_{1L}	I_{1S}	I_{2L}	I_{2S}	value
	f=0.8, χ=0.5				
lower bound	1.00	0.13	1.00	0.00	0.16
	1.00	0.17	1.00	0.00	0.17
	1.00	0.22	1.00	0.00	0.18
	1.00	0.26	1.00	0.00	0.19
	1.00	0.30	1.00	0.00	0.20
	1.00	0.34	1.00	0.00	0.21
	1.00	0.38	0.50	0.00	0.22
	1.00	0.42	0.50	0.00	0.23
	1.00	0.46	0.50	0.00	0.24
	1.00	0.50	0.50	0.00	0.25
	1.00	0.53	0.50	0.00	0.27
	1.00	0.57	0.50	0.00	0.28
	1.00	0.61	0.50	0.00	0.29
	1.00	0.64	0.50	0.00	0.30
	1.00	0.68	0.50	0.00	0.31
	1.00	0.72	0.50	0.00	0.33
	1.00	0.75	0.50	0.00	0.34
	1.00	0.75	0.50	0.00	0.35
	1.00	0.75	0.50	0.00	0.36
	1.00	0.75	0.50	0.00	0.37
upper bound	1.00	0.75	0.50	0.00	0.38

γ	I_{1L}	I_{1S}	I_{2L}	I_{2S}	value
	f=0.6, χ=0.4				
lower bound	1.00	0.34	0.50	0.00	0.15
	1.00	0.39	0.50	0.00	0.16
	1.00	0.43	0.50	0.00	0.17
	1.00	0.47	0.50	0.00	0.19
	1.00	0.52	0.50	0.00	0.20
	1.00	0.56	0.50	0.00	0.21
	1.00	0.61	0.50	0.00	0.22
	1.00	0.65	0.50	0.00	0.23
	1.00	0.70	0.50	0.00	0.24
	1.00	0.74	0.50	0.00	0.25
	1.00	0.79	0.50	0.00	0.26
	1.00	0.84	0.50	0.00	0.27
	1.00	0.89	0.50	0.00	0.28
	1.00	0.94	0.50	0.00	0.29
	1.00	0.99	0.50	0.00	0.30
	1.00	0.98	0.50	1.00	0.31
	1.00	0.95	0.50	1.00	0.33
	1.00	0.93	0.50	1.00	0.34
	1.00	0.90	0.50	1.00	0.36
	1.00	0.88	0.50	1.00	0.37
upper bound	1.00	0.85	0.50	1.00	0.38

γ	I_{1L}	I_{1S}	I_{2L}	I_{2S}	value
	f=0.8, χ=0.4				
lower bound	1.00	0.34	0.50	0.00	0.20
	1.00	0.39	0.50	0.00	0.22
	1.00	0.43	0.50	0.00	0.23
	1.00	0.47	0.50	0.00	0.25
	1.00	0.52	0.50	0.00	0.26
	1.00	0.56	0.50	0.00	0.28
	1.00	0.61	0.50	0.00	0.29
	1.00	0.65	0.50	0.00	0.31
	1.00	0.70	0.50	0.00	0.32
	1.00	0.74	0.50	0.00	0.34
	1.00	0.79	0.50	0.00	0.35
	1.00	0.84	0.50	0.00	0.36
	1.00	0.89	0.50	0.00	0.38
	1.00	0.94	0.50	0.00	0.39
	1.00	0.99	0.50	0.00	0.40
	1.00	0.98	0.50	1.00	0.42
	1.00	0.95	0.50	1.00	0.44
	1.00	0.93	0.50	1.00	0.45
	1.00	0.90	0.50	1.00	0.47
	1.00	0.88	0.50	1.00	0.49
upper bound	1.00	0.85	0.50	1.00	0.51

Figure 5: Industry profit of the specialization problem

Note that the measure of D_1 advantage, $\frac{\phi_i}{\phi_i+\psi}(\phi_i - \phi_j)$ is increasing in ϕ_i and $(\phi_i - \phi_j)$ and decreasing in ψ. Higher prior advantage (captured by a larger loyalty base γ_i relative to γ_j) and better information product on the loyalty dimension (captured by higher I_1^L relative to I_2^L) increase D_1's market share, while an increase in either I_1^S or I_2^S also increases D_1's market share. The positive effect of I_2^S on s_1 is worth further elaboration. Intuitively, an increase in I_2^S intensifies the competition on the switchers region, which decreases its relative importance (by reducing the surplus at stake). The multiplier $\frac{\phi_1}{\phi_1+\gamma}$ which measures the relative importance of the loyalty region increases. For D_1's market share, the multiplier is multiplied by a positive term, so the effect on market share is positive.

Industry profit From the analysis above we can calculate the upstream profit when $(I_1^L, I_1^S, I_2^L, I_2^S)$ are optimally chosen. The table below presents the optimal choices of I and the value function, for various values of f and χ. The columns vary γ_1 from just above $\frac{1}{2}(1-\chi)$ to just below $(1-\chi)$.

A few remarks are in order.

First, when γ increases, total industry profit generates increases. Loosely speaking, this corresponds to the implicit incentive for the upstream party cultivate a downstream leader. Intuitively, it is the asymmetric between D_1 and D_2 (reflected in the difference between $\gamma_!$ and γ_2) that allows the upstream party to extract surplus. For similar reasons, a smaller χ, all else equal, corresponds to larger industry profit (c.f. Panel 1 and 3, Panel 2 and 4).

Since our assumption implies D_1 is more lucrative party, greater value of f accentuates this advantage and makes it easier for the upstream party to extract surplus. So industry profit is increasing in f.

The highlighted yellow area corresponds to the region of specialization: where D_1 obtains perfect information on its loyalty base $I_1^L = 1$ and D_2 obtains imperfect information on I_2^L but perfect information on the switcher region $I_2^S = 1$. This occurs due to the pattern of binding participation constraint, as discussed previously.

The highlighted green area corresponds to the case where both D_1 and D_2 achieves perfect targeting on its respective loyalty base. This occurs when the asymmetry between D_1 and D_2 is small. From the previous discussion, we know that this case only occurs when one of the two participation constraints binds.

Information sharing between downstream parties Another interesting aspect of the model is the incentive for the firms to manage its information asset. In this subsection, we explore the firms' incentives to guard its customer information.

An important aspect of information is that it is cost-less to share and disseminate (once produced).

Note that firms have incentives to participate in information exchange if

$$\frac{\partial V_i}{\partial I_i} + \frac{\partial V_i}{\partial I_j} > 0 \; for \; i = 1,2 \; and \; i \neq j$$

Roughly, this condition is equivalent to existence of a Pareto improvement with transfer, where the transfer price from j to i is set to be $\frac{\partial V_i}{\partial I_i}$ for $i = 1,2$ (the price is not unique).

A stronger condition, as derived in Chen et al. (2001), is for D_i to voluntarily give away information:

$$\frac{\partial V_i}{\partial I_j} > 0$$

In this subsection, we will explore when cross-firm sharing is likely to occur, in terms of parameter configurations of the model.

We will then study the implication of privacy restrictions and exclusions on industry profit and welfare.

First, we observe that sharing is more likely to occur (in the sense that either $\frac{\partial V_i}{\partial I_j} > 0$ or the weaker condition $\frac{\partial V_i}{\partial I_i} + \frac{\partial V_i}{\partial I_j} > 0$ or both are more likely to hold) when one firm's targeting technology is sufficiently inferior (to be made precise later) and the relative magnitudes of the loyal bases of the two firms are not too different. We also observe that the low type D_j (i.e., the firm with less perfect targeting technology) has stronger incentive to sell than the high type D_i (i.e., the firm with better targeting technology) in the sense that

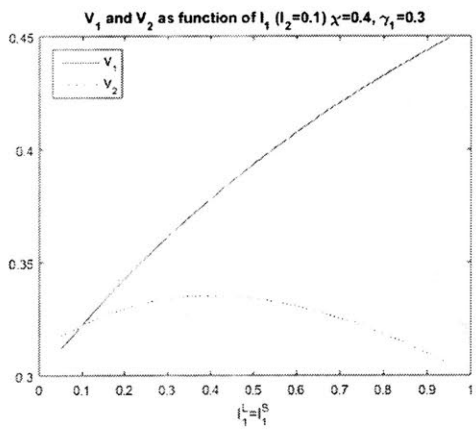
$$\frac{\partial V_i}{\partial I_j} > \frac{\partial V_j}{\partial I_i}$$

Intuitively, an increase in either firm's targetability benefits both firms when the starting point is low. The firm with low starting point has incentive to sell but the firm with high starting point doesn't – since only the high type can pay a (positive) price to compensate for the potential loss.

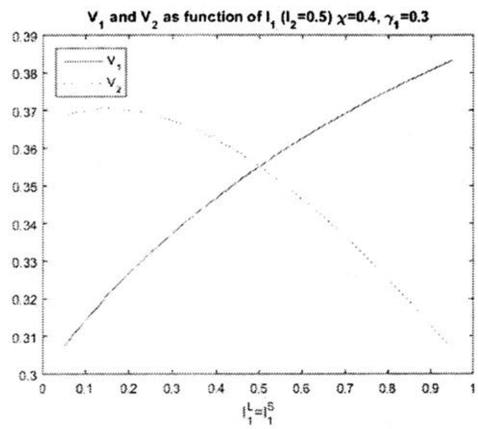
Further, when the sizes of the loyalty bases are sufficiently different, the firm with the smaller loyalty base never wants to improve the opponent's targetability due to the large loss at stake in the switchers' turf.

The figure below illustrates the argument above.

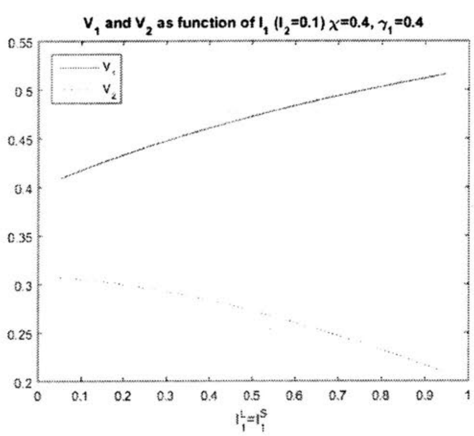
The intuition is above is imprecise, because we have implicitly collapsed the two-dimensional information product



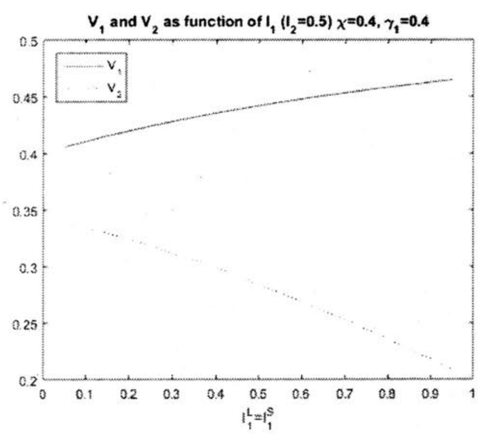
(a) Small I_2; Small γ_1



(b) Large I_2; Small γ_1

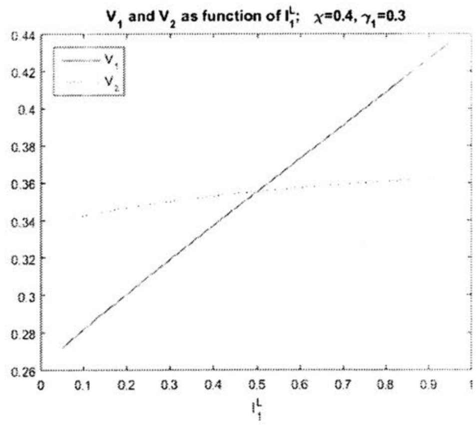


(c) Small I_2; Large γ_1

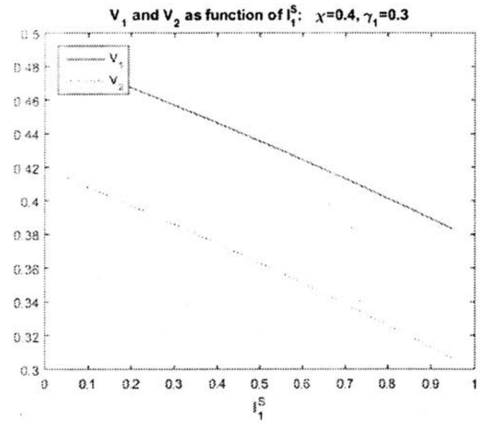


(d) Large I_2; Large γ_1

Figure 6: Incentives to share targeting technology



(a) V_i as function of I_1^L



(b) V_i as function of I_1^S

Figure 7: Incentives to share targeting technology, two dimensional product

into one. The figure below briefly explores the intricacies involved in two-dimensionality of information product. In general, the high type party has incentive to sell I^L but not I^S, since an improvement in the latter worsens the value V to both parties. In two-dimensional models, the targeting technology for the switchers is used mainly for screening purposes.

Alternative mode of upstream-downstream relationships Our discussion so far assumes the monopolist can design a menu of differentiated information products and ask the downstream parties to select their most desired one (if any) from the menu. In this subsection, we consider several alternative modes of upstream-downstream relationships. In the case of nonexclusive selling as discussed in Villas-Boas (1994), the seller produces a single information product $\bar{I}$ and makes a take-it-or-leave-it offer to both downstream parties. The monopolist's surplus from D_1 is given as $V_1(\bar{I},\bar{I}) - V_1(\bar{I},0)$ and the monopolist's surplus from D_2 is given by $V_2(\bar{I},\bar{I}) - V_2(\bar{I},0)$. From our discussion in the previous section, it is easy to see that non-exclusive contracting is equivalent to the monopolist's problem with the additional restriction that the menu can consist of only one product (of potentially multiple dimensions), and that both incentive constraints bind.

In the case of exclusive contracting, on the other hand, the product can only be given to one party, so the maximum price the monopolist can charge is given by the firm's willingness to pay for the product when the opponent does not have access to the product, minus the firm's value when the product in question is given to the rival. Formally, a monopolist's profit under exclusive contracting is given by $V_1(\bar{I},0) - V_1(0,\bar{I})$ from D_1 and $V_2(0,\bar{I}) - V_2(\bar{I},0)$ from D_2.

2.5 Extension: general Hotelling style competition

So far, we have assumed that the the switchers are completely indifferent between D_1 and D_2 and the loyalty customers are perfectly attached to their preferred party. In this subsection, we relax this assumption in the following sense. We assume the customer base can be divided into $2n+1$ segments. ordered by $\chi_1, \chi_2, ..., \chi_{2n+1}$. Customers in $\chi_1$is loyal to D_1, customers in χ_{2n+1} is loyal to D_2. Customers in χ_k $(2 \leq k \leq n)$ purchase from D_1 if $p_1 < p_2 + c_k$ and $p_1 \leq 1$, purchase from D_2 if $p_1 \geq p_2 + c_k$ and $p_2 \leq 1$ and purchase neither if $p_1 > 1$ and $p_2 > 1 - c_k$. Customers in χ_k $(n+1 \leq k \leq 2n)$ purchase from D_2 if $p_2 < p_1 + b_{2n+1-k}$ and $p_2 \leq 1$, purchase from D_1 if $p_2 \geq p_1 + b_{2n+1-k}$ and $p_1 \leq 1$ and purchase neither if $p_1 > 1 - b_{2n+1-k}$ and $p_2 > 1$. Assume c_k is decreasing in k and b_{2n+1-k} is increasing in k. Customers in χ_n purchase from D_1 if $p_1 \geq p_2 + d$ and $p_2 \leq 1$ and purchase from neither if $p_1 > 1$ and $p_2 > 1 - d$.

Note that the formulation in the previous section can be viewed as the special case with $n = 3$ and $d = 0$. In this case, $\chi_1 = \gamma_1$, $\chi_3 = \gamma_2$ and $\chi_2 = \chi$.

We will consider a case where there are three segments: χ_1, χ_2, χ_3. Let $\Pi_i = (\pi^i_{jk})_{jk}$ denote the information product of D_i.

$$\Pi_i = \begin{bmatrix} \pi^i_{11} & \pi^i_{12} & \pi^i_{13} \\ \pi^i_{21} & \pi^i_{22} & \pi^i_{23} \\ \pi^i_{31} & \pi^i_{32} & \pi^i_{33} \end{bmatrix}$$

In particular,

$$Pr_i(s = \chi_1 | \omega = \chi_1) = \pi^i_{11}$$

$$Pr_i(s = \chi_2 | \omega = \chi_2) = \pi^i_{22}$$

$$Pr_i(s = \chi_3 | \omega = \chi_3) = \pi^i_{33}$$

The off-diagonal entries are defined analogously.

Given a pricing strategy, let $G^k_i(p) = Pr_i(p^k_i < p)$ denote the CDF of price on χ_k.

Define

$$G_2(p) = \begin{bmatrix} 0 \\ G^2_2(p-d) \\ G^3_2(p) \end{bmatrix}$$

Note on the segment χ_2, customer purchases from D_1 when faced with price $p < p_2 + d$, or $p_2 > p - d$, which happens with probability $(1 - G^2_2(p-d))$.

Similarly, define

$$G_1(p) = \begin{bmatrix} G^1_1(p) \\ G^2_1(p-d) \\ 0 \end{bmatrix}$$

Note on the segment χ_2, customer purchases from $D_2$when faced with price $p > p_2 + d$ which happens with probability $(1 - G^2_1(p+d))$.

On perceived segment χ_1, at price p charged by D_1, total expected profit is given by

$R_1^1(p) = p \cdot \{\chi_1 \cdot \pi_{11}^1 + \chi_2 \cdot \pi_{21}^1 \cdot [\pi_{22}^2 \cdot (1 - G_2^2(p-d)) + \pi_{23}^2 \cdot (1 - G_2^3(p))]\}$

On perceived segment χ_2, at price p charged by D_1, total expected profit is given by

$R_1^2(p) = p \cdot \{\chi_1 \cdot \pi_{12}^1 + \chi_2 \cdot \pi_{22}^1 \cdot [\pi_{22}^2 \cdot (1 - G_2^2(p-d)) + \pi_{23}^2 \cdot (1 - G_2^3(p))]\}$

For D_2, on perceived segment χ_3, at price p charged by D_2, total expected profit if given by

$R_2^3(p) = p \cdot \{\chi_3 \cdot \pi_{33}^2 + \chi_2 \cdot \pi_{23}^2 \cdot [\pi_{21}^2 \cdot (1 - G_1^1(p)) + \pi_{22}^2 \cdot (1 - G_1^2(p+d)]\}$

For D_2, on perceived segment χ_2, at price p charged by D_2, total expected profit if given by

$R_2^2(p) = p \cdot \{\chi_3 \cdot \pi_{32}^2 + \chi_2 \cdot \pi_{22}^2 \cdot [\pi_{21}^2 \cdot (1 - G_1^1(p)) + \pi_{22}^2 \cdot (1 - G_1^2(p+d)]\}$

First consider the case there there's no targeting technology available. The following can be established as shown in Naraishm (1998):

If $(\chi_2 + \chi_3)(1-d) < \chi_2$, then $(1,1)$ is a pure strategy equilibrium. This corresponds to the case where lowering price to $1-d$ and gaining the entire market is less profitable than charging the reservation price (and obtaining only the loyalty base).

Assume $(\chi_2+\chi_3)(1-d) > \chi_2$ and $d > \hat{p}_1 - \hat{p}_2$ where $\hat{p}_1 = \frac{\chi_1}{\chi_1+\chi_2}$ and $\hat{p}_2 = \frac{\chi_3}{\chi_2+\chi_3}$. In this case, the mixed-strategy equilibrium exists.

Note that D_2 will not charge price in $(1-d, 1)$ and D_1 will not charge below $\hat{p}_2 + d$. In equilibrium, D_1's price support is $(\hat{p}_2 + d, 1)$ and D_2's price support is $(\hat{p}_2, 1-d)$ and 1.

The equilibrium pricing functions can be obtained by solving

$$\chi_1 p + [1 - G_2(p-d)]\chi_2 p = (\chi_1 + \chi_2)(\hat{p}_2 + d),\ if\ \hat{p}_2 + d \le p \le 1$$

$$\chi_3 p + [1 - G_1(p+d)]\chi_2 p = \chi_3,\ if\ \hat{p}_2 \le p \le 1-d$$

which yields

$$G_1(p) = \begin{cases} 0 & if\ p \le \hat{p}_2 + d \\ 1 + \frac{\chi_3}{\chi_2} - \frac{\chi_3}{\chi_2(p-d)} & if\ \hat{p}_2 + d \le p \le 1 \\ 1 & if\ p \ge 1 \end{cases}$$

$$G_2(p) = \begin{cases} 0 & if\ p \le \hat{p}_2 \\ 1 + \frac{\chi_1}{\chi_2} - \frac{(\chi_1+\chi_2)(\hat{p}_2+d)}{\chi_2(p+d)} & if\ \hat{p}_2 \le p \le 1-d \\ 1 + \frac{\chi_1}{\chi_2} - \frac{(\chi_1+\chi_2)(\hat{p}_2+d)}{\chi_2} & if\ 1-d \le p \le 1 \\ 1 & if\ p \ge 1 \end{cases}$$

Now suppose targeting technology is available. Assume $(\chi_2+\chi_3)(1-d) > \chi_2$ and $d > \hat{p}_1 - \hat{p}_2$ where $\hat{p}_1 = \frac{\chi_1}{\chi_1+\chi_2}$ and $\hat{p}_2 = \frac{\chi_3}{\chi_2+\chi_3}$. In this case, there exists a unique mixed-strategy equilibrium. It is easy to see that in any mixed strategy equilibrium, the joint support of $p_1^1 \cup p_1^2$ is continuous and the joint support of $p_2^2 \cup p_2^3$ is continuous. Neither party's support of pricing strategy can have mass point below the reservation price 1. The joint support is of the from $[\underline{p} + d, 1]$ for D_1 and $[\underline{p}, 1 - d] \cup \{1\}$ for D_2.

The boundary conditions are: $G_1^1(\bar{p}) = 0$, $G_1^1(1) = 1 - \frac{\chi_3 d}{\chi_2(1-d)}$, $G_1^2(\underline{p} + d) = 0$, and $G_1^2(\bar{p}) = 1$. $G_2^3(\bar{p}) = 0$, $G_2^3(1 - d) = 1 - \frac{(\chi_1+\chi_2)(\hat{p}_2+d)}{\chi_2} + \frac{\chi_1}{\chi_2}$ or $(1 - q)$, $G_2^2(\underline{p}) = 0$, and $G_2^2(\bar{p}) = 1$.

By construction, any price in the support yields the same expected revenue. So we have the indifference conditions: $R_1^1(\bar{p}) = R_1^1(1)$, $R_1^2(\underline{p} + d) = R_1^2(\bar{p})$, and $R_1^1(\bar{p}) = R_1^2(\bar{p})$; $R_2^3(\bar{p}) = R_2^3(1 - d)$, $R_2^2(\underline{p}) = R_2^2(\bar{p})$, and $R_2^3(\bar{p}) = R_2^2(\bar{p})$.

3 Application: Upstream IT infrastructure provider and downstream corporate customers

In this section, we consider a model with slightly different focus. We maintain the two-step composition structure of the problem, where the outer problem features multi-dimensional screening and the inner problem features ex post interactions. Instead of focusing on specific form of downstream competition, we will use a conflict matrix to capture generic form of downstream interactions and shift our attention to the outer stage, where we generalize the screening model introduced in the previous section.

3.1 Background

The field of information transaction frequently involves an upstream information service center and multiple downstream firms which provide services in multiple fields and may be involved in direct or indirect competition among each other. A prominent example is AWS, which is an IT infrastructure platform that "provides a broad set of infrastructure services, such as computing power, storage options, networking and databases, delivered as a utility: on-demand, available in seconds, with pay-as-you-go pricing." According to its website introduction, AWS is a "platform for virtually every use case", including data warehousing, deployment tools, directories to content delivery, the purpose of which is to allow "enterprises, start-ups, SMBs and customers in the public sector to access the building blocks they need to respond quickly to changing business requirements".

To illustrate, consider the cloud computing service offered by AWS. The IT infrastructure provides virtualization service, i.e., the creation of a virtual environment which operates like a high-performing computer and supervises program scheduling through "cloud fabric". It also provides network communication protocol inter- and intra- data centers.

To our knowledge, there is little prior work systematically studying the customer base of Amazon Web Service. According to the website's introduction, AWS serves customers "of all types and sizes" including enterprises, startups and public sectors. Menu of service provided includes big data, data center solutions, financial services, healthcare

Figure 9: AWS Customer Stories Website

and life sciences, internet of things (IoT), machine learning and artificial intelligence, and web and mobile apps.

(a) Customer Success by Solution Area

(b) Customers by types and sizes

Figure 8: AWS Customer Stories site

To get a better understanding of the downstream customers of AWS, we scrape the website of AWS customer stories to identify the latent structures of its customer bases. (The code and procedures involved in generating these figures can be found in the appendix.)

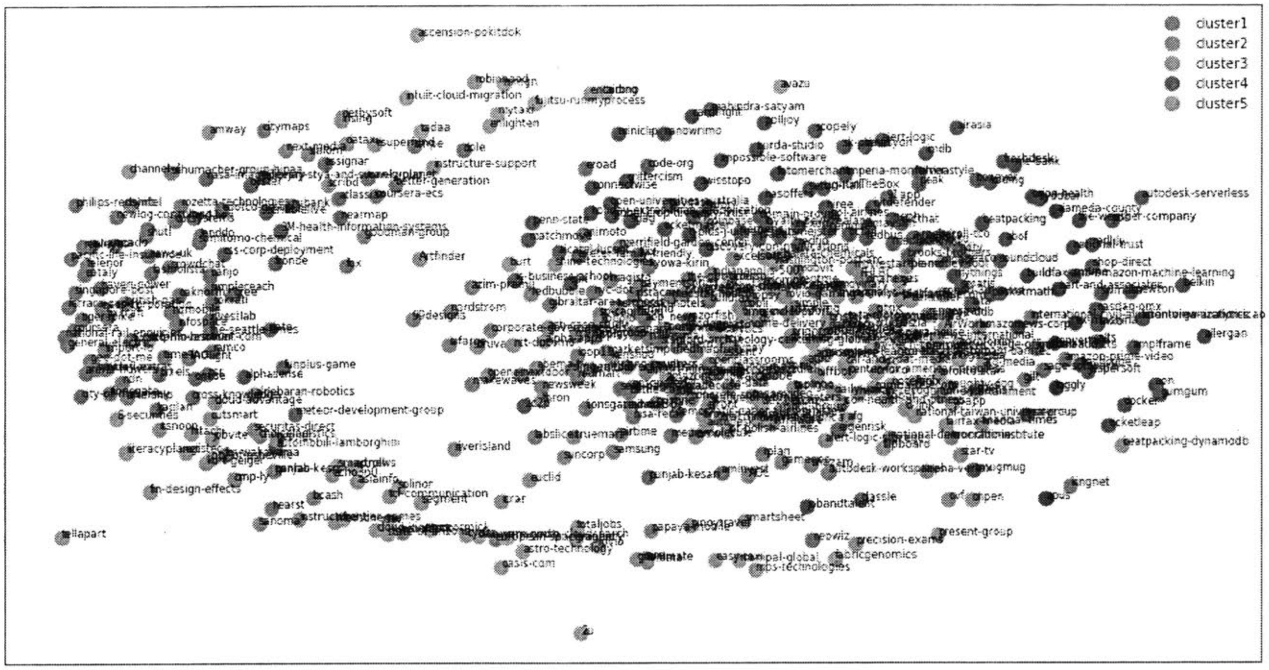


Figure 10: AWS Customer Stories: Clustering result, 500 companies sample

We further conduct a hierarchical clustering on the corpus using Ward clustering plotting a Ward dendrogram topic modeling using Latent Dirichlet Allocation (LDA)

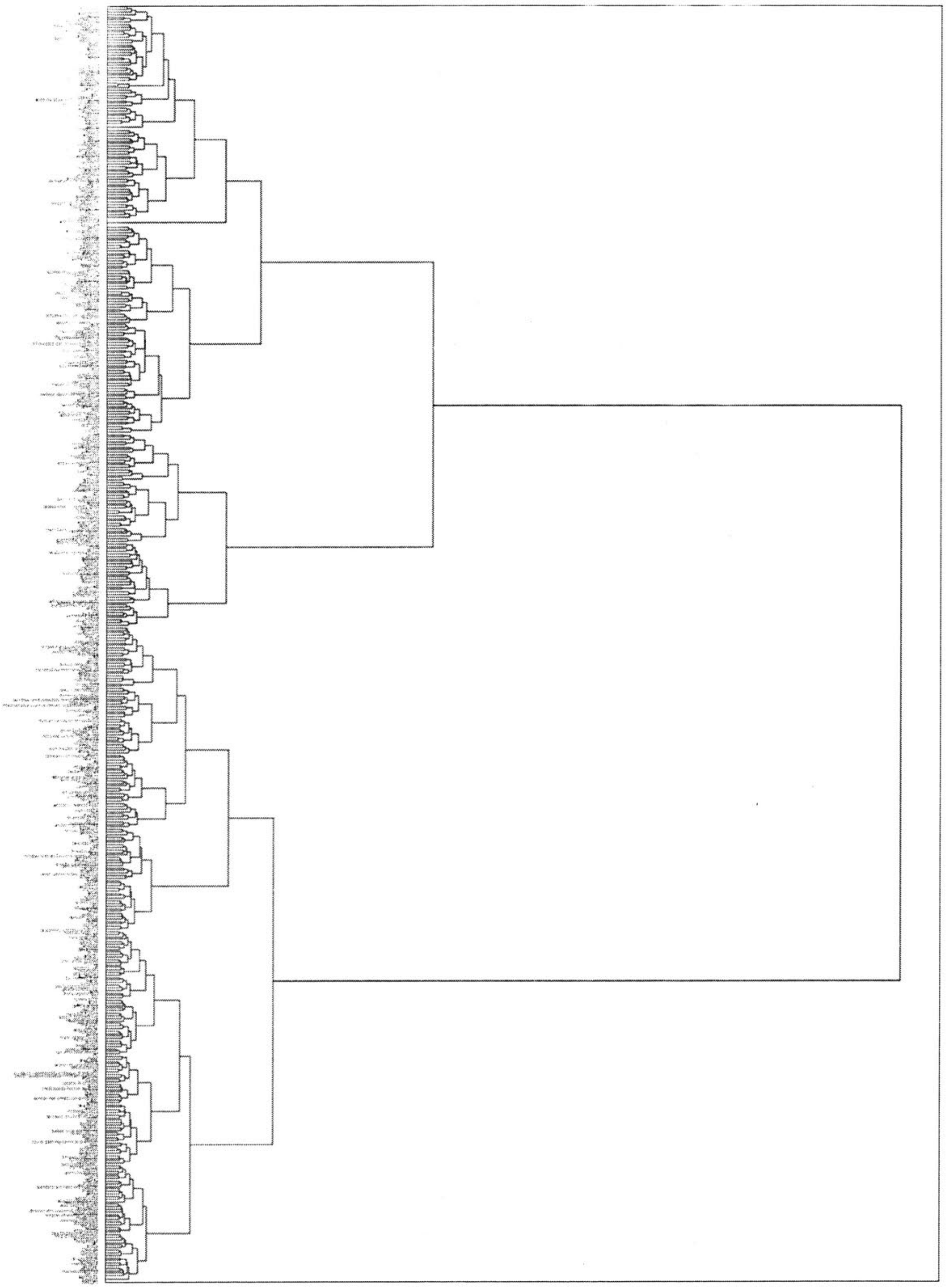


Figure 11: AWS Customer Stories: Hierarchical Ward dendrogram result, 500 companies sample

In the remaining part of this section, we will formalize the business model of AWS. We will consider AWS's corporate customers as downstream parties in the baseline models. These firms have types which could reflect firms' production technology, which we view as determined by the countless engineering decisions made in the past, all of which are proprietary information privately known to the firm. In addition, the private information could also reflect a firm's preferences, composition of customer base, etc. Endowed with such private types, firms choose from a menu of service products. A performance menu consists of a set of VMs with different CPU, memory and disk configurations. In the case of AWS, the firm offers about 20 different VM configurations, ranging from low performance "micro" to high performance "extra large." Firms' valuations of these products differ as a result of varying sensitivity to job completion time, completion valuation, workload requirement, etc.

3.2 Setup

In addition to the interaction of I_1 and I_2, another departure from standard screening model is the multi-dimensional nature of the information product, which we subsumed in the above example for the sake of simplicity. To simplify, we first consider a monopolist's problem with four possible types of downstream parties.

Suppose there are two states $\Omega = \{\omega_1, \omega_2\}$. Information provided by the upstream party add incremental value to each state. Specifically, assume the downstream party's additional valuation in state ω_i obtained from full information can take on two values $v_i \in \{v_i^L, v_i^H\}$. Write $\Delta_i = v_i^H - v_i^L > 0$. So there are 4 possible combination of values in the presence of perfect information in state (ω_1, ω_2): (v_L, v_L), (v_L, v_H), (v_H, v_L) and (v_H, v_H). The seller's prior distribution over the buyer's type is $Pr(v_i, v_j) = \beta_{ij}$ where $i, j \in \{L, H\}$. We will make the simplifying assumption that $\beta_{LH} = \beta_{HL} = \beta$.

The seller's problem is to design service menu $I = \{(x_1^{ij}, x_2^{ij})\}_{ij}$, where $x_{ij} \in [0, 1]$ parameterized the informativeness of the information device in stage ij ($x_{ij} = 0$ means no additional information provided, $x_{ij} = 1$ means perfect information) and the corresponding price T_{ij} for all types ij. The seller's expected profit is

$$\sum_{ij} \beta_{ij} T_{ij}$$

Using a slightly different notation, we can represent the information product provided to type ij as

$$I_{ij} = \begin{bmatrix} x_1^{ij} & 1 - x_1^{ij} \\ 1 - x_2^{ij} & x_2^{ij} \end{bmatrix}$$

where the rows represent (ω_1, ω_2) and the columns represent (s_1, s_2). The valuation is given by the weighted sum of the diagonal elements, i.e., $v_1^i \cdot x_1^{ij} + v_2^j \cdot x_2^{ij}$.

The buyer's net surplus from the information product is

$$R_{ij} \equiv R(I_{ij}) = x_1^{ij} v_1^i + x_2^{ij} v_2^j - T_{ij}$$

We want to maximize the seller's surplus subject to IR and IC constraints.

As in the one-dimensional problem, the IR constraint that is binding is $R_{LL} \geq 0$. The IC constraints are

$$R_{ij} \geq R_{kl} + x_1^{kl}(v_1^i - v_1^k) + x_2^{kl}(v_2^j - v_2^l) \; \forall ij, \; kl$$

which imply the monotonicity conditions:

$$x_1^{HH} \geq x_1^{LH}, \; x_1^{HL} \geq x_1^{LL}, \; x_2^{HH} \geq x_2^{HL}, \; x_2^{LH} \geq x_2^{LL}$$

We will first consider the relaxed problem where only the downward incentive constraints are relevant. After substituting in the IR constraint $R_{LL} = 0$ we have

$$R_{HL} = x_1^{LL}\Delta_1$$

$$R_{LH} = x_2^{LL}\Delta_2$$

$$R_{HH} = x_1^{LL}\Delta_1 + x_2^{LL}\Delta_2 + max\{(x_1^{LH} - x_1^{LL})\Delta_1, \; (x_2^{HL} - x_2^{LL})\Delta_2, \; 0\}$$

Consider the case where $x_1^{LL} = x_1^{LH}$ and $x_2^{LL} = x_2^{HL}$, so the relevant constraint for type (v_H, v_H) is

$$R_{HH} = x_1^{LL}\Delta_1 + x_2^{LL}\Delta_2$$

Substituting into the objective, we have

$$\max_{\{x_{ij}\}} \sum_{ij} \beta_{ij} \left[x_1^{ij} v_1^i + x_2^{ij} v_2^j\right] - (\beta_{HH} + \beta)\left[x_1^{LL}\Delta_1 + x_2^{LL}\Delta_2\right]$$

s.t monotonicity constraints.

If on the other hand, the relevant downstream constraint for R_{HH} is

$$R_{HH} = x_1^{LL}\Delta_1 + x_2^{LL}\Delta_2 + (x_2^{HL} - x_2^{LL})\Delta_2$$

or

$$R_{HH} = x_1^{LL}\Delta_1 + x_2^{LL}\Delta_2 + (x_1^{LH} - x_1^{LL})\Delta_1$$

Substituting into the objective, we have

$$\max_{\{x_{ij}\}} \sum_{ij} \beta_{ij} \left[x_1^{ij} v_1^i + x_2^{ij} v_2^j\right] - (\beta_{HH} + \beta) \left[x_1^{LL} \Delta_1 + x_2^{LL} \Delta_2\right]$$

$$-\beta_{HH} max \left\{(x_2^{HL} - x_2^{LL})\Delta_2,\ (x_1^{LH} - x_1^{LL})\Delta_1\right\}$$

s.t monotonicity constraints.

3.3 Baseline: Two-dimensional information with independent downstream actions

Continuing our discussion in the previous section, let's first consider the symmetric case where $\Delta \equiv \Delta_1 = \Delta_2$, $v^i = v_1^i = v_2^i$, $x^{LH} \equiv x_1^{LH} = x_2^{HL}$, $x^{HH} = x_k^{HH}$, $x^{LL} = x_k^{LL}$, etc.

The incentive constraint becomes

$$R_{HH} = 2x^{LL}\Delta + max\{(x^{LH} - x^{LL})\Delta, 0\}$$

The relaxed problem is

$$\max_{\{x_{ij}\}} {}_{ij} 2\left[(\beta_{HH} x^{HH} + \beta x^{HL}) v^H + (\beta x^{LH} + \beta_{LL} x^{LL}) v^L\right]$$

$$-\left[2\beta x^{LL} + \beta_{HH}(2x^{LL} + max\{(x^{LH} - x^{LL}), 0\})\right]\Delta$$

s.t.

$$x^{HH} \geq x^{LH}$$

$$x^{HL} \geq x^{LL}$$

It's easy to see that $x^{HH} = x^{HL} = 1$.

The coefficient of x^{LH} is $2\beta v^L - \beta_{HH}\Delta$. So if

$$\frac{\Delta}{v^L} \geq 2\frac{\beta}{\beta_{HH}}\ (i)$$

then $x^{LH} = x^{LL}$. Otherwise $x^{LH} = x^{HH} = 1$.

The coefficient of x^{LL} is $2\beta_{LL}v^L - (2\beta + \beta_{HH})\Delta$. The seller minimizes $x^{LL} = x^{LH}$ if

$$\frac{\Delta}{v^L} \geq 2\frac{\beta_{LL}}{1 - \beta_{LL}}\ (ii)$$

Case 1: Both (i) and (ii) hold. Set $x^{LL} = x^{LH} = 0$.

Case 2: (i) fails but (ii) holds. Set $x^{LH} = x^{HH} = 1$ and $x^{LL} = 0$.

Case 2A: (i) holds but (ii)fails. Set $x^{LL} = x^{LH} < 1$.

Case 3: (i) fails and (ii) fails, and $\frac{\beta}{\beta_{HH}} \geq \frac{\beta_{LL}}{1-\beta_{LL}}$. Set $x^{LL} = x^{LH} = x^{HH} = 1$.

Case 4: (i) fails and (ii) fails, and $\frac{\beta}{\beta_{HH}} < \frac{\beta_{LL}}{1-\beta_{LL}}$, and $\frac{\beta_L}{\beta} \geq \frac{\Delta}{v_L}$. Set $x^{LL} = x^{LH} = x^{HH} = 1$.

Case 5: (i) fails and (ii) fails, and $\frac{\beta}{\beta_{HH}} < \frac{\beta_{LL}}{1-\beta_{LL}}$, and $\frac{\beta_L}{\beta} < \frac{\Delta}{v_L}$. Set $x^{LL} = x^{LH} = 0$.

The above simple example illustrates a key feature of multidimensional type model: the structure of the optimal menu depends on (1) the retailer's valuations across states; and (2) the size of the information rent necessary to induce truth telling. If the size of information rent is too high, the seller may exclude the low type (v_L, v_L). If the retailer's valuations not too positively correlated, that is when the term

$$\frac{\beta_{LL}\beta_{HH} - \beta^2}{\beta_L \beta_H}$$

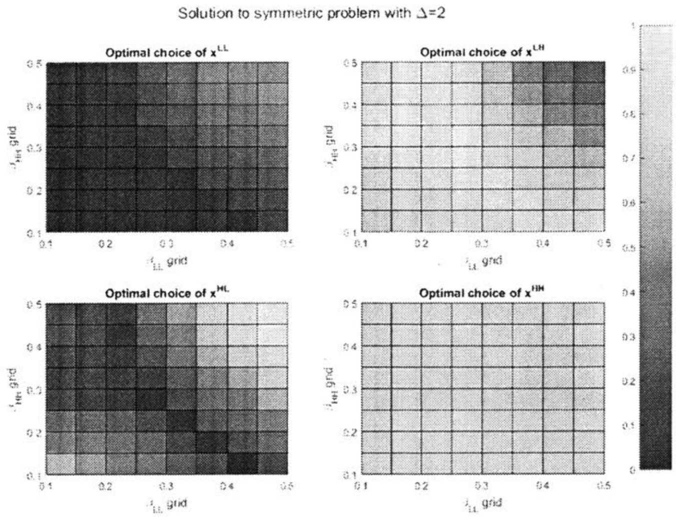
is small relative to $\frac{\beta_H}{1-\beta}$, then we will have $x^{LH} = x^{HL} = x^{HH} = 1$ and exclusion $x^{LL} = 0$.

The intuition is related to the bundling literature, c.f., Armstrong and Rochet (1999)'s intuition that bundling can be used to effectively screen buyers' of multidimensional types, whenever the correlation coefficient of valuations across goods are sufficiently small.

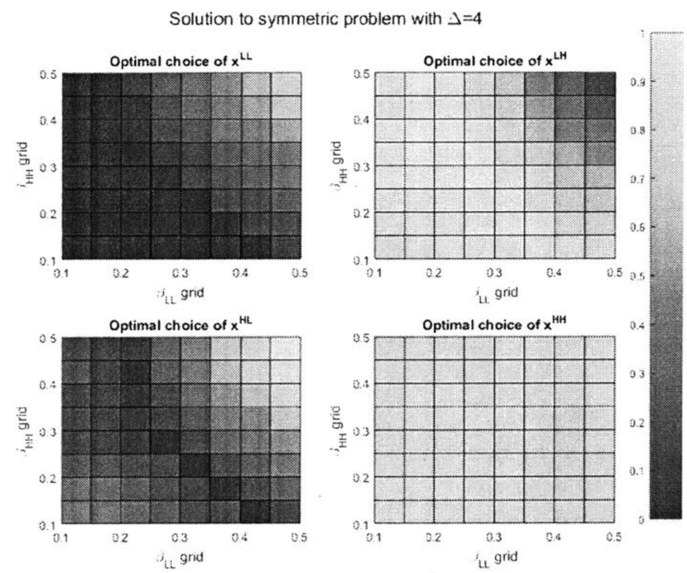
Another key aspect of the multidimensional model is that incentive constraints besides downward ones may be binding. If this is the case, then solution to the above relaxed problem does not coincide with the solution to the underlying problem. This problem does not arise in the symmetric model, since, by construction $\Delta \equiv \Delta_1 = \Delta_2$. But in the asymmetric version of the same model, if Δ_1 is sufficiently larger than Δ_2 and the retailer's valuation are negatively correlated, then the optimal menu may feature "distortion at the top", i.e., (v_H, v_H) may be allocated $x_{HH} < 1$.

Note that the reason for "distortion at the top" is different than the one mentioned in the previous section. There, "distortion" arises from negative externalities in downstream competition: higher precision of one's information technology may intensity price competition and eat up downstream profits, which reduces the retailers' willingness-to-pay. Here, distortion arises from incentive reasons: multi-dimensionality puts more restrictions on the set of incentive constraints. Unlike the one-dimensional model where types can be ordered according to their relative valuation for the product, here the structure of "neighboring types" is more complicated.

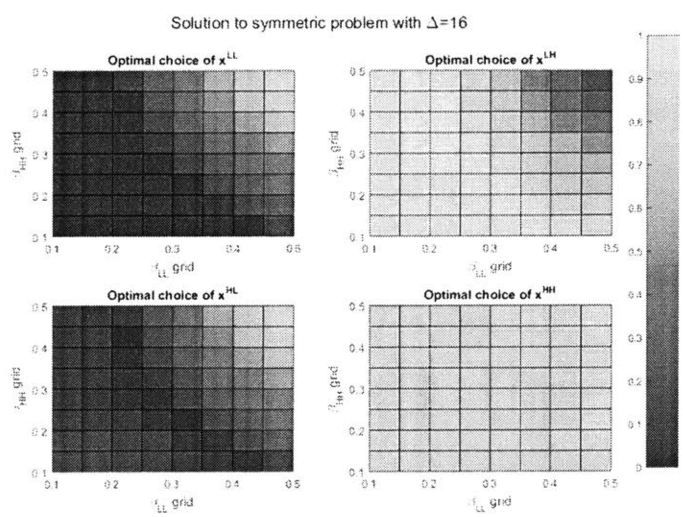
The solutions to the problem are illustrated in the figure below for various combinations of (β_{LL}, β_{HH}) and various values of Δ.



(a) $\Delta = 2$



(b) $\Delta = 4$



(c) $\Delta = 16$

Figure 12: Solution to two-dimensional information, independent action problem

3.4 General Case: Two-dimensional information with interdependent downstream actions

Consider adding an ex post stage to the model in the previous section.

As in the previous section, suppose there are two states $\Omega = \{\omega_1, \omega_2\}$. Valuation of in state ω_i can take on two values $v_i \in \{v_i^L, v_i^H\}$ with $\Delta_i = v_i^H - v_i^L > 0$ denote the incremental value of state i. The seller's prior distribution over the buyer's type is $Pr(v_i, v_j) = \beta_{ij}$ where $i, j \in \{L, H\}$. We will restrict our attention to symmetric distributions where $\beta_{LH} = \beta_{HL} = \beta$.

The seller's problem is to design service menu $I = (x_1^{ij}, x_2^{ij})_{ij}$ and the corresponding price T_{ij}. The seller's expected profit is

$$\sum_{ij} \beta_{ij} T_{ij}$$

Unlike the previous example where the buyer's net surplus is

$$R_{ij} = x_1^{ij} v_1^i + x_2^{ij} v_2^j - T_{ij}$$

we assume there's an additional stage where a random state occurs (each with probability $\frac{1}{2}$) and two types of players are paired randomly. The value of the state is only realized if the party is the sole owner of the product (the snob effect).

Formally,

$$R_{ij} = \frac{1}{2} \cdot \sum_{(k,m) \neq (i,j)} \beta_{km} I(v_1^k \neq v_1^i) \cdot x_1^{ij} v_1^i + \frac{1}{2} \cdot \sum_{(k,m) \neq (i,j)} \beta_{km} I(v_2^k \neq v_2^i) \cdot x_2^{ij} v_2^j - T_{ij}$$

Or

$$R_{LL} = \frac{1}{2} \cdot (\beta + \beta_{HH}) \cdot x_1^{LL} v_1^L + \frac{1}{2} \cdot (\beta + \beta_{HH}) \cdot x_2^{LL} v_2^L - T_{LL}$$

$$R_{LH} = \frac{1}{2} \cdot (\beta + \beta_{HH}) \cdot x_1^{LH} v_1^L + \frac{1}{2} \cdot (\beta_{LL} + \beta) \cdot x_2^{LH} v_2^H - T_{LH}$$

$$R_{HL} = \frac{1}{2} \cdot (\beta_{LL} + \beta) \cdot x_1^{HL} v_1^H + \frac{1}{2} \cdot (\beta + \beta_{HH}) \cdot x_2^{HL} v_2^L - T_{HL}$$

$$R_{HH} = \frac{1}{2} \cdot (\beta_{LL} + \beta) \cdot x_1^{HH} v_1^H + \frac{1}{2} \cdot (\beta_{LL} + \beta) \cdot x_2^{HH} v_2^H - T_{HH}$$

We want to maximize the seller's surplus subject to IR and IC constraints.

Rewrite

$$\tilde{v}_1^L \equiv \frac{1}{2} \cdot (\beta + \beta_{HH}) v_1^L$$

$$\tilde{v}_2^L \equiv \frac{1}{2} \cdot (\beta + \beta_{HH}) v_2^L$$

$$\tilde{v}_1^H \equiv \frac{1}{2}(\beta + \beta_{LL}) v_1^H$$

$$\tilde{v}_2^H \equiv \frac{1}{2}(\beta + \beta_{LL}) v_2^H$$

If $\beta_{HH} \leq \beta_{LL}$, then the relative ordering of $\tilde{v}_i^L$ and $\tilde{v}_i^H$ is preserved. Define

$$\tilde{\Delta}_1 = \tilde{v}_1^H - \tilde{v}_1^L = \frac{1}{2}(\beta + \beta_{LL}) v_1^H - \frac{1}{2} \cdot (\beta + \beta_{HH}) v_1^L$$

and

$$\tilde{\Delta}_2 = \tilde{v}_2^H - \tilde{v}_2^L = \frac{1}{2}(\beta + \beta_{LL}) v_2^H - \frac{1}{2} \cdot (\beta + \beta_{HH}) v_2^L$$

As in the one-dimensional problem, the IR constraint that is binding is $R_{LL} \geq 0$. The IC constraints are

$$R_{ij} \geq R_{kl} + x_1^{kl}(\tilde{v}_1^i - \tilde{v}_1^k) + x_2^{kl}(\tilde{v}_2^j - \tilde{v}_2^l) \ \forall ij,\ kl$$

which imply the monotonicity conditions:

$$x_1^{HH} \geq x_1^{LH},\ x_1^{HL} \geq x_1^{LL},\ x_2^{HH} \geq x_2^{HL},\ x_2^{LH} \geq x_2^{LL}$$

The problem is essentially the same after renaming the variable. The relaxed problem where only the downward incentive constraints are relevant have the following constraints (after substituting in the IR constraint $R_{LL} = 0$):

$$R_{HL} = x_1^{LL} \tilde{\Delta}_1$$

$$R_{LH} = x_2^{LL} \tilde{\Delta}_2$$

$$R_{HH} = x_1^{LL} \tilde{\Delta}_1 + x_2^{LL} \tilde{\Delta}_2 + max\{(x_1^{LH} - x_1^{LL})\tilde{\Delta}_1,\ (x_2^{HL} - x_2^{LL})\tilde{\Delta}_2,\ 0\}$$

First, for the case where $x_1^{LL} = x_1^{LH}$ and $x_2^{LL} = x_2^{HL}$ and binding constraint for type (v_H, v_H) is

$$R_{HH} = x_1^{LL} \tilde{\Delta}_1 + x_2^{LL} \tilde{\Delta}_2$$

we obtain after substitution

$$\max_{\{x_{ij}\}} \sum_{ij} \beta_{ij} \left[x_1^{ij}\tilde{v}_1^i + x_2^{ij}\tilde{v}_2^j\right] - (\beta_{HH} + \beta)\left[x_1^{LL}\tilde{\Delta}_1 + x_2^{LL}\tilde{\Delta}_2\right]$$

s.t monotonicity constraints.

For the case where the relevant downstream constraint for R_{HH} is

$$R_{HH} = x_1^{LL}\tilde{\Delta}_1 + x_2^{LL}\tilde{\Delta}_2 + (x_2^{HL} - x_2^{LL})\tilde{\Delta}_2 = x_1^{LL}\tilde{\Delta}_1 + x_2^{HL}\tilde{\Delta}_2$$

or

$$R_{HH} = x_1^{LL}\tilde{\Delta}_1 + x_2^{LL}\tilde{\Delta}_2 + (x_1^{LH} - x_1^{LL})\tilde{\Delta}_1 = x_1^{LH}\tilde{\Delta}_1 + x_2^{LL}\tilde{\Delta}_2$$

we obtain after substitution

$$\max_{\{x_{ij}\}} \sum_{ij} \beta_{ij} \left[x_1^{ij}\tilde{v}_1^i + x_2^{ij}\tilde{v}_2^j\right] - (\beta_{HH} + \beta)\left[x_1^{LL}\tilde{\Delta}_1 + x_2^{LL}\tilde{\Delta}_2\right]$$
$$-\beta_{HH} \max\left\{(x_2^{HL} - x_2^{LL})\tilde{\Delta}_2,\ (x_1^{LH} - x_1^{LL})\tilde{\Delta}_1\right\}$$

s.t monotonicity constraints.

For the symmetric case where $\tilde{\Delta} \equiv \tilde{\Delta}_1 = \tilde{\Delta}_2$, $\tilde{v}^i = \tilde{v}_1^i = \tilde{v}_2^i$, $x^{LH} \equiv x_1^{LH} = x_2^{HL}$, $x^{HH} = x_k^{HH}$, $x^{LL} = x_k^{LL}$, etc. The incentive constraint becomes

$$R_{HH} = 2x^{LL}\tilde{\Delta} + \max\{(x^{LH} - x^{LL})\tilde{\Delta}, 0\}$$

The relaxed problem is

$$\max_{\{x_{ij}\}} 2\left[(\beta_{HH}x^{HH} + \beta x^{HL})\tilde{v}^H + (\beta x^{LH} + \beta_{LL}x^{LL})\tilde{v}^L\right]$$
$$-\left[2\beta x^{LL} + \beta_{HH}(2x^{LL} + \max\{(x^{LH} - x^{LL}), 0\})\right]\tilde{\Delta}$$

s.t.

$$x^{HH} \geq x^{LH}$$

$$x^{HL} \geq x^{LL}$$

As with the previous case, we have $x^{HH} = x^{HL} = 1$. The coefficient of x^{LH} is $2\beta v^L - \beta_{HH}\tilde{\Delta}$. $x^{LH} = x^{LL}$ if

$$\frac{\tilde{\Delta}}{\tilde{v}^L} \geq 2\frac{\beta}{\beta_{HH}} \ (i')$$

Otherwise $x^{LH} = x^{HH} = 1$.

The coefficient of x^{LL} is $2\beta_{LL}v^L - (2\beta + \beta_{HH})\Delta$. The seller minimizes $x^{LL} = x^{LH}$ if

$$\frac{\tilde{\Delta}}{\tilde{v}^L} \geq 2\frac{\beta_{LL}}{1-\beta_{LL}} \; (ii')$$

Note that (i') is equivalent to

$$\frac{(\beta + \beta_{HH})v^H - (\beta + \beta_{HH})v^L + (\beta_{LL} - \beta_{HH})v^H}{(\beta + \beta_{HH})v^L} \geq 2\frac{\beta}{\beta_{HH}}$$

or equivalently

$$\frac{\Delta}{v^L} + \frac{(\beta_{LL} - \beta_{HH})v^H}{(\beta + \beta_{HH})v^L} \geq 2\frac{\beta}{\beta_{HH}}$$

Since $\beta_{HH} \leq \beta_{LL}$,

$$\frac{\Delta}{v^L} + \frac{(\beta_{LL} - \beta_{HH})v^H}{(\beta + \beta_{HH})v^L} \geq \frac{\Delta}{v^L}$$

(i') is more easily satisfied than (i)

Similarly, we can show that (ii') is more easily satisfied than (ii).

Depending on the sign of the inequalities in (i') and (i''), the solution to the problem can be divided into five cases, as in the previous example.

Case 1': Both (i') and (ii') hold. Set $x^{LL} = x^{LH} = 0$.

Case 2': (i') fails but (ii'') holds. Set $x^{LH} = x^{HH} = 1$ and $x^{LL} = 0$.

Case 2A': (i) fails but (ii) holds. Set $x^{LH} = x^{HH} = 1$ and $x^{LL} = 0$.

Case 3': (i') fails and (ii'') fails, and $\frac{\beta}{\beta_{HH}} \geq \frac{\beta_{LL}}{1-\beta_{LL}}$. Set $x^{LL} = x^{LH} = x^{HH} = 1$.

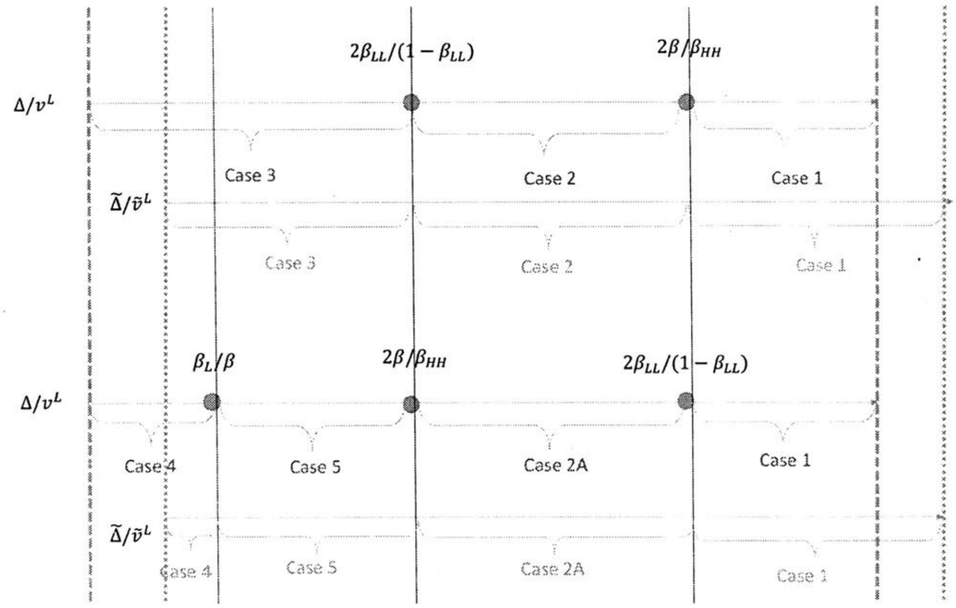
Case 4': (i') fails and (ii'') fails, and $\frac{\beta}{\beta_{HH}} < \frac{\beta_{LL}}{1-\beta_{LL}}$, and $\frac{\beta_L}{\beta} \geq \frac{\tilde{\Delta}}{\tilde{v_L}}$. Set $x^{LL} = x^{LH} = x^{HH} = 1$.

Case 5': (i') fails and (ii'') fails, and $\frac{\beta}{\beta_{HH}} < \frac{\beta_{LL}}{1-\beta_{LL}}$, and $\frac{\beta_L}{\beta} < \frac{\tilde{\Delta}}{\tilde{v_L}}$. Set $x^{LL} = x^{LH} = 0$.

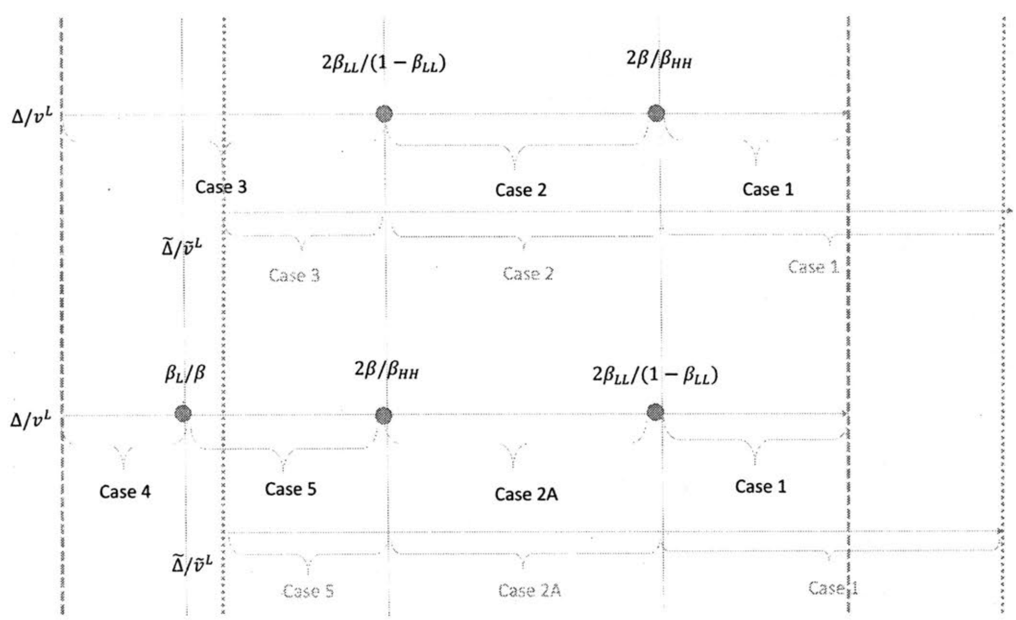
We will focus on the change of solution region of Case 1', 2', 2' and 5'.

Given the same prior configurations (β_{ij}, v_k), the area for Case 1' is greater than that for Case 1. The comparison of Case 2 v.s 2', Case 5 v.s 5' is ambiguous.

In general, we can show that the exclusion region (where at least one type is served $x = 0$) is enlarged. See illustrations below.



(a) Small shift

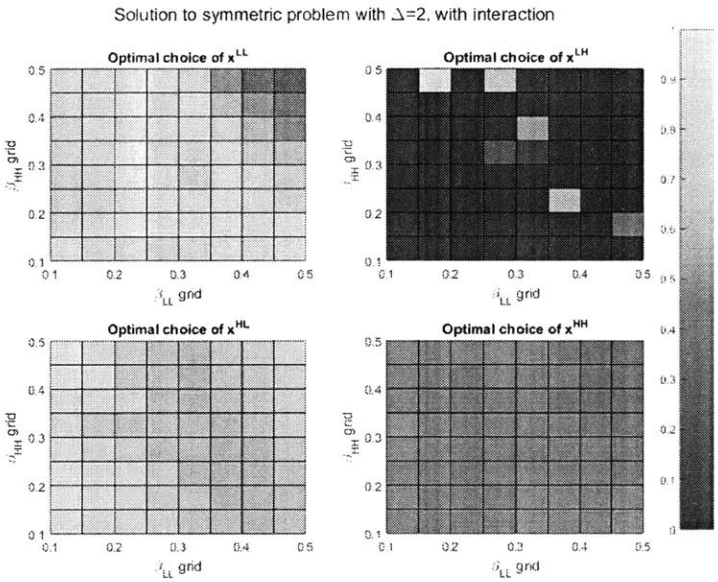


(b) Large shift

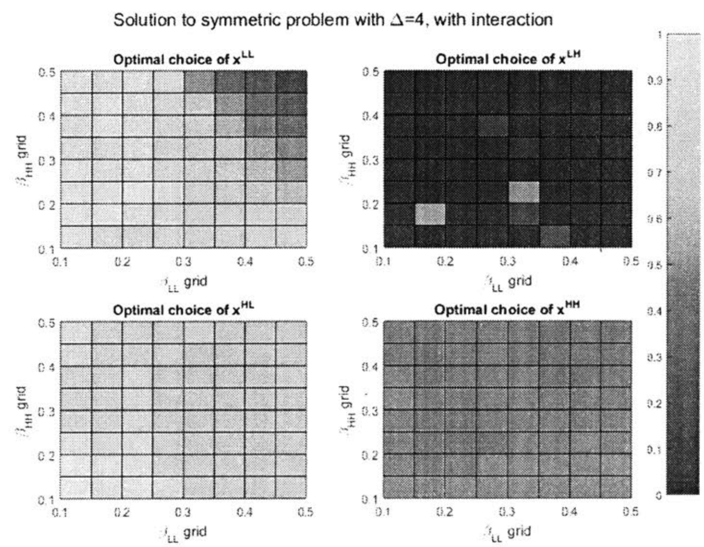
Figure 13: Illustration of endogenous action

A few remarks are in order. First, the addition of the ex post stage changes the exclusion region. Because of the symmetry assumption and the linear structure of the model, endogenous ex post interaction changes the prediction of the model through the term $\frac{\tilde{\Delta}}{\tilde{v}^L}$, which measures the relative increment of the high v.s low valuation in a given state. The difference between the ex-post augmented $\frac{\tilde{\Delta}}{\tilde{v}^L}$ and the ex ante $\frac{\Delta}{v^L}$ is $\frac{(\beta_{LL}-\beta_{HH})v^H}{(\beta+\beta_{HH})v^L}$, which reflects the probability of downstream "clashes". Note that since $\tilde{v}_1^L \equiv \frac{1}{2}\cdot(\beta+\beta_{HH})v_1^L$ and $\tilde{v}_2^L \equiv \frac{1}{2}\cdot(\beta+\beta_{HH})v_2^L$, higher β_{HH} improves the value of v_i^L. On the other hand, $\tilde{v}_1^H \equiv \frac{1}{2}(\beta+\beta_{LL})v_1^H$, $\tilde{v}_2^H \equiv \frac{1}{2}(\beta+\beta_{LL})v_2^H$, so higher β_{LL} improves the value of v_i^H. This is intuitive: we assume players are paired randomly and the value of the state is only realized if the party is the sole owner of the product (the snob effect), so a higher $\beta_{HH}(\beta_{LL})$ decreases the chances of clashes with v_i^L (v_i^H). If the ex post interaction effect is moderate (see sub-figure a: small shift), the structure of the 5 cases are maintained but the relative magnitudes are distorted. If the ex post interaction effect is large (see sub-figure b: large shift), the structure of the cases may change. In our example, case 4 (the full service case) is completely eliminated. In our example, because of the snob effect, the value of information is dampened and the screening is made harder, which enlarges the exclusion region (case 1). Although the example is based on a particular form of downstream interaction, the intuition extends to more general settings where competition among downstream parties eat away the value of the upstream information provision.

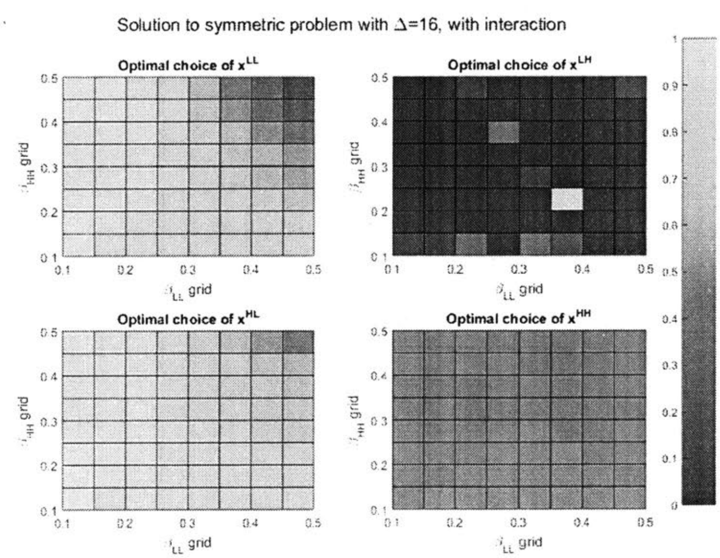
The figure below illustrates the change to the solution, where we add additional snob effect constraint $R_{HH} = R_{LH}$ on the second dimension of the problem.



(a) $\Delta = 2$



(b) $\Delta = 4$



(c) $\Delta = 16$

4 Conclusion

In this paper, we consider a model of an economic transaction between an upstream monopolist and several downstream oligopolists. The upstream monopolist supplies a menu of multi-dimensional intermediate goods from which the downstream oligopolists select. The oligopolists then use the previously purchased intermediate goods to produce the final products and compete with each other. The key feature of the model is that the intermediate good is multi-dimensional, whose value is affected by the extent of ex post competition among the downstream players. Such scenarios arise naturally in the real world, especially in the emerging sectors involving IT products in its production. For example, the downstream parties may be E-commerce retailers who compete over a heterogeneous customer base. Each party may have some prior assessment over the distribution of customer types, but would benefit from incremental knowledge on customer information. The upstream party, in this scenario, is an information vendor, who has access to technology required to develop a targeting device. Since information is valuable, to extract surplus the upstream party would like to improve the quality of information. Such motive is counterbalanced by the incentive to manage competition. In the extreme case where the downstream parties engage in pure Bertrand competition, perfect knowledge about customer types reduces the size of downstream profit, driving price to the marginal cost. In such case, it might be desirable for the upstream party to deliberately hold back and coarsify information on some segments of the market. Another key aspect of the model is that the downstream parties are ex-ante different in their preferences, and such preferences are private information. A menu designed by the upstream party must induce self-selection, taking into account of the possibility that some parties might misrepresent themselves to land a better deal.

In such market transactions, information which is an crucial aspect of production is imperfect, the market structures are distinctively not perfectly competitive (as opposed to the models studied by Arrow and Debreu). Firms directly involved in competition strive to enhance their market power over each other, giving rise to distortions deviating from the first best. Upstream and downstream interactions are partially but not completely managed through the menu design of intermediate goods.

Our model also differs from the canonical model of monopolistic screening in several important ways. We do not take a customer as the basic unit of analysis. In our framework, a customer represents a bundle of information, each dimension of which may be relevant to some but not all firms. Customer preferences and types are only partially revealed through behaviors observed by the firms. The remaining part cannot be made available to the firms free of charge. Another key aspect captured by the model is the indirect externalities conferred in the market for information. Specifically, the value of customer information to a given firm is no longer determined solely by the customer-firm pair's own characteristics. Instead, the value depends on the market competition structure among all downstream firms. [4] For example, we show in Section 2 a model where competition of customer information

[4]The intuition is related to the literature on contracting with externality. See Segal (1999) for the original exposition and Segal (2003) and Segal and Whinston (2003) for extensions.

has features similar to an arms race: having better information over the opponent allows one to better engage in better price discrimination, but it also increases the value to the opponent and induces more aggressive demand for information on the part of the opponents.

Our analysis yields several results which hold in general in this type of models. First, exclusion and specialization are generic. Second, the "distortion-at-the-top" result from standard screening literature no longer holds: the highest-value downstream party's menu is distorted downward (relative to the first-best) when such distortion softens the competition with its opponents. Third, bunching where different but similar types are offered the same bundle of information products arise in equilibrium. Fourth, information foods feature cross-externality, the sign and magnitude of such externality have implications on the optimal design of privacy protection regulations.

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6 Appendix

6.1 Derivation for Section 2.4

Case 1: $[IC1]$ and $[IR1]$ bind. We have

$$t_1 = \frac{\phi_1}{\phi_1 + \psi}(\phi_1 + \psi + \chi - \rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}\right\}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$$

$$t_2 = \frac{\phi_{12}}{\phi_{12} + \psi_2}(\phi_{12} + \psi_2 + \chi - \rho_2) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}\right\}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$$

$$\begin{aligned} t_1(I_1^L, I_2^L, I_1^S, I_2^S) &= \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(\phi_1(I_1^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\ &- max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}(I_2^S)}, \frac{\tilde{\phi}_2(I_2^L)}{\tilde{\phi}_2(I_2^L) + \tilde{\psi}(I_2^S)}\right\}(\tilde{\phi}_1 + \tilde{\psi}(I_2^S) + \chi - \tilde{\rho}(I_2^S)) \end{aligned}$$

$$\frac{\partial t_1}{\partial I_1^L} = \phi_1'(I_1^L) + \frac{\partial}{\partial I_1^L}\left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}(\chi - \rho(I_1^S, I_2^S)) > 0$$

$$\begin{aligned} \frac{\partial t_1}{\partial I_1^S} &= \frac{\partial}{\partial I_1^S}\left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}(\chi - \rho(I_1^S, I_2^S)) \\ &+ \left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}\frac{\partial}{\partial I_1^S}(\chi - \rho(I_1^S, I_2^S)) \\ &= \frac{\phi_1(I_1^L)\left[-\frac{\chi\gamma_1}{\gamma_1+\chi}(\frac{\gamma_2}{\gamma_2+\chi}(1 - I_2^S))\right]}{(\phi_1(I_1^L) + \psi(I_1^S, I_2^S))^2}(\chi - \rho(I_1^S, I_2^S)) \\ &+ \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(-\frac{\chi\gamma_1}{\gamma_1 + \chi}(\frac{\chi}{\gamma_2 + \chi}(1 - I_2^S) + I_2^S))) \\ &< 0 \end{aligned}$$

$$\begin{aligned} t_2(I_2^L, I_2^S) &= \frac{\phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}(\phi_{12}(I_2^L) + \psi_2(I_2^S) + \chi - \rho_2(I_2^S)) \\ &- max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}(I_2^S)}, \frac{\tilde{\phi}_2(I_2^L)}{\tilde{\phi}_2(I_2^L) + \tilde{\psi}(I_2^S)}\right\}(\tilde{\phi}_1 + \tilde{\psi}(I_2^S) + \chi - \rho(\tilde{I}_2^S)) \end{aligned}$$

$$\frac{\partial t_2}{\partial I_2^L} = \phi_{12}'(I_2^L) + \frac{\partial}{\partial I_2^L}\left\{\frac{\phi_{12}(I_2^L)}{\phi_{12}(I_2^L)+\psi_2(I_2^S)}\right\}(\chi - \rho_2(I_2^S))$$
$$+ \frac{\partial}{\partial I_2^L}\left\{max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}(I_2^S)}, \frac{\tilde{\phi}_2(I_2^L)}{\tilde{\phi}_2(I_2^L)+\tilde{\psi}(I_2^S)}\right\}\right\}(\tilde{\phi}_1 + \tilde{\psi}(I_2^S) + \chi - \rho(\tilde{I}_2^S)) > 0$$

$$\frac{\partial t_2}{\partial I_2^S} = \frac{\partial}{\partial I_2^S}\left\{\frac{\phi_{12}(I_2^L)}{\phi_{12}(I_2^L)+\psi_2(I_2^S)}\right\}(\chi - \rho_2(I_2^S))$$
$$+\left\{\frac{\phi_{12}(I_2^L)}{\phi_{12}(I_2^L)+\psi_2(I_2^S)}\right\}(-\rho_2'(I_2^S))$$
$$+ \frac{\partial}{\partial I_2^S}\left\{max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}(I_2^S)}, \frac{\tilde{\phi}_2(I_2^L)}{\tilde{\phi}_2(I_2^L)+\tilde{\psi}(I_2^S)}\right\}\right\}(\tilde{\phi}_1 + \tilde{\psi}(I_2^S) + \chi - \rho(\tilde{I}_2^S))$$
$$+\left\{max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}(I_2^S)}, \frac{\tilde{\phi}_2(I_2^L)}{\tilde{\phi}_2(I_2^L)+\tilde{\psi}(I_2^S)}\right\}\right\}(\tilde{\psi}'(I_2^S) - \tilde{\rho}'(I_2^S)) < 0$$

Hence, the solution will feature the largest possible values of I_1^L, I_2^L and the smallest possible I_1^S, I_2^S while maintaining $[IR2]$ and $[IC2]$.

It's easy to see $[IC2]$ is satisfied. Suppose $\gamma_1 > \gamma_2$, $[IR2]$ implies

$$\frac{\gamma_1}{\gamma_1 + \chi\cdot(\frac{\gamma_2}{\gamma_2+\chi})^2}(\gamma_1 - \gamma_2)$$
$$\leq -\frac{\gamma_1}{\gamma_1+\chi(\frac{\gamma_2}{\gamma_2+\chi})^2}\left(\frac{\gamma_1^2}{\gamma_1+\chi} + \chi\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi} - \chi\frac{\chi}{\gamma_1+\chi}\frac{\chi}{\gamma_2+\chi} - \gamma_2 - \chi(\frac{\gamma_2}{\gamma_2+\chi})^2 + \chi(\frac{\chi}{\gamma_2+\chi})^2\right)$$

A contradiction.

So the optimum solution must have $[IR2]$ binding as well.

Case 2: $[IC1]$ and $[IR2]$ bind.

In this case, we have

$$t_1 = \frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - \frac{\phi_{12}}{\phi_{12}+\psi_2}(\phi_{12}+\psi_2+\chi-\rho_2)$$
$$+\frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1+\hat{\psi}}(\hat{\phi}_2+\hat{\psi}+\chi-\hat{\rho})$$

$$t_2 = \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1+\hat{\psi}}(\hat{\phi}_2+\hat{\psi}+\chi-\hat{\rho})$$

$$
\begin{aligned}
t_1(I_1^L, I_1^S, I_2^L, t_2^S) = & \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(\phi_1(I_1^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
& - \frac{\phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}(\phi_{12}(I_2^L) + \psi_2(I_2^S) + \chi - \rho_2(I_2^S)) \\
& + \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(\phi_2(I_2^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
& - \frac{\hat{\phi}_1(I_1^L)}{\hat{\phi}_1(I_1^L) + \hat{\psi}(I_1^S)}(\hat{\phi}_2 + \hat{\psi}(I_1^S) + \chi - \hat{\rho}(I_1^S))
\end{aligned}
$$

$$
\begin{aligned}
t_2(I_1^L, I_1^S, I_2^L, t_2^S) = & \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(\phi_2(I_2^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
& - \frac{\hat{\phi}_1(I_1^L)}{\hat{\phi}_1(I_1^L) + \hat{\psi}(I_1^S)}(\hat{\phi}_2 + \hat{\psi}(I_1^S) + \chi - \hat{\rho}(I_1^S))
\end{aligned}
$$

Let

$$
\begin{aligned}
t(I_1^L, I_1^S, I_2^L, t_2^S) = & f \cdot t_1 + (1 - f) \cdot t_2 \\
= & \frac{f \cdot \phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(\phi_1(I_1^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
& - \frac{f \cdot \phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}(\phi_{12}(I_2^L) + \psi_2(I_2^S) + \chi - \rho_2(I_2^S)) \\
& + \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}(\phi_2(I_2^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
& - \frac{\hat{\phi}_1(I_1^L)}{\hat{\phi}_1(I_1^L) + \hat{\psi}(I_1^S)}(\hat{\phi}_2 + \hat{\psi}(I_1^S) + \chi - \hat{\rho}(I_1^S))
\end{aligned}
$$

We solve for the partial derivatives of t with respect to I_1^L, I_1^S, I_2^L and I_2^S. First note that

$$
\begin{aligned}
\frac{\partial t}{\partial I_1^L} = & f \cdot \phi_1'(I_1^L) + \frac{\partial}{\partial I_1^L}\left\{\frac{f \cdot \phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}(\chi - \rho(I_1^S, I_2^S)) \\
& + \frac{\partial}{\partial I_1^L}\left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}(\phi_2(I_2^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
& - \frac{\partial}{\partial I_1^L}\left\{\frac{\hat{\phi}_1(I_1^L)}{\hat{\phi}_1(I_1^L) + \hat{\psi}(I_1^S)}\right\}(\hat{\phi}_2 + \hat{\psi}(I_1^S) + \chi - \hat{\rho}(I_1^S)) > 0
\end{aligned}
$$

So the solution will feature the highest possible I_1^L that is consistent with the omitted two constraints $IC2$ and $IR1$.

We have

$$
\begin{aligned}
\frac{\partial t}{\partial I_1^S} &= \frac{\partial}{\partial I_1^S}\left\{\frac{f\cdot\phi_1(I_1^L)}{\phi_1(I_1^L)+\psi(I_1^S,I_2^S)}\right\}(\chi-\rho(I_1^S,I_2^S)) \\
&+\left\{\frac{f\cdot\phi_1(I_1^L)}{\phi_1(I_1^L)+\psi(I_1^S,I_2^S)}\right\}\frac{\partial}{\partial I_1^S}(\chi-\rho(I_1^S,I_2^S)) \\
&+\frac{\partial}{\partial I_1^S}\left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L)+\psi(I_1^S,I_2^S)}\right\}(\phi_2(I_2^L)+\psi(I_1^S,I_2^S)+\chi-\rho(I_1^S,I_2^S)) \\
&+\left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L)+\psi(I_1^S,I_2^S)}\right\}\frac{\partial}{\partial I_1^S}(\phi_2(I_2^L)+\psi(I_1^S,I_2^S)+\chi-\rho(I_1^S,I_2^S)) \\
&-\frac{\partial}{\partial I_1^S}\left\{\frac{\hat{\phi}_1(I_1^L)}{\hat{\phi}_1(I_1^L)+\hat{\psi}(I_1^S)}\right\}(\hat{\phi}_2+\hat{\psi}(I_1^S)+\chi-\hat{\rho}(I_1^S)) \\
&-\left\{\frac{\hat{\phi}_1(I_1^L)}{\hat{\phi}_1(I_1^L)+\hat{\psi}(I_1^S)}\right\}\frac{\partial}{\partial I_1^S}(\hat{\phi}_2+\hat{\psi}(I_1^S)+\chi-\hat{\rho}(I_1^S))
\end{aligned}
$$

The effect is ambiguous. An increase in I_1^S reduces the first three terms in the expression of t_1 but increases the last term (which enters the expression with a negative sign). This term corresponds to the effect of I_1^S on $V(I_1, 0)$. Increasing I_1^S is valuable for D_1 if D_2 has no access to information technology (because the competition effect on the switcher region is moot).

At $I_1^L = 1$,

$$\frac{\partial t}{\partial I_1^S} = \left\{ \frac{-\chi(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi})\cdot f\cdot\gamma_1}{(\gamma_1+\chi(1-I_1^S)(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi})^2} \right\}$$

$$(\chi - \chi(I_1^S + (1-I_1^S)\cdot\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\cdot\frac{\chi}{\chi+\gamma_2}))$$

$$-\left\{ \frac{f\cdot\gamma_1}{\gamma_1+\chi(1-I_1^S)(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi}} \right\}(\frac{\chi\gamma_1}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\cdot\frac{\chi}{\chi+\gamma_2})$$

$$+\left\{ \frac{\gamma_1\chi(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi}}{(\gamma_1+\chi(1-I_1^S)(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi})^2} \right\}$$

$$(\gamma_2(I_2^L + (1-I_2^L)\cdot\frac{\gamma_2}{\gamma_2+\chi})) + \chi(1-I_1^S)(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi}$$

$$+\chi - \chi(I_1^S + (1-I_1^S)\cdot\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\cdot\frac{\chi}{\chi+\gamma_2}))$$

$$+\left\{ \frac{\gamma_1}{\gamma_1+\chi(1-I_1^S)(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi}} \right\}$$

$$(-\chi(1-I_2^S)\frac{\gamma_1}{\gamma_1+\chi}\frac{\gamma_2}{\gamma_2+\chi} - \chi(\frac{\gamma_1}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\cdot\frac{\chi}{\chi+\gamma_2}))$$

$$+\left\{ \frac{-\gamma_1\chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{(\gamma_1+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}))^2} \right\}$$

$$(\gamma_2(I_2^L + (1-I_2^L)(\frac{\gamma_2}{\gamma_2+\chi})) + \chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})$$

$$+\chi - \chi(I_1^S + (1-I_1^S)(\frac{\chi}{\gamma_1+\chi}))(\frac{\chi}{\gamma_2+\chi}))$$

$$+\left\{ \frac{\gamma_1\chi(\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \right\}$$

Note at $I_1^S = 0$, $I_2^S = 0$, the above expression simplies to

$$\frac{\partial t}{\partial I_1^S} = \frac{\chi^2 f(\chi+\gamma_1)(-\gamma_2^2+\gamma_1\gamma_2+\chi\gamma_1)}{(\chi\gamma_1+2\chi\gamma_2+\gamma_1\gamma_2+\chi^2)^2}$$

The sign of which depends on

$$-\gamma_2^2 + \gamma_1\gamma_2 + (1-\gamma_1-\gamma_2)\gamma_1 = \gamma_1 - \gamma_1^2 - \gamma_2^2$$

We have

$$\frac{\partial t}{\partial I_1^S} > 0 \iff max\{0, \frac{1-\sqrt{1-4\gamma_2^2}}{2}\} < \gamma_1 < min\{1, \frac{1+\sqrt{1-4\gamma_2^2}}{2}\}\ if\ \gamma_2 \le \frac{1}{2}$$

Note that when $\gamma_1 = \frac{1}{2}$, $\frac{\partial t}{\partial I_1^S} > 0$ if $\gamma_2 < \frac{1}{2}$, which is always satisfied.

When $\gamma_1 = \gamma_2 = \chi = \frac{1}{3}$, the above condition is satisfied and we have $\frac{\partial t}{\partial I_1^S} > 0$.

When $\gamma_1 = \frac{3}{4}$, on the other hand, the above condition is violated and we have $\frac{\partial t}{\partial I_1^S} < 0$.

Now consider the sign of $\frac{\partial t}{\partial I_1^S}$ at $I_1^S = 1$.

At $\gamma_1 = \gamma_2 = \chi = \frac{1}{3}$ and $f = \frac{1}{2}$, the expression simplies to

$$\frac{\partial t}{\partial I_1^S} = -\frac{90 I_2^S + (2I_2^L - 29)(I_2^S)^2 + 30 I_2^L I_2^S - 25}{75(I_2^S - 5)^2}$$

which has roots $\frac{3(5I_2^L - \sqrt{5})(5(I_2^L)^2 - 4I_2^L + 58)^{\frac{1}{2}}}{2I_2^L - 29}$ and $\frac{3(5I_2^L + \sqrt{5})(5(I_2^L)^2 - 4I_2^L + 58)^{\frac{1}{2}}}{2I_2^L - 29}$. Note the denominator $2I_2^L - 29$ is always negative. So the second root (if exists and evaluates to a real number) is always positive. The first root is positive if $0 < I_2^L < \frac{\sqrt{5}}{5}$. At $I_2^L = 0$, The two roots are ± 1.7617, and $\frac{\partial t}{\partial I_1^S} = \frac{29(I_2^S)^2 - 90 I_2^S + 25}{75(I_2^S - 5)^2} < 0$. At $I_2^L = \frac{\sqrt{5}}{5}$, $\frac{\partial t}{\partial I_1^S} = \frac{-(2\sqrt{5} - 145)(I_2^S)^2 - (450 + 30\sqrt{5}) I_2^S + 125}{375(I_2^S - 5)^2} > 0$ if and only if $I_2^S > 0.2601$.

So, the optimal value of I_1^S is strictly between 0 and 1, and it displays complementarity with I_2^S.

To see the effect of I_2^L on t, note that

$$\begin{aligned}\frac{\partial t}{\partial I_2^L} = &-\frac{\partial}{\partial I_2^L}\left\{\frac{f \cdot \phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}\right\}(\phi_{12}(I_2^L) + \psi_2(I_2^S) + \chi - \rho_2(I_2^S)) \\ &- \left\{\frac{f \cdot \phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}\right\}\phi_{12}'(I_2^L) \\ &+ \frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)} \cdot \phi_2'(I_2^L)\end{aligned}$$

Or equivalently,

$$\begin{aligned}\frac{\partial t}{\partial I_2^L} = &-\frac{\partial}{\partial I_2^L}\left\{\frac{f\gamma_1(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1 - I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2}\right\} \\ &\left\{\gamma_1(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1 - I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2 + \chi - \chi(I_2^S + (1 - I_2^S)\frac{\chi}{\gamma_2+\chi})^2\right\} \\ &- \left\{\frac{f\gamma_1(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1 - I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2}\right\}\left\{\frac{\gamma_1\chi}{\gamma_2+\chi}\right\} \\ &+ \frac{\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) \cdot \gamma_2(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(I_1^S + (1 - I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1 - I_2^S)\frac{\chi}{\chi+\gamma_2})}\end{aligned}$$

The effect is ambiguous. An increase in I_2^L increases the value of $IR2$. When the constraint is binding, the increase translates to an increase in the value of t_2. On the other hand, an increase in I_2^L tightens D_1's incentive constraint and reduces the gap in $IC1$.

At $I_1^L = 1$, this expression can be written as

$$\frac{\partial t}{\partial I_2^L} = -\left\{\frac{f\gamma_1\frac{\chi}{\gamma_2+\chi}\left\{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2\right\} - 2f\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})\frac{\chi}{\chi+\gamma_2}}{\left\{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2\right\}}\right\}$$

$$\left\{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2+\chi-\chi(I_2^S+(1-I_2^S)\frac{\chi}{\gamma_2+\chi})^2\right\}$$

$$-\left\{\frac{f\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2}\right\}\left\{\frac{\gamma_1\chi}{\gamma_2+\chi}\right\}$$

$$+\frac{\gamma_1\cdot\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1+\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})}$$

$$\frac{\partial t}{\partial I_2^L} = -f\frac{\chi}{\gamma_2+\chi}$$

$$\left\{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2+\chi-\chi(I_2^S+(1-I_2^S)\frac{\chi}{\gamma_2+\chi})^2+2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})\right\}$$

$$-\left\{\frac{f(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2}\right\}\left\{\frac{\gamma_1\chi}{\gamma_2+\chi}\right\}$$

$$+\frac{1}{\gamma_1+\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})}\cdot\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})$$

At the symmetric case $\gamma_1 = \gamma_2 = \chi = \frac{1}{3}$ and $f = \frac{1}{2}$, this evaluates to

$$\frac{\partial t}{\partial I_2^L} = \frac{(I_2^S-1)^2(\frac{I_2^L}{3}+\frac{I_2^S}{4}-\frac{2}{3})}{12} - \frac{I_2^L}{24} - \frac{I_2^L+1}{24(\frac{I_2^L}{6}+\frac{(I_2^S-1)^2}{12}+\frac{1}{6})} + \frac{\frac{I_2^L+1}{2}}{(\frac{I_1^S+1}{2})(\frac{I_2^S+1}{2})+1} - \frac{1}{24}$$

At $I_2^L = 0$, the expression simplies to

$$\frac{\partial t}{\partial I_2^L} = \frac{(I_2^S-1)^2(\frac{I_2^S}{4}-\frac{2}{3})}{12} - \frac{1}{2(I_2^S-1)^2+4} + \frac{1}{2(\frac{I_1^S+1}{2})(\frac{I_2^S+1}{2})+2} - \frac{1}{24}$$

When $I_1^S = 0$,

$$\frac{\partial t}{\partial I_2^L} = \frac{(I_2^S-1)^2(\frac{I_2^S}{4}-\frac{2}{3})}{12} - \frac{1}{2(I_2^S-1)^2+4} + \frac{1}{(\frac{I_2^S+1}{2})+2} - \frac{1}{24} > 0$$

When $I_1^S = 1$,

$$\frac{\partial t}{\partial I_2^L} = \frac{(I_2^S-1)^2(\frac{I_2^S}{4}-\frac{2}{3})}{12} - \frac{1}{2(I_2^S-1)^2+4} + \frac{1}{(\frac{I_2^S+1}{2})+2} - \frac{1}{24}$$

which is decreasing in I_2^S and crosses the zero once. (See figure below).

Above this point, $\frac{\partial t}{\partial I_2^L} > 0$, so a higher I_2^S implies higher I_2^L (complementarity). Below this point, the opposite holds and I_2^S and I_2^L are (gross) substitutes.

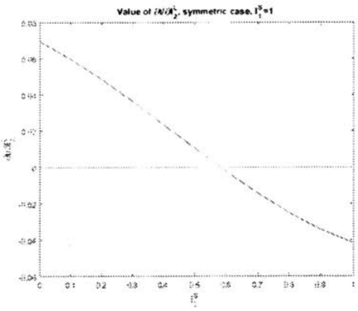


Figure 15: $\frac{\partial t}{\partial I_2^L}(I_2^S)$, $\gamma_1 = \gamma_2 = \chi = \frac{1}{3}$, $f = \frac{1}{2}$ $I_1^L = 1$, $I_1^S = 1$

$$\begin{aligned}
\frac{\partial t}{\partial I_2^S} &= \frac{\partial}{\partial I_2^S}\left\{\frac{f \cdot \phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}(\phi_1(I_1^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
&+ \left\{\frac{f \cdot \phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}\frac{\partial}{\partial I_2^S}(\psi(I_1^S, I_2^S) - \rho(I_1^S, I_2^S)) \\
&- \frac{\partial}{\partial I_2^S}\left\{\frac{f \cdot \phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}\right\}(\phi_{12}(I_2^L) + \psi_2(I_2^S) + \chi - \rho_2(I_2^S)) \\
&- \left\{\frac{f \cdot \phi_{12}(I_2^L)}{\phi_{12}(I_2^L) + \psi_2(I_2^S)}\right\}\frac{\partial}{\partial I_2^S}(\psi_2(I_2^S) - \rho_2(I_2^S)) \\
&+ \frac{\partial}{\partial I_2^S}\left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}(\phi_2(I_2^L) + \psi(I_1^S, I_2^S) + \chi - \rho(I_1^S, I_2^S)) \\
&+ \left\{\frac{\phi_1(I_1^L)}{\phi_1(I_1^L) + \psi(I_1^S, I_2^S)}\right\}\frac{\partial}{\partial I_2^S}(\psi(I_1^S, I_2^S) - \rho(I_1^S, I_2^S))
\end{aligned}$$

At $I_1^L = 1$

$$
\begin{aligned}
\frac{\partial t}{\partial I_2^S} = & \left\{ \frac{f\cdot\gamma_1\cdot\chi(1-I_1^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})}{\left(\gamma_1+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})\right)^2} \right\} \\
& (\gamma_1+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})) \\
& +\left\{ \frac{f\cdot\gamma_1}{\gamma_1+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \\
& \left\{ -\chi(1-I_1^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})+\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2} \right\} \\
& -\left\{ \frac{f\cdot\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})\cdot 2\chi(1-I_2^S)(\frac{\gamma_2}{\gamma_2+\chi})^2}{\left(\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2\right)^2} \right\} \\
& \left\{ \gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_2^S)^2(\frac{\gamma_2}{\gamma_2+\chi})^2+\chi-\chi(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})^2 \right\} \\
& -\left\{ \frac{f\cdot\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_1(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \\
& \left\{ -2\chi(1-I_2^S)(\frac{\gamma_2}{\gamma_2+\chi})^2-\chi(\frac{\gamma_2}{\chi+\gamma_2})^2 \right\} \\
& +\left\{ \frac{\gamma_1\chi(1-I_1^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})}{\left(\gamma_1+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})\right)^2} \right\} \\
& \left\{ \gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2}) \right\} \\
& +\left\{ \frac{\gamma_1}{\gamma_1+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \\
& \left\{ -\chi(1-I_1^S)(\frac{\gamma_1}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2}) \right\}
\end{aligned}
$$

Again, the effect is ambiguous. The second term (which enters with a negative sign) is evaluated against the sum of the first and the third terms. At $\chi = \gamma_1 = \gamma_2 = \frac{1}{3}$ and $f = \frac{1}{2}$, this simplifies to

$$
\frac{\partial t}{\partial I_2^S} = \frac{(\frac{I_2^S-1}{72})(\frac{(I_2^S-1)^2}{12} - \frac{1}{3}(\frac{I_2^S+1}{2})^2 + \frac{1}{2})}{(\frac{(I_2^S-1)^2+2}{12})^2} - \frac{I_2^S}{12} + \frac{1}{8} > 0
$$

for $I_2^S \in [0,1]$.

To sum up, for the symmetric case with $\chi = \gamma_1 = \gamma_2 = \frac{1}{3}$ and $f = \frac{1}{2}$,

$$
\frac{\partial t}{\partial I_2^S} > 0 \; and \; \frac{\partial t}{\partial I_1^L} > 0
$$

Increases in I_1^L and I_2^S are beneficial to the upstream party. Suppose the omitted constraints do not bind (we will come back and check this), we set $I_1^L = 1$ and $I_2^S = 1$.

From the discussion above, $\frac{\partial t}{\partial I_1^S} > 0$ when $I_2^S = 1$. Complementarity requires setting I_1^S to the maximal value compatible with the omitted constraints. Suppose $I_1^S = 1$. We then have I_2^L from the dicussions above.

It remains to check the omitted constraints $IC2$ and $IR1$ are satisfied.

$$t_1 \leq \frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2+\tilde{\psi}}\right\}(\tilde{\phi}_1+\tilde{\psi}+\chi-\tilde{\rho})\ [IR1]$$
$$= (\gamma_1\frac{\chi}{\gamma_2+\chi})$$

$$t_1 - t_2 \geq \frac{\phi_1}{\phi_1+\psi_1}(\phi_2+\psi_1+\chi-\rho_1) - \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho)\ \ [IC2]$$
$$= 0$$

Note

$$t_2 = 0\ [IR2]$$

$$t_1 = (\gamma_1\frac{\chi}{\gamma_2+\chi})\ [IC1]$$

which satisfy $IR1$ and $IC2$.

Case 3: $[IC2]$ and $[IR2]$ bind.

In this case, we have

$$t_1 = \frac{\phi_1}{\phi_1+\psi_1}(\phi_2+\psi_1+\chi-\rho_1) - \frac{\hat{\phi}_1}{\hat{\phi}_1+\hat{\psi}}(\hat{\phi}_2+\hat{\psi}+\chi-\hat{\rho})$$

$$t_2 = \frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1+\hat{\psi}}(\hat{\phi}_2+\hat{\psi}+\chi-\hat{\rho})$$

Let

$$
\begin{aligned}
t(I_1^L, I_1^S, I_2^L, t_2^S) &= f \cdot t_1 + (1-f) \cdot t_2 \\
&= \frac{f \cdot \phi_1}{\phi_1 + \psi_1}(\phi_2 + \psi_1 + \chi - \rho_1) \\
&+ \frac{(1-f) \cdot \phi_1}{\phi_1 + \psi}(\phi_2 + \psi + \chi - \rho) - \frac{\hat{\phi}_1}{\hat{\phi}_1 + \hat{\psi}}(\hat{\phi}_2 + \hat{\psi} + \chi - \hat{\rho}) \\
=& \frac{f \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2} \\
&\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2 + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})^2\right\} \\
&+ \frac{(1-f) \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \\
&\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1 - I \right. \\
&- \frac{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \\
&\left\{\gamma_2\frac{\gamma_2}{\gamma_2+\chi} + \chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2}\right\}
\end{aligned}
$$

Take the partial derivative of t with respect to I_2^L to obtain

$$
\begin{aligned}
\frac{\partial t}{\partial I_2^L} =& \frac{f \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2}\gamma_2(\frac{\chi}{\gamma_2+\chi}) \\
&+ \left\{\frac{(1-f) \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\} \\
&\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2} \right. \\
&+ \left\{\frac{(1-f) \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}\right\}\left\{\gamma_2(\frac{\chi}{\gamma_2+\chi})\right\}
\end{aligned}
$$

which has the same sign as

$$
\begin{aligned}
&\frac{f}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2} \\
&+ \left\{\frac{(1-f)(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\} \\
&\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\} \\
&+ \left\{\frac{(1-f)}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}\right\}
\end{aligned}
$$

For the symmetric case $\chi = \gamma_1 = \gamma_2 = \frac{1}{3}$ and $f = \frac{1}{2}$, this evaluates to

$$\frac{1}{\frac{I_1^L+1}{3} + \frac{(I_1^S-1)(I_2^S-1)}{6}} + \frac{1}{I_1^L + \frac{(I_1^S-1)^2}{6} + \frac{5}{3}} - \frac{(I_1^S - 1)(\frac{I_2^L}{6} + \frac{(I_2^S-1)^2}{12} - (\frac{I_1^S+1}{6})(\frac{I_2^S+1}{6}) + \frac{1}{2}}{4(\frac{I_1^L}{6} + \frac{(I_1^S-1)(I_2^S-1)}{12} + \frac{1}{6})^2}$$

This expression is positive throughout.

Second, we take the partial derivative of t with respect to I_2^S to obtain

$$\frac{\partial t}{\partial I_2^S} = \left\{ \frac{(1-f)\cdot\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2} \right\}$$
$$\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2}\right.$$
$$+ \left\{ \frac{(1-f)\cdot\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \right\}$$
$$\left\{-\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})\right\}$$

which has the same sign as

$$\left\{ \frac{(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2} \right\}$$
$$\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$
$$+ \left\{ \frac{1}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \right\}$$

For the symmetric case $\chi = \gamma_1 = \gamma_2 = \frac{1}{3}$ and $f = \frac{1}{2}$, this evaluates to

$$\frac{1}{\frac{I_1^L+1}{6} + \frac{(I_1^S-1)(I_2^S-1)}{12}} - \frac{(I_1^S - 1)(\frac{I_2^L}{6} - (\frac{(I_1^S+1)}{6}\frac{(I_2^S+1)}{6} + \frac{(I_1^S-1)(I_2^S-1)}{12} + \frac{1}{2})}{2(\frac{I_1^L}{6} + \frac{(I_1^S-1)(I_2^S-1)}{12} + \frac{1}{6})^2}$$

This expression is positive throughout.

We will set $I_2^L = I_2^S = 1$, which is valid as long as the omitted constraints are satisfied, which we assume to be the case for now. Turning to (I_1^L, I_1^S) , we obtain

$$
\begin{aligned}
\frac{\partial t}{\partial I_1^L} &= \gamma_1(\frac{\chi}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi}) \\
&\left\{\frac{f\cdot(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2\right)^2}\right\} \\
&\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})^2\right\} \\
&+\gamma_1(\frac{\chi}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi}) \\
&\left\{\frac{(1-f)\cdot(1-I_2^S)(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\} \\
&\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\} \\
&-\gamma_1(\frac{\chi}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi}) \\
&\left\{\frac{\left((\frac{\gamma_2}{\gamma_2+\chi})\right)}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\} \\
&\left\{\gamma_2\frac{\gamma_2}{\gamma_2+\chi}+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2}\right\}
\end{aligned}
$$

which has the same sign as

$$
\begin{aligned}
\frac{\partial t}{\partial I_1^L} &= \left\{\frac{f\cdot(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2\right)^2}\right\} \\
&\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})^2\right\} \\
&+\left\{\frac{(1-f)\cdot(1-I_2^S)(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\} \\
&\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\} \\
&-\left\{\frac{(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\} \\
&\left\{\gamma_2\frac{\gamma_2}{\gamma_2+\chi}+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2}\right\}
\end{aligned}
$$

For the symmetric case $\chi=\gamma_1=\gamma_2=\frac{1}{3}$ and $f=\frac{1}{2}$, at $I_1^L=0$, this evaluates to

$$
\frac{\frac{I_1^S}{6}-\frac{1}{2}}{2(\frac{I_1^S}{12}-\frac{1}{4})^2}-\frac{(I_2^S-1)(\frac{I_2^L}{6}-\frac{(\frac{I_1^S+1}{2})(\frac{I_2^S+1}{2})}{3}+\frac{(I_1^S-1)(I_2^S-1)}{12}+\frac{1}{2})}{4(\frac{(I_1^S-1)(I_2^S-1)}{12}+\frac{1}{6})^2}-\frac{(I_1^S-1)(\frac{I_2^L}{6}+\frac{(I_1^S-1)^2}{12}-\frac{1}{3}(\frac{I_1^S+1}{2})^2+\frac{1}{2})}{4(\frac{((I_1^S-1)(I_2^S-1))}{12}+\frac{1}{6})^2}
$$

	$I_1^S = 0$	$I_1^S = 1$
$I_1^L = 0$	+	-
$I_1^L = 1$	+	-

Table 3: Sign of $\frac{\partial t}{\partial I_1^L}$

When $I_1^S = 0$, $\frac{\partial t}{\partial I_1^L} > 0$. When $I_1^S = 1$, $\frac{\partial t}{\partial I_1^L} < 0$.
At $I_1^L = 1$ and $I_1^S = 1$, $\frac{\partial t}{\partial I_1^L} < 0$.
At $I_1^L = 1$ and $I_1^S = 0$, $\frac{\partial t}{\partial I_1^L} > 0$ if $I_2^L = 1$ and $\frac{\partial t}{\partial I_1^L} < 0$ if $I_2^L = 0$.
At $I_2^S = 1$ and $I_2^L = 1$, this simplifies to

$$\frac{72(\frac{I_1^S}{6} - \frac{1}{2})}{(2I_1^L - I_1^S + 3)^2} + \frac{3(I_1^S)^2 - 9I_1^S + 6}{(I_1^L + 1)^2}$$

whose sign depends on (I_1^S, I_1^L), as shown in the following table.

$$\begin{aligned}
\frac{\partial t}{\partial I_1^S} = & \left\{ \frac{2f \cdot \chi\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})^2}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2\right)^2} \right\} \\
& \left\{ \gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2 + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})^2 \right\} \\
& + \left\{ \frac{f \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2} \right\} \\
& \left\{ -2\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})^2 + \chi - 2\chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\gamma_1}{\chi+\gamma_1} \right\} \\
& + \left\{ \frac{(1-f) \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2} \right\} \\
& \left\{ \gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2}) \right\} \\
& + \left\{ \frac{(1-f) \cdot \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \right\} \\
& \left\{ -\chi(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(\frac{\gamma_1}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2}) \right\} \\
& - \left\{ \frac{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}))^2} \right\} \\
& \left\{ \gamma_2\frac{\gamma_2}{\gamma_2+\chi} + \chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2} \right\} \\
& - \left\{ \frac{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})} \right\} \\
& \left\{ -\chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}) + \chi - \chi(\frac{\gamma_1}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2} \right\}
\end{aligned}$$

	$I_2^S=0$	$I_2^S=1$
$I_2^L=0$	0	-
$I_2^L=1$	+	+

(a) $I_1^L=0$

	$I_2^S=0$	$I_2^S=1$
$I_2^L=0$	-	+
$I_2^L=1$	+	-

(b) $I_1^L=1$

Table 4: Sign of $\frac{\partial t}{\partial I_1^L}$

which has the same sign as

$$\left\{\frac{2f\cdot\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})^2}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2\right)^2}\right\}$$
$$\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})^2\right\}$$
$$+\left\{\frac{f}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\gamma_1}{\gamma_1+\chi})^2}\right\}$$
$$\left\{-2\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})^2+\chi-2\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\gamma_1}{\chi+\gamma_1}\right\}$$
$$+\left\{\frac{(1-f)\cdot(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})\right)^2}\right\}$$
$$\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$
$$+\left\{\frac{(1-f)\cdot}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}\right\}$$
$$\left\{-\chi(1-I_2^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(\frac{\gamma_1}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$
$$-\left\{\frac{\chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}{(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi}))^2}\right\}$$
$$\left\{\gamma_2\frac{\gamma_2}{\gamma_2+\chi}+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2}\right\}$$
$$-\left\{\frac{1}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})}\right\}$$
$$\left\{-\chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})+\chi-\chi(\frac{\gamma_1}{\chi+\gamma_1})\frac{\chi}{\chi+\gamma_2}\right\}$$

For the symmetric case $\chi=\gamma_1=\gamma_2=\frac{1}{3}$ and $f=\frac{1}{2}$, at $I_1^S=0$, the expression simplifies to (up to a multiplicative term)

$$\frac{\partial t}{\partial s_1}=\frac{(I_1^L+1)}{12I_1^L-6I_2^S+18}-\frac{(I_1^L+1)}{(2I_1^L+3)^2}-\frac{(I_1^L+1)}{(6I_1^L+9)}+\frac{(I_1^L+1)(I_2^L+3)}{3(2I_1^L+3)^2}-\frac{(I_1^L+1)(I_2^S-1)(\frac{I_2^L-I_2^S}{6}+\frac{1}{2})}{(2I_1^L-I_2^S+3)^2}$$

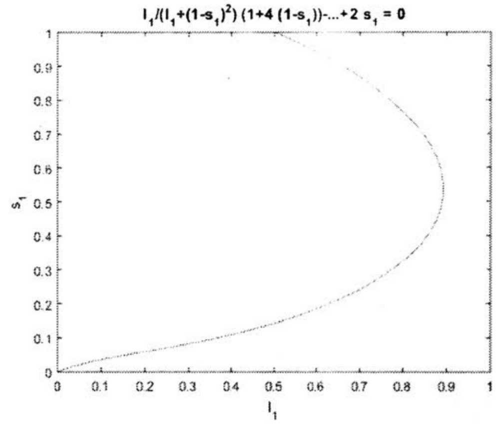


Figure 16: Plot of $\frac{l_1}{l_1+(1-s_1)^2}(1+4(1-s_1)) - \frac{2l_1}{l_1+(1-s_1)^{\frac{1}{2}}}(2-s_1) - 2l_1 + 2s_1 \leq 0$

At $I_1^S = 1$, the expression simplies to (up to a multiplicative term)

$$\frac{\partial t}{\partial s_1} = -\frac{2I_1^L - I_2^L + 3I_2^S - (I_2^S)^2 + I_2^L I_2^S + 4}{24(I_1^L + 1)}$$

which is negative. Note that when $I_2^S = 1$ and $I_2^L = 1$, $IR1$ and $IC1$ simplify to

$$t_1 \leq \phi_1 + \frac{\phi_1}{\phi_1 + \psi}(\chi - \rho) - \gamma_1 \; [IR1]$$

$$t_1 - t_2 \leq \phi_1 + \frac{\phi_1}{\phi_1 + \psi}(\chi - \rho) - \gamma_1 \; [IC1]$$

or simply

$$t_1 \leq \phi_1 + \frac{\phi_1}{\phi_1 + \psi}(\chi - \rho) - \gamma_1$$

At

$$t_1 = \frac{\phi_1}{\phi_1 + \psi_1}(\phi_2 + \psi_1 + \chi - \rho_1) - \frac{\hat{\phi}_1}{\hat{\phi}_1 + \hat{\psi}}(\hat{\phi}_2 + \hat{\psi} + \chi - \hat{\rho})$$

The omitted constraints are satisfied if

$$\frac{\phi_1}{\phi_1 + \psi_1}(\phi_2 + \psi_1 + \chi - \rho_1) - \frac{\phi_1}{\phi_1 + \hat{\psi}}(\hat{\phi}_2 + \hat{\psi} + \chi - \hat{\rho}) \leq \phi_1 + \frac{\phi_1}{\phi_1 + \psi}(\chi - \rho) - \gamma_1$$

Let $l_1 = Pr_1(l_1|L)$ and $s_1 = Pr_1(s_1|S)$. We have

$$\frac{l_1}{l_1 + (1-s_1)^2}(1 + 4(1-s_1)) - \frac{2l_1}{l_1 + (1-s_1)^{\frac{1}{2}}}(2 - s_1) - 2l_1 + 2s_1 \leq 0$$

This constraint corresponds to the right half of the graph below.

For the symmetric case,

$$t_1 = \frac{l_1}{l_1 + (1-s_1)^2}(\frac{1}{3} + \frac{1}{3}(1-s_1)^2 + \frac{1}{3} - \frac{1}{3}s_1^2) - \frac{l_1}{l_1 + \frac{1}{2}(1-s_1)}(\frac{1}{6} + \frac{1}{6}(1-s_1) + \frac{1}{3} - \frac{1}{6}s_1)$$

$$t_2 = (\frac{1}{3} + \frac{1}{3} - \frac{1}{3}s_1) - \frac{l_1}{l_1 + \frac{1}{2}(1-s_1)}(\frac{1}{6} + \frac{1}{6}(1-s_1) + \frac{1}{3} - \frac{1}{6}s_1)$$

$$\begin{aligned} t = \frac{1}{2}t_1 + \frac{1}{2}t_2 &= \frac{\frac{1}{2}l_1}{l_1 + (1-s_1)^2}(\frac{1}{3} + \frac{1}{3}(1-s_1)^2 + \frac{1}{3} - \frac{1}{3}s_1^2) \\ &+ \frac{1}{2}(\frac{1}{3} + \frac{1}{3} - \frac{1}{3}s_1) - \frac{l_1}{l_1 + \frac{1}{2}(1-s_1)}(\frac{1}{6} + \frac{1}{6}(1-s_1) + \frac{1}{3} - \frac{1}{6}s_1) \end{aligned}$$

It's easy to see that this expression is maximized when $l_1 = s_1 = 0$.

Case 4: $[IC2]$ and $[IR1]$ bind.

$$t_1 = \frac{\phi_1}{\phi_1 + \psi}(\phi_1 + \psi + \chi - \rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}\right\}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho})$$

$$\begin{aligned} t_2 = &-\frac{\phi_1}{\phi_1 + \psi_1}(\phi_2 + \psi_1 + \chi - \rho_1) + \frac{\phi_1}{\phi_1 + \psi}(\phi_2 + \psi + \chi - \rho) \\ &+ \frac{\phi_1}{\phi_1 + \psi}(\phi_1 + \psi + \chi - \rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}\right\}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho}) \end{aligned}$$

$$\begin{aligned} t &= f \cdot t_1 + (1-f) \cdot t_2 \\ &= \frac{\phi_1}{\phi_1 + \psi}(\phi_1 + \psi + \chi - \rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1 + \tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2 + \tilde{\psi}}\right\}(\tilde{\phi}_1 + \tilde{\psi} + \chi - \tilde{\rho}) \\ &- \frac{(1-f)\phi_1}{\phi_1 + \psi_1}(\phi_2 + \psi_1 + \chi - \rho_1) + \frac{(1-f)\phi_1}{\phi_1 + \psi}(\phi_2 + \psi + \chi - \rho) \end{aligned}$$

Equivalently

$$
\begin{aligned}
t(I_1^L, I_1^S, I_2^L, I_2^S) = &\left(\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1 + \chi})\right) \\
&+ \left\{ \frac{(2f-1)\left(\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)}{\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1 - I_1^S)(1 - I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right\} \\
&\left\{ \chi - \chi(I_1^S + (1 - I_1^S)\frac{\chi}{\chi + \gamma_1})(I_2^S + (1 - I_2^S)\frac{\chi}{\chi + \gamma_1}) \right\} \\
&- max \left\{ \frac{\gamma_1}{\gamma_1 + \chi(1 - I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})}, \frac{\gamma_2(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_2(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi\frac{\gamma_1}{\gamma_1+\chi}(1 - I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \\
&\left\{ \gamma_1\frac{\gamma_1}{\gamma_1 + \chi} + \chi(\frac{\gamma_1}{\gamma_1 + \chi})(\frac{\gamma_2}{\gamma_2 + \chi})(1 - I_2^S) + \chi - \chi(\frac{\chi}{\gamma_1 + \chi})(I_2^S + (1 - I_2^S)\frac{\chi}{\chi + \gamma_2}) \right\} \\
&+ \left\{ \frac{(1-f)\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1 - I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2} \right\} \\
&\left\{ \gamma_2(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2 + \chi}) + \chi(1 - I_1^S)^2(\frac{\chi}{\chi + \gamma_1})^2 + \chi - \chi(I_1^S + (1 - I_1^S)\frac{\chi}{\chi + \gamma_1})^2 \right\} \\
&- \left[\frac{(1-f)\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1 - I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1 - I_1^S)(1 - I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right] \\
&\left\{ \gamma_2(I_2^L + (1 - I_2^L)\frac{\gamma_2}{\gamma_2 + \chi}) + \chi(1 - I_1^S)(1 - I_2^S)(\frac{\chi}{\chi + \gamma_1})(\frac{\chi}{\chi + \gamma_2}) + \chi - \chi(I_1^S + (1 - I_1^S)\frac{\chi}{\chi + \gamma_1})(I_2^S + (1 - \right.
\end{aligned}
$$

The partial derivative of t with respect to I_2^S is

$$\frac{\partial t}{\partial I_2^S} = \left\{ \frac{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right\}$$

$$\left\{\chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_1})\right\}$$

$$+ \left\{ \frac{(2f-1)\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right\}$$

$$\left\{-\chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_1}{\chi+\gamma_1})\right\}$$

$$- \frac{\partial}{\partial I_2^S} max \left\{ \frac{\gamma_1}{\gamma_1 + \chi(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})}, \frac{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} \right\}$$

$$\left\{\gamma_1\frac{\gamma_1}{\gamma_1+\chi} + \chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})(1-I_2^S) + \chi - \chi(\frac{\chi}{\gamma_1+\chi})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$

$$- max \left\{ \frac{\gamma_1}{\gamma_1 + \chi(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})}, \frac{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \left\{-\chi(\frac{\chi}{\gamma_1+\chi})(\frac{\gamma_2}{\chi+\gamma_2})\right\}$$

$$+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right]$$

$$\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_1}\right.$$

$$+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right]$$

$$\left\{\chi(1-I_1^S)(-1)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})\right\}$$

First, let's consider the case where $\frac{\gamma_1}{\gamma_1+\chi(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} \geq \frac{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})}$, or equivalently,

$$\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) \leq \frac{\gamma_1^2}{\gamma_1+\chi}$$

When the condition

$$\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) \leq \frac{\gamma_1^2}{\gamma_1+\chi}$$

holds, we have

$$\frac{\partial t}{\partial I_2^S} = \left\{ \frac{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right\}$$

$$\left\{\chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_1})\right\}$$

$$+ \left\{ \frac{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right\}$$

$$\left\{-\chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_1}{\chi+\gamma_1})\right\}$$

$$- \left\{ \frac{\gamma_1\chi(\frac{\gamma_2}{\chi+\gamma_2})}{\left(\gamma_1 + \chi(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})\right)^2} \right\}$$

$$\left\{\gamma_1\frac{\gamma_1}{\gamma_1+\chi} + \chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})(1-I_2^S) + \chi - \chi(\frac{\chi}{\gamma_1+\chi})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$

$$+ \left\{ \frac{\gamma_1}{\gamma_1 + \chi(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \left\{\chi(\frac{\chi}{\gamma_1+\chi})(\frac{\gamma_2}{\chi+\gamma_2})\right\}$$

$$+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right]$$

$$\left\{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_1}\right.$$

$$+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right]$$

$$\left\{\chi(1-I_1^S)(-1)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2})\right\}$$

It can be shown that this expression is negative throughout.

In this case, we have

$$\frac{\partial t}{\partial I_2^L} = -\left\{ \frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2} \right\} \left\{\gamma_2(\frac{\chi}{\gamma_2+\chi})\right\}$$
$$+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right] \gamma_2(\frac{\chi}{\gamma_2+\chi})$$

which has the same sign as

$$\frac{\partial t}{\partial I_2^L} = -\frac{1}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2}$$
$$+ \frac{1}{\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}$$

We have

$$\frac{\partial t}{\partial I_2^L} > 0 \iff$$

$$\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})$$
$$\le \gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2$$

Or

$$\frac{\partial t}{\partial I_2^L} \le 0 \iff \frac{1-I_2^S}{\chi+\gamma_2} \ge \frac{1-I_1^S}{\chi+\gamma_1}$$

Note that this always satisfied at $I_2^S = 0$.

Next, let's turn to the case with $\frac{\gamma_1}{\gamma_1+\chi(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} < \frac{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})}$. We have (up to a multiplicative term)

$$\frac{\partial t}{\partial I_2^L} = \left\{ \frac{\left(\chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})\right)}{\left(\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})\right)^2} \right\}$$
$$\left\{\gamma_1\frac{\gamma_1}{\gamma_1+\chi} + \chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})(1-I_2^S) + \chi - \chi(\frac{\chi}{\gamma_1+\chi})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$
$$- \left\{ \frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2} \right\}$$
$$+ \left[\frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right]$$

Consider the partial derivatives with respect to the other components.

$$
\begin{aligned}
\frac{\partial t}{\partial I_2^S} = &\left\{ \frac{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right\} \\
&\left\{ \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_1}) \right\} \\
&+ \left\{ \frac{\left(\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right\} \\
&\left\{ -\chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_1}{\chi+\gamma_1}) \right\} \\
&- \left\{ \frac{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})\chi\frac{\gamma_1}{\gamma_1+\chi}(\frac{\gamma_2}{\chi+\gamma_2})}{\left(\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})\right)^2} \right\} \\
&\left\{ \gamma_1\frac{\gamma_1}{\gamma_1+\chi} + \chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})(1-I_2^S) + \chi - \chi(\frac{\chi}{\gamma_1+\chi})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_2}) \right\} \\
&- \left\{ \frac{\gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})}{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})} \right\} \left\{ -\chi(\frac{\chi}{\gamma_1+\chi})(\frac{\gamma_2}{\chi+\gamma_2}) \right\} \\
&+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right] \\
&\left\{ \gamma_2(I_2^L + (1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) + \chi - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S + (1-I_2^S)\frac{\chi}{\chi+\gamma_1} \right. \\
&+ \left[\frac{(1-f)\gamma_1(I_1^L + (1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi}) + \chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right] \\
&\left\{ \chi(1-I_1^S)(-1)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) - \chi(I_1^S + (1-I_1^S)\frac{\chi}{\chi+\gamma_1})(\frac{\gamma_2}{\chi+\gamma_2}) \right\}
\end{aligned}
$$

which is negative throughout.

$$\frac{\partial t}{\partial I_1^L} = \left(\gamma_1 \frac{\chi}{\gamma_1+\chi})\right)$$

$$+ \left\{ \frac{\left(\gamma_1 \frac{\chi}{\gamma_1+\chi}\right)\left(\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right\}$$

$$\left\{\chi - \chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_1})\right\}$$

$$- \left\{ \frac{(1-f)\gamma_1\frac{\chi}{\gamma_1+\chi}\left(\chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2\right)}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2\right)^2} \right\}$$

$$\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})^2\right\}$$

$$+ \left[\frac{(1-f)\gamma_1\frac{\chi}{\gamma_1+\chi}\left(\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right]$$

$$\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$

which is positive throughout.

$$\frac{\partial t}{\partial I_1^S} = \left\{ \frac{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)\chi(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right\}$$

$$\left\{\chi - \chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_1})\right\}$$

$$+ \left\{ \frac{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\right)}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right\}$$

$$\left\{-\chi(\frac{\gamma_1}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_1})\right\}$$

$$- \left\{ \frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2} \right\}$$

$$\left\{-2\chi(1-I_1^S)(\frac{\chi}{\chi+\gamma_1})^2 - 2\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})\frac{\gamma_1}{\chi+\gamma_1}\right\}$$

$$+ \left[\frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})\chi(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}{\left(\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})\right)^2} \right]$$

$$\left\{\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})+\chi-\chi(I_1^S+(1-I_1^S)\frac{\chi}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$

$$+ \left[\frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})} \right]$$

$$\left\{-\chi(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2}) - \chi(\frac{\gamma_1}{\chi+\gamma_1})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$

which is negative throughout at $I_1^S = 1$.

At $I_2^S = 0$ and $I_1^L = 1$, the expression is positive.

Finally

$$\frac{\partial t}{\partial I_2^L} = \left\{ \frac{\left(\chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})\right)}{\left(\gamma_2(I_2^L+(1-I_2^L)\frac{\gamma_2}{\gamma_2+\chi})+\chi\frac{\gamma_1}{\gamma_1+\chi}(1-I_2^S)(\frac{\gamma_2}{\chi+\gamma_2})\right)^2} \right\}$$

$$\left\{\gamma_1\frac{\gamma_1}{\gamma_1+\chi}+\chi(\frac{\gamma_1}{\gamma_1+\chi})(\frac{\gamma_2}{\gamma_2+\chi})(1-I_2^S)+\chi-\chi(\frac{\chi}{\gamma_1+\chi})(I_2^S+(1-I_2^S)\frac{\chi}{\chi+\gamma_2})\right\}$$

$$-\left\{\frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)^2(\frac{\chi}{\chi+\gamma_1})^2}\right\}$$

$$+\left[\frac{(1-f)\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})}{\gamma_1(I_1^L+(1-I_1^L)\frac{\gamma_1}{\gamma_1+\chi})+\chi(1-I_1^S)(1-I_2^S)(\frac{\chi}{\chi+\gamma_1})(\frac{\chi}{\chi+\gamma_2})}\right]$$

When $I_2^S = 0$, the expression is positive at $I_2^L = 0$ and negative at $I_2^L = 1$.

At $I_1^S = 0$, this expression simplifies to

$$\frac{\partial t}{\partial I_2^L} = \frac{1}{24(\frac{I_2^L}{6}+\frac{1}{4})^2} - \frac{5}{18}$$

which has zero at $I_2^L = 0.82379$

Finally, we check the omitted constraints at $I_1^L = 1$, $I_1^S = 0$, $I_2^S = 0$ and $I_2^L = l_2$.

At

$$t_2 = -\frac{\phi_1}{\phi_1+\psi_1}(\phi_2+\psi_1+\chi-\rho_1)+\frac{\phi_1}{\phi_1+\psi}(\phi_2+\psi+\chi-\rho)$$
$$+\frac{\phi_1}{\phi_1+\psi}(\phi_1+\psi+\chi-\rho) - max\left\{\frac{\tilde{\phi}_1}{\tilde{\phi}_1+\tilde{\psi}}, \frac{\tilde{\phi}_2}{\tilde{\phi}_2+\tilde{\psi}}\right\}(\tilde{\phi}_1+\tilde{\psi}+\chi-\tilde{\rho})$$

It is easy to check the omitted constraints are satisfied at $l_2 = 1$ but not when $l_2 < 1$.

Illustrations of constraints

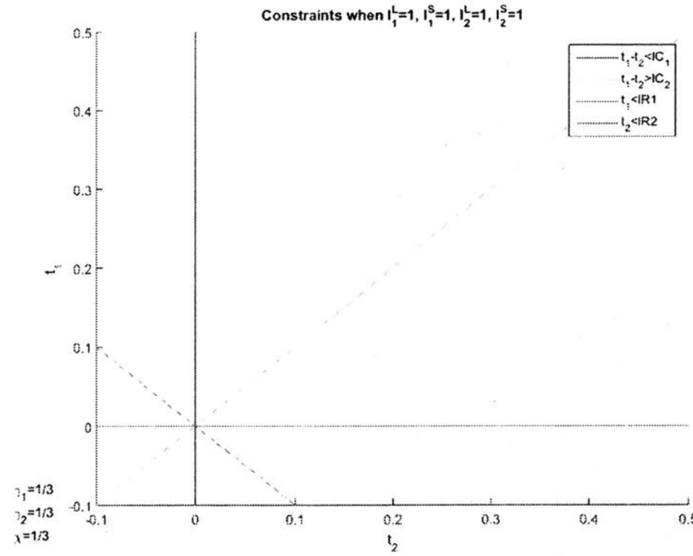
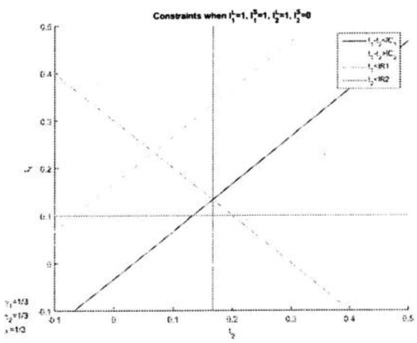
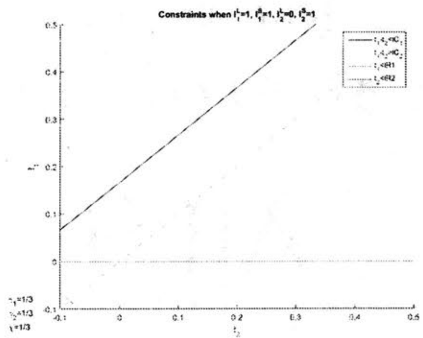


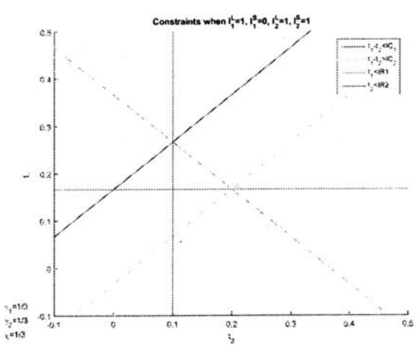
Figure 17: Illustrations of the constraints; symmetric case; full information; $I_1^S = I_1^L = 1,\ I_2^S = I_2^L = 1$



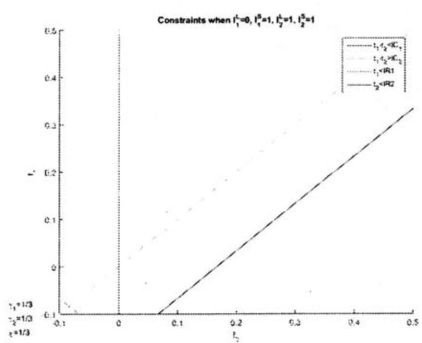
(a) $I_1^S = I_1^L = 1,\ I_2^L = 1, I_2^S = 0$



(b) $I_1^S = I_1^L = 1,\ I_2^L = 0, I_2^S = 1$

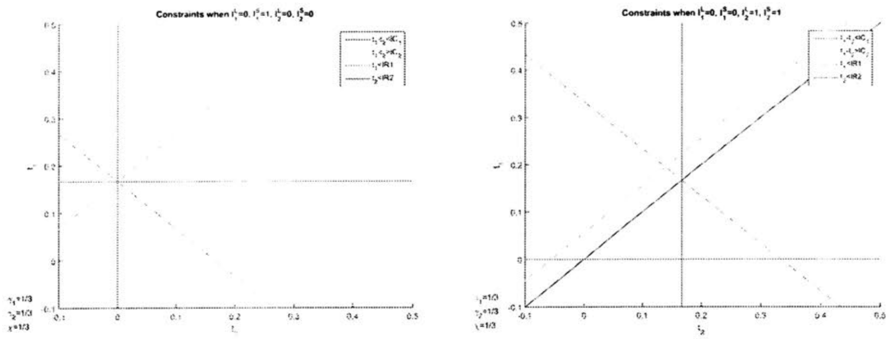


(c) $I_1^L = 1,\ I_1^S = 0,\ I_2^S = I_2^L = 1$



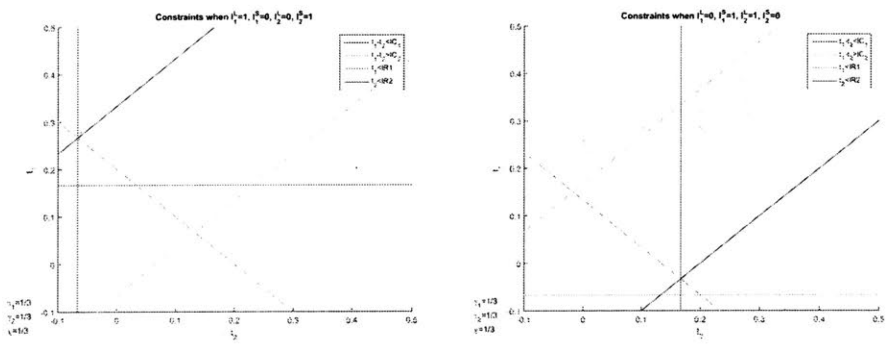
(d) $I_1^L = 0,\ I_1^S = 1,\ I_2^S = I_2^L = 1$

Figure 18: Illustrations of the constraints; symmetric case; one party with full information



(a) $I_1^S = I_1^L = 1,\ I_2^L = 0, I_2^S = 0$ (b) $I_1^S = I_1^L = 0,\ I_2^L = 1, I_2^S = 1$

Figure 19: Illustrations of the constraints; symmetric case; one party with zero information (exclusion)



(a) $I_1^L = 1,\ I_1^S = 0,\ I_2^L = 0, I_2^S = 1$ (b) $I_1^L = 0,\ I_1^S = 1,\ I_2^L = 1, I_2^S = 0$

Figure 20: Illustrations of the constraints; symmetric case; parties specialized in L and S dimensions

6.2 Description of procedure used to generate Figure 11 and 10.

1. Scrape customer case study from the AWS website.

2. Tokenize and stem each case study document using NLTK's list of English stop words and Snowball stemmers.

3. Transform the corpus into vector space using tf-idf calculating cosine distance between each document as a measure of similarity. We define term frequency-inverse document frequency (tf-idf) vectorizer parameters and then convert the text list into a tf-idf matrix. To get a Tf-idf matrix, first count word occurrences by document. This is transformed into a document-term matrix (dtm). This is also just called a term frequency matrix. Then apply the term frequency-inverse document frequency weighting: words that occur frequently within a document but not frequently within the corpus receive a higher weighting as these words are assumed to contain more meaning in relation to the document.

4. Cluster the documents using the k-means algorithm. Using the tf-idf matrix, we initializes K-means with a pre-determined number of clusters (e.g., 5 clusters as is the case here). Each observation is assigned to a cluster (cluster assignment) so as to minimize the within cluster sum of squares. Next, the mean of the clustered observations is calculated and used as the new cluster centroid. Then, observations are reassigned to clusters and centroids

recalculated in an iterative process until the algorithm reaches convergence.

5. Use multidimensional scaling to reduce dimensionality within the corpus.

The graphs plot the clustering output from the above procedure using matplotlib and mpld3.

Chapter 3: Information Theory Foundation of Propaganda

1 Introduction

What is propaganda? Canonical political theory holds that propaganda is restriction of information. Hence, it can be effective only when the regime can control the media outlets, restricting the number of information sources (Stockmann 2012, Chapter 8). According to this view, when the number of media sources proliferate and goes beyond the control of the state, citizens will select out of biased state media sources into reliable information. An alternative theory proposes that propaganda serves as a signal, creating norms that citizens are *trained* to follow (Brady 2008). As a form of "cultural governance" (Perry 2013), propaganda encourages compliance via establishing rituals and statndards (Brady 2008). These theories were challenged by empirical works by King, Pan and Roberts (2013, 2014) who found that government restriction on information largely doesn't target criticism of government policies, but remove all post related to collective action events, activists regardless of their support or criticism of the government.

Consider a society consisting of a continuum of citizens. Each citizen would like to take actions that represent their best assessment of an ex-ante state. We interpret the state as the degree of underlying social tension and the action as the strength of protests. In the case where the action is binary, it could represent the choice between revolt or no revolt, as in the original Angeletos et al. (2005) model on regime change. Citizens would also like their actions to fit in with each other. This could reflect the psychological tendency of herding behavior, fear of a crackdown (as Roberts (2018) documentts, small groups of individuals are more likely to be targeted by the authoritarian state to "make an example" and deter others from taking similar actions), or fixed cost hurdles of collective action (similar to the idea of critical hurdle in Pavan's model).

The authoritarian regime who controls the propaganda machine (the official mouthpiece) seeks to disrupt citizens' collective action and to mislead (bias the citizens' actions away from the real state). The question we are interested in answering are the following: Can the regime achieve its goal? When is propaganda most effective? When will propaganda backfire? How does changes in environment (e.g., a reduction in cost of information production thanks to social media) affect the regime's incentive to manipulate? Our baseline model features rational decision makers who understand the regime's objective perfectly. In other words, we want to understand the role of propaganda from a pure strategic information transmission point of view, without relying assumptions of credulity or inattention, as in Little (2012, 2015). We find that the effectiveness of propaganda depends crutially on the availability and quality of an alternative, untampered information source. This is unsurprising: In the extreme case where citizens can *costlessly* obtain undistorted information which reflects the true state of the world, the authoritarian regime's propaganda is completely useless. It should come as no surpise that governments who seek to manipulate information use tools to increase the cost of obtaining outside information, a practice which Roberts (2018) refers to as creating information "friction". By imposing small taxes on information access, the government sets up a threshold for citizens to cross. Should they find it too costly to resort to those outside sources, they will choose to listen to the biased but costly sources. In other words, citizens' attention to propaganda reflects an optimal tradeoff in information acquisition, not just a behavioral anomaly.

This paper is related to several strands of literature.

Political science analysis of the mechanism of censorship and media bias

The study of media bias often assumes that citizens are subject to behavioral bias, e.g., confirmation bias, herding tendency, so that the equilibrium information sent departs from the true state of the world (Gentzkow and Shapiro (2006), Allcott and Gentzkow (2017), Gentzkow, Shapiro and Stone (2015)). On the other hand, the study of propaganda, which by definition represents media bias in its most extreme form, focuses on the departure of the sender's objective from the receivers (citizens).

Traditional political science theory posits that authoritarian regime's control of information is based on deterrence. By definition, this can be very constraining in that threat must be observable and credible in order to have an impact. On the other hand, high visibility of deterrence may draw attention to authoritarian weakness and create backlash. Edmond (2007) studies a regime-change model where the dictator exerts costly efforts to downplay the regime's weakness, thereby diluting citizens' tendency to protest. Egorov, Guriev and Sonin (2009) propose a model in which a watchdog media helps the central government to keep local officials in check, while steering away from reporting on wrongdoing at higher levels. Lorentzen (2012) develops a model inwhich the regime chooses how much free journalism to permit, balancing the desire to minimize local corruption and the desire to shield the regime's core interests from public scrutiny. Gehlbach and Sonin (2013) explore cases where the authoritarian regime skews news reporting to affect citizens' investment behaviors. These models tend to focus on the role of propaganda to mislead citizens, which, by definition only works if citizens are misled.

Roberts (2018) posits that propaganda can also be effective as a form of censorship on the online environment by influencing the relative costs of information; formalizing the connection between propaganda and information theory. She points out information flooding can be effective in changing citizens' action by creating distraction and confusion. Unlike deterence, information manipulation through changing the noise in the system does not need to be obviously observed to have an impact on information consumption and dissemination.

Roberts (2018) proposes that part of the strategy of information flooding is issuing propaganda from many different sources, so as to disguise the fact that the information orignated with the government, so that the authoritarian regime's credibility is not compromised. This is related to the facts that propaganda is subject to citizens' interpretation, and it is only effective to the extent that the bias in information conveyed is not completely undone. Using leaked archives from government sources, Roberts and Stewart (2016) show that the regime uses propaganda to distract citizens from valuable information by producing detailed coverage of mundane routine party meetings.

Empirical analysis by King et al. (2014) find that authoritarian governments who have control over internet censorship focus the efforts on removing those posts related to collection actions, e.g., protest events or those who could organize protests. They conclude that the goal of modern autocrats is to nudge citizens away from "focal points" surrounding which citizens can coordinate actions.

Another important empirical analysis on the mechanism of propaganda is by Roberts and Stewarts (2016), who reverse engineer propaganda by identifying days when all newspapers publish the same or nearly identical articles. Such irregular repetitions are detected using plagarism detection software, combined with leaked directives from the authorities. The paper finds that flooding begets more flooding (shares, likes, etc.), which the authors term the multiplier effect. Propaganda is effective because what began as a propaganda can seem like an online event created by citizens as more people read and share the story.

Empirical works on citizens' reactions to propoganda inspires several modeling assumptions of the paper. First, empirical analysis shows that citizens do not react to propaganda passively. Roberts et al. (2016) conduct a matched pair study using data made available by Fu, Chan and Chau (2013) of users who experience the regime's restriction on free expression versus other who do not. They find that the former are aware of censorship (when their online posts are taken down) and react by more actively re posting their censored content. Interestingly, as they become more targeted by the censors, they also become more proactive in expressing their frustration with censorship.

These findings are further corroborated by survey studies, in which Chinese internet users are asked about how they would feel if their online activities were blocked or deleted. Among those who are aware of propaganda and who have experienced censorship, many report feeling angry and being more likely to seek out untampered information, rather than feeling passively worried.

One interesting online experiement randomly assigns participants to come across a blocked webpage. The researchers find that observation of government intervention in the flow of information generates more, not less, interest in the topic in question.

Game theoretic models of strategic information transmission The model features a sender and a unit mass of ex-ante identical receivers. The preferences of the informed sender and the uninformed receivers are misaligned, as in the classic model of Crawford and Sobel (1982). But in our model, signaling is costly (we have costless signaling as a special case). In our model, strategic signal transmission works through changing the information structure of the game, not just biasing the receivers' belief. In fact, in equilibrium, the posterior beliefs of the receivers are unbiased. This is the key distinction between this model and Little (2017) who studies the mechanism of propaganda through the lens of game theory, but his model has a fraction of credulous receivers who are inclined to take the sender's word for granted. In contrast to this "self-fulfilling promise" strand of literature, our model features correct equilibrium belief, but manipulation still works for the sender because it changes other aspects of the receivers' information structure. Following the intuition of Kamenica and Gentzkow (2011), our model shows that manipulating the mean of the belief is not the only tool that the sender has. If he can credibly manipulate other moments of the information structure (e.g., the variance), he can still be made better off. However, our work is distinct from Kamenica and Gentzkow (2011) in that the sender is not allowed to commit to an information structure: the sender chooses the message (in our setting, a propaganda) after becoming informed about the true state of the world. In this sense, the sender-receiver problem is one of strategic equilibrium rather than a single agent problem.

Organizational Economics literature of leadership in large organizations

Since Rotemberg and Saloner (1993), various organizational economic models have been proposed in which the leader's preferences affect the followers' incentives to achieve organizational goals. Hermalin (1998) considers a model where the leader has superior information than the followers and aims to overcome the problem of "free-riding" in a setting where the aggregate output depends on all members' efforts but the leader cannot correctly give individual follower's credit upon observing the final output. In equilibrium, the leader who has private information about the return to effort will overstate it so as to incentivize more efforts and thus mitigate the free-riding problem. Rotemberg and Saloner (2000) propose a model where leaders can "sharpen the focus" by ruling out certain actions, so that employees are better able to coordinate on their course of actions. Dessein and Santos (2006) consider an organization trading off coordination and adaptation through signal communication. However in their model the leader (i.e., the manager) does not have information superiority and differs from the followers (i.e., the employees) in that he has superior rights to decide. Dewan and Myatt (2007) point out that a leader's communication both provides direction and clarifies the state of the world. Bolton, Brunnermeier and Veldkamp (2009) consider a model in which a leader's personal trait of "resoluteness" helps overcome the organization's incentives misalignment problem and facilitate coordination.

2 Model

There is a unit mass of ex-ante identical receivers (the citizens), indexed by $i \in [0, 1]$, and a single sender (the authoritarian's mouthpiece) who attempts to achieve its goal through manipulating the receivers' beliefs.

There is underlying state $\theta \in \mathbb{R}$ about which the receivers are imperfectly informed. The sender, on the other hand, knows the state perfectly but might not find it beneficial to disclose the state to the receivers.

Each receiver i chooses an action $a_i \in \mathbb{R}$ to balance two objectives. First, each receiver wants to coordinate with each other. Specifically, receiver i wants her own action to stay as close as possible to the aggregate action $\int_j a_j d_j$. Second, each receiver wants to match her action to the true state of the world θ. Since θ is unknown, she will base her decision on her best assessment of the underlying state θ, conditional on the information she receives (which we will elaborate on further). As in Morris and Shin (2002), we adopt a quadratic-loss formulation of receiver's objective. Specifically, if the state θ and the aggregate action $\int_j a_j d_j$ are known with certainty, receiver i chooses a_i to minimize the quadratic loss, or

$$min\ \lambda(a_i - \int_j a_j dj)^2 + (1 - \lambda)(a_i - \theta)^2$$

where the parameter $\lambda \in [0, 1]$ measures the importance of the coordination motive, relative to the goal of matching one's action to the true state. In the case where either aggregate action $\int_j a_j d_j$ or true state θ is not known, the receiver will minimize the expectation of the above expection, where the expectation is conditional on the information (manipulated or not) received.

Before turning to the sender's problem and the information structure, let's consider a few special cases.

If $\lambda = 0$, there is no coordination motive, each receiver will simply set a_i to their expected θ. If $\lambda = 1$, each receiver i will set a_i to match the expected aggregate action $\int_j a_j d_j$. When $\lambda \in (0, 1)$ but θ is known with certainty, receiver i will take the optimal action

$$a_i = \lambda A + (1 - \lambda)\theta$$

where $A = \int_j a_j d_j$

Given the assumption that θ is known, in equilibrium, each individual will take $a_i = \theta = A$ hence the two objectives are perfectly aligned. When there is uncertainty regarding θ, individuals' actions might not be perfectly aligned, a case which we will turn to later.

Next, we turn to the objective of the sender. The sender knows the value of θ, but he has incentives to distort and disguise the true θ from the receivers. This captures the idea that θ which measures the underlying strength of the regime is known by the regime but not by the citizen. The regime may seek to mislead the citizens into believing the state of affairs is $\tilde{\theta}$, a value which is different than the actual θ. Specifically, we model the sender's objective, given a manipulated state $\tilde{\theta}$ and a true state θ as

$$V(\tilde{\theta}) = \gamma \int_i (a_i - \theta)^2 di + (1 - \gamma)(A - \tilde{\theta})^2 - c(\tilde{\theta} - \theta)^2$$

The first term in the expression $\int_i (a_i - \theta)^2 di$ measures the dispersion of citizens' action around the true state. As we discussed above, $a_i = A = \theta \ \forall i$ is the receivers' equilibrium action profile in the case of no-uncertainty. The interpretation of this term is the innate variation in the citizens' action profile, independent of the information structure of the environment. Also note that this term is the aggregation of $(a_i - \theta)^2$ over all i, while $(a_i - \theta)^2$ (multiplied by $-(1 - \lambda)$) enters receiver i's objective. This is a stark contrast to the signaling literature on leadership, where the leader's and the followers' goals are aligned (partially), and the leader's objective has a term which is the aggregation of the followers' objective, analogous to a welfare function.

The second term in the expression measures the sender's incentive to achieve his own political agenda: he wants the average of the receivers' actions to be as close to his desired state $\tilde{\theta}$ as possible. In the special case where the sender has no specific agenda, i.e, $\tilde{\theta} = \theta$, the objective reduces to

$$\gamma \int_i (a_i - \theta)^2 d_i + (1-\gamma)(A-\theta)^2$$

Interestingly, this is the same expression as the weighted average of variance and (quadratic) bias calculation from the statistics literature. The differences here are two-fold. First, the sender seeks to maximize this expression, instead of minimizing it. Second, the sender cannot dictate the receivers' actions.

Finally, the last term measures the cost of manipulation. To illustrate the idea, define $b = \tilde{\theta} - \theta$ which is the bias of the regime (relative to the true state). The last term says the cost of introducing a bias of size b is quadratic with coefficient c. In the special case where $c = 0$, this corresponds to the cheap talk model.

More generally, the cost term can be flexibly dependent on $\tilde{\theta}$ and θ, which we write as $c(\tilde{\theta}, \theta)$. This allows for the case where the cost of manipulation is asymmetric: it's easier to manipulate when the actual state is higher than the anchor $\tilde{\theta}$ and harder to manipulate otherwise.

In the limiting case where $c \to \infty$, the cost of manipulating the state away from θ is so high that the sender will choose $\tilde{\theta} = \theta$.

Also note that $\int_i (a_i - \theta)^2 d_i = \int_i (a_i - A)^2 + (A - \theta)^2$. So the sender's objective can be rewritten as

$$\gamma \int_i (a_i - A)^2 + (A-\theta)^2 - c(\tilde{\theta} - \theta)^2$$

When $\gamma = 1$, the sender's objective reduces to

$$V(\tilde{\theta}) = \int_i (a_i - \theta)^2 di - c(\tilde{\theta} - \theta)^2$$

In this case, the sender only cares about forestalling coordination, which is the idea captured by Gary King's model of propaganda as distraction and disruption of mass action.

When $\gamma = 0$, the sender's objective reduces to

$$V(\tilde{\theta}) = (A - \tilde{\theta})^2 - c(\tilde{\theta} - \theta)^2$$

In this case, the sender only cares about distorting aggregate behavior away from θ to its own desired state $\tilde{\theta}$. This corresponds to the influencing model of costly signal transmission.

Next, we turn to the information structure of the model. In our baseline model, we assume that the receivers have perfect information about the sender's objective function, but are not perfectly informed about the realization of θ.

There is an uncontaminated source of public information z about θ which has variance $1/\alpha_z$ (so the precision of z is α_z). This can be seen as either a common prior or some verifiable, baseline assessment of the state. In addition, each receiver receives idiosyncratic signal which is subject to manipulation $x_i = \tilde{\theta} + \epsilon_i = \theta + b + \epsilon_i$.

We assume that ϵ_i is independently and identically distributed across all i, with mean zero and variance $1/\alpha_\epsilon$ (so the precision of ϵ_i is α_i).

To sum up, given a true state θ, each receiver receives two signals which she uses to base her beliefs on. A tampered private signal $x_i \sim N(\tilde{\theta}, 1/\alpha_\epsilon)$, where $\tilde{\theta}$ is a tampered version of θ, as we discussed above, and an untampered public signal $z \sim N(\theta, 1/\alpha_z)$. From this formulation, it is easy to see that the mean of the tampered signal x_i is endogenously determined. For our baseline model, we assume that the precision of the untampered signal is fixed, this corresponds to the interpretation z as a common prior. More generally, α_x and α_z can be endogenized. For example, we might assume that citizens may collectively pay $c(\alpha_z)$ to improve on the precision of z. This cost could be either real monetary cost (purchase of VPN, etc.), monetarized time cost (waiting time for throttled web traffic), or attention cost (a la Myatt (2011)) associated with information acquisition. Importantly, because the signal z is public, we naturally have the free-riding problem as with any public good purchase had this purchase been private instead of public. We will discuss variants of this assumption later in this paper.

Formally, the game proceeds as follows.

In stage 0, nature draws θ and z. θ is observed only by the sender while z is observed by both the sender and the receiver.

- In stage 1, the sender privately chooses an propaganda/information manipulation policy at a cost proportional to the size of manipulation.
- In stage 2, the sender's manipulated signal is realized and privately observed by each citizen.
- In stage 3, receivers take actions.
- In stage 4, payoffs are realized.

We define a symmetric perfect Bayesian equilibrium of the propaganda model as consisting of the following:

1. a profile of receiver actions $a(x_i, z)$ and subjective assessments;

2. a sender's manipulation policy $\tilde{\theta}(\theta, z)$;

3. an aggregate action $A(X, z) \equiv \int_i a(x_i, z)di$, where X denotes the entire distribution of x_i;

such that:

1. Each receiver's action a_i maximizes her objective, given her assessment of the true state θ and the aggregate action A;

2. Sender's manipulation function maximizes the sender's objective, given the receiver's action profile and the corresponding aggregate action;

3. The equilibrium assessment of aggregate action is consistent with the individual action profile;

4. Receiver's beliefs are rational and consistent with the sender's manipulation function.

3 Analysis

In this section, we first solve for the sender's and receivers' equilibrium strategies, holding model parameters and the other party's strategy as fixed. Then we characterize the equilibrium of the model. Finally, we discuss the changes to the equilibrium solutions when environmental parameters change. We also study the welfare implications of the model. Before we solve for the general case, we start with a special case where manipulation is infinitely costly.

3.1 Case 1: $c \to \infty$ Equilibrium without information manipulation

First we consider the baseline case when the sender cannot manipulate information. One way to think about the model is that this corresponds to the special case where $c \to \infty$ so that the sender always optimally chooses $\tilde{\theta} = \theta$. In this case, the receiver's problem is exactly the same in as the setting of Morris and Shin (2002).

Specifically, given a public signal z and a private signal x_i, write the receiver's interim belief about aggregate action $A(\theta, z)$ and the true state θ as, respectively, $\mathbb{E}[A(\theta, z)|x_i, z]$ and $\mathbb{E}[\theta|x_i, z]$.

Since both x_i and z follow normal distributions, the posterior belief of θ has a mixed normal distribution as a weighted average of the distribution of x_i and z, where the weights are given by the relative precisions of x_i and z. We have

$$\mathbb{E}[\theta|x_i, z] = \frac{\alpha_x}{\alpha_x + \alpha_z} x_i + \frac{\alpha_z}{\alpha_x + \alpha_z} z$$

Following Morris and Shin (2002), it is without loss of generality to focus on symmetric linear strategies of the form

$$a(x_i, z) = kx_i + hz$$

where the coefficients k and h will be solved in equilibrium.

By assumption, $\tilde{\theta} = \theta$, so the aggregate action, defined as $A = \int_j a(x_j, z) d_j$ is

$$A = k\tilde{\theta} + hz = k\theta + hz$$

Taking expectations on both sides, we have

$$\mathbb{E}[A|x_i, z] = k\mathbb{E}[\theta|x_i z] + hz$$

Substituting into the receiver's optimal strategy,

$$a(x_i, z) = \lambda \mathbb{E}[A|x_i, z] + (1 - \lambda)\mathbb{E}[\theta|x_i, z]$$

we have

$$\begin{aligned} a(x_i, z) &= \lambda(k\mathbb{E}[\theta|x_i z] + hz) + (1 - \lambda)\mathbb{E}[\theta|x_i, z] \\ &= (\lambda k + 1 - \lambda)\frac{\alpha_x}{\alpha_x + \alpha_z} x_i + ((\lambda k + 1 - \lambda)\frac{\alpha_z}{\alpha_x + \alpha_z} + \lambda h)\text{z} \end{aligned}$$

Match coefficients to the postulated linear strategy, we have

$$k = (\lambda k + 1 - \lambda)\frac{\alpha_x}{\alpha_x + \alpha_z}$$

or

$$k^*_{MS} = \frac{(1 - \lambda)\alpha_x}{(1 - \lambda)\alpha_x + \alpha_z}$$

and

$$h = 1 - k^*_{MS}$$

The expression for k is intuitive. First note that if there is no coordination motive, this expression reduces to $k = \frac{\alpha_x}{\alpha_x+\alpha_z}$, and in this case, $a_i = \mathbb{E}[\theta|x_i, z]$ which is just receiver i's interim expectation of the true state, weighted by the relative precision of the two signals, as we discussed above.

When there is nontrivial coordination motive, agents will give more weight to the public signal z, since this is the signal received by all receivers and the signal which conveys not only information about the true state θ but also the actions by the other agents.

3.2 Case 2: c is finite, Equilibrium with information manipulation

Now, let's turn to the case where the sender may manipulate information at a cost. Conditional on the signals x_i and z, the receiver's problem is essentially the same as the baseline case we discussed above. Specifically, the receiver's objective (gross of cost of information acquisition) is

$$min\ \lambda(a_i - \int_j a_j dj)^2 + (1-\lambda)(a_i - \theta)^2$$

Following our discussion in the previous section, the solution to the receiver's problem is

$$a(x_i, z) = \lambda E\left[A(\theta, z)|x_i, z\right] + (1-\lambda)\mathbb{E}\left[\theta|x_i, z\right]$$

The difference is we can no longer write the conditional expection of θ in terms of exogenous parameters, as we did in the previous section

$$\mathbb{E}[\theta|x_i, z] = \frac{\alpha_x}{\alpha_x+\alpha_z}x_i + \frac{\alpha_z}{\alpha_x+\alpha_z}z$$

The reason is that since x_i is subject to manipulation, the relative precision of that signal is determined in equilibrium, which we wil elaborate on further.

However, it is easy to show that it is without loss of generality to focus on linear weighted strategy of the form $a(x_i, z) = kx_i + (1-k)z$. (The proof is tedious, but the argument is essentially the same as the proof in Morris and Shin (2001)) where k (with mild abuse of notation) is endogenously determined.

Taking expectation of the expression for $a(x_i, z)$, we have an expression for the aggregate receiver's action

$A(\tilde{\theta}, z) = k\tilde{\theta} + (1-k)z$ecalling that $x_i = \tilde{\theta} + \epsilon_i$ where $\epsilon_i \sim N(0, 1/\alpha_\epsilon)$ is the distribution of the manipulated signal.

The sender's objective is

$$V(\tilde{\theta}) = \gamma(k\tilde{\theta} + (1-k)z - \theta)^2 - \gamma\frac{k^2}{\alpha_\epsilon} + (1-\gamma)\left((k-1)\tilde{\theta} + (1-k)z\right)^2 - c\left(\tilde{\theta} - \theta\right)^2$$

The sender chooses a manipulation policy $\tilde{\theta}$ to maximize the above expression, which yields the first order condition

$$\gamma k(k\tilde{\theta} + (1-k)z - \theta) + (1-\gamma)(k-1)\left((k-1)\tilde{\theta} + (1-k)z\right) - c\left(\tilde{\theta} - \theta\right) = 0$$

and the second order condition

$$\gamma k^2 + (1-\gamma)(1-k)^2 - c < 0$$

Solving for the first order condition, we have[1]

$$\tilde{\theta} = \frac{(\gamma k - c)}{(\gamma k^2 + (1-\gamma)(1-k)^2 - c)}\theta + \frac{((1-\gamma)(1-k)^2 - \gamma k(1-k))}{(\gamma k^2 + (1-\gamma)(1-k)^2 - c)}z$$

Let

$$\beta(k \mid c, \gamma) = \frac{((1-\gamma)(1-k)^2 - \gamma k(1-k))}{(\gamma k^2 + (1-\gamma)(1-k)^2 - c)}$$

denote the relative weight on z. We have that $\beta(k|c,\gamma) \leq 1$if and only $\gamma k \leq c$ and $\beta(k \mid c, \gamma) > 0$ if and only if $\gamma(k - k^2) - (1-\gamma)(1 - 2k + k^2) \geq 0$, or equivalently, $(1-k)(1-k-\gamma) \leq 0$.

The above expression for $\tilde{\theta}$ can be written as $\tilde{\theta} = (1-\beta)\theta + \beta z$. To see an interpretation for β, note

[1]Note that

$$\begin{aligned}&(\gamma k^2 + (1-\gamma)(1-k)^2 - c)\tilde{\theta} \\ =&(\gamma k - c)\theta + ((1-\gamma)(1-k)^2 - \gamma k(1-k))z\end{aligned}$$

that $(\tilde{\theta} - \theta) = \beta(z - \theta)$, hence

$$\beta = \frac{(\tilde{\theta} - \theta)/\theta}{(z - \theta)/\theta} = \frac{\Delta\tilde{\theta}\%}{\Delta z\%}$$

In other words, β measures the **relative bias** of the tampered signal (relative to the fundamental bias captured by $\Delta z\%$.

For the receiver's problem, note that

$$\begin{aligned} a_i &= (1 - \lambda)E[\theta|x_i, z] + \lambda E[A|x_i, z] \\ &= [1 - \lambda(1 - k(1 - \beta))]\, E[\theta|x_i, z] + \lambda(1 - k(1 - \beta))z \end{aligned}$$

Importantly, we cannot write $\mathbb{E}[\theta|x_i, z]$ using our previous formulation, since x_i and z are not independent: x_i comes from the sender's manipulation policy, which takes into account z to the extent that z affects receivers' actions and those actions enter the sender's objective function.

A first step toward solving for a_i is to decompose x_i into two orthogonal parts, one related to z but not x_i, the other related to x_i but not z[2].

Define

$$s_i \equiv \frac{1}{1-\beta}(x_i - \beta z) = \theta + \frac{1}{(1-\beta)}\epsilon_i$$

This constructed signal is independent from z and "up-weigh" the noise ϵ_i by a fraction $\frac{1}{1-\beta}$. Given this formulation, we have

$$E[\theta|s_i, z] = \frac{(1-\beta)^2\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z}s_i + \frac{\alpha_z}{(1-\delta\beta)\alpha_x + \alpha_z}z$$

$$E[\theta|x_i, z] = \frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z}x_i + (1 - \frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z})z$$

3

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[2]Note that

$$\begin{aligned} x_i &= \tilde{\theta} + \epsilon_i \\ &= (1-\beta)\theta + \beta z + \epsilon_i \\ &= (1-\beta)\theta + \beta(\theta + \epsilon_z) + \epsilon_i \\ &= \theta + \beta\epsilon_z + \epsilon_i \end{aligned}$$

[3]Note that this technique can be easily extended to the case with n untampered signals $Z = (z_1, ..., z_n)$ and m potentially tampered signals $X = (x_1, ..., x_m)$. Denote the pdf of a $n + m$ dimensional Gaussian random vector y as:

$$f(\vec{y}) = \frac{1}{(2\pi)^{(n+m)/2}|\Gamma|^{\frac{1}{2}}} exp\left(-\frac{1}{2}(\vec{y} - \bar{y})^T\Gamma^{-1}(\vec{y} - \bar{y})\right)$$

Substituting into the expression for a_i, we have

$$a_i = [1 - \lambda(1 - k(1-\beta))]\left(\frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z}x_i + (1 - \frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z})z\right)$$
$$+ \lambda(1 - k(1-\beta))z$$

Matching coefficients, we have

$$k = (\lambda k(1-\beta) + 1 - \lambda)\frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z}$$

Let $\alpha = \alpha_x/\alpha_z$ and $m(\beta, \alpha) = \frac{(1-\beta)\alpha}{(1-\beta)^2\alpha+1}$. We have $k = (\lambda k(1-\beta) + 1 - \lambda)m(\beta, \alpha)$. Or equivalently $k(\beta|\ \alpha, \lambda) = \frac{(1-\lambda)m(\beta,\alpha)}{1-\lambda(1-\beta)m(\beta,\alpha)}$. Intuitively, m is the weight of $E[\theta|x_i, z]$ on x_i relative to z, which is a function of the endogenous relative bias β and the relative precision α.

For another interpretation, note that

$$x_i = (1-\beta)\theta + \beta z + \epsilon_i = \theta + \beta\epsilon_z + \epsilon_i$$

$$z = \theta + \epsilon_z$$

so β also measures the covariance of x_i and z. We have

$$dm/d(1-\beta) = \frac{\alpha - (1-\beta)^2\alpha^2}{((1-\beta)^2\alpha + 1)^2} > 0$$

if and only if $(1-\beta)^2 \leq 1/\alpha$. Intuitively, the weight of $E[\theta|x_i, z]$ on x_i is increasing in the (partialed out) covariance between x_i and θ and decreasing in the covariance between x_i and z as long as the precision of x is sufficiently small (not too large relative to the precision of z), i.e., when $1/\alpha = \alpha_z/\alpha_x$ is sufficiently large.

On the other hand, when $1/\alpha = \alpha_z/\alpha_x$ is small, or when α_x is sufficiently large, the covariance between x_i and θ is decreasing. This seemingly counterintuitive prediction arises from the endogeneity of x_i. To

where $\bar{y}$ is the mean of the vector $\vec{y}$, Γ is the $(n+m) \times (n+m)$ dimensional covariance matrix, and $|\Gamma|$ is the determinant of Γ.

We write $Z \sim N(\vec{z}|\bar{z}, \Gamma_z)$ and $X \sim N(\vec{x}|\bar{x}, \Gamma_x)$. Then the conditional distribution of y, has the following form:

$$\vec{y}|X, Z \sim N(\vec{y}|m, \Gamma)$$

where $\Gamma^{-1} = \Gamma_x^{-1} + \Gamma_z^{-1}$and $m = \Gamma\Gamma_x^{-1}X + \Gamma\Gamma_z^{-1}Z$

see this, compare the expression for $E[\theta|s_i, z]$

$$E[\theta|s_i, z] = \frac{(1-\beta)^2\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z}s_i + \frac{\alpha_z}{(1-\beta)^2\alpha_x + \alpha_z}z$$

to the expression for $E[\theta|x_i, z]$

$$E[\theta|x_i, z] = \frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z}x_i + (1 - \frac{(1-\beta)\alpha_x}{(1-\beta)^2\alpha_x + \alpha_z})z$$

where $m(\beta, \alpha) = \frac{(1-\beta)}{(1-\beta)^2+\alpha}$ and $\alpha = \alpha_z/\alpha_x$ captures the weight on x_i in the second expression and $\tilde{m}(\beta, \alpha) \equiv \frac{(1-\beta)^2}{(1-\beta)^2+\alpha_z}$ captures the weight on s_i (the constructed signal) in the first expression.

Note that

$$d\tilde{m}/dh = \frac{2h\alpha}{(h^2+\alpha)^2} > 0$$

unambiguously. This is because, by construction, s_i is orthogonal of z, and hence is independent of the endogenous manipulation motive which is present in x_i. This intuition will be clearer after we solve for the equlibrium of the model.

To summarize our discussion thus far, the equilibrium of the model is determined by the intersection of a manipulation policy

$$\beta(k \mid c, \gamma) = \frac{((1-\gamma)(1-k)^2 - \gamma k(1-k))}{(\gamma k^2 + (1-\gamma)(1-k)^2 - c)}$$

and an interpretation policy

$$k(\beta| \, \alpha, \lambda) = \frac{(1-\lambda)m(\beta, \alpha)}{1 - \lambda(1-\beta)m(\beta, \alpha)}$$

where $m(\beta, \alpha) = \frac{(1-\beta)}{(1-\beta)^2+1/\alpha}$ and $\alpha = \alpha_x/\alpha_z$.

The specific forms of those two functions depend on the parameters of the environment. Specifically, c measures the cost of manipulation, γ measures the relative importance of the sender's motive to disrupt coordination, λ measures the receivers' coordination motive and α measures the relative precision of the public signal.

3.3 Equilibrium Characterization

To characterize the equilibrium of the model, first, we study the domain of $k(\beta)$ and $\beta(k)$ (holding fixed the shifters).

The natural domain for β is $\mathbb{R}$, while the natural domain for k is given by the second order condition:

$$c - (\gamma k^2 + (1-\gamma)(1-k)^2) \geq 0$$

or

$$k^2 - 2(1-\gamma)k + (1-\gamma-c) \leq 0$$

Note that is a quadratic expression in k for parameter values of c and γ. In the figure in the Appendix, we plot the domain for k, $K(c,\gamma)$ for various combinations of (c,γ).

The domain is symmetric around $k = 1-\gamma$ and is unempty if

$$\gamma^2 - \gamma + c \geq 0$$

which is guaranteed to hold since $\gamma \in [0,1]$.

$$\beta'(k|\, c,\gamma) = \frac{[-2(1-k)+\gamma]\left[\gamma k^2 + (1-\gamma)(1-k)^2 - c\right]}{[(\gamma k^2 + (1-\gamma)(1-k)^2 - c)]^2} \\ \frac{-(1-k-\gamma)(1-k)\left[2\gamma k - 2(1-\gamma)(1-k)\right]}{[(\gamma k^2 + (1-\gamma)(1-k)^2 - c)]^2} > 0$$

if and only if

$$\gamma k^2 - 2ck + 2c - \gamma - c\gamma + \gamma^2 \geq 0$$

In this special case $\gamma = 0$, this reduces to $2ck - 2c \leq 0$, or $k \leq 1$. From the second order condition $k^2 - 2(1-\gamma)k + (1-\gamma-c) \leq 0$, we have another condition:

$$k^2 - 2k + (1-c) \leq 0$$

or $k \in [1-\sqrt{c}, 1+\sqrt{c}]$

An immediate result is that when $\gamma = 0$, $\beta(k)$ is nonmonotonic. As long as c does not equal zero, the domain for k will overlap with both $k \leq 1$ and $k \geq 1$. This result says when the sole intention of the sender is to bias the aggregate action away from the true state, then there is an interior value of $\bar{k}$ such that for all $k \geq \bar{k}$, the more attention the receivers give to the biased signal, the less (relative) bias the sender introduces into the signal, in the sense that $\beta(k)$ is decreasing.

To see the intuition for this result, it's helpful to rewrite the sender's objective as

$$\left[(1-k)^2(1-\beta)^2 - c\beta\right](z-\theta)^2$$

The first term measures the benefit from manipulation: increasing the distance between A and $\tilde{\theta}$ through changing receivers attention paid to z (with coefficient $(1-k)$) and changing the gap between $(\tilde{\theta}-z)$.

Note that the cross partial of the above expression with respect to β and k is $4(1-\beta)(1-k)$. For the case of $0<\beta<1$ (the relative bias in $\tilde{\theta}$ is less than the relative bias in z), this expression is positive when $k<1$ and negative otherwise.

In other words, k and β are strategic complements when $k<1$.

When $\gamma=1$, the condition for $\beta'<0$ reduces to $k^2-2ck+c\leq 0$, or $k\in[c-\sqrt{c^2-c}, c+\sqrt{c^2-c}]$ if $c(c-1)>0$ and $k\in\emptyset$ if $0<c<1$[4].

From the second order condition, we have $k^2-c\leq 0$, or $k\in[-\sqrt{c},\sqrt{c}]$. The condition for $[c-\sqrt{c^2-c}, c+\sqrt{c^2-c}]\cap[-\sqrt{c},\sqrt{c}]$ being nonempty is $\sqrt{c}>c-\sqrt{c^2-c}$. (Note that $[c-\sqrt{c^2-c}, c+\sqrt{c^2-c}]$ is symmetric around $c>0$ and $[-\sqrt{c},\sqrt{c}]$ is symmetric around 0, so a sufficient condition for the overlap being nonempty is $\sqrt{c}>c-\sqrt{c^2-c}$), or $\sqrt{c^2-c}>c-\sqrt{c}$, which reduces to $c>1$. Hence, as long as $c>1$, there will be certain region where $\beta'<0$.

When $c=0$, the condition for $\beta'<0$ reduces to $k^2\leq 1-\gamma$ and the second order condition can be written as $(k-1)^2\leq 0$ which admits only $k=1$. This is intuitive, when cost of manipulation is zero, k is set to the boundary value.

Also not that given the second order condtion, the expression for β

$$\beta(k\mid c,\gamma)=\frac{((1-\gamma)(1-k)^2-\gamma k(1-k))}{(\gamma k^2+(1-\gamma)(1-k)^2-c)}$$

is negative if $\gamma k(1-k)-(1-\gamma)(1-k)^2<0$ or $(k-(1-\gamma))(k-1)>0$. So $\beta<0$ if $k>1$ or $k<1-\gamma$.

From the second order condition $\gamma k^2-2ck+2c-\gamma-c\gamma+\gamma^2\geq 0$, we see that the evaluation of the inequality at $k=1$ is $\gamma\geq c$ and the evaluation of the second order condition at $k=1-\gamma$ is $c-\gamma+\gamma^2\geq 0$. If $\gamma=0$, we have $\beta<0$ if $k<1$. If $\gamma=1$, we have $\beta<0$ if $k>1$ and $1\geq c$, or if $k<0$ and $c\geq 0$.

Next we turn to the shape of k function. In the following graphs, we plot the shapes of the k function

[4] An immediate result is that when $0<c<1$, $\beta'\geq 0$ always.

as a function of β, for various combinations of α and λ parameter values, where we recall that

$$k(\beta|\,\alpha,\lambda) = \frac{(1-\lambda)m(\beta,\alpha)}{1-\lambda(1-\beta)m(\beta,\alpha)}$$

where $m(\beta,\alpha) = \frac{(1-\beta)}{(1-\beta)^2+1/\alpha}$ and $\alpha = \alpha_x/\alpha_z$.

From the corresponding graphs in the appendix, it is obvious that the k is a cubic function in β which in general has two local extrema. Formally,

$$k'(\beta|\,\alpha,\lambda) = \frac{(1-\lambda)\frac{\partial}{\partial\beta}m(\beta,\alpha)\,[1-\lambda(1-\beta)m(\beta,\alpha)]}{[1-\lambda(1-\beta)m(\beta,\alpha)]^2}$$
$$\frac{-(1-\lambda)m(\beta,\alpha)\left[-\lambda(1-\beta)\frac{\partial}{\partial\beta}m(\beta,\alpha)+\lambda m(\beta,\alpha)\right]}{[1-\lambda(1-\beta)m(\beta,\alpha)]^2} > 0$$

if and only if $\frac{\partial}{\partial\beta}m(\beta,\alpha) > \lambda m^2(\beta,\alpha)$ where $m(\beta,\alpha) = \frac{(1-\beta)}{(1-\beta)^2+1/\alpha}$ and $\alpha = \alpha_x/\alpha_z$.

We have

$$(1-\beta)^2 - 1/\alpha > \lambda(1-\beta)\left[(1-\beta)^2+1/\alpha\right]$$

or eqivalently

$$\alpha\lambda(1-\beta)^3 - \alpha(1-\beta)^2 + \lambda(1-\beta) + 1 < 0$$

Note that when $\lambda = 0$, i.e, when there's no coordination motive, the expression for k coincides with the expression for m, i.e., $\frac{(1-\beta)}{(1-\beta)^2+1/\alpha}$, and $k' > 0$ if $1 < \alpha(1-\beta)^2$or $(1-\beta)^2 > 1/\alpha$.

This expression says the greater the precision of x (the larger the α_x or the smaller the $1/\alpha$ term), the more likely that greater β will induce greater k, and this condition is more easily satisfied when β is small. This intuition carries over to the general case.

In general, the condition can be rewritten as

$$\lambda\left[(1-\beta)^3 + \frac{1}{\alpha}(1-\beta)\right] < (1-\beta)^2 - \frac{1}{\alpha}$$

Consider the regular case $0 < (1-\beta) < 1$. We have $\lambda < \frac{(1-\beta)^2-\frac{1}{\alpha}}{(1-\beta)^3+\frac{1}{\alpha}(1-\beta)} \equiv Q(\frac{1}{\alpha})$. When $(1-\beta)^2 - \frac{1}{\alpha} < 0$, the above admits no solution for λ. Otherwise, when $(1-\beta)^2 - \frac{1}{\alpha} > 0$, the right hand side $Q(\frac{1}{\alpha})$ has derivative with respect to $\frac{1}{\alpha}$ as

$$Q'(\frac{1}{\alpha}) = \frac{-1((1-\beta)^3 + \frac{1}{\alpha}(1-\beta)) - ((1-\beta)^2 - \frac{1}{\alpha})(1-\beta)}{((1-\beta)^3+\frac{1}{\alpha}(1-\beta))^2} < 0$$

Hence the condition gets harder to satisfy when $1/\alpha$ increases, or when α_x becomes small.

Next, we give a full characterization of the domain and range of the functions $k(\beta)$ and $\beta(k)$. From the first order condition $\beta'(k) = 0$ or

$$\begin{aligned}\beta'(k|\, c, \gamma) &= \frac{[-2(1-k)+\gamma]\left[\gamma k^2 + (1-\gamma)(1-k)^2 - c\right]}{[(\gamma k^2 + (1-\gamma)(1-k)^2 - c)]^2} \\ &\quad \frac{-(1-k-\gamma)(1-k)\left[2\gamma k - 2(1-\gamma)(1-k)\right]}{[(\gamma k^2 + (1-\gamma)(1-k)^2 - c)]^2} \\ &= 0\end{aligned}$$

we have the equation $\gamma k^2 - 2ck + 2c - \gamma - c\gamma + \gamma^2 = 0$. Solve for the set of loal minima and local maxima of β by setting the first order condition to zero and finding the roots for k, we have

$$k_1 = \frac{1}{\gamma}(c + (c-\gamma)(\gamma^2 - \gamma + c))^{\frac{1}{2}}$$

$$k_2 = \frac{1}{\gamma}(c - (c-\gamma)(\gamma^2 - \gamma + c))^{\frac{1}{2}}$$

It's easy to show that $\beta(k_1) = 0$ and $\beta(k_2) = 0$. Hence the local maximum for β when k has interior value 0. Therefore, the range of β is given by the boundary values of k (derived from the second order condition) $\beta \in [0, 1]$. From the first order condition $k'(\beta) = 0$ or $\alpha\lambda(1-\beta)^3 - \alpha(1-\beta)^2 + \lambda(1-\beta) + 1 = 0$, we can solve for the set of local minima and local maxima of k between $\beta = 0$ and $\beta = 1$. It's easy to see that $k(1) = 0$ and $k(0) = \frac{1-\lambda}{1-\lambda+\frac{1}{\alpha}}$.

In the special case of $\gamma = 1$, the equilibrium characterization is especially transparent. In this case, the second order condition takes the simple form of $k \in [-\sqrt{c}, \sqrt{c}]$ and we have $\beta(k) < 0$ if $k > 1$ and $1 \geq c$, or if $k < 0$ and $c \geq 0$. Recall that $\beta(k \mid c, \gamma = 1) = \frac{k-k^2}{c-k^2}$. We have $\beta'(k) = 0 \iff k^2 - 2ck + c = 0$. The two corresponding roots are $k_1 = c - (c(c-1))^{\frac{1}{2}}$ and $k_2 = c + (c(c-1))^{\frac{1}{2}}$. The local extrema of β are given by $\beta(k_1) = 0$ and $\beta(k_2) = 0$. The boundary value of β are given by $\beta(0) = \frac{1}{c}$.

Consider the case $c > 1$. Note $\sqrt{c} < c$. $\beta(k) < 0$ if $k \in (1, \sqrt{c}]$, $\beta(k) < 0$ if $k < 0$ and $\beta \in [0, \frac{1}{c}] \subset [0, 1]$ if $k \in [0, 1]$. But from $k(\beta|\, \alpha, \lambda) = \frac{(1-\beta)(1-\lambda)}{(1-\beta)^2(1-\lambda)+\alpha}$and $\lambda \in [0, 1)$ and $\alpha > 0$, $k < 0$ iff $\beta > 1$. So all possible equilibria must lie in the region $\beta \in [0, 1]$ and $k \in [0, 1]$. Now for the case $c < 1$. Note $c < \sqrt{c}$. $\beta(k) < 0$ if $k < 0$ and $\beta(k) > 0$ if $k \in (0, \sqrt{c})$. We have $\beta > 1$ if $k \in (0, c)$ and $0 < \beta < 1$ if $k \in (c, \sqrt{c})$. So all possible equilibria must like in the region $\beta \in [0, 1]$ and $k \in (c, \sqrt{c}) \subset [0, 1]$.

3.4 Comparative statics

In this section, we examine the comparative statics of the policy functions β and k with respect to the parameters of the model, namely, c, γ, α and λ.

First note that any change in c and γ represents a shift in the function $\beta(k)$ and a movement along the curve $k(\beta)$. On the other hand, any change in α or λ is a shift in the $k(\beta)$ schedule and a movement along $\beta(k)$.

$$\frac{d}{dc}\beta(k|\ c,\gamma) = \frac{(1-k-\gamma)(1-k)}{(\gamma k^2 + (1-\gamma)(1-k)^2 - c)^2}$$

which has the same sign as $(1-k-\gamma)(1-k)$. Hence, we have $\frac{d\beta}{dc} < 0$ if $1-\gamma < k < 1$ (which is equivalent to the condition that $\beta \geq 0$) .

Note that if we limit our attention to the regular case where $0 < k < 1$, then this inequality is automatically satisfied at $\gamma = 1$ and admits no solution for k at $\gamma = 0$.

The first case is intuitive. If $\gamma = 1$, the sender only cares about disrupting coordination (through exposing receivers to aggregate noise). If c becomes large, the marginal cost of manipulation jumps up discretely while the marginal benefit of manipulation has no first order change. So on the margin, the optimal choise of β must derease (assuming continuity and differentiability on the margin).

For the second case, recall that $\beta(k \mid c, \gamma = 0) = \frac{-(1-k)^2}{c-(1-k)^2}$ which always has negative slope (and an essential discontinuity at $c = (1-k)^2$. To see why, recall that the expression for β comes from the first order condition from the sender's optimization problem: s

$$V(\tilde{\theta}) = \left((k-1)\tilde{\theta} + (1-k)z\right)^2 - c\left(\tilde{\theta} - \theta\right)^2$$

The sender chooses a manipulation policy $\tilde{\theta}$ to maximize the above expression, which yields the first order condition

$$(k-1)\left(\left(\tilde{\theta}-\theta\right)/(z-\theta) - 1\right) - c\left(\tilde{\theta}-\theta\right)/(z-\theta) = 0$$

Or $(k-1)^2(\beta-1) - c\beta = 0$. Equivalently, $(k-1)^2\left(1-\frac{1}{\beta}\right) - c = 0$.

An interpretation of this result is that when choosing for relative bias β, the sender balances the increase in cost off the change in marginal benefit. Recall that the aggregate action $A = k\tilde{\theta} + (1-k)z$, so

$$\begin{aligned}(A-\tilde{\theta})^2 &= (k-1)^2(\tilde{\theta}-z)^2 \\ &= (k-1)^2(\beta-1)^2(\theta-z)^2\end{aligned}$$

where $(\theta - z)^2$ is a structural noise exogenous to the model's choice parameters.

When c increases, the direction of change in β required to keep the first order condition hold is positive. (Note $\beta > 1$ in this scenario.) Intuitively, when it becomes more expensive to induce bias in $(\tilde{\theta} - \theta)$, the marginal benefit in inducing bias must be greater to justify the same size of bias. And in the case of $\gamma = 0$, this increase in marginal benefit is induced by inducing a larger bias size.

Next, we examine the comparative statics of β with respect to γ. Taking derivatives of β with respect to γ we have

$$\frac{d}{d\gamma}\beta(k|\, c,\gamma) = \frac{(k-1)(c-k(k-1))}{(\gamma k^2 + (1-\gamma)(1-k)^2 - c)^2}$$

Importantly, the sign of this expression is independent of γ (as it open appears in the denominator which is a squared term).

The expression has three roots, $k = 1$ and roots to $k^2 - k - c = 0$ or $k = \frac{1+\sqrt{1+4c}}{2}$ and $k = \frac{1-\sqrt{1+4c}}{2}$. When $0 < k < 1$ (the regular case), this expression is negative, so β is decreasing in γ.

When $k > 1$, the sign of the expression depends on the relative magnitude of c and $k(k-1)$. When c is sufficiently large, this expression becomes positive, i.e., β increases as γ increases. Intuitively, when γ is large, the motive to disrupt coordination increases, which means the sender wants to introduce more aggregate noise into individuals' decision profile. When $k > 1$, receivers put more weights on x_i and negative weight on z, so fixing k and c, the incentive to manipulate is greater under $\gamma = 1$.

Recall that $k(\beta|\, \alpha,\lambda) = \frac{(1-\lambda)m(\beta,\alpha)}{1-\lambda(1-\beta)m(\beta,\alpha)}$ where $m = \frac{(1-\beta)}{(1-\beta)^2+1/\alpha}$. Taking the derivatives with respect to α, we have

$$\begin{aligned}\frac{dk}{d\alpha} &= \frac{dk}{dm}\frac{\partial m}{\partial(1/\alpha)} \\ &= \frac{-(1-\beta)(1-\lambda)}{(1-\lambda(1-\beta)m)^2((1-\beta)^2+1/\alpha)^2}\end{aligned}$$

This expression is intuitive. When $\beta < 1$, $\frac{dk}{d(1/\alpha)} < 0$, so weight put on x_i decreases as the relative precision on z increases (noting that $1/\alpha = \alpha_x/\alpha_z$ by definition). Next, we turn to the comparative statics of k with respect to λ

$$\begin{aligned}\frac{dk}{d\lambda} &= \frac{\partial k}{\partial \lambda} + \frac{\partial k}{\partial m}\frac{\partial m}{\partial \lambda} \\ &= \frac{m((1-\beta)m-1)}{(1-\lambda(1-\beta)m)^2}\end{aligned}$$

Note that the sign of this expression depends on m. Specifically, when $\beta < 1$, the expression is positive if $m > \frac{1}{1-\beta}$ and negative if $0 < m < \frac{1}{1-\beta}$. When $\beta > 1$, the expression is positive if $m < \frac{1}{1-\beta}$.

There is an easier way to characterize the comparative statics of k. Rewrite

$$\begin{aligned}k(\beta|\,\alpha,\lambda) &= \frac{(1-\lambda)m(\beta,\alpha)}{1-\lambda(1-\beta)m(\beta,\alpha)} \\ &= \frac{(1-\lambda)\alpha(1-\beta)}{(1-\lambda)\alpha(1-\beta)^2+1}\end{aligned}$$

Define $\tilde{\alpha} = (1-\lambda)\alpha$, the above can be rewritten as (with slight abuse of notation)

$$k(\beta|\tilde{\alpha}) = \frac{(1-\beta)\tilde{\alpha}}{(1-\beta)^2\tilde{\alpha}+1}$$

Hence, we may proceed with comparative statics of k with respect to λ and α in two steps. First, we derive the comparative statics of k with respect to $\tilde{\alpha}$. Then we derive the comparative statics of $\tilde{\alpha}$with respect to λ and α. The overall effects are the composite of the outer effect and the inner effect. For the outer effect, we have

$$\frac{d}{d\tilde{\alpha}}k(\beta|\tilde{\alpha}) = \frac{(1-\beta)}{((1-\beta)^2\tilde{\alpha}+1)^2}$$

which has the same sign as $(1-\beta)$.

For the inner effect, we have

$$\frac{d}{d\alpha}\tilde{\alpha} = (1-\lambda) \geq 0$$

$$\frac{d}{d\lambda}\tilde{\alpha} = -\alpha < 0$$

For the special case $\gamma = 1$, as we discussed above, $\beta \in [0, 1]$ in any equilibrium. So we have $\frac{d}{d\tilde{\alpha}}k > 0$ and $\frac{d}{d\alpha}k > 0$ and $\frac{d}{d\tilde{\alpha}} < 0$.

Note that an immediate corollary of the above analysis is that the effect of $1/\alpha$ and the effect of λ on k are of the same sign. This is intuitive. Recall that $1/\alpha$ measures the relative precision of x_i, while λ measures the weights on coordination. and (1-λ)/$\alpha$is the coordination-weighted precision of signal x. A higher weight on coordination reduces the incentive to listen to x (and increases the incentive to listen to z), while a higher precision of the former does exactly the opposite.

3.5 Welfare comparison

For welfare analysis, we are interested in examing how the sender's utility change with manipulation. Let $k^*(\beta|\alpha, \lambda) \equiv \underset{k}{argmin} L(k|\beta, \alpha, \lambda)$ where

$$L(k|\beta, \alpha, \lambda) = \lambda(a_i - \int_j a_j dj)^2 + (1-\lambda)(a_i - \theta)^2$$

represents the receiver's objective (quadratic loss).

Similarly, let $\beta^*(k|c, \gamma) \equiv \underset{\beta}{argmax} V(\beta|k, c, \gamma)$ where

$$\begin{aligned} V(\beta|k, c, \gamma) &= \gamma(k\tilde{\theta} + (1-k)z - \theta)^2 - \gamma\frac{k^2}{\alpha_\epsilon} \\ &+ (1-\gamma)\left((k-1)\tilde{\theta} + (1-k)z\right)^2 - c\left(\tilde{\theta} - \theta\right)^2 \end{aligned}$$

represents the sender's objective.

The two formula above give a system of two equations in (β, k). The intersections of them represent the equilibria of the model. Let solution to

$$\beta^*(k^*(\beta|\alpha, \lambda)|c, \gamma) = \beta$$

$$k^*(\beta^*(k|c, \gamma)|\alpha, \lambda) = k$$

be $\beta^{eqm}(\alpha, \lambda, c, \gamma)$ and $k^{eqm}(\alpha, \lambda, c, \gamma)$.

Note that in general β^{eqm} and k^{eqm} are sets which may contain more than one element or be empty. For the former case, we shall focus on the equlibria that's optimal for the sender. For tractability, in our analysis we focus on the case where both sets are singleton. The general case carries through with appropriate notational modifications. Let $V^{eqm}(\alpha, \lambda, c, \gamma)$ denote the value of V evaluated at $\beta^{eqm}(\alpha, \lambda, c, \gamma)$ and $k^{eqm}(\alpha, \lambda, c, \gamma)$. Let $V^{nm}(\alpha, \lambda, \gamma)$ denote the value of V in the absence of manipulation. We are interested in the relative magnitudes of V^{eqm} and V^{nm} for various combinations of parameters values $(\alpha, \lambda, c, \gamma)$, and especially, for what combinations of parameter values does manipulation backfire in the sense that $V^{eqm} < V^{nm}$.

Since λ and γ are from the sender's and the receivers' innate preferences, we treat them as given. α and c on the other hand, are environmental parameters that are potentially responsive to exogenous shocks (e.g., a technological improvements which reduces the cost of manipulation), so we treat them as model shifters.

The analysis shall proceed in the following steps. First, we identify the shape of β^{eqm} and k^{eqm} functions. We will focus on the case where there is at most one interior extremum. (As we saw in the previous section, this restriction is not binding for γ sufficiently large.) Let $\bar{\beta}(\alpha, \lambda)$ be the interior point (if any) such that k is first increasing and then decreasing. Consider the equilibrium k and β under no manipulation, denoted as $k^{nm}(\alpha, \lambda, \gamma)$ and $\beta^{nm}(\alpha, \lambda, \gamma)$ respectively. We solve for the implicit equation

$$k^{eqm}(\alpha, \lambda, \gamma, c) = k^{nm}(\alpha, \lambda, \gamma)$$

First, we find the c that satisfies $\beta(k^{nm}(\alpha, \lambda, \gamma)|c, \gamma) = \beta^{nm}(\alpha, \lambda, \gamma)$ This c answers the question: given the equilibrium under no manipulation (β^{nm}, k^{nm}), if the receivers use the same interpretation policy k^{nm}, under what level of c will the sender use the same manipulation policy as β^{nm}. Let this c be denoted as $c^{nm}(\alpha, \lambda, \gamma)$. We have that

$$k^{eqm}(\alpha, \lambda, \gamma, c) < k^{nm}(\alpha, \lambda, \gamma)$$

if and only if $c < c^{nm}(\alpha, \lambda, \gamma)$. This is true because $k(\beta|\alpha, \lambda)$ is decreasing in β for all $\beta > \hat{\beta}(\alpha, \lambda, \gamma)$ where $\hat{\beta}(\alpha, \lambda, \gamma)$ solves

$$k(\hat{\beta}(\alpha, \lambda, \gamma)|\alpha, \lambda) = k^{nm}(\alpha, \lambda, \gamma)$$

So $k^{eqm}(\alpha, \lambda, \gamma, c) < k^{nm}(\alpha, \lambda, \gamma)$ if and only if $\beta^{eqm}(\alpha, \lambda, \gamma, c) > \beta^{nm}(\alpha, \lambda, \gamma)$.

For the special case $\gamma = 1$, we have the following simplifying result. Let $\tilde{\alpha} = (1-\lambda)/\alpha$. In the absence of manipulation, we have

$$k^{nm}(\tilde{\alpha}) = \frac{(1-\lambda)\alpha_x}{(1-\lambda)\alpha_x + \alpha_z} = \frac{\tilde{\alpha}}{\tilde{\alpha}+1}$$

Recall that $k(\beta|\tilde{\alpha}) = \frac{(1-\beta)\tilde{\alpha}}{(1-\beta)^2\tilde{\alpha}+1}$. So $\beta^{nm}(\tilde{\alpha})$ which is defined implicitly via $k(\beta^{nm}(\tilde{\alpha})|\tilde{\alpha}) = k^{nm}(\tilde{\alpha})$

solves

$$\frac{(1-\beta^{nm}(\tilde{\alpha}))\tilde{\alpha}}{(1-\beta^{nm}(\tilde{\alpha}))^2\tilde{\alpha}+1} = \frac{\tilde{\alpha}}{\tilde{\alpha}+1}$$

i.e., [5]

$$\beta^{nm}(\tilde{\alpha}) = \frac{\tilde{\alpha}-1}{\tilde{\alpha}}$$

Here we focus on the case $\tilde{\alpha} > 1$. Next we solve for the level of $c^{nm}(\tilde{\alpha})$ implicitly defined via$\beta(k^{nm}(\tilde{\alpha})|c) = \beta^{nm}(\tilde{\alpha})$[6].

Note that the critical point of β given by the solution to $k'(\beta|\tilde{\alpha}) = \tilde{\alpha}\frac{(1-\beta)^2\tilde{\alpha}-1}{((1-\beta)^2\tilde{\alpha}+1)^2} = 0$ is $1-\beta^{crit} = \frac{1}{\sqrt{\tilde{\alpha}}}$ whereas $1-\beta^{nm}(\tilde{\alpha}) = \frac{1}{\tilde{\alpha}} < \frac{1}{\sqrt{\tilde{\alpha}}}$. So $\beta^{nm}(\tilde{\alpha}) > \beta^{crit}$ and $k(\beta|\tilde{\alpha})$ is decreasing in β at $\beta^{nm}(\tilde{\alpha})$.

Hence,

$$\begin{aligned} k^{eqm}(\tilde{\alpha}, c) &\equiv k^*(\beta^{eqm}(\tilde{\alpha}, c)|\tilde{\alpha}) \\ &< k^*(\beta(k^{nm}(\tilde{\alpha})|c)|\tilde{\alpha}) \\ &= k^*(\beta^{nm}(\tilde{\alpha})|\tilde{\alpha}) = k^{nm}(\tilde{\alpha}) \end{aligned}$$

if and only if $\beta^{eqm}(\tilde{\alpha}, c) > \beta^{nm}(\tilde{\alpha})$. From the comparative statistics analysis above,

$$\begin{aligned} \frac{\partial}{\partial c}\beta^{eqm}(\tilde{\alpha}, c) &= \frac{d}{dc}k^{*-1}(\beta^*(k|c)|\tilde{\alpha})|_{k=k^*} \\ &= \frac{\frac{\partial}{\partial c}\beta^*(k^*|c)}{1-k'(\beta^*)\beta'(k^*)} \end{aligned}$$

Given that $k'(\beta) < 0$ and $\beta'(k) > 0$ at (k^{eqm}, β^{eqm}) and that $\frac{\partial}{\partial c}\beta^*(k^*|c) < 0$, we have $\frac{\partial}{\partial c}\beta^{eqm}(\tilde{\alpha}, c) < 0$. As we proved in the previous section, any equilibrium of this special case lies in the region $\beta \in [0,1]$ and $k \in [0,1]$. We have $k'(\beta) > 0$ if $0 < \beta < 1 - 1/\sqrt{\tilde{\alpha}}$. So $k^*(\tilde{\alpha}, c) < k^*_{nm}(\tilde{\alpha})$ iff $\beta^*(\tilde{\alpha}, c) > \beta^*_{nm}(\tilde{\alpha})$.

[5] Note that from

$$(1-\beta^{nm}(\tilde{\alpha}))^2\tilde{\alpha} - (\tilde{\alpha}+1)(1-\beta^{nm}(\tilde{\alpha})) + 1 = 0$$

we have

$$1-\beta^{nm}(\tilde{\alpha}) = \frac{1}{\tilde{\alpha}}$$

[6] Recall that

$$\frac{k^{nm}(\tilde{\alpha}) - k^{nm}(\tilde{\alpha})^2}{c - k^{nm}(\tilde{\alpha})^2} = \frac{\tilde{\alpha}-1}{\tilde{\alpha}}$$

or

$$\begin{aligned} c^{nm}(\tilde{\alpha}) &= \frac{\tilde{\alpha}}{\tilde{\alpha}-1}(k^{nm}(\tilde{\alpha}) - k^{nm}(\tilde{\alpha})^2) + k^{nm}(\tilde{\alpha})^2 \\ &= \frac{\tilde{\alpha}}{\tilde{\alpha}-1}\frac{\tilde{\alpha}}{(\tilde{\alpha}+1)^2} \end{aligned}$$

4 Applications

4.1 Propaganda and censorship in the digital age

Although our model is simple, it captures several keep aspects of propaganda in the digital age documented in Roberts (2018). We illustrate a few examples here.

Roberts (2018) posits that the internet, while making it easier to get access to untampered information, can also help spreading propaganda. First, authoritarian regimes might use a firewall to filter information or impose friction by throttling websites, making them slow to load. Essentially, such practice changes the relative precision of information obtained from various sources (measured in unit time). A citizen who is indifferent between two information sources in the absense of manipulation will reorient her attention if one particular source is made marginally faster than the others. Roberts (2018) also documents evidence that government censors affect information consumption by influencing the relative ease of access to information. By repeating information from multiple sources, the regime does not so much to signal power or to deter as to prioritize the consumption of one source of information over the other. In our model, this boils down to changing the relative precision α. From our analysis in the comparative statics section, such practice can affect the effectiveness of propaganda and thus affect the equilibrium outcome.

Another way in which internet changes the mechanism of spreading propaganda is by changing the cost of manipulation. Nowadays, group or even algorithms can create information and disseminate it at virtually zero marginal cost. The decrease in cost of manipulation (measured by c), makes it cheaper for the regime to spread propaganda. For example, the Chinese government is known to hire "Fifty Cent Party", social media users who spread information at the government's directive (King, Pan and Roberts (2017)).

4.2 Effectiveness of propaganda depends on the degree of herding tendency

A key implication of the model is that confusion and noisiness in the information environment increases the variance of return to actions. Given that citizens value coordination, a noisier environment makes it harder for citizens to act in unison with each other. As Barbera et al. (2015), Steinert-Threlkeld (2017) and Chenoweth and Stephan (2011) show, the "periphery" of the society is crutial to sucessful protests. The degree of herding tendency is captured by the λ parameter of the model, which measures the degree of strategic complementarity or substitutability among citizens' actions. In the special case where $\lambda = 0$, the citizen's quadratic loss term is $(a_i - \theta)^2$, in which case each citizen wants her own action to be aligned with the true state but does not take into account what the others do. In contrast to the "individualistic" case, the model can also be used to study a society where herding tendency is extremely high, i.e.g, when $\lambda = 1$ and citizens' quadratic loss term reduces to $(a_i - \int_j a_j)^2$. In this case, citizens only care about matching their actions with each other, regardless of the underlying true state of the world.

Note that although in the previous discussions, we have assumed that $\lambda > 0$, i.e., citizens' actions are strategic complements, there's nothing in the model that prevents λ from taking on negative values. In this case, receivers seek to differentiate their actions from others.

Because of the quadratic formulation of the model, λ affects the model through affecting the "coordination adjusted relative precision", namely $\tilde{\alpha} = \frac{(1-\lambda)}{\alpha}$. It is easy to see that a smaller λ is effectively the same as an increase in the precision of x: the more the receivers want to differentiate from others, the more weights they will give to the private, idiosyncratic signal x_i, so that in equilibrium, their action profile will be the same as if the precision of x_i has been increased.

5 Extensions

5.1 Citizens may choose to reduce the noise of the private signal

In reality, we might think that citizens can commit to a noise-reduction investment prior to observing the signal and taking action. For example, a citizen who is blocked by the regime from visiting a website may purchase a VPN to bypass information censorship. In our model, the signal z is observed by all. So investment in noise reduction is similar to investment in public goods. To bypass this issue, we assume each private signal has idiosyncratic precision α_{x_i}. Write the net payoff from observing better information on x_i as

$$L(a_i, A, \theta) - C(\alpha_{x_i})$$

where $C(\alpha_{x_i})$ denotes the cost of private information acquisition.

We will focus on the symmetric equilibrium of the problem. It can be shown that (c.f. Angeletos and Pavan (2007)) in the unique symmetric equilibrium, equilibrium private information precision can be derived by holding the action rule fixed and equating each agent's marginal benefit of more precise private information to its cost.

The marginal benefit of more precise private information is given by the residual variance in $(a_i - A)$, conditional on the action profile, so the characterizing equation for equilibrium α_{x_i} is given by

$$\frac{\partial}{\partial \alpha_x}\sigma^2_{(a_i-A)^2|a,\alpha_x} = C'(\alpha_x)$$

Note that uniqueness of symmetric equilibrium follows directly from the convexity of the cost function.

5.2 Citizens have access to more than two signals

The techniques of the model can be easily extended to the case with n untampered signals $Z = (z_1, ..., z_n)$ and m potentially tampered signals $X = (x_1, ..., x_m)$. Denote the pdf of a $n+m$ dimensional Gaussian random vector y as:

$$f(\vec{y}) = \frac{1}{(2\pi)^{(n+m)/2}|\Gamma|^{\frac{1}{2}}} exp\left(-\frac{1}{2}(\vec{y}-\bar{y})^T\Gamma^{-1}(\vec{y}-\bar{y})\right)$$

where $\bar{y}$ is the mean of the vector $\vec{y}$, Γ is the $(n+m)\times(n+m)$ dimensional covariance matrix, and $|\Gamma|$ is the determinant of Γ.

We write $Z \sim N(\vec{z}|\bar{z}, \Gamma_z)$ and $X \sim N(\vec{x}|\bar{x}, \Gamma_x)$.

Then the conditional distribution of y has the following form:

$$\vec{y}|X, Z \sim N(\vec{y}|m, \Gamma)$$

where

$$\Gamma^{-1} = \Gamma_x^{-1} + \Gamma_z^{-1}$$

and

$$m = \Gamma\Gamma_x^{-1}X + \Gamma\Gamma_z^{-1}Z$$

The analysis of the model goes through with little modification.

5.3 Citizens are attention constrained

Relatedly, one way to extend the model will incorporate the citizens' information-acquisition policy, in addition to information-interpretation policy (the focus of this paper). For example, we might assume that a individual's move takes three steps: first, receiver i chooses the amount of attention paid to each of the n information sources. Write this vector as k_i, where the p^{th} coordinate measures the amount of attention paid to the p^{th} information source. After this step, receiver i will observe a vector of signals x_i about the unobserved state θ. Some coordinates represent untampered information; other coordinates represent manipulated information. The precisions of each signal is a function of receiver i's information acquisition choice in the first stage. Finally, player i takes action and the payoffs are the same as the baseline model. The distribution of the noise in the p^{th} information source, x_{ip}, has the following distribution

$$x_{ip} \sim N(\tilde{\theta}, \epsilon_{ip})$$

where $\epsilon_{ip} = \frac{1/\alpha_{xp}}{k_{ip}}$, i.e., more attention k_{ip} paid to the p^{th} information source reduces the noise. This model with information acquisition is formally equivalent to the model in which $X \sim N(\theta, \Gamma_x)$ where Γ_x has diagonal entries σ_p^2 and covariance entries $cov(x_p, x_q) = \rho_{pq}\sigma_p\sigma_q$ where

$$\sigma_p^2 = (\frac{1}{\alpha_z}) + \frac{1/\alpha_{xp}}{z_p}$$

and

$$\rho_{pq} = (\frac{1}{\alpha_z}) \left[(\frac{1}{\alpha_z} + \frac{1/\alpha_{xp}}{z_p})(\frac{1}{\alpha_z} + \frac{1/\alpha_{xq}}{z_q}) \right]^{-\frac{1}{2}}$$

It remains to posit a functional form of the cost function or budget constraint. Here the choices are flexible. Depending on the specific application, one may argue that the cost represents the real cost of acquiring information (e.g., purchasing internet services, subscribing to various media outlet). Alternatively, one could borrow models from behavioral economics and propose an attention span constraint.

5.4 Citizens may choose to communicate with each other

This extension can be easily incorporated into our model in reduced form. One way to incorporate strategic interaction is by assuming that better interactions among the citizens allow them to reduce the noise in x_i by a factor of $(1 - \zeta)$. One might assume that such interactions are costly. For example, the authoritarian regime might crack down on citizens who spread information that the authorities deem dangerous. We might model this as a cost term $C(\zeta)$, where ζ is the equilibrium level of communication. Alternatively, we might decompose aggregate action into two terms: A' and A''. The former represent the aggregate actions of agents that communicate with each other and A'' represent those who don't. Then the model can be rewritten in the following manner: In the sender's problem, replace aggregate action with the weighted average of A' and A'' and replace variance of A with within-variance of A'' composed with between-variance between A' and A''. In the receivers' problem, replace aggregate action with A''.

In more complicated models, one might assume that the communication network among citizens also matter. For example, we might assume that citizens in the model are divided into a finite number of clusters. The easiness of communication on this network is determined by the centrality of the network, where, following the literature on network, for a given graph $G = (V, E)$, we define closeness of x as

$$C(x) = \frac{1}{\sum_y d(x, y)}$$

where $d(x, y)$ is distance between x, y. Again, more complicated models with network structures can be subsumed into our model in reduced form by modeling cost of communication as a function of certain properties of the communication network, e.g., centrality, clusterness, etc.

5.5 Citizens have heterogenous degree of sophistication

The model can be easily extended to the case with heteogenous receivers. An especially interesting scenario is the following: what if a fraction of the society is "sophisticated", in that they know the sender's bias beforehand? For simplicity, we assume a $(1 - \epsilon)$ fraction of the receivers directly observes whether the government has manipulated or not, while the remaining ϵ fraction of the receivers do not observe manipulation.

Following Little (2017), we adopt a slightly different notion of equilibrium. We define an equilibrium of sophistication q as the outcome of the game when receivers' and sender's action profiles are consist with a Perfect Bayesian Equilibrium, the uninformed type's prior assessed probability of manipulation is q. The key step in solving the problem is to derive the uninformed type's belief of manipulation as a function of signals x and a common conjecture about β, $\hat{\beta}$.

Letting f denote the prior distribution of $\theta|z$, we can write the uninformed type's belief about manipulation as function of x and $\hat{\beta}$ as

$$q(x, \hat{\beta}) = \frac{qf(\theta|\hat{\beta}, x)}{qf(\theta|\hat{\beta}, x) + (1 - q)f(\theta|\beta = 0, x)}$$

The remaining part of the problem is identical, with the only difference being that the uninformed type's belief about manipulation is based on $f(\theta|\hat{\beta}, x)$ which takes the form $q(x, \hat{\beta})$ as we derived above.

6 Conclusion

In this paper we develop a model of strategic information signaling with an informed sender and a continuum of imperfectly informed receivers. The sender costly sends a signal to disrupt receivers' coordination action and to bias their aggregate action away from the true state to the sender's desired state. The receivers want to match their actions to the true state and also seek to coordinate with each other. The leading application of the model is an authoritarian regime sending a propaganda to its citizens to prevent them from learning the true strength of the regime and taking collective actions. In equilibrium the sender's manipulation does not succeed in changing the mean of the the receivers' beliefs, but manipulation makes their interpretation of the signal noisier. This model is helpful for us to resolve the empirical puzzle: does propaganda work, even if the citizens who see it know it is biased information? The model answers the question in the affirmative. Such propaganda works not through changing beliefs per se, but through adding noise and confusion into the communication structure, so that citizens, who value coordination, are more likely to redirect their attention across various sources.

It is worth mentioning that the framework of this paper does not capture the long-term effects of propaganda. The model is static. Yet the long-run economic and ideological impacts of propanda unfold over time. We focus on the informational implication of propaganda while ignoring the broader political implications sin authoritarian environments. Understanding the ever-changing scope of authoritarianism and the specific methods the governments use to control the agenda are beyond the scope of this paper.

Another limitation of the model is that it is silent on who operates the propaganda platform. In our model, the regime is equivalent to the mouthpiece. Yet in the real world, information distortion occurs on domestic platforms e.g., Sina Weibo, Renren and Baidu (the Chinese equivalent of Twitter, Facebook and Google, respectively). How does information restriction skew the market for information and create inefficiencies? Do domestic companies benefit from the regime's subsidy in the same way that domestic firms benefit from international trade protections? Do restrictions and censorship associated with propaganda and censorship impose higher costs on innovation, since firms cannot fully recuperate the returns from innovation?

Finally, the model does not allow the regime to selectively target certain segment of the citizen population. As Wu and Meng (2016) argue, propaganda is especially effective in creating discrepancies in access to and reach of information across socioeconomic classes. While propaganda is less effective on the elite, who can bypass information barriers and obtain outside information at low cost, it is much more likely to be effective on the rural population who do not have the resources to circumvent government propaganda. Will selective propaganda exacerbate the wealth and knowledge chasm among classes? We leave the question to future researches.

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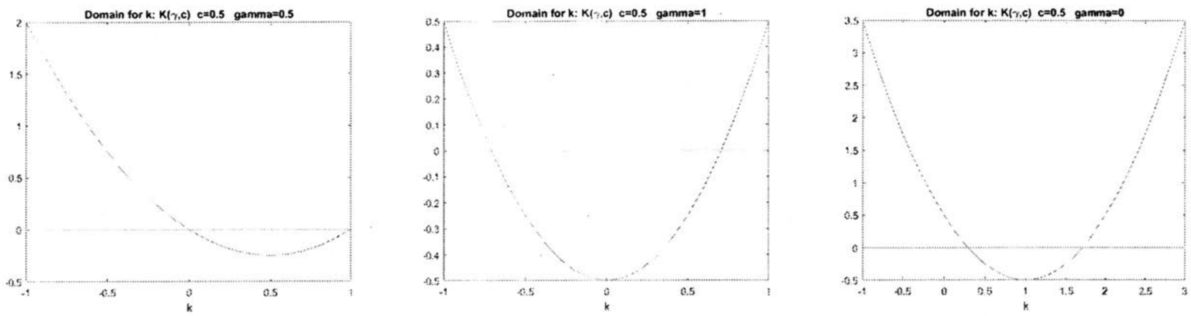
8 Appendix



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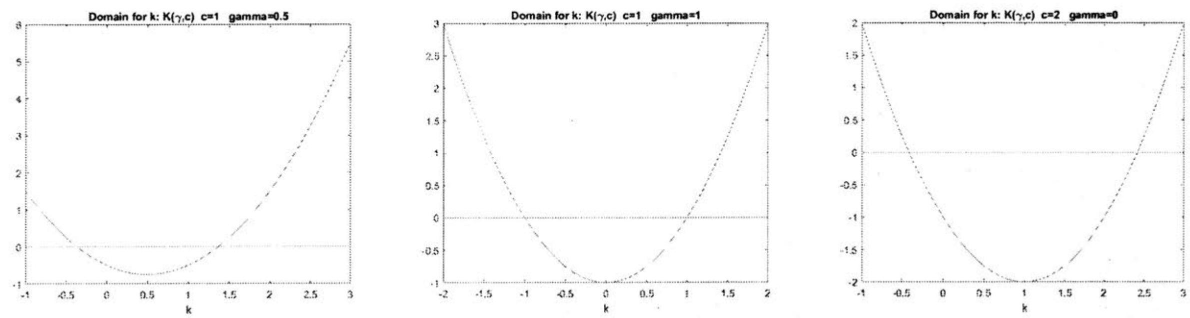

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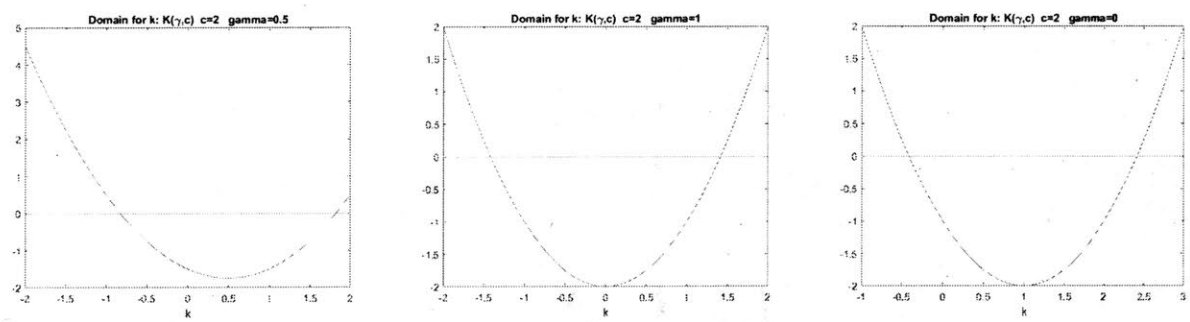
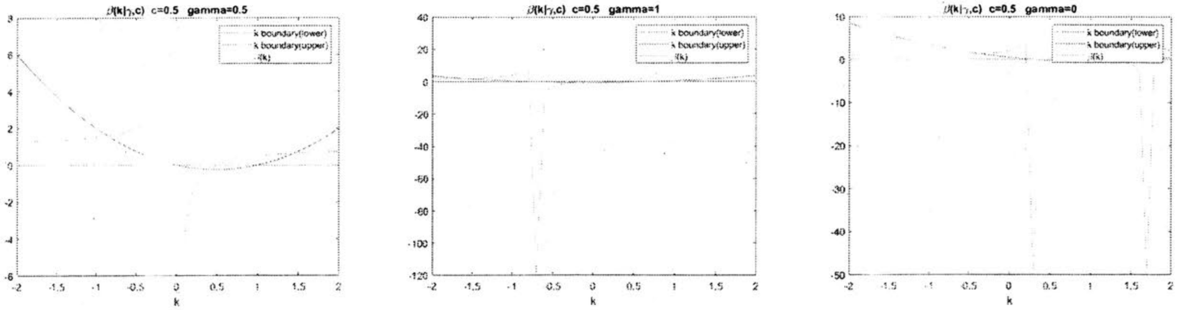


(c) $c = 2$

Figure 1: Domain for k: $K(\gamma, c)$, for different values of c and γ



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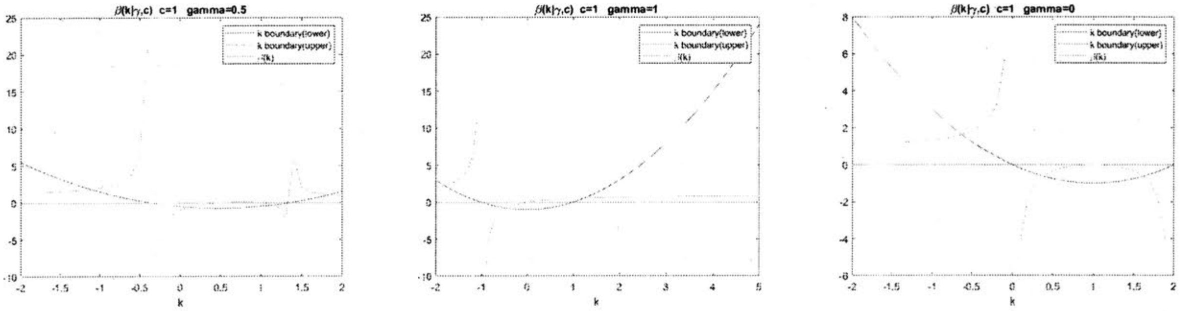

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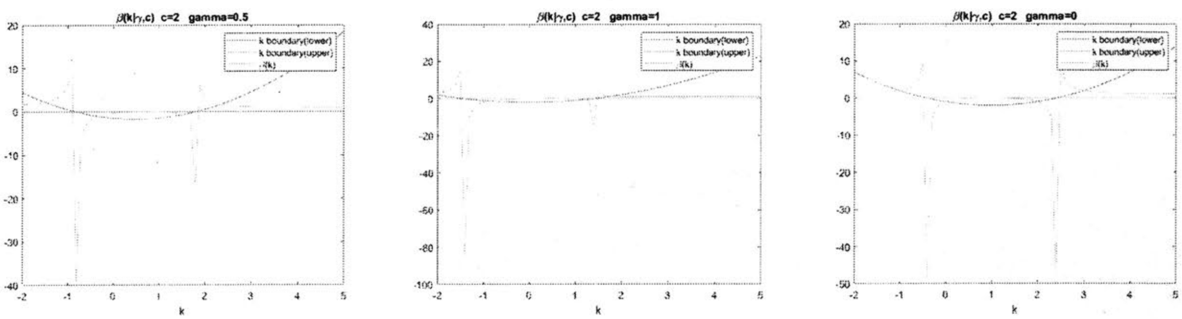
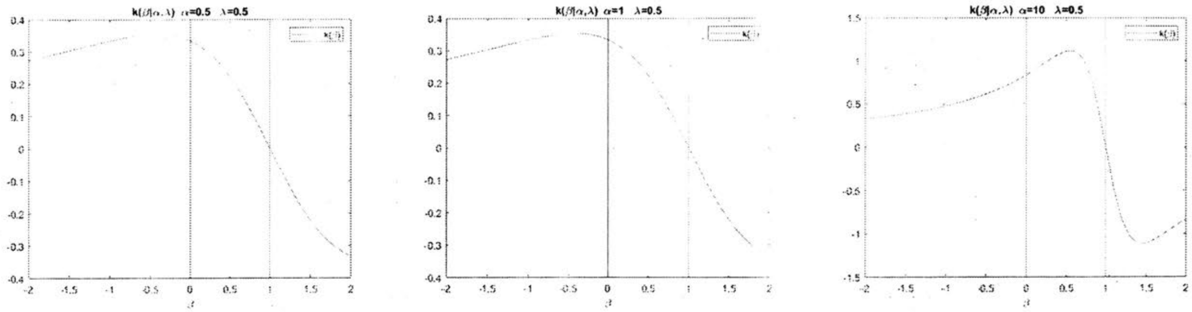
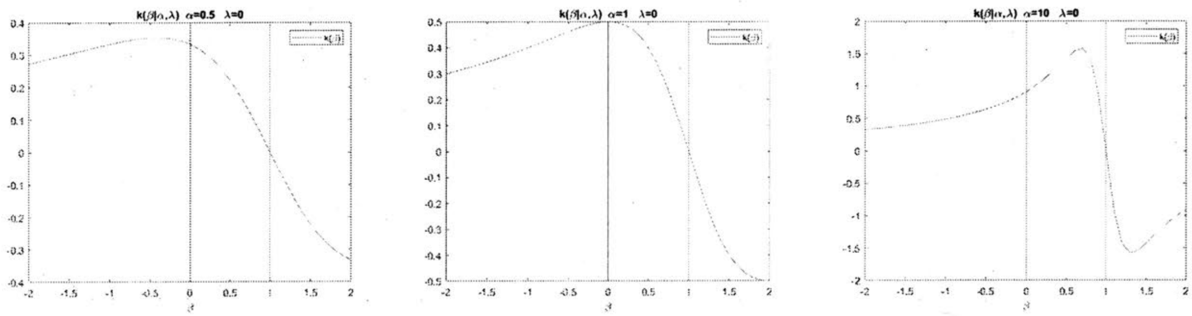


(c) $c = 2$

Figure 2: value of $\beta(k|\gamma, c)$, for different values of c and γ



(a) $\lambda = 0.5$



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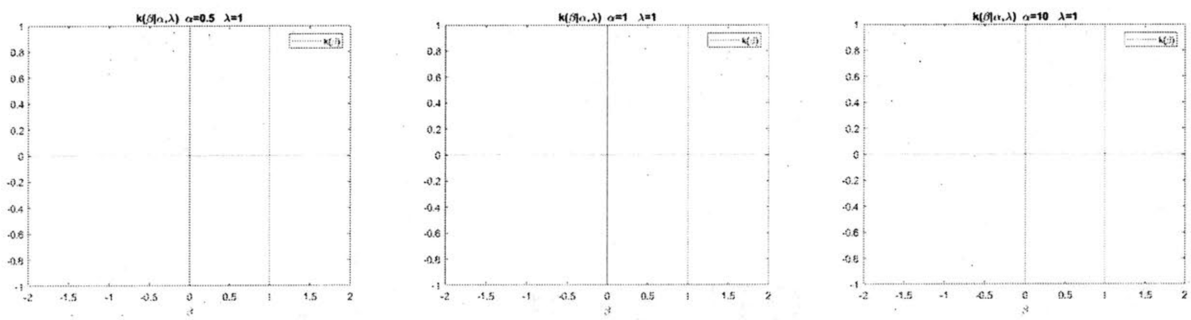

(c) $\lambda = 1$

Figure 3: value of $k(\beta|\alpha, \lambda)$, for different values of α and λ