YIELD MANAGEMENT FOR THE MARITIME INDUSTRY

by

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Abstract

The success of the application of yield management to the seat inventory control of the major airlines, has forced several leading liner shipping companies to consider the incorporation of yield management tools into their capacity control and reservation systems. The ordered arrival of the classes of customers, and other conventional assumptions of the airline yield management problem, are not realistic enough to describe the booking process of containerships. Furthermore, the airline yield management literature offers accurate solution only for the one leg trip. The published solutions for multi-leg itineraries are heuristic, with tedious calculations and questionable accuracy as the number of the legs of the itinerary increases.

A dynamic programming (DP) model of the booking process with time variable arrival rates, with the expected revenue as cost functional is introduced. The Hamilton-Jacobi-Bellman (HJB) partial differential equation corresponding to the above DP formulation is derived. It is shown that the solution of the HJB equation, reduces to the solution of a parametric linear programming (LP) model. It is also shown that the optimal LP solution gives both the optimal expected revenue and the optimal control (accept/do not accept a customer) of the original stochastic DP model. Therefore, the decision tool presented, is endowed with both the modeling accuracy of a DP and the facility of solution of an LP model. This facility allows for real life yield management applications on multi-leg vessel itineraries, and on networks with overlapping vessel itineraries.

The DP booking model is expanded to include booking cancelations and capacity overbooking. Once more, the solution of the HJB equation derived by the DP booking model is given by an LP model. Simulations show that when the LP is used instead of the DP, as the decision tool, the operational results are inferior by at most 1%.

Closer examination of the LP models shows that a careful interpretation of the output of the LP can improve the operational results further. The discrepancy of the output of the above simulations becomes now less than 10^{-3}.

In addition to being used for yield management, the LP models can be used for optimal pricing and/or optimal fleet deployment. The LP models can be used to determine the level of the freight rates that would maximize the expected revenue/profit of the shipping liner. They can also be used to determine the optimal equipment allocation of the shipping company (i.e. allocate the company vessels or other equipment to particular itineraries).

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Αφιερώνεται

Στούς Γονείς μου

Μαρία καί Αποστόλη

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Chapter 1

Introduction

The container cargo tariffs are invariably value based. These different freight rates are justified by the fact that the different customers of the shipping company are offered different levels of service. For instance, the electronics manufacturer who incurs capital costs for every day his high value product spends in the warehouse, or the fruit producer whose product loses part of its value for every day of delay before the fruit reaches the consumers, need and value the regular, frequent and reliable transoceanic transportation service a liner containership offers. At the other end of the spectrum, we have the shippers of raw material, semifinished goods, and other low value commodities who consider liner shipping to be one of several alternatives for their transportation needs. The different groups or classes of customers get different levels of utility from the same service. As we argue later, the only viable pricing scheme is for the freight rates to reflect the level of service each class of containers is given. As a result, the price paid by a container of high value cargo is greater than the price paid by a similar container filled with low value cargo.

One of the main challenges for containerships operators is the matching of the capacity they offer to the demand they can generate for their services. There are several ways they could try and balance the two. The first way is to adapt the level of supply to the level of demand. Addition and subtraction of vessels on the different itineraries in order to follow the fluctuations of the demand is not a viable alternative, since the decision horizon of the allocation of a vessel at an itinerary is larger than short term demand fluctuations.

The vessel operators could set a level of freight rates such that it regulates the demand for their services and brings it to the level of the transportation capacity. Variable freight rates would defy the purpose of liner shipping. Freight rate stability is sought by both the shipping conferences (the associations of vessel operators who have agreed to provide regular service at published and stable rates) and the users of the services offered by liner shipping. As a result, a "dynamic" pricing of the conference services is out of the question.

An alternative way for capacity and demand balance, is the direct control of the demand. The operators can control the transportation capacity inventory, and when the demand for transportation is strong they reserve the capacity for the high freight rate customers, whereas they refuse transportation capacity to lower yield customers. This is the method of Yield Management. Yield management has both short and long term effects. In the short term, yield management maximizes the revenues or profits of the vessel operator. In the long run, yield management creates favorable conditions for profitability. In more general terms, yield management is ([1]):

> The integrated management of price and space to maximize revenues in the short term and to enhance profitability in the longer term

In other words, the application of yield management techniques in liner shipping will assure that the capacity of the vessels is mainly used by these customers who fully use the services of the conference and can afford the cost of liner shipping. Yield management makes certain that the low freight rate classes of customers who use liner shipping as one of many alternative ways for transportation and cannot really afford the cost of liner shipping, use the services of the containerships occasionally, and only when the demand from the high freight rate customers is low enough to make the acceptance of the low freight rate customers profitable.

The success of the application of yield management to the seat inventory control of the major airlines, has forced several leading liner shipping companies to consider the incorporation of yield management tools into their capacity control and reservation systems. There are several differences in the nature of the two industries that they do not permit the direct transfer of the yield management models from the airline to the shipping industry. In the following, we study the nature of the service industries in general, and the shipping industry in particular with reference to the airline industry.

1.1 Characteristics of the Shipping Liner Industry

The total volume of shipping capacity demanded is subject to cyclical and seasonal fluctuations. The carrier is expected to be able to satisfy this demand in order to maintain customer good will, even though the supply of shipping capacity is constant. That means that the need to supply high-quality service would require the capacity to be sufficient to cover the seasonal and other fluctuations. The above explain why overcapacity is a structural problem of liner shipping.

The unit costs of shipping depend inversely on the utilization of the vessel capacity. Carrying reserve capacity, imposes additional costs. In such a case, the carrier would have to decide what amount of reserve capacity would be required to balance, in the long run, the costs of holding idle capacity, the cost of chartering extra capacity to meet the peak season requirements, the price and service combinations the market would accept, and the costs all of the above alternatives would have on the return leg of the trip.

An additional characteristic of shipping is that vessels are long lived assets, and when their capital costs are written off, and they are crewed with cheap labor from the developing world, old vessels can be competitive with more sophisticated vessels. The long life of vessels means that, even under very competitive conditions, there is no guarantee that that surplus tonnage can be eliminated very easily. Under conditions of intensive competition many shipping firms might not survive. Through their liquidation, the physical capital will continue to exist, offering thus the opportunity for cheap entry to others. It is rather certain that the capacity will not decline to a level that would allow balance between capacity supply and demand for transportation.

The alternative to carrying extra capacity would be to apply methods of capacity control that would guarantee optimal allocation of the capacity. During times of weak demand from the cargo that pays high freight rates, it is easier for the low freight rate containers to be offered transportation capacity. When the demand from the high freight rate products becomes stronger again, the low freight rate cargo should not have unrestricted access to the vessel capacity. The access should be given conditionally and only to the extend that offering of shipping services to the low freight rate customers does not deprive the same capacity from the high freight rate customers. The alternative of the low freight rate customers becomes preferable only when the demand from the high end customers is weak.

1.1.1 Differences between Liner Shipping and Manufacturing

The main difference between the operational characteristics of liner shipping and manufacturing is that the output of liner companies cannot be stored, and as a result, liner shipping, like most service industries, lacks production flexibility. Transportation services, which is the output produced by liner companies, cannot be put into inventory, for later resale to the market. The revenue potential of shipping space cannot be exploited if unsold ("spoilage"). That situation has the following implications [2]:

- Marketing experiments become much more risky, than in manufacturing. When a manufacturer prices his products too high, and part of his products remain unsold, the manufacturer can always lower the price and put the inventory back on the market, with no additional cost to the storage cost they have incurred so far. If a liner has priced its services too high, and its product remains unsold, it loses all its value after the sailing of the vessel. In order for the liner operator to offer again his product to the market, i.e. to offer transportation services, he has to incur the costs of a new trip.

- The possibility of a vessel that leaves the port of origin with unsold space can force the operator into attempts to find the right price at which some customers would respond and would give the vessel enough cargo to fill the empty space. The fact that the marginal cost of transporting an extra container is negligible, if we subtract the handling costs, is tempting for vessel operators to engage in bidding wars and gravitate towards low prices in the vicinity of the short term marginal cost. That pricing policy can very well be rational, if someone's horizon is the short term revenue maximization. On the other hand, it can prove to be disastrous for the long term profitability and viability of the shipping company. Neglect of the long term implications of pricing with the rule of the short term marginal capacity cost, does not take into consideration the fixed costs of setting up a regular shipping service. Usually the persistent existence of unsold space, encourages programs like freight rate differentiation or pricing initiatives and rebates wherever this practice is tolerated by regulatory authorities.

With reference to the flexibility of production, manufacturing industries are better able to respond to the needs of the market, by both the scheduling of their production and the management of their inventories. Fluctuations in demand can be satisfied with constant production output, and therefore constant/optimal utilization of the plant, and satisfaction of the extra demand through the inventories. Alternatively, in periods of low demand, the inventories can grow, and there is no need to adjust the production capacity. On the contrary, a liner operator, can sell only the capacity that he has available. Any changes in capacity utilization mean changes in the unit production costs.

In manufacturing, the producers can always add (or subtract) some capacity in order to meet the demand of the market. This option increases the flexibility they already have through the manipulation of the inventories. In shipping, short term changes of capacity availability are much more difficult to obtain, and the requirement for a demanded frequency of service limit the short term elasticity of supply.

Furthermore, there is an optimal size of containership, or a vessel in general, although vessels vary in size. The addition of an extra vessel at a particular itinerary will increase the capacity supply unnecessarily, whereas taking a vessel out of the same itinerary will cause transportation capacity supply shortages.

Reduction of the capacity available on a given route, usually means reduction of the frequency of the sailings and consequently, reduction of the quality of service. On the contrary, in manufacturing, a reduction of the capacity of the plant is not necessarily associated with any fluctuation of the quality of the products.

We have discussed above the different ways in which manufacturing firms and service firms (in our case shipping companies) match their supply to the demand from their customers. Manufacturing firms have the flexibility of production, and they can further regulate their supply by increasing or decreasing their inventories. As a result we saw that manufacturing firms try to adjust their supply to the demand from their customers.

When shipping companies make capacity changes decisions, they have to take into consideration more parameters rather than the matching between capacity and current demand. Changes in the shipping capacity that serves an itinerary affect the level of quality of service. We therefore see that shipping companies do not have the flexibility of manufacturing companies, when it comes to the addition or subtraction of capacity at their itineraries.

What the shipping companies do is that they try to match the demand to their capacity. They offer their services to a multitude of customers and they give priority to those customers who pay higher freight rates. In conclusion we see that in liner shipping the operators are faced with an inherent difficulty to make short term changes in the supply of their capacity. As a result they try to match the demand through pricing and selective acceptance of the potential customers.

In manufacturing a typical product has a life cycle that consists of four major faces: Market Introduction, Market Growth, Market Maturity, and Sales Decline [3]. In each of these faces the typical purchaser of the product and user of its services is different.

The utility the typical customer gets and consequently the price he is willing to pay for the product is different in each subsequent face. As a matter of fact both the utility and the price decline through the life cycle of the product. By charging different prices at each face of the product life cycle, the manufacturing company break down the markets into separate segments. They charge each segment the price that would maximize the company profitability.

In her Ph.D. thesis, Williamson [4] discusses how service companies (airlines in her case) segment their market. Once they have segmented the market, the airlines have to apply yield management techniques, in order to offer capacity to the high paying customers, and decline the low revenue customers.

We see that in manufacturing it is relatively easy for the producer to segment the markets of the potential customers through out the life cycle of the product. In the service industries, where the separation among the markets is not as easy, and the customers compete at the same time for the restricted service capacity that when stays unsold suffers "spoilage", we need to apply sophisticated Yield Management techniques.

1.1.2 Pricing Issues

We have to assume that the objective of a Shipping Company is the maximization of its long term profits. The Conference system exhibits the following characteristics (see Gardner [5])

- Liner Shipping is a multi-product business
- The market structure is oligopolistic
- Prices are calculated with reference to long term costs, taking into consideration the competitive pressures exercised by alternative forms of transportation
- The supply of space is elastic, at prices that cover the average cost of the Shipping Company

- The long-run costs of a shipping Company are decreasing, as a function of capacity
- There is a freedom for a shipping company to enter liner trades, but not a right to become member of the conference that serves the trade
- There is need to reserve extra capacity to meet seasonal demand fluctuations, to prevent disruption of service in case of repairs and to allow some flexibility in the mix of the products carried

The quality of service a shipping liner and the conference to which it belongs, offer to their customers, is proportional to the frequency of sailings. At the same time, the main cost of liner shipping is the frequency of service. The liner companies that comprise the conference must take into account the fact that the frequency, and therefore the quality of service they provide is not desired by all customers. As a result, the prices charged at these customers have to reflect the level of utility the liner service offers them. The liner companies services are also subject to competition from other modes of transportation. These alternative forms include air freight transportation for the high end of the market, tramp shipping for the low end of the market, transportation by land (land-bridges) and competition from other liners for all other combinations of the cargo matrix.

Furthermore, the prices charged by the shipping company are part of the cost of a product that has to be shipped, usually overseas, before it reaches its customers. Higher freight rates mean that these products become less competitive and therefore their demand and consequently their derived demand for transportation capacity decreases. As a result, the freight rates a shipping company charges their cargo, especially the higher or the lower end of the product mix they carry, has to reflect this reality, an the pricing should be done accordingly.

The above considerations influence the prices paid by the different customers to a degree greater than the need for the even spreading of the costs among customers. Even spreading would require every customer to make a contribution to the overhead and profit of the company that would be proportional to the volume transported. Instead, the mix of the products bears unevenly the cost of the service, and makes a disproportional contribution to the revenues of the shipping company.

The above considerations explain why containers with apparently identical cost characteristics are charged different freight rates. This business practice can be argued to be to the benefit of all the involved parties, because, although some products are carried at a cost less than the proportional cost of the offered services, they still contribute to the revenue of the company. Alternatively, if the customers who pay less than their proportional share had to be charged their fair share of the liner cost, they could have chosen a competing mode of transportation, as we mentioned above. When that happens, the liner loses their moderate contribution to the company revenues, without having them replaced with the volume generated by other customers.

In the context of liner shipping, and for that matter in many other capital intensive industries, it would be neither fair, nor economically viable to set prices equal to the short term marginal costs of production. Instead, it would be more viable for the rates to cover at least the long term marginal unit cost. That means that a shipper should not just pay the handling costs for his container (i.e short term marginal cost) but the cost of having the container slot available for the potential users (i.e long term marginal cost).

Historically, the role shipping conferences came to play was a direct consequence of the predominance of the indirect (overhead) costs in liner shipping operations [5, p. 206]. One of the most important functions of shipping liners, is to maintain pricing discipline among member lines during trade recessions.

The kind of service provided by conferences is typically long-term in nature. As such, it shares some common characteristics with telephone service, in the sense that someone has access to it, whether he needs it or not. In the case of the telephone service, the user pays a monthly fee for the right to have immediate access to the telephone. A potential shipper enjoys the same immediate access, without paying any fee for the guaranteed service. From the carrier's point of view that means that some of the costs of the service provided are independent of use and that justifies the inclusion in the payment of both direct user costs and the "service" costs, in the sense we saw them in the case of the telephone service. Therefore, the users of the liner operator, would have to pay an increased freight rate every time they use the services of the shipping company.

If long run marginal cost is accepted as the rule for the pricing of the freight of a container, then pricing would be straightforward. In the case of shipping, with the apparent economies of scale that a greater containership can offer, it is not clear that long run marginal cost would be the appropriate rule for pricing. In fact, long run marginal cost pricing leads to systematic losses because the cost of the marginal container slot gets lower and lower as containerships become larger and larger [6]. As a result, the price sensitive customers of a liner, i.e. the low end of the freight rate spectrum that have as an alternative for their transportation the use of tramp vessels, do not have to pay freight rates equal or greater than the long run average cost of the service. It is not fair or feasible to make those customers subsidize the cost of the frequency of the liner service. A frequency that is not vital for the nature of their business and that does not add to the attributes of their low specific value products. It would be enough for them to cover the long run marginal cost of the capacity they use. It is only realistic that those low end customers pay freight rates at the vicinity of the long run marginal cost. Greater rates could make them defect to the competing modes of transportation.

If, for example, the rates for containers containing electronics and machinery where decreased and the freight rates of the low revenue containers were increased, there would be a substantial decline in traffic, since the price elasticity of the former group of products is much smaller than the price elasticity of the latter group. In this example of the even distribution of the freight rates, in order for the operator to maintain the same frequency of service, the liner company would have to use smaller vessels with higher unit operating costs. Many argue, that in a case described by the above scenario, all participants, the liner company and the shippers of all the spectrum of product unit value would not benefit.

On the other hand, it would be fair for the low end customers to pay only a percentage of the long run average unit cost of the service. The major cost in liner shipping is the maintenance of frequency of sailings. Since the low freight rate customers are not concerned with, do not value and do not get much utility out of the frequency of service, they should not pay for this component of the service. Seeing the same problem from a different angle, it is the users of refrigeration that should pay for the refrigerated space and equipment and not the average user of the liner company services who would have to do that.

An alternative pricing method would be for the shipping liners to apply peak pricing techniques. That mostly applies to markets where the demand shows a periodicity. Shipping fulfills this prerequisite. In the case of liner shipping though, the very existence of liner shipping and the conference system is connected to rate stability. If the rates of good customers are increased in times of high demand, the company will suffer loss of good will on the part of these customer.

What the shipping liner could do is to manage its capacity in an optimal way. The operators should offer capacity to the low rate customers only when they expect that there will have ample capacity to cover all the demand. On the contrary, when the anticipated demand outstrips the available capacity, the vessel operators should control their capacity in a way that guarantees that the high end customers of the shipping company have their transportation needs satisfied and the relations between these shippers and the liners kept at a satisfactory level.

1.2 Yield Management in Liner Shipping

The problem can be described as follows: The operator announces the itinerary adequate time before departure and he waits for the orders of the shippers. The shippers cover a wide range of cargo value. For example, for the same itinerary we have containers loaded with computers, as well as containers loaded with small quantities of semi- finished materials and crude commodities.

As we have already mentioned, the tariff structure is given and for the time scope of our problem, it can be considered as constant. As a result, the operator does not have the option of negotiating the prices with the customers. Therefore, the operator does not have the ability to increase his revenues by changing the prices. Each customer pays a freight rate equal to a percentage of the value of the cargo he ships. This percentage is constant across the board.

The operator has limited capacity to offer, and this capacity can be easily exceeded by the demand for transportation. Therefore, he can increase his revenues by being selective with reference to the customers he accepts. An obvious choice for the operator is to offer capacity to the high value-high freight rate customers, and turn down other shippers with low value cargo. In this case, the shippers who are not accepted for transportation ask other vessel operators for capacity. The assumption is that they are lost revenues for our operator. Whether they are offered transportation capacity by other operators on not, they never return to our operator to ask again for capacity. That does not pose a problem for the ship operator if he is able to fill his vessel to capacity with high value customers. Nevertheless, this is not always possible and the operator, has to weigh the benefits of a success of his policy, when the vessel departs with full load, against an unsuccessful implementation of his policy when the low value cargo has been turned down and the ship leaves port with cargo below its capacity. That can happen when the operator does not get enough orders from high value customers that would enable him to fill his vessel.

An additional complicating feature of the liner booking pattern is that the low value cargo asks for reservations well before departure, whereas the high value cargoes arrange for their transportation at dates close to departure. From the description above, it seems that the problem at hand is similar to the yield management problem of the airlines. The major similarity is the pattern with with the customers arrive. The high fare customers start asking for transportation capacity after the low fare customers have decided on their transportation needs and they do not shop for transportation capacity any more. Capacity that has not been booked does not generate revenue for the operator.

1.2.1 Yield Management Practices in Liner Shipping

There are already a few Ocean Shipping Companies that use yield management techniques in areas of their operations. Usually, they apply yield management on their refrigerated cargo traffic. The market segment of refrigerated cargo is among the most dynamic as well as most profitable parts of transoceanic shipping. The freight rates per container paid by the refrigerated cargo are probably the highest, certainly among the higher freight rates paid by shippers, placing the market segment of refrigerated cargo (usually fruit) among the most lucrative in the shipping business. On the other hand the refrigerated containers need specially converted container slots in order to accommodate their refrigeration needs. The container slots that are equiped with electricity outlets and the other necessities for the refrigerated containers represent less than 10% of the total number of the container slots of a typical containership.

A factor that complicates the issue further is the unpredictability of the demand for the refrigerated cargo container slots as well as refrigerated containers. The refrigerated containers are appropriately modified containers with insulation, power generators etc. These containers represent a higher value investment for the shipping company. The weather influences greatly the time of the fruit harvest, and some unexpected weather pattern could mean the difference between shipping the refrigerated containers with the next vessel or some future departure.

The result is that refrigerated cargo shippers often have to cancel their orders, many times with sort notice, at small or no cost to them. Even when they have to pay some cancellation fee, this fee is nowhere near the cost of opportunity of the empty refrigerated cargo slots that the vessel operator has to suffer. The high cancellation rate of orders, of high and low value containers alike, and the subsequent cost of opportunity that they create is one of the problems that Shipping Firms executives have to take into consideration in their booking practice.

The results of the application of yield management for refrigerated cargo have encouraged the executives of Ocean Shipping firms to test whether this success can be repeated with other types of cargo. Nevertheless, there is a concern among Shipping executives that the Shipping Liner business is an environment where the assumptions of Airline yield management do not hold, and its practices cannot be applied. In other words, the size of the average customer of an Airline is much smaller than that of the average Shipping Liner customer. Therefore, the short term revenue maximization, that can very well guarantee long term profitability for a passenger Airline, does not necessarily explain, it is argued, how the Shipping Company that is bound by contracts and has to deal with a few big customers that have negotiating power, can profitably apply yield management to their operations.

Furthermore, Shipping Companies value service contracts in key trades and they feel that the application of yield management does not address strategic concerns of the Company. In addition, the shippers who ask for the services of a Shipping Liner company in the form of service contracts, prefer low prices and freight rate stability. These customers cannot relate to the capacity utilization problem of the operator, and they will be alienated if they perceive the yield management practices of the operator to offer them a lower level of service.

An answer to the above questions and worries of the shipping companies is that yield management can be applied in two faces. The first face of yield management application allows for application of yield management and capacity and equipment control, for the part of the vessel capacity that the operator is not bound by contracts to have available to any of the few big shippers. This is the part of the vessel that is available to the smaller shippers who do not generate enough transportation volume to have contracts with the operator. The relation between these shippers and the operator is similar to the relation that an airline has with its customers. As a result, the application of yield management at this segment of the market could result in increased profitability.

The second face of yield management application can start when the shipping company, with the application of yield management tools estimates what is the contribution of the big customers to the bottom line of the company. These estimations can serve as an internal tool for the company to negotiate the new contracts with the big customers. In this second face, yield management can serve as an analytical tool to access individual transactions and contracts, and figure ways to make them more profitable for the Carrier. Yield management/capacity control techniques are used by Shipping Liners. Sometimes, they are used under different names. Some other times they are used in a different form. Often they are not very sophisticated. Often, Shipping Companies that are members of the same Conference agree to withhold voluntarily freight carrying capacity from the market. This kind of capacity control is a not a sophisticated but it is a form of yield management, nevertheless.

Other times, shipping liners sell part of their capacity to non-vessel operators. These non-vessel operators are agents who "rent" part of the vessel space and they try to woo customers to buy transportation capacity through their own marketing networks. Often these non-vessel operators, while soliciting customers, are in direct competition with the vessel operator from whom they have leased transportation capacity. All of the above mentioned variations of capacity control techniques are forms of yield management.

Some shipping operators hesitate to apply capacity control methods, fearing that there is a fundamental difference between airlines and shipping companies. They argue that these differences would impede the application and dampen the impact of the application of yield management techniques on the results of their operations.

Airlines operate in a deregulated environment (at least in the U.S.), whereas the Shipping Liners operate in a regulated environment with many of the executive decisions made at a Conference rather than Company level, with pricing being the most characteristic among them. Price inflexibility is an inherent characteristic of Shipping Conferences. Although every conference member has the right to independent action on any tariff item, Shipping Conferences try to work towards greater cohesion on pricing and not many of the operators look forward to start price wars with their fellow Conference members.

The above arguments, are essentially arguments in favor of the application of yield management in shipping. Airlines have the option of controlling their prices, and regulating their capacity availability through yield management. On the other hand, since Shipping Liners operate in an environment where capacity control through pricing is not always feasible, they have to use yield management techniques as a straightforward way of controlling their capacity and their equipment allocation.

1.2.2 Shipping Liners Reservation Systems

The reservation systems of Shipping companies are not as advanced as the reservation systems of the Airlines. The reservation systems of Shipping Companies range in sophistication. One factor among many that influences the achieved sophistication of the reservation system is the size of the company. The reservation system of a typical company is very simple.

A major deterrent that keeps shipping lines from testing the effectiveness of yield management is the need for computerization. Besides sophisticated computer software, the successful use of a good yield management decision tool needs sizable databases with reliable past reservation booking information that would reveal the behavior of the market within which the shipping company operates. Many shipping companies do not have the above prerequisites for an immediately successful application of capacity control methods. The building of the databases necessary for the use of the software, takes time and the profits from the use of a computerized reservation system are not immediately apparent, when the Shipping Liner lacks the data that would streamline the capacity reservation system. There are many shipping companies waiting in the sidelines to see the results from the reservation systems of other companies that use yield management for their operations.

The use of a computerized system for capacity control can be very helpful even in cases when the database of the company is not sufficient to support sophisticated yield management tools. It can bring the information that floats around in the company in a central unit. That will benefit the decision makers who will have immediate access to all the information, that seems relative to the booking process (i.e. booking levels at the different legs of the trip and estimation for future bookings).

It is typical of the local offices of many Shipping Companies to have three different departments for sales/marketing, booking and equipment control. The sales/marketing department solicits offers from customers, who when persuaded to transport their containers with the vessels of the company call the booking department to place their orders. The booking department accepts the order without even knowing whether they have the available capacity for this booking or not. Furthermore, they do not know what the capacity commitment of the other booking departments are for the legs of the trip that are under the booking responsibility of the other local booking department. It is possible for one Far East office of the Shipping Company to have accepted a booking from Asia to Vancouver, on an itinerary that calls first in Los Angeles, and the Los Angeles Sales and Booking departments do not know that the vessel has already committed some of its capacity at the Los Angeles-Vancouver leg of the itinerary to a shipper.

It is not often that such a lack of coordination ends in overbooking and straining of the relations of the company with its customers, but the above example is indicative of the room for improvement that the booking systems of shipping companies have. An other possibility, especially during seasons of high demand, is that the Sales department solicits orders, the booking department accepts them, and only then the equipment department informs sales and bookings that the vessel has reached the capacity level and they cannot accept any more bookings. Situations like that, no matter how infrequent, can easily strain the relations with customers and affect the quality of service offered by the company, its reliability and its image.

The equipment department always knows, with some delay, what the available capacity at the different legs of the vessel itinerary is. In addition, they know what other equipment, like refrigerated containers and track chassis is available and where. What the shipping company lacks, is a system of communication between the different departments that will help each one of them to optimize their operations with information coming from the other departments of the company. Computerization will help the operations of the company and will increase the demand for the services of the operator, in the following way. There are shippers who are sensitive to the timely arrival of their products and they want assurances from their contacts in the company (usually the sales department) that there is sufficient capacity for their containers. Before the company representative checks with the equipment department and gets back to the customer, usually in less than 15 to 20 minutes, the potential customer has already given the order to an other shipping liner. This case is not very frequent either, but along with the other possible scenaria that we described above, it is indicative of the point that we try to get across. A reservation system that is not modern and does not use the advantages of modern day computers can be the bottleneck for the expansion of a shipping company and can inhibit the optimal utilization of the company resources.

A reservation system that employs yield management techniques, can very well bridge the gap that there is between the different departments of the company and will not allow for counterproductive decisions that are based on outdated or false information.

1.3 Differences between Airlines and Shipping Liners

So far, yield management has been mainly used by the most successful Airline operators. The experience with yield management tools is mainly concentrated in the airline industry. The success of the operating results of the Airlines has motivated several Shipping Liner Companies to change and update their reservation systems as well as explore potential changes of their booking practices and operating systems. Many shipping company executives wonder whether the nature of the shipping business is related closely enough to the nature of the airlines for the lessons learned in the practice of the airlines to be valid for shipping companies too. Furthermore, are the characteristics of the operations of the two different industries close enough for the transfer of experience and expertise between the two areas to be fruitful ? Can the models of the airline yield management be useful to the vessel operators?

At this point we have to describe the operations of a shipping company and compare them against the operations of a typical airline.

Shipping Liners do not have the advanced reservation systems of their counterparts in the airline industry. The booking systems of many Shipping Companies are fragmented and the Sales/Booking/Equipment and Capacity Control functions are distributed among several departments.

The yield management problem of the Shipping Liners is different than the yield management problem of the airlines. The demand for capacity, which is cyclical in both industries, exhibits shorter cycles in the case of the shipping industry. For instance, the fruit harvest season which creates demand for refrigerated containers and vessel space is much shorter than any vacation period that would create demand for passenger aircraft capacity.

The space of an aircraft is separated with permanent or movable bulkheads, that separate the different classes of aboard offered service. On the contrary, the carrying capacity of a container is much larger than that of an aircraft, and uniform. Nevertheless, the containership yield management problem is complicated by the fact that many different classes of customers need specialized equipment (i.e. refrigerated containers etc.). As a result, the booking decisions for these classes of customers are complicated by the availability of the specialized equipment.

The many classes of containers with the different freight rates that depend on the value of the cargo is a feature similar to the different classes of customers of the airline industry. In other words, there is a parallel between the business people/backpackers market segmentation of the demand for the Airlines, and the equivalent computers/semifinished products segmentation in the liner shipping industry.

The different classes of goods that use the services of a ocean liner receive different levels of service. They do occupy the same space aboard the vessel but the regular reliable service that a Shipping Liner/Conference provides, addresses mainly the needs of the high value products, or the products that are perishable (fruit that need refrigerated containers). The products of high specific value (i.e. relative to their volume unit), need the services of the Shipping Liners because it is too costly for them to stay idle in the warehouses. Fruit on the other hand has to reach the consumers within a limited amount of time before it loses part of its value. At the other end of the spectrum, the semifinished goods do not make full use of the services a liner company offers. The specific value of these products is relatively low and the high frequency of the Conference service does not decrease significantly the inventory costs for the owner of the semifinished good. Furthermore, the services of a Shipping Liner is one of several alternative ways for the transportation of these goods. The owner can always offer his cargo to a tramp vessel that visits both the port of origin and the port of destination of the cargo.

Additionally, the nature of the flow of the demand for the Shipping Liners is different than the flow of the demand for the Airlines. In the case of Airlines, the demand between two cities is balanced on average. The passengers who fly form on city to another, eventually return to the original city. There can exist some imbalances before and after major holidays (Thanksgiving etc.), but they are not characteristic of the year round nature of the business.

On the other hand, the demand for capacity on the one of the legs of the trip of the Shipping Liner is usually stronger than the demand for the returning leg of the vessel. That imbalance of demand at the two legs of the trip of a vessel is directly proportional to the trade imbalance between the two regions that are the hinterlands of the two ports. For instance the demand for capacity on the incoming leg from Japan is much stronger than the demand for capacity at the outgoing leg. This characteristic of the international trade that the Shipping Companies have to live with give a greater relative significance of the application of yield management at the case of the Shipping Liners.

In other words, when the shipping liners consider placing a vessel at a certain itinerary with imbalanced demand, they try to capture the demand of the "strong" leg. This demand is more profitable not only because it has a larger volume to offer but also because the unit value of the products transported in this direction is higher. As a result, the operators employ ample capacity that allows for them to capture the demand at the "strong" leg, whereas their capacity is underutilized at the other direction.

The costs of maintaining liner service at some trade route, are proportional to the number of the vessels employed at the service, i.e. the frequency of the service. The cost of the service also depends, but in a weaker fashion, on the size of the vessels. On the other hand, the costs are not very sensitive to the amounts of cargo they carry. With all the costs made with the establishment of the liner service and the commitment to the regular service, the operators are obliged to utilize their capacity fully in order to maximize their profits. In order for the operators to utilize their capacity fully, they have to employ capacity inventory control methods.

The alternative to optimal utilization for revenue increase would be to put in service larger vessels that could generate more revenues at the stronger leg. Such vessels would also have increased operating costs, that would offset part of the increased profits from the trade in the strong leg. In short, the bottleneck of the vessel itinerary is the available capacity at the strong leg. Capacity is the limited resource that has to be utilized wisely in order for the Shipping Company to maximize its revenues/profits.

We have stated above that volume capacity is a constraint at the strong leg of the trip. As expected, the capacity is not a constraint at the weaker or returning leg of the trip. The nature of liner shipping is such that usually the cargo traffic at the weak leg consists of items that are of lower value, compared to the goods that are transported over the strong leg, and often times they are heavier. It is not unusual for a vessel on the returning leg of the trip to have reached its dead weight limit before it has exhausted the available capacity. In cases like that the vessel is forced to decline extra cargo, exactly because they have reached the allowed carrying weight capacity.

We have therefore stated that we always have two kinds of constraints. The one constraint is the volume or the number of container slots offered by the vessel. This is the constraint that we are more concerned about because the nature of the containership cargo is such that most of it is relatively light and occupying large volume. As a result, the volume constraint is the constraint that is reached more frequently and therefore of major concern. Next to it we have a weight constraint that is reached often enough at the returning leg of the trip.The control of the weight at the weak leg of the trip is a rather subtle form of yield management not really understood by many Shipping Liners.

Both areas of capacity and weight control should be taken into consideration during the booking process of the shipping companies. In addition they should do management of the available space for specialized containers and other equipment.

Despite the many similarities between the Containership and the Airline yield problems, the liner shipping yield management problem can be more complicated. One source of complication is the number of the ports that the containership includes in its schedule. The number of ports a Containership visits is greater than the number of airports that an airplane includes at its itinerary.

A further complication arises from the time span that the trip of the containership covers relatively to the time it takes from the announcement of the trip to the departure of the vessel. An airline typically announces the flight two months before it takes place. The airplane visits the two or more airports of the itinerary within at most twenty-four to forty-eight hours from departure from the airport of origin. Consequently, we can argue that the customers who belong at the same customer class (i.e. high-value) ask for transportation the same period of time, whether the origin of their flight is the first airport or an intermediate one. On the contrary, when we consider the Containership problem we could have a difference of e.g. twenty days between the time the vessel sails from ports A and the time it sails from a following port B (see Figure 1-1). The week preceding the departure of the vessel from port A we have offers of high-value cargo for transportation from port A to port D.

At the same time the operator accepts offers for transportation of goods from port B to port C. It is almost a month before departure from port C, and the offers that the operator gets for transportation from port C to port D, are offers for low-value cargo. The operator is offered at the same time high value cargo for transportation from port A to port D and low value cargo for transportation from port C to port D. These two classes of goods are competing for the same capacity constraint (namely the capacity constrain at the leg C-D). At the same time the operator has to reserve some space for the high-value cargo originating at port C with destination D.

An additional reason of complication is the lack of a clear origin and a final destination for the Liner. The liner visits the consecutive ports of the itinerary and every port of the itinerary could be considered to be the origin of the itinerary (rolling horizon problem). The capacity available thought, is reduced by the capacity that is already occupied from goods transported from the previous ports and the capacity that has been committed for transportation from the following ports. The horizon (i.e. the number of ports) over which we want to optimize depends on the confidence the operator has in the estimation of the demand at the following ports.

A final reason of potential complication is the large number of different freight rates that have to correspond to all the possible combinations of cargo class and origin destination combinations.

1.3.1 Airline Market Segmentation

A seller can maximize his revenues, and consequently his profits when he causes the market to segment, so as to be able to charge each segment of the market as much as this segment of the market can bear. A successful implementation of the segmentation of the market does not "allow" customers who belong in segments of the market that are able to pay a higher price for the service or product, to purchase the service or the product that is offered at a lower price at the lower end of the market.

The airline industry implements the above principle as follows: They start accepting reservations for a flight, as soon as the schedule is announced. They offer a variety of fares that go with certain constraints. The more constraints they impose on the fares,

One week before departure from Port A.

Twenty-seven days before departure from Port C.

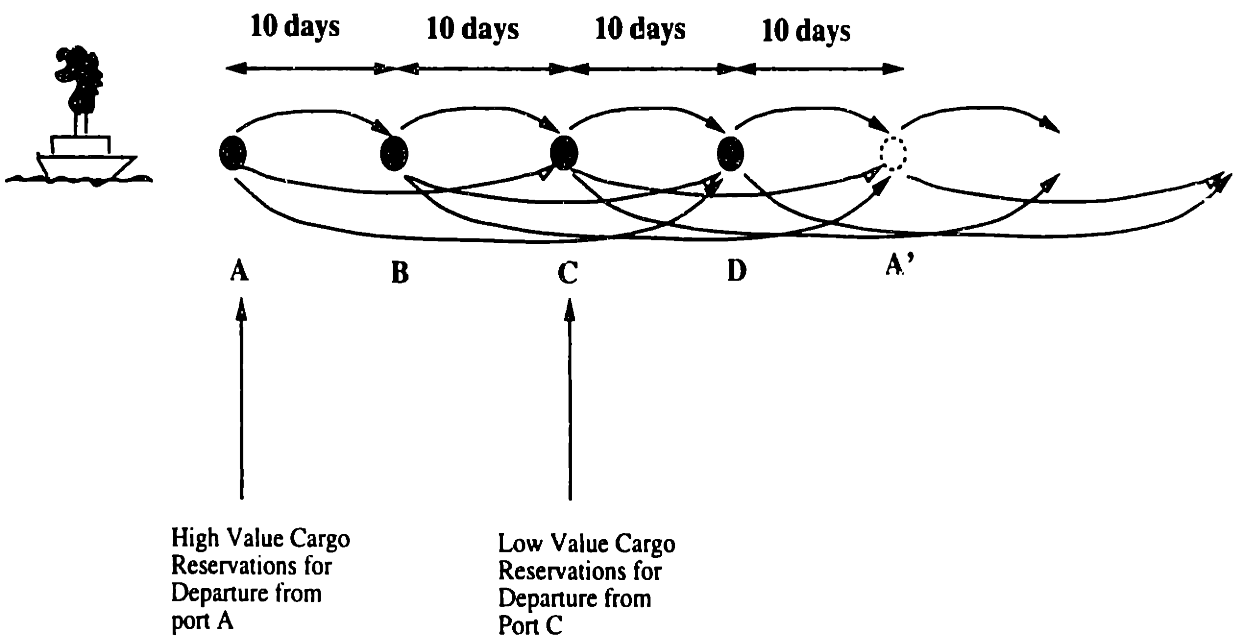


Figure 1-1: Vessel Itinerary example

the cheaper the fares are. Some of the classes of the fares they offer are so low that they cover only the variable cost of the passenger, and in some they are unable to sustain the overhead costs that correspond to the resources they utilize. It is of no surprise that these fares are the most restrictive. The more usual form of constraint is the date of the purchase of the ticket. The further away from the departure day the purchase takes place, the lower the fare is. The mechanics of this strategy work as follows:

The higher end of the airline market, namely the business travelers, usually travel on a short notice. As a result they do their reservations for a flight a few days before departure. Since the demand of the business market for airline transportation is relatively inelastic, the airlines are able to charge business travelers heavily. This segment of the market, that does the late reservations is ensuring the profitability of the airlines. Not all of the capacity offered by the airlines can be covered by customers belonging at this segment of the market. That means that if the airlines want to increase their revenues they have to offer the remaining seats at prices affordable by the other segments of the market. The operator has to make sure that the prices at which he offers the airline seats to the low end of the market are high enough to cover at least the variable cost of these seats.

The travelers who belong to the lower end of the market, namely backpackers, students, seniors etc., do not have the need or the economic ability to travel frequently, and they are able to make their travel plans far in advance. As a result, they know their travel needs well ahead in time, and they can purchase their tickets long before departure.

Therefore, the pattern for airline reservations emerges as follows: The low end of the market has the ability and it is encouraged by the constraints of the low fares, to make their purchase well ahead of time. The passengers belonging at the upper end of the market know their travel plans only a short time ahead of departure. Since they are able to pay the increased fares charged to them, the operators are able to recover their costs and make their profits.

The operators face a problem that is a result of their booking policies. The demand from the lower end market is much stronger than the demand from the high end of the market and the demand for seats offered at low prices, on peak flights, outstrips the capacity the operators can offer. If there were no further constraints imposed, the airplanes could end up being filled with low end customers who pay enough to cover only their variable costs. In this case the operator would be burdened with the deficit of the overhead costs. The operators have solved this problem by imposing restrictions on the number of the seats they offer at the discounted rates. The next problem is to define the maximum number of the seats they offer to the low end customers so as to maximize their revenues.

1.3.2 Liner Company Market Segmentation

We have seen that the airline operators have segmented their market by imposing restrictions on the date of the purchase of the sale of the ticket. They have also imposed other restrictions with reference to the length of stay, the days of the week the traveler can fly etc. As a result they have created the pattern where all the low end customers come enough time before departure and they are given seats, up to a certain number predefined by the operator and the remaining the seats are preserved for the high end customers.

The way a Liner Conference can discriminate among the different segments of the market is different from the way the airlines the Conference Liners have segmented their market. That makes the Conference Liner problem to be different than the problem faced by the airline operators.

The Liner Conferences discriminate with reference to the value of the cargo of the container and this distinction would be enough for the markets to be segmented. Nevertheless, it is true that the lower end cargo asks for transportation capacity earlier than the high end cargo. A possible reason that forces the low value customer to seek transportation capacity long time before departure might be routed into the fear that the limited capacity offered to the low end customers is not enough to satisfy all the demand for low value cargoes and therefore early action would be appropriate.

The fact, though, is that the timeliness of the purchase of the transportation capacity is not a factor affecting the price paid by the low value customers. A low value cargo shipper will pay the same price irrespective of whether he asks for capacity two months or only a week before departure. Therefore, the pattern that we observed in the case of the airlines is not very strict in the case of the Liner Conferences. Nevertheless, the vessel operator is faced with a booking pattern that is similar to the booking pattern an airliner operator is faced with. The low fare customers in both cases, ask for transportation far in advance, whereas the high fare customers ask for it shortly before departure.

In conclusion, the main differences between the airline and the shipping liner yield management problem are the following:

1. One source of complication is the number of the ports that the containership includes in its schedule. That, combined with the large number of different classes of goods, each of which has its own freight-rate, give us a fairly large number of possible combinations of cargo classes and origin-destination pairs. On the other hand, the shipping company network is smaller than the network of a typical airline. In addition, the number of vessels a liner company operates, is smaller than the number of the aircrafts of a typical airline operator.

2. Shipping Liners do price discrimination on the basis of the value of the cargo that asks for transportation. The freight-rate is a percentage of the value of the cargo, and not a function of the timing of the arrival.

3. A further complication arises from the ratio between the time length of a round trip of the containership, and the time it takes from the announcement of the trip, to the departure of the vessel. We could have a substantial time difference of i.e. twenty days between the time the vessel sails from ports A and the time it sails from a following port B. Therefore, the problem can not be solved statically. It is a dynamic problem, the nature of which is sequential.

4. There is a lack of a clear origin and a clear final destination for the Liner. The liner visits the consecutive ports of the itinerary and at no port the vessel becomes totally empty. Therefore, every port of the itinerary could be considered as the origin of the itinerary, since all ports are in a circle (rolling horizon problem). The decisions that we make for some leg of the itinerary, can and do influence all subsequent decisions. The consequences of a decision that we make now, can influence our revenue potential for more than one round trip.

1.4 Goal of the Dissertation

As we have already explained, the main function of Liner shipping is to offer transportation services to customers who want to be offered frequent and reliable transportation services. The customers also have come to expect rate stability. The demand for transportation capacity from these customers is usually seasonal and certainly not constant over time. Furthermore there is a spectrum of rates that the liner company charges the potential customers.

To a great extent the price charged reflects the utility the shipper gets from the liner company and the conference system. In order to reconcile the seasonality of the demand from the customers that represent the core of their business, ocean liners have to employ extra capacity. Otherwise they risk the loss of goodwill from their customers. That need of liner shipping to have extra capacity so as to service their main customers and preserve their high quality of service, contributes to the overcapacity of liner shipping.

In order to combat chronic overcapacity, liner companies have expanded the scope of their service and they offer transportation capacity to customers who do not utilize fully the services of liner shipping, and who can choose alternative modes of transportation. Those customers are mainly transporters of semifinished goods, raw materials etc, and other products of low specific value. In order to offer transportation capacity to theses shippers, liner companies compete with tramp ships. Since those shippers get practically the same service utility from both the liner and the tramp, they can be attracted by the liners only if the prices they are offered are competitive, and comparable to the prices of the others forms of shipping.

We see that by offering capacity to the low end of the freight rate spectrum, shipping liners can combat their overcapacity problem. When the transportation demand for high freight rate products is high, the vessel operator cannot afford to offer transportation to the low fare customers. They have to allocate their capacity in a way such that the low end customers do not deprive the high freight rate customers from the services of the vessel and the vessel operator from the revenue that he can obtain by allocating the vessel space to the high end customers.

In order for the operators to do an optimal allocation of their capacity, they have to have access to the appropriate demand statistics and use optimization models which, through optimal allocation of the vessel capacity, will guarantee the operator the maximization of the revenues of the shipping company in the short term and its profitability in the long term. In the following chapters, we will develop models that try to do exactly that.

1.5 Thesis Outline

In Chapter Two we present a literature review of the yield management problem. Since there is non existent literature on Yield Management for Shipping, our literature review focuses heavily on Yield Management for Airlines. In addition we present some of the research done on Yield Management for Hotels.

In the following Chapter Three we introduce the dynamic programming (DP) model for the simplest case of the one leg ocean yield management problem, and we study the properties of the model.

The solution to the linearized form of the model that we presented in Chapter Three is given in Chapter Four. A DP model for the multi-leg multi-class ocean yield management problem is presented in Chapter Five. In the same Chapter we also give the solution to the linearized form of this dynamic programming model.

In Chapters Six and Seven we modify the assumptions and the boundary conditions of the DP model for the network yield management problem, in order to incorporate cancelations of orders and the practice of overbooking into the yield management model. In both chapters we give solutions to the linearized versions of the modified DP models.

We run several simulations in Chapter Eight. We confirm the satisfactory agreement between the results of the DP models and the results from simulations where we use the solutions of the linearized DP's as decision criteria. We introduce a variation to the linear programming models that give the solution to the linearized form of the DP models that we presented before.

In the following Chapter Nine we use the linear model developed for the yield management problem for pricing and capacity allocation. We give the necessary conditions for optimal pricing.

All the models so far have assumed arrival of orders consisting of one container. We extend our decision support method for the case of orders consisting of many containers in Chapter Ten. As a further extension we introduce a method of evaluation for long term contracts. We examine the yield management model in the framework of the reservation system of the shipping company. We make a suggestion about the reservation system that would best exploit the potential of our yield management model. We additionally comment on the database needed for the support of the yield management model.

Finally, in the last chapter, we give a summary of the thesis and a section on the contributions of this work.

Chapter 2

Literature Review

The Yield Management Literature on Shipping Liners is virtually non existent. As a result, the following literature review will primarily be a review of the existing literature for Airline Yield Management. A brief review of the Hotel capacity control literature is also offered.

2.1 Airline Yield Management Review

Before 1970 most of the work done in the area of capacity control for Airlines had focused in the direction of the maximization of the aircraft load factors. By and large, maximization of the load factor would mean maximization of the revenues of the operator. As a result, the research of the operators was in the direction of overbooking.

Since then, changes in the structure of the airline fares and the introduction of multiple fares, the optimizing criterion has shifted from the load factor maximization to revenue maximization. This change had an obvious result. The scope of the research conducted on revenue management broadened in order to include seat allocation to the different fare classes of customers.

The research started with the simplest form of seat allocation, the one leg trip (non-stop flight between two cities). From then on the research included methods of Operation

Research techniques. Mathematical Programming and Network flow analysis were used in order to include all the possible passenger itineraries and the fare classes.

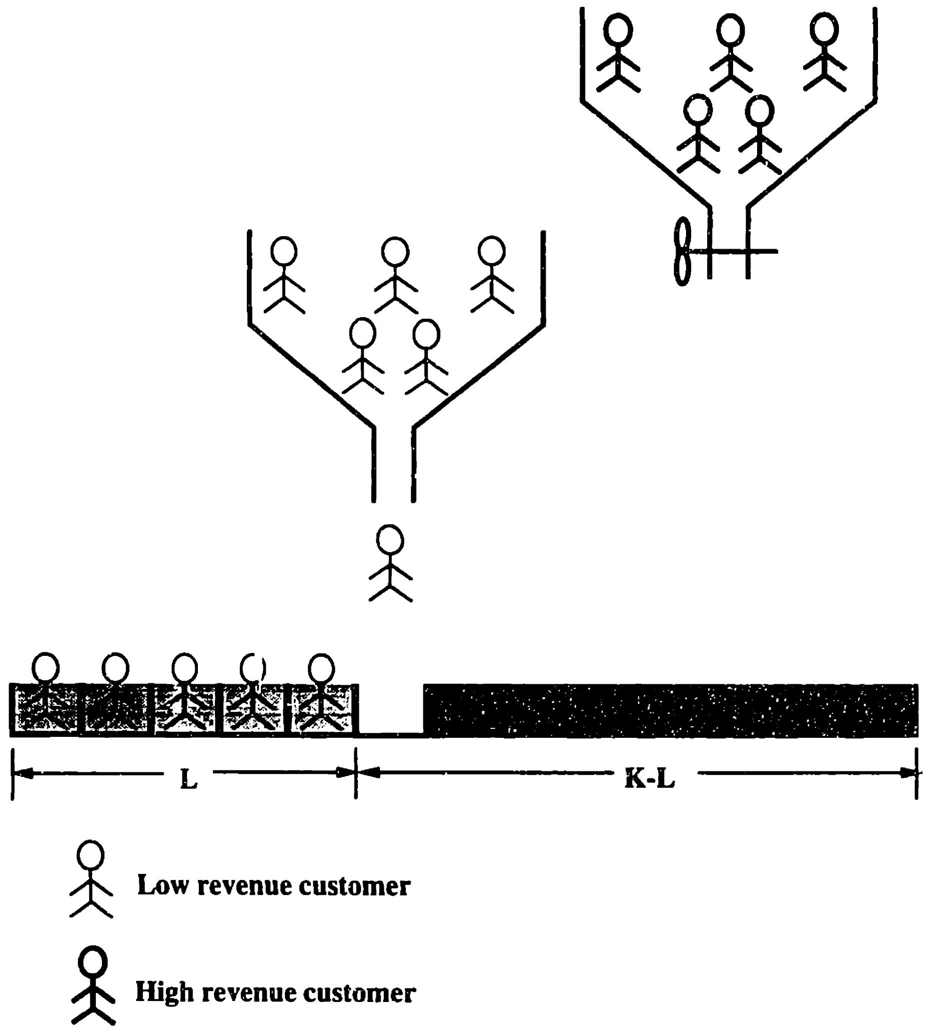
In 1972 Littlewood [7] introduced the model of a flight with one leg and two classes of customers. The aircraft was assumed to have remaining capacity C. The basic assumptions that Littlewood made, assumptions that are kept on in the later literature are:

- The low revenue customers ask for transportation capacity before the high revenue customers
- The demand for one class of customers is independent from the other
- The probability distribution of arrivals is a continuous distribution
- There are no cancellations of bookings

Littlewood also introduced the concept of the expected marginal seat revenue, and he used is as his criterion for the rejection or the acceptance of a reservation from the class with the lower fare. When a lower fare customer asks for a reservation the revenue from this customer is equal to the fare f_L. This reservation decreases the available seats (capacity) by one unit. Alternatively this seat is not given to the customer from the lower paying class, but instead is reserved for the last customer from the high fare class (it is easier to think of the last rather than any other high fare customer). It is an implicit assumption that once we close our capacity to the low fare class we do not open it again. In other words, if we decline a booking from the low fare class, we will expect to accept only high fare customers until the end of the booking process. In this case, the seat that we have left empty now will be used only if we have the arrival of more or equal than C higher paying customers. If we reserve the seat for the high fare customer, the expected revenue is

$$f_H \cdot P(x_H \geq C)$$

The above formula states that the revenue from a particular seat, if we reserve it for



ASSUMPTIONS:

1) Low revenue customers arrive before high revenueue customers
2) A refused low value customer means that we stop accepting low value customers
3) No cancelations of bookings
4) Demand of different classes is independent
5) Denied booking is lost revenue for the operator

Figure 2-1: The assumptions of the model by Littlewood the higher paying is equal to the fare that the higher fare class pays multiplied by the probability that this seat will be used. As a result, Littlewood concluded that the operator should accept the lower revenue class customer if the marginal seat revenue from this customer is greater (or equal) than the expected marginal revenue from a future high revenue customer.

$$f_L \geq f_H \cdot P(x_H \geq C) \tag{2.1}$$

In conclusion, Littlewood suggests that the revenue is maximized when the operator accepts low fare passengers up to the point where the probability of selling all the remaining capacity to high fare customers is equal to the ratio of the low fare to the high fare f_L/f_H

Several authors have worked on Littlewood's model and have proposed several variations of the original model. Bhatia and Parekh [8] in 1973 proved that Littlewood's criterion for capacity allocation is the condition that maximizes the revenues. This theoretical result was the main contribution of their work.

Richter [9] in 1982 introduced the concept of "differential revenue". He defined it as the difference between the revenue realized from selling a seat to a low fare customer and the expected revenue from the same seat had it been kept for a high fare customer. It was Richter's turn now to find that Littlewood's allocation criterion is in fact the revenue maximization criterion for the two fare classes, single leg problem that Littlwood presented.

An observation that we can make here is that the seat allocation proposed by equation 2.1 is only a function of the demand for the high revenue customers. It is independent from the demand for the low fare customers. That means that whether the demand for the low fare class is unlimited or weak, the number of seats that we preserve for the higher paying customers is the same. The intuition behind this result can be found in the above analysis. If the low revenue customers ask for seats before any of the high fare customers, then by turning down a low fare customer, we implicitly mean that we do the same to all the low fare customers who ask for transportation between that moment and departure. The observation that we mentioned, is a direct result of the assumptions of the model.

The reservation process is a dynamic process, and the criterion of Littlewood, along with the formulations of the problem by the above mentioned authors reveal that their formulation is a static one that ignores the dynamic aspect of the booking process. In practice, this theoretical shortcoming of the above models can be overcome if the operators update "constantly" their accessment of the expected demand for high fare customers (as well as the low fare customers), and recalculate the optimal capacity allocation.

Mayer [10] has tested the sensitivity of the main assumptions of the Littlewood model. He assumed that low fare customers book first in each of many periods before departure. His analysis suggested that the smaller the ratio $\frac{f_L}{f_H}$ is, the more sensitive the expected revenue from the flight is to the non-optimal allocation of seats. Mayer showed that a heuristic application of Littlewood's formula results in only a slight revenue decrease, relatively to more complex optimization methods.

Titze and Griesshaber (1983) [11] have used simulation for the same one leg problem with the basic assumptions that have been stated earlier. Their work shows that in practice the strict booking sequence can be relaxed somehow without any major reduction of the optimal revenues.

Buhr in 1982 [12] examined a multi-sector flight where the demand on certain flight segments competes with the demand on other flight segments for seats on the aircraft. This study considers only one fare class. Buhr proposes a criterion according to which the seat allocation should be done in such a way that the difference between expected revenues should be minimum. An iterative solution method was used to find the optimal values of the capacity allocation.

With reference to seat allocation between different fare classes on the same itinerary, Buhr suggested a two step approach. At the first step he finds the seat allocation for every itinerary. The second step is to allocate the capacity given to each itinerary, to the different fare classes that use this itinerary. Although Buhr describes the process, he does not implement it. Furthermore, he offers no proof for his criterion nor does he generalize it for an itinerary with more than two legs.

Wang [13] (1983) extended Buhr's model. His goal was to model the Yield Management problem in the case of an itinerary with multiple legs and multiple fare classes. According to his model, a particular seat on a multi-leg itinerary, if necessary, can be allocated to several different origin-destination and fare combinations. He developed a method for the maximization of the expected marginal revenue of each seat across the multi-leg flight path. His suggestion was that the revenue is maximized when the seat is allocated to the origin-destination and fare combination that offers the highest expected marginal revenue. As a result, Wang suggests that we compare the expected marginal revenue of all the origin-destination and fare combinations, and reserve the seat for the combination with the highest marginal revenue. Although this method can be applied to relatively small networks, with few fare classes, it is not very practical to be used when we have a multi-leg multi-class real life application.

Hersh and Ladany [14] (1978), developed a sequential decision tool for an airline reservation system, for an aircraft flying on a two leg itinerary. In their model, the airline has the right to transport passengers over either the first or both legs of the trip. It cannot board passengers for only the second leg of the trip. This is a realistic situation often encountered in international flights, as a result of bilateral regulatory agreements. In their dynamic programming formulation, the authors have given consideration to the effects of waiting lists, stand-by customers and overbookings. They further incorporate a Bayesian reassessment of probabilities in the decision process. Despite the above realistic representation of the booking process, the authors restrict the fare classes to just one. As a result, the long haul and the short haul passengers do not compete for the same capacity, in the sense that the long haul customers are always given priority over the short haul customers. The Bayesian reassessment of probabilities, despite its theoretical intrigue, is doubtful if it can be useful (it needs an extended database to become meaningful), and it is questionable if the improved results are worth the added computational intensity. Furthermore, the proof of the inherent complexity of their approach is the fact that the authors run an example with aircraft capacity equal to just 6 customers. Nevertheless, we have to keep in mind, that the authors ran their examples back in 1978 with computers of equivalent capabilities. Anyhow the work by Hersh and Ladany is valuable for the questions they raise, rather than the answers they give.

Glover et al. [15] (1982) formulated and developed the capacity control problem as a large network flow problem. The scope of the authors was to maximize the revenue of all the itineraries of an airline network. The task of the model was to find the optimal allocation to the different origin-destination and fare combinations over the network of a particular airline. The system built for Frontier Airlines, based on the above model, could handle a network of up to 600 flights and 30,000 passenger itineraries (origin-destination combinations) with up to five fare classes per passenger itinerary. The number of the side constraints of the model would be at the range 1,800-2,400. The side constraints were the aircraft capacity limits and the deterministic demand limits.

Wollmer [16] developed a large scale multi-leg multi-class network model into which he incorporated the probabilistic demands of the different classes. Each origin-destination and fare combination is represented as a binary variable. The objective of the model is the maximization of the expected revenue of the airline from the operation of the whole airline network. Wollmer's probabilistic formulation, combined with the binary representation of every variable of the model produce an extremely large problem, that can be unmanageable under non favorable conditions. Wollmer argues [17] in a subsequent paper, that despite this potential difficulty, only a small subset of the variables needs to be considered at any particular time. As a result, the solution to the general model can be found by solving a shortest path problem on a number of relatively small networks.

Dror et al. [18] present a formulation of the airline yield management problem that emphasizes passenger cancellations, multi-leg flights and a rolling planning horizon for the Airline. The authors present several network formulations in an order of increasingly realistic assumptions and therefore complication. The objective is to obtain flow (revenue) maximization. The first model they present is a simple network flow problem of a single flight with several intermediate stops. The basic formulation is expanded in order to include many flights. This second network model is a structured version of a maximal flow, multi-commodity problem with sources and sinks at the nodes of the network.

The model can be implemented for relatively simple cases, when no switch-over passengers are allowed. Incorporation of the switch-over passengers into the model, transforms the network into a more complex multi-commodity flow problem with additional constraints. The authors do not address the following questions (i) the calculation of the parameters for this type of network, and (ii) presentation of the algorithms necessary for the solution of such network problems.

Belobaba [19] (1987) presented a heuristic approach to the one leg multi-class booking problem with ordered arrival of the customers according to the fare class where they belong. The heuristic is called the Expected Marginal Seat Revenue Method (EMSR) method. The objective is to calculate the number of the seats the operator must preserve for the higher fare classes, by protecting them from the lower fare classes. At each stage of his method, Belobaba calculates the number of seats he has to preserve for a fare-class over all the lower fare classes. According to the assumptions of the nested yield management problem lower fare classes ask for transportation capacity earlier than high fare classes.

Belobaba separates each stage into several steps. At every step he calculates how many seats he has to preserve for the higher fare class by protecting them over each one lower fare class. In order to find how many seats should be protected for a higher fare class over a lower fare class, the author treats the two classes in isolation, and he applies Littlewood's methodology. By adding up the number of the seats that should be protected over all the lower fare classes, he finds the total number of seats that should be protected for the higher fare class. Belobaba repeats the same algorithm for all the classes of the flight, and he finds the protection levels for each one of them.

Littlewood gave the accurate solution to the one leg, two fare classes problem under a set of assumptions that became almost synonymous to the yield management problem for airlines. Other models have extended Littlewood's model to include more than two nested fare classes. Under the assumption of the independence of demand for the nested classes of customers, the above generalization of Littlewood's model has been solved independently by Curry [20], Brumelle and McGill [21], Wollmer [22], and Maragos [23].

In his work, Curry assumes continuous distributions of demand for the different fare classes and he has expressed the expected revenue from each fare class in the form of a series of recursive integral equations. These recursive equations suggest convolved multiple integrals of order up to the number of the fare classes of the model.

Curry formulates the problem as an optimization problem with non-linear objective function, and linear constraints. The non-linear objective function is the sum of the expected revenues from every fare class. Each one of the expected revenue expressions is formulated like the above mentioned convolved multiple integrals. The value of each one of the integrals is a function of the protection levels of all the lower fare classes and the class itself. The linear constraints of the problem are the capacity constraints of the vessel.

Curry's model is an accurate formulation for the multi-class, one leg problem, where the nesting of the classes is clear. Curry extends the model for the multi-class, multi-leg problem. That problem does not fulfill the basic assumption of the nested ordered fare classes. In other words, a businessman who books a seat shortly before departure in order to fly from San Francisco via Minneapolis to Boston, and a businessman who books a seat shortly before departure in order to fly from San Francisco to Minneapolis, on the same aircraft, book their seats at around the same time, compete for the same capacity and they do not belong to the same fare class.

The above example shows that the assumptions of Curry's (and Littlewood's) model are not met in the case of multi-leg itineraries. Nevertheless, Curry mentions that his multi-leg model is accurate only when the different origin-destination pairs do not share the same seat inventory. A practical problem that we encounter if we attempt to find the optimal capacity allocation using the model by Curry, is that we will have to calculate the multiple integrals (or their derivatives) of the objective function, at each step of the iterative nonlinear optimization method.

Brumelle and McGill [21], work on the one leg multiple fare classes problem, with the usual airline yield management assumptions. The formulation of the model is done through dynamic programming. The major contribution of the paper by Brumelle and McGill, is the proof that the fixed-limit booking policies are optimal within the class of all admissible policies that depend only on the observed number of current bookings. The proof shows that the policies of protection levels for the higher fare classes studied by Littlewood and the other authors are optimal. The formulation by Brumelle and McGill accepts either discrete or continuous demand distributions.

Through subdifferential optimization within a stochastic dynamic programming framework, the optimality conditions are reduced to a set of probability statements that become equivalent to the Littlewood optimality condition for the simple case of the two fare classes. Despite the simplicity of their formulation and their theoretical value, the usefulness of the optimality criteria is hindered by the same drawbacks that limit the work by Curry. The authors consider just the case of the one leg itinerary, but the optimality criteria involve calculation of multiple convoluted integrals (or multiple convoluted sums in the case of the discrete demand distributions). These integrals are the same as the integrals in the model by Curry. The reservations that were expressed before about the usefulness of the Curry model, are expressed here as well.

Wollmer [22], assumes that the demand is described by discrete rather than continuous functions and he reaches results that are similar to the results by Curry, and Brumelle and McGill. Wollmer presents the optimal booking limits as decreasing functions of the fare price and increasing functions of the available capacity. He also presents an algorithm for the calculation of both the optimal protection levels and the optimal expected revenue. Brumelle and McGill as well as Wollmer present simulations of both their methods and EMSR method by Belobaba. The comparison of EMSR against the respective methods shows that although EMSR is clearly a suboptimal method, the slight revenue improvement (in the range of 1%) obtained by the Brumelle and McGill or Wollmer methods, does not always justify the use of the later methods over EMSR.

Maragos [23], in his Master's thesis employs the same assumptions as the above authors and he considers the demand to be a continuous function. First, he examines the one leg itinerary. He, as Curry before him, formulates the problem as a static non-linear optimization problem, with linear constraints. The objective function of the optimization model is again the sum of the expected revenues from the different freight rate classes. Maragos proved that any capacity allocation that fulfills the necessary conditions for unconstrained optimality, also fulfills the sufficient conditions for optimality. From this observation he concludes that the necessary condition of optimality has a unique solution that corresponds to the global maximum of the optimization problem. He shows that the necessary conditions for optimality that he derived, are equivalent to the necessary conditions of optimality derived by Brumelle and McGill [21].

The model presented in this thesis, is basically similar to the model presented by Curry. An improvement over the model by Curry is that the formulation of the nonlinear optimization model that describes the yield management capacity allocation problem, is notationally and computationally more compact. Therefore there is no need to retreat to Curry's recursive equations. The compact notation allows for the derivation of the gradient of the objective function, which is used at the gradient method algorithm for the derivation of the optimal booking limits for each freight rate class. Nevertheless, the bottleneck of this method is again the calculation of the multiple convolved integrals, of both the objective function and its gradient.

In order to bypass this problem, Maragos simplified the convoluted integrals down to single and double integrals. At the calculation of the expected revenue for a particular higher freight rate class, he approximates the demand distributions of all the lower freight rate classes, with the cumulative demand distribution of the lower fare classes.

If it is assumed that the probabilistic demand for all classes is normal distributed, the cumulative distribution of several classes is still normal with mean and variance equal to the sum of the means and the sum of the variances respectively. As a result of this property of the Normal distribution, the derivation of the distribution function of the cumulative demand, does not need any extra calculations.

Because of the above simplifications the integrals of the objective function, and the gradient, become at most double convoluted integrals. The author simplifies the model even further and he gets a system of equations involving only single integrals that represent the simplified necessary conditions for optimality. He also developed a criterion for the acceptance or rejection of orders that consist of more than one containers. The author extends his model for the multi-leg case. The shortcomings of this extension are the shortcomings of the model by Curry. Maragos extended his model for the case of standby customers (non ordered customer arrivals).

Brumelle et al. [24], examine the two fare classes-one leg flight problem, and they relax the assumption of the statistical independence between the demands for the two fare classes. They allow the unconditional probability demand of the higher fare class to become conditional on the fact that the lower fare class bookings have exceeded a booking limit, and they generalize the optimal criterion derived by Littlewood. The authors remark that:

> With small cabin capacities relatively to demand, the booking limits in the dependent case are the same as those in the independent case, as there is no revenue benefit from taking dependency into account.

The success of Yield Management depends on the implementation of the following routine [1]:

- Demand forecasting for all the Classes of Customers
- Optimization of the Booking limits for each and every Class of Customers
- Revision of forecasts and booking limits as departure approaches

We therefore see that the development of good statistical methods that can adequately describe the reservations demand distributions is a prerequisite for the optimization process to be successful.

The derivation of the Bivariate statistics for the two classes of customers case is more challenging than the derivation of two independent distributions, one for each of the two classes of goods. That means that we need more data in order to derive the bivariate demand distribution for the two fare classes.

The model proposed by Brumelle et al. tries to incorporate dynamic elements with the introduction of the correlation between the two classes of customers. It is, nevertheless static in its nature. The user of the model would always have to update periodically the estimation of the demand and the booking limits for the model. It is doubtful whether the application of the method suggested by Brumelle et al. can be effective in any aspect except for the decreased frequency with which the user would have to update the model results.

Robinson [25], relaxes the assumption of the monotonic increase of the fare classes arrival. His model allows for the fare classes to arrive in an arbitrary order. Robinson uses the optimality conditions to show that under the optimal booking limits, the ratio of the "current" fare to the highest remaining fare, is equal to the probability of filling the aircraft. Using this observation, Robinson shows that one can find good approximations to the optimal booking limits by employing Monte Carlo integration.

Williamson [26], focuses on the model of an airline itinerary network. The author introduced a nesting method based on shadow prices. The shadow prices are the change in revenue if one additional unit of capacity (seat) is allocated to a given origin-destination and fare class combination. In the case of deterministic demand these shadow prices are the shadow prices for the demand constraints of the given origin-destination and fare class combination.

As a variation of the above method, Simpson [27], developed a similar approach with shadow prices that pertain to the capacity constraints, rather than the origin-destination and fare class combinations. In Simpson's work it is the shadow prices that determine which origin-destination and fare class combinations should be accepted or not, and the level at which the booking limits of the different combinations should be set.

Pfeifer [28], studies the one leg two fare classes problem. In an interesting departure from the typical airline yield management assumptions, he assumes that both classes of customers come from the same pool. He assumes that after the low fare class exhausts the space allocated to it, the customers who wanted to purchase low fare seats, accept to buy a high fare seat with some probability p_1.

There is one weak point in Pfeifer's analysis. It has to do with the fact that Pfeifer wants to find the probability that the $(q+1)$th customer (where q is the maximum number of low fare customers that the operator accepts) is a customer who will agree to a sellup and buy a high fare ticket. In order to find that probability one would need a definition of the demand as a function of time, whereas Pfeifer treats the problem as a static one that is updated periodically.

Furthermore, Pfeifer gives a solution to the airline yield management problem as an analogy of the newsboy inventory problem. He does not elaborate on the similarities of the two problems and why the solution to the newsboy problem could be used with no major modifications, to find the solution to Pfeifer's version of the yield management problem.

Alstrup et al. [29] present an overbooking model for a fixed non-stop flight with two types of passengers. Their model includes reservations, cancellations, as well as "no shows" (passengers who fail to arrive for the flight without notice). The authors incorporated denied boardings and downgrading of passengers. The model presents the airline booking process as a Markovian non-homogeneous sequential decision process. The model is solved by a Stochastic Dynamic Programming formulation. The model presented in the paper by Alstrup et al. is very similar to the two-variable Dynamic Programming model developed by Hersh et al. ([14]) and described earlier in the current literature review. The only difference between the two models is the initial conditions in the model by Alstrup et al. which takes into consideration up-grading and down-grading of passengers. The objective of the authors was to minimize the difference between the maximum expected potential gain and the actual gain. Several

adjustments with reference to the grouping of seat reservations and the reduction of the range of the state variables was able to make the running time of the model short enough so that it could be applicable in airline yield management practice.

Weatherford et al. [30], feel that the Yield Management problem should be renamed Perishable-Asset Revenue Management (PARM) in order to describe more accurately the nature as well as the goals of the problem. They present a taxonomy (classification scheme) and a research overview of the Yield Management problem. They analysed situations in which Yield Management has been practiced, and they concluded in some common characteristics for all Yield Management applications. Those common characteristics are:

- One date on which the product is available and after which it is either not available or it ages.
- A fixed number of units.
- The possibility of segmenting price sensitive customers.

The authors discuss the different possible management objectives from the application of yield management, and they conclude that in most cases, a risk neutral management would use the criterion of maximizing expected profit.

An interesting classification of fourteen distinguishing elements is presented. Some of these elements pertain to the nature of the specific yield management problem or situation. Some characteristics depend on the decision maker, and others on the formulation of the model and the assumptions of the booking process.

A review of the published research on Yield management problems is presented. Each variation is described briefly and then the optimal rules, along with the literature for this problem are presented.

At an other paper, Weatherford et al. [31], present a model for the customer arrivals pattern, which is presented as a non-homogeneous Poisson process. The model allows evaluation of different decision rules for yield management situations. It is also used to derive the probability distributions that are necessary for the operational implementation of the optimal decision rules for yield management problems with diversion and two classes of customers.

The authors present and compare several heuristic approaches to the optimal acceptance rules. The improvement to the expected contribution is the main criterion for the usefulness of the approach. Finally the sensitivity of the different models to the changes of the model's input parameters is examined. The paper focuses heavily on the airline models, data and experience.

Lee et al. [32], develop a discrete time dynamic programming model for finding an optimal booking policy, that would ideally be reduced to a set of critical values. The formulation of the model is valid for any pattern of arrivals and not only the usual o.dered arrival pattern of the airline models. Multiple seat bookings are also incorporated in the model. The basic properties of the model are studied. An attempt for the derivation of a demand probability distribution as a function of time is presented.

The authors give some limits for the case of multiple bookings. We have many bookings when one ordering is placed for several bookings. The operator either accepts them all or none. The option to accept only a few of them is not possible.

The limits for multiple bookings given by the authors for the cases of Non-nested Seat-Allocation Approach or the Nested Booking-Limit Approach, are simple extensions of the case of single bookings. The rules offered are the best that can be offered when the criteria for single bookings have to be extended for the case of many bookings, but they do not necessarily reflect the theory from which the criteria have been developed. (See [23] for an alternative treatment of the multiple bookings problem).

In order for a theory about multiple bookings to be developed, we have need of statistics that pertain to the probabilities of an ordering to be an ordering of multiple bookings. Given that we have a multiple bookings order, we need to define the probability distribution of the number of the bookings. It is rather difficult to find statistics that satisfy the above demands. On the other hand it would be rather costly for companies to keep records of offered bookings and sort them according to the number of the offered bookings. It is doubtful that the increased complication of the model would benefit its user with substantially increased expected revenues, that would justify the increased number of computations.

2.2 Hotel Yield Management Review

Rothstein [33], presents a sequential decision model for the hotel overbooking problem and the determination of the booking policies. He assumes that each booking is for only one day. He treats a booking for S days as a series of S independent bookings. He formulates the reservation system as a Markov process with transition probabilities that depend on the booking policy. The model presented by Rothstein has the added notion of economic rewards for transition from state to state. A dynamic programming algorithm is developed for the computation of the optimal booking policies. According to the author, this paper has been mainly written in order to demonstrate the nature of the hotel booking problem, demonstrate one solution approach and indicate its shortcomings rather than provide a model for decision making.

Ladany [34] presents a Dynamic programming model for Hotel rooms reservations and capacity control. He treats bookings as a sequential dynamic process. The objective of his model is the maximization of expected profit contribution per rental day. The model includes the effects of cancellations and overbookings. It accepts Poisson or Normal distribution demand, but a non-fitted empirical distribution could be used without complicating the utilization of the model. The fact that the decision tool relies exclusively on the Dynamic programming model, confines the use of the model to the management of a small number of rooms.

Liberman et al. [35] address hotel overbooking, which is a form of the capacity control problem. The modeling of the process is as follows: M hotel rooms are available at a date n periods from now. Reservations are made by the customers for only one day. Reservations for more than one day are treated as independent events. Customers may cancel their previously confirmed reservations at any time prior to their arrival, with no penalty. On the other hand, new requests for rooms for the target day are generated randomly.

The authors want to find an optimal booking policy that will maximize the expected net profit (or the discounted net profit) realized over a time period. The problem is formulated as a Markovian Sequential Decision process, and they give the structure of an optimal booking policy. It is shown that the optimal strategy is a 3-region policy. Depending on the region, overbooking is either encouraged (unconditionally or up to some level) or discouraged.

2.3 Summary

In the previous two sections we overviewed the literature for yield management in the airline industry and the hotel industry respectively.

The yield management research for the airline industry can be approached from two perspectives. The research focuses either on the allocation of the capacity among the different classes of customers who want to travel on a particular itinerary or on the allocation of the capacity among the different itineraries. The one leg problem has been treated as a stochastic optimization problem, whereas the network problem has been treated mainly as a deterministic flow problem. All of the airline yield management models focus either on the one or the other aspect of the problem.

The combination of the two problems in a single model has not been done in a satisfactory manner, due to a large extent to the very assumptions of the airline yield management problem. The ordered arrival of the different classes of customers is an assumption that helps the formulation of the one leg airline yield management model. This same assumption complicates the formulation of the network problem, because some of the classes of customers arrive concurrently. These classes of customers might not travel on the same itinerary, but they eventually compete for the same capacity. The network model is just a collection of all the possible itineraries that are treated as independent booking processes. The only constraint that is imposed on the different itineraries is that they satisfy collectively the capacity constraint of the aircraft. The extension of the one leg model to the case of the network, gives a model whose accuracy is questionable. Additionally, the computations involved in the solution of the network model are tedious. This is the criticism for the airline model as far the airline practice is concerned. With reference to the liner shipping problem, we could add that the airline model is based on assumptions that fit the airline practice, but could not be enforced in the setting of a liner shipping company.

The hotel yield management literature review is a collection of rather short lived dynamic programming formulations. The assumptions of the models, although leading to an oversimplification of the problem, need a considerable amount of computer power in order to give meaningful solutions to problems that have real life proportions.

In conclusion, the assumptions of the airline yield management problem are not representative of the ocean liner yield management problem. Furthermore, the network version of the airline yield management problem is increasingly inaccurate as the number of the airports (or ports) increases. At the model for shipping yield management, we cannot adopt the simplifying assumption of the ordered arrivals of the different classes of customers that travel on the same itinerary. The fact that the customers have the option of asking for capacity at any point in time makes the ordered arrivals assumption void. The alternative is to formulate the problem as a dynamic programming model. The dynamic programming model in itself cannot be a satisfying solution for the network problem. The curse of dimensionality of dynamic programming can easily turn any dynamic programming formulation into a model of limited usefulness.

In the following chapters we will give the dynamic programming formulation of the various forms of ocean yield management that we will examine. Additionally, we will show that the optimal solution (and therefore optimal booking policy) to the dynamic programming is given by a linear programming formulation. As a result, the models that we will develop will have the modeling accuracy of a dynamic programming model and the facility of solution of a linear programming model. Furthermore, these linear programming models are nothing more than descriptions equivalent to maximum flow network models for the yield management problem of a network of ports. As a result, we show that the solution of the dynamic programming model is as fast as the solution of a linear programming model with special structure.

Chapter 3

Dynamic Programming Model for Two Ports, M Classes of Goods

The booking of the capacity of a containership involves many decisions that are made in stages. Customers arrive at all times and the vessel operator has to make his decision of whether to accept of reject each customer, without having full knowledge of the results of each decision and before the next decision has to be made. What the vessel operator knows, is that his objective is to maximize the revenues from each particular sailing.

Each booking decision cannot be viewed in isolation, since the operator wants to balance his desire for high present revenue with the possibility of low future revenue. This tradeoff can be captured with a dynamic programming formulation. The booking process has two prominent features:

- An underlying dynamic system. The dynamic system at hand is the booking process. Depending on wheather the model is discrete or continuous, the underlying system is discrete or continuous respectively.

- A revenue function that is additive over time. The revenue that the operator accrues is additive in the sense that it accomulates over time, as more bookings take place.

3.1 DP Model and Boundary Conditions

In this section we will model the booking process of a ocean liner as a dynamic programming model. We will assume that the vessel serves only two ports. The first port is the port of origin, and the second and last port of the itinerary is the destination of all the cargo that vessel carries from the port of origin.

The vessel itinerary has been announced T time units before departure. For the sake of simplicity we consider the booking process to be discrete. We assume that we have possible transportation requests from customers only on discrete time points, i.e. we can have arrivals at exactly t or $t-1$ time units before departure but we cannot have the arrival of a potential customer at the time interval between t and $t-1$. We can have at most one arrival at each time t. Figure 3-1 is an illustration of the discrete time assumption. The discrete time assumption is not an impediment for the realistic description of the booking process. We can always increase the number of the time intervals, if we want to increase the accuracy of the dynamic programming (DP) model. At the limit, when the number of time intervals goes to infinity, our model becomes a continuous time dynamic programming model.

If we go back to the discrete DP model, the time points of potential customer arrivals, do not have to be equidistant. We can have a dense distribution of time points where the frequency of customer arrival is higher, and sparser where the rate of arrival of customers is relatively low.

We consider the probability of the arrival of more than two customers at the same time, to be of higher order, and therefore we have incorporated in the model the assumption that the probability of the arrival of more than two customers at the same time is equal to zero.

The total capacity of the vessel is equal to K container slots. We know that the total number of the classes of the potential customers of the vessel is equal to M. We have announced the freight rates for all the M classes of customers. These freight rates are $f_1 < f_2 < \ldots < f_M$, where f_1 is the freight rate of the containers with the lowest value cargo, and f_M the freight rate of the highest value cargo. We also know what the probability distribution of the demand for transportation from the different classes of containers is. We assume that the probability of arrival of a customer belonging to class m at time t is equal to $\lambda_m(t)$. Of course $\sum_{m=1}^{M} \lambda_m(t) < 1$.

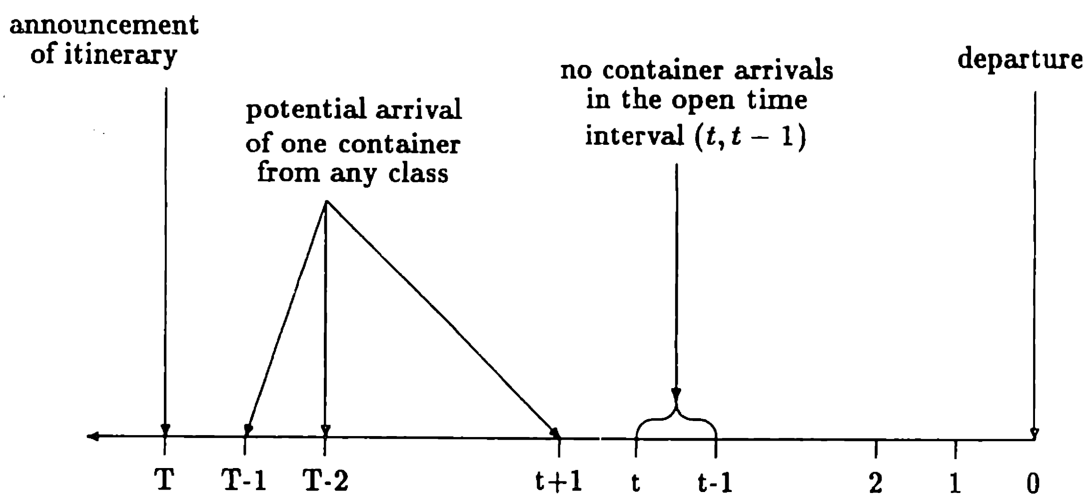


Figure 3-1: Discrete time arrival pattern for all classes of customers

In the introduction of the present chapter, we have shown that the booking process of a vessel can be adequetely described with dynamic programming. We will show here that the booking process can have what is called *the essential structure* for Dynamic Programming. We assume that the booking process is going on for some time, that we are at time t before departure, and the remaining non booked vessel capacity is equal to C. The model of the booking process includes:

1. A discrete time system of the form $n_{t+1} = n_t - w_{m,t} \cdot I_{m,t}$, where:

   n_t is the remaining capacity at time t.

   $I_{m,t}$ is the decision to accept ($I_{m,t} = 1$), or not to accept ($I_{m,t} = 0$), a customer who arrives at time t and who belongs to class m.

2. Independent random arrivals of customers. The probability for a customer of class m to arrive at time t ($w_{m,t} = 1$) is equal to $\lambda_m(t)$. In general, the probability distribution of customer arrivals could be a function of the remaining capacity n_t and the decision variable $I_{m,t}$. There is no need for such a dependence to be incorporated in our model.

4. A control constraint. If a customer arrives, then $I_{m,t} \in \{0, 1\}$. If no customer arrives ($w_{m,t} = 0$), then $I_{m,t} = 0$.

5. An additive cost of the form

$$E\left\{\sum_m I_{m,T} \cdot f_m + \sum_{t=1}^{T-1} \sum_m I_{m,t} \cdot f_m\right\} \tag{3.1}$$

   Each term of the above sum is a function of the remaining capacity C_t at time t, and the probability distribution of the random variable $w_{m,t}$.

From the above discussion, we see that the yield management problem for an ocean carrier, can be formulated as a dynamic programming model.

The stage of our cost functional is the remaining time units t until departure. The state is the number of the remaining container slots (n) that are still available, i.e. the container slots that have not been assigned yet to any shipper. The cost functional has as follows:

$$V_m(t,n) = \max_{I \in \{0,1\}} [I \cdot f_m + W(t, n - I)] \quad \forall\, m \in \{1, M\} \tag{3.2}$$

where

$$W(t,n) = \sum_{i=1}^{M} [\lambda_i(t-1) \cdot V_m(t-1,n)] + \left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] \cdot W(t-1,n) \tag{3.3}$$

$V_m(t,n)$ is the maximum expected value at time t of the revenue that will be realized between time t and time $t = 0$, when the vessel departs from the port of origin. The maximum expected value $V_m(t,n)$ is conditional on the fact that we have an arrival of class m customer at time t. The cost functional is also a function of the available container slots (i.e. a function of n) at time t.

$W(t,n)$ is the maximum expected revenue at time t, when there are n container slots available. $W(t,n)$ is conditional on the event that we have no arrival of any customer at time t.

$\lambda_i(t)$ is the rate of arrival (or the probability of arrival) for a class i customer at time t.

I is the control variable when we have the arrival of a class m customer who asks for transportation capacity. The optimal control variable at stage t and state n, is $I_m(t,n)$. i.e. $V_m(t,n) = I_m(t,n) \cdot f_m + W(t, n - I_m(t,n))$. $I_m(t,n) \in \{0,1\}$. When we have no customer arrival, $I_m(t,n) = 0$ by default.

f_m is the freight rate for containers that belong to class m.

The expected future revenue is zero, when there is no remaining capacity that could be sold to future customers. Also the future revenues are zero at the time of the departure of the vessel. As a result, the boundary conditions become:

$$V_m(t, n=0) = 0.0\,, \quad \forall\ t \in \{0,T\} \text{ and } m \in \{1,M\} \tag{3.4}$$

$$W(t, n=0) = 0.0\,, \quad \forall\ t \in \{0,T\} \tag{3.5}$$

$$W(t=0, n) = 0.0\,, \quad \forall\ n \in \{0,K\} \tag{3.6}$$

3.2 Properties of the cost functionals and the control variable

First, we should mention that $V_m(t,n)$, and $W(t,n)$ are increasing functions of t. That means that the further away we get from the day of departure, and for the same available capacity, the greater the expected revenue is, when we apply optimal booking policy. The fact that $V_m(t,n)$, and $W(t,n)$ are increasing functions of t is an expression of the evident, yet very important monotonicity property of the DP algorithm.

Definition: The first forward difference $\Delta g(x)$ of a function $g(x)$ whose domain is the set of non-negative integers is defined as: $\Delta g(x) = g(x+1) - g(x), \quad x = 0, 1, 2, \ldots$. The function $g(x)$ is concave, if its first forward difference is nonincreasing. That is: $\Delta g(x) \geq \Delta g(x+1), \quad x = 0, 1, 2, \ldots$ [36, p. 49] . Therefore:

$$g(x): \ concave \iff g(x+1) \geq \frac{1}{2} \cdot g(x+2) + \frac{1}{2} \cdot g(x), \quad x \in \mathcal{N} \tag{3.7}$$

Lemma 1 *$V_m(t,n)$ and $W(t,n)$ are concave functions of n.*

Proof: The full proof of this lemma is given in the appendix A. Here, we will just give a sketch of the proof. We do a simultaneous and interdependent proof of the concavity of the two functions, $V_m(t,n)$ and $W(t,n)$. $W(t,n)$, is defined for all the positive and integer values of n. In order to do the proof of the concavity of $V_m(t,n)$, we extend the definition of $W(t,n)$, at the domain of the real and positive values of n. The value of $W(t,x)$ for the non-integer value of x, is given as the linear intepolation of the values of $W(t,n)$ for the two integers n and $n+1$ such that $n < x < n+1$.

The following three Theorems will show that :

1. When the available capacity decreases, it is increasingly difficult to accept a low freight rate customer.
2. When the time of departure approaches, it becomes easier to accept a low freight rate customer (alll other things being equal, i.e. for the same capacity)
3. The higher the freight rate paid by a container, the easier it is to accept this container for transorpation.

We remind that from equation 3.2 we can have:

$$I_m(t,n) = \arg \max_{I \in \{0,1\}} [I \cdot f_m + W(t, n - I)] \tag{3.8}$$

In other words, $I_m(t,n)$ is the optimal decision variable for a container that belongs to class m, when there are t remaining units of time untill the departure of the vessel, and there are n remaining container slots.

Theorem 2 *$I_m(t,n)$ is a non-decreasing function of n.*

Proof: We have to prove that $I_m(t,i) \leq I_m(t,j), \ \forall\, i < j$.
If $I_m(t,i) = 0$, it is trivial to prove that $I_m(t,i) \leq I_m(t,j), \ \forall\, i < j$.
If now $I_m(t,i) = 1$, then we have to prove that $1 = I_m(t,i) \leq I_m(t,j), \forall\, i < j$. In other words, we have to prove that when $I_m(t,i) = 1$ then
$I_m(t,j) = 1, \ \forall\, j > i$.
By substituting $I_m(t,i) = 1$ in equation 3.2 we get that:

$$f_m + W(t, i-1) > W(t,i) \implies f_m > W(t,i) - W(t,i-1) \tag{3.9}$$

We will prove Theorem 2 by contradiction. We assume that $I_m(t,j) = 0, \ \forall\, j > i$, in which case from equation 3.2 we will get:

$$f_m + W(t, j-1) < W(t,j) \implies f_m < W(t,j) - W(t,j-1) \tag{3.10}$$

From equations 3.9 and 3.10 we have

$$W(t,j) - W(t,j-1) > W(t,i) - W(t,i-1), \ \forall\, j > i \tag{3.11}$$

From Lemma 1 we know that $W(t,n)$ is a concave function of n. From the concavity we get that:

$$W(t,j) - W(t,j-1) < W(t,i) - W(t,i-1), \ \forall\, j > i \tag{3.12}$$

The contradiction between equations 3.11 and 3.12 shows that the assumption $I_m(t,j) = 0$ is false. Therefore, we conclude that if $I_m(t,i) = 1$ then $I_m(t,j) = 1, \ \forall\, j > i$. **Q.E.D.**

Theorem 4, suggests that it becomes easier to accept a customer, as the time for the departure of the vessel approaches. Before we continue with the proof of the Theorem 4, we will prove the following Lemma 3 which will be used in the proof of Theorem 4.

Lemma 3 *If* $I_m(t,n) = 0 \Longrightarrow f_m + W(t,n-1) \leq W(t,n)$ *and* $f_m + V_m(t,n-1) \leq V_m(t,n)$

Proof: If $I_m(t,n) = 0$, equation 3.2, gives that $f_m + W(t,n-1) \leq W(t,n)$.
Again from equation 3.2, we get:

$$V_m(t,n-1) = I_m(t,n-1) \cdot f_m + W(t,n-1-I_m(t,n-1)) \tag{3.13}$$

As a result,

$$\begin{aligned} f_m + V_m(t,n-1) &= f_m + I_m(t,n-1) \cdot f_m + W(t,n-1-I_m(t,n-1)) \\ &= f_m + W(t,n-1) \\ &\leq V_m(t,n) \\ \Longrightarrow f_m + V_m(t,n-1) &\leq V_m(t,n) \end{aligned} \tag{3.14}$$

The second equality of equation 3.14 comes from the fact that
$I_m(t,n-1) = 0$ when $I_m(t,n) = 0$ (from Theorem 2). The inequality comes from equation 3.2.
Therefore, when $I_m(t,n) = 0$ we have $f_m + V_m(t,n-1) \leq V_m(t,n)$. **Q.E.D.**

Theorem 4 $I_m(t,n)$ *is a non-increasing function of* t.

Proof: We have to prove that $I_m(t+1,n) \leq I_m(t,n)$.
If $I_m(t,n) = 1$, it is trivial to prove that $I_m(t+1,n) \leq I_m(t,n)$.

We also want to prove that when $I_m(t,n) = 0 \Longrightarrow I_m(t+1,n) = 0$.

Instead, we will prove that

$$f_m + W(t+1, n-1) \leq W(t+1, n) \tag{3.15}$$

which is equivalent to proving that $I_m(t+1,n) = 0$. From equation 3.3 we get:

$$\begin{aligned} f_m + W(t+1,n-1) &= f_m + \sum_{i=1}^{M}[\lambda_i(t) \cdot V_m(t,n-1)] + \left[1 - \sum_{i=1}^{M}\lambda_i(t)\right] \cdot W(t-1,n-1) \\ &= \sum_{i=1}^{M}\lambda_i(t) \cdot [f_m + V_m(t,n-1)] + \\ &+ \left[1 - \sum_{i=1}^{M}\lambda_i(t)\right] \cdot [f_m + W(t,n-1)] \end{aligned} \tag{3.16}$$

Since $I_m(t+1,n) = 0$, from Lemma 3 we have that:

$$\sum_{i=1}^{M}\lambda_i(t) \cdot [f_m + V_m(t,n-1)] \leq \sum_{i=1}^{M}\lambda_i(t)V_m(t,n) \tag{3.17}$$

and

$$\left[1 - \sum_{i=1}^{M}\lambda_i(t)\right] \cdot [f_m + W(t,n-1)] \leq \left[1 - \sum_{i=1}^{M}\lambda_i(t)\right] \cdot W(t,n) \tag{3.18}$$

Therefore:

$$\begin{aligned} f_m + W(t+1,n-1) &\leq \sum_{i=1}^{M}\lambda_i(t)V_m(t,n) + \left[1 - \sum_{i=1}^{M}\lambda_i(t)\right] \cdot W(t,n) \\ &= W(t+1,n) \end{aligned}$$

The above equality is a result of equation 3.3. Therefore:

$$f_m + W(t+1, n-1) \leq W(t+1, n) \tag{3.19}$$

If we remember the definition of $I_m(t,n)$, the above inequality shows that $I_m(t+1,n) = 0$. We have therefore shown that if $I_m(t,n) = 0 \Longrightarrow I_m(t+1,n) = 0$. **Q.E.D.**

The following Theorem 5 shows that if we accept a customer from a particular fright rate class, we would also accept any customer who belongs to a higher freight rate class.

Theorem 5 *$I_m(t,n)$ is a non-decreasing function of m.*

Proof: We want to prove that $I_i(t,n) \leq I_j(t,n), \;\; \forall\, i < j$.
If $I_i(t,n) = 0$, it is trivial to prove that $I_i(t,n) \leq I_j(t,n), \;\; \forall\, i < j$
For $I_i(t,n) = 1$, we have to prove that $I_j(t,n) = 1 \;\; \forall\, i < j$.
If $I_i(t,n) = 1$, we have that

$$f_i + W(t,n-1) > W(t,n) \implies W(t,n) - W(t,n-1) \leq f_i \tag{3.20}$$

For $i < j$ we have that $f_i < f_j$. From the two previous equations we conclude that:
$W(t,n) - W(t,n-1) < f_j \Longrightarrow I_j(t,n) = 1$ **Q.E.D.**

The results from the above Theorems 2, 4, and 5 are compatible with the intuitive understanding that:

- The more capacity available the operator has, the easier it is to accept a low revenue customer.
- The closer to departure we get, the easier it is to accept low revenue customers (for the same remaining capacity).
- The higher the freight rate of a container is, the easier it is to accept this container for transportation.

3.2.1 Numerical Results

In the attached graph (Figure 3-2), we present with a numerical example some of the ideas that we have presented in this chapter.

The assumption of the numerical model is that we have a vessel that serves an itinerary between only two ports. The vessel operator accepts ten (10) classes of container for transportation. The freight rates are such that $f_1 < \ldots < f_M$. The vessel has an original capacity of $3,000$ container slots, and the itinerary is announced $2,000$ time units before departure. The point that represents the original stage ($t = 2,000$) and state ($K = 3,000$) of our DP model, is the upper right corner. The departure of the vessel takes place at stage $t = 0$.

Figure 3-2, shows nine curves, that correspond to the nine lower freight rate classes of containers. Each of these curves seperates the possible stage-state combinations in two areas. Let us consider the curve that corresponds to class m. If we are at any of the combinations to the left of the curve that corresponds to class m, we accept any container that belongs to class m. In other words for these (t, n) combinations we have that $I_m(t, n) = 1$. If we are at any of the combinations to the right of the curve, we do not accept the class m container, or $I_m(t, n) = 0$.

At figure 3-2, we can verify that:

1. $I_m(t, n)$ is a non-decreasing function of n.

2. $I_m(t, n)$ is a non-increasing function of t.

3. $I_m(t, n)$ is a non-decreasing function of m.

Policy Areas: From the theory that we have developed in this chapter we understand, and by inspecting figure 3-2 we can verify, that there are severals areas (i.e. sets of neighboring time-capacity combinations) where the optimal policy, is common. In other words, throught out each of these areas we have the common optimal policy of accepting some classes of goods, and rejecting all the others.

For instance, we can see at figure 3-2 that the area between the curves I_4 and I_5 is the area where it is optimal to accept the classes $5, \ldots, 10$ for transportation, and reject the classes $1, 2, 3$ and 4. If, for example, we are at the area to the left of curve I_1, we accept all classes of customers for transportation, whereas if we are at the area below curve I_9, we accept only class 10 customers. Figure 3-2 is divided into 10 such areas. Actually, the number of these areas, for the two ports, M classes of goods problem is equal to M. These areas we call *Policy areas.*

3.3 Summary

In this chapter we introduced the dynamic model for the two port multi-commodity booking problem. We introduced two cost functionals. $V_m(t, n)$ is the expected revenue until departure, when we have the arrival of an m class container at time t and $W(t, n)$ is the expected revenue until departure, when we do not have the arrival of any container at time t.

Lemma 1 shows that both $V_m(t, n)$ and $W(t, n)$ are concave functions of the remaining capacity. That means that the marginal value of the vessel capacity is decreasing for increasing values of the capacity.

Theorems 2, 4 and 5 prove the monotonicity of the control variable (accept / do not accept a customer). In other words, Theorems 2, 4 and 5 show that with reference to a class of containers, the remaining time-remaining capacity space is separated in two areas. If we are within the one area we accept containers from this class, whereas if we are in the other we do not accept these containers. The overlap of these accept / do not accept areas for all the different classes of containers creates the *Policy Areas.* In each Policy Area, we accept the associated high freight rate classes of containers, whereas we reject all the others. If we know the policy area within which we are located, we know the optimal policy. Because of the fact that the policy areas are compact, knowledge of the outer border of the policy areas is enough to give the policy area within which we are located. Once we know the borders of the policy areas, we can cofidently apply the optimal policy of the current policy area.

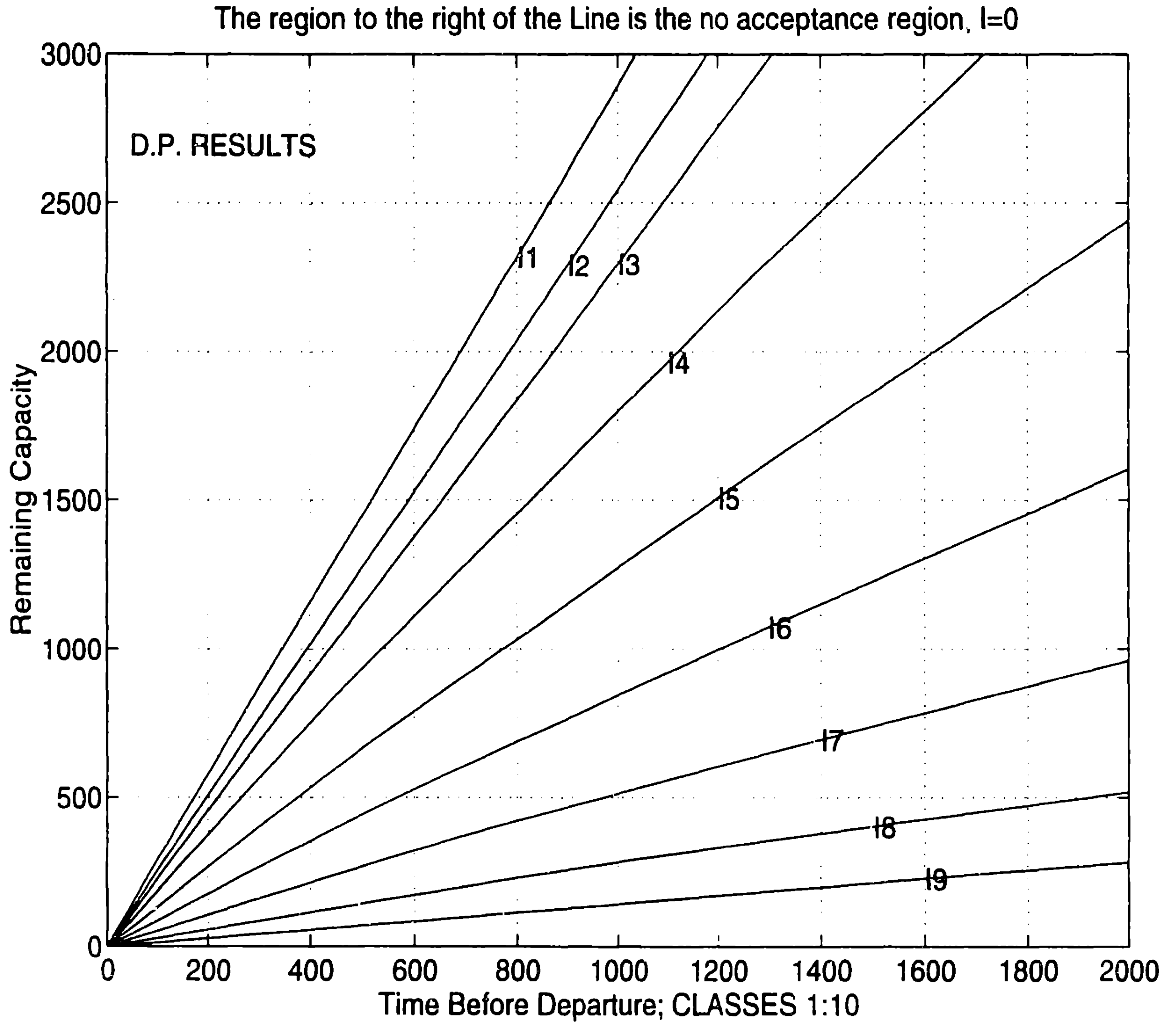


Figure 3-2: CAPACITY = 3,000, TIME = 2,000 Constant Arrival Rates

Chapter 4

Continuous model for the two ports, M classes problem

In the previous chapter we studied the behavior of the cost functional of the dynamic programming formulation for the two ports, M classes of goods problem. We concluded that there are exactly M policy areas. Within each of these policy areas we accept some classes of goods, and we reject all the others.

The original problem was to find when it is optimal to accept for transportation an arriving customer who belongs to class m. An alternative, and as we have explained, equivalent expression of the problem is to find the boundaries of the M Policy areas. When we define the boundaries of the policy areas, we can safely apply the policy of the area in which the remaining time, remaining capacity combination belongs, with the certainty that the policy we apply, is the optimal.

In the first part of this chapter we use the dynamic programming formulation of the booking problem, in order to derive the differential equation of the linearized maximum expected revenue function. This differential equation is the Hamilton-Jacobi-Bellman equation of control theory. For constant arrival rates, we show that the solution of the HJB differential equation is the solution of a linear program given by formula 4.10. With the assistance of the *Equivalence Theorem*, this linear program suggests as optimal policy, the same policy that is suggested by the linearized dynamic programming.

Before we continue, we will give a brief overview of the structure of the theory that we develop at this chapter. We do it for two reasons. The first reason is to give a description of the steps that we follow, to make them easier to understand, and to provide the reader with an epitome of the various proofs that we do through out the chapter.

The second reason is that in this chapter as well as the following ones, we will be developing a model of the booking process, and we will be working with it through out the chapter. Each of the booking models that will be presented will be an enriched version, of the previous models. Several of the assumptions of one model, will be relaxed at the subsequent models. The process that will be followed here, and the flow of the theory (the lemmas, the theorem and the corollaries included) will be similar from the one chapter to the next. Since the same type of theory structure is going to appear more than once it is worth talking about it and showing it in a figure (see 4-1).

Summary of the structure of the chapter

1. We state the assumptions of the DP model. We present a discrete time form of the model. We give the boundary conditions of the model.

2. From the discrete time DP model, we derive the continuous time DP model. We linearize the expected revenues $W(t, C)$, and we get the Hamilton-Jacobi-Bellman equation of control theory.

3. We formulate the linear programming model. The maximum of the objective function is $z(t, C)$.

4. From the linear programming model, we derive the governing equation of the maximum of the objective function ($z(t, C)$).

5. The theorem shows that the solutions of the two differential equations coincide when and only when the two methods suggest the same reservation policies.

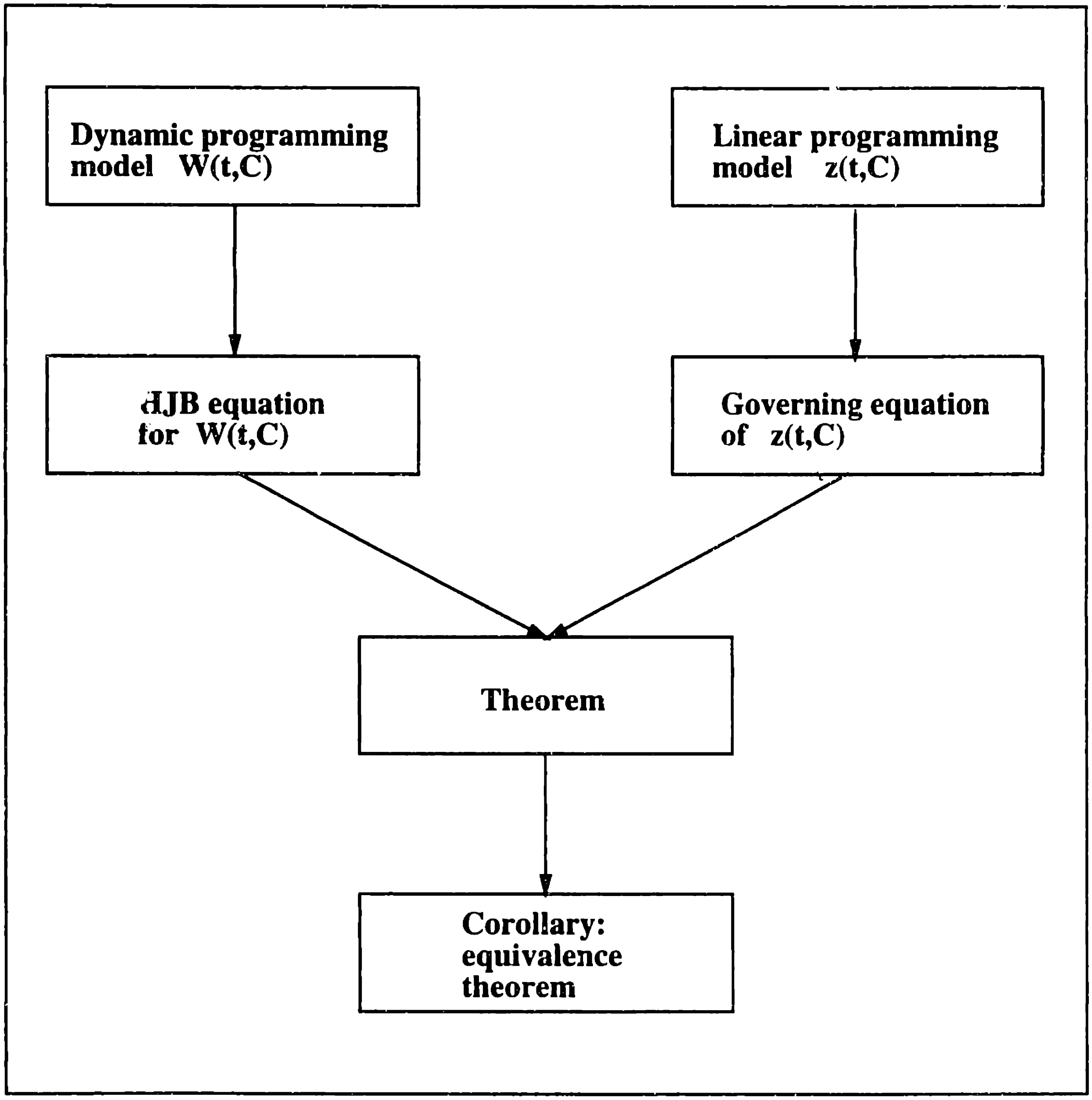


Figure 4-1: Summary of the structure of the chapter

6. A natural outcome of the theorem is the corollary, with is called *equivalence theorem.* The equivalence theorem, suggests that the output (expected future revenues and optimal policy) of the linearized dynamic programming and the equivalent linear programming are the same.

4.1 Derivation of the HJB equation

The DP modeling formulation that we have used for the booking process of the two ports, M classes of goods problem, is a discrete time model. The total time from the announcement of the itinerary until the departure of the vessel is equal to T. We divide this time T, into N equal time intervals, and we can have potential arrivals only at the beginning (or expiration) of these N intervals. If we want increased accuracy of the model, we can increase the number of the intervals into which we have divided the original time. (We have to keep in mind, that this increase of the time intervals, increases proportionally the computational intensity needed for the solution of the DP program.) As the number N of the time intervals increases the model becomes more accurate. The limiting case is for the number of time intervals $N \longrightarrow \infty$. The assumption of an infinite number of time intervals is, of course, a mathematical formalization which cannot be satisfied in practice. It can, nevertheless, be a reasonable approximation for a problem similar to our booking problem that involve a finite, but very large number of stages (time intervals).

The reason why we let the number of the intervals to go to infinity is that usually the limiting cases, give elegant analytical expressions (and sometimes solutions) for the models that we study, and the implementation of the optimal policy can be simple. The same happens in the current situation, but not before we make the assumption that the remaining capacity C, is $C >> 1$ (C is much larger that 1).

When the number of time intervals $N \longrightarrow \infty$, then the duration of each time interval $\Delta t = \frac{T}{N} \longrightarrow 0$. Our assumptions are modified in order to include the event that at the beginning (or expiration) of each time interval Δt we have a possible arrival of a container belonging to class m, with probability $\Delta p_m(t) = \Delta t \cdot \lambda_m(t)$.

The booking process, is described by equations 3.2 and 3.3 that will be repeated here, with the modification that the time intervals will be changed from being equal to 1, to be equal to Δt.

$$V_m(t, C) = \max_{I \in \{0,1\}} [I \cdot f_m + W(t, C - I)] \ , \ \forall\, m \in \{1, M\} \tag{4.1}$$

where

$$W(t, C) = \sum_{i=1}^{M} \Delta t \cdot \lambda_i(t - \Delta t) \cdot V_i(t - \Delta t, C) + \left[1 - \sum_{i=1}^{M} \Delta t \cdot \lambda_i(t - \Delta t)\right] \cdot W(t - \Delta t, C) \tag{4.2}$$

The following lemma 6 gives the differential equation that describes the linearized form of the maximum expected revenue, as it is given by the dynamic programming formulation. The differential equation given by the lemma is the Hamilton-Jacobi-Bellman equation of control theory.

Lemma 6 *$W(t, C)$ is the maximum expected revenue of the continuous time model under optimal policy . The optimal policy is such that $I_m(t, C) = 0$, $\forall\, m = 1, \ldots, i - 1$, and $I_m(t, C) = 1$, $\forall\, m = i, \ldots, M$. We assume that the partial derivatives of $W(t, C)$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t, C)$ is the equation:*

$$\frac{\partial W(t, C)}{\partial t} + \left[\sum_{m=i}^{M} \lambda_m(t)\right] \cdot \frac{\partial W(t, C)}{\partial C} = \sum_{m=i}^{M} \lambda_m(t) \cdot f_m \tag{4.3}$$

The boundary conditions, for $W(t, C)$ are given by equations 3.5 and 3.6.

The proof of the above lemma 6, is given in the appendix B.

The above partial differential equation 4.3, is the continuous-time analog of the DP algorithm. The solution of this equation is the limit of the solution of the DP, when the number of the stages $N \longrightarrow 0$. The boundary conditions are the same for both the discrete and the continuous version of the model. The boundary conditions are given by equations 3.5 and 3.6.

Equation 4.3 is the so called Hamilton-Jacobi-Bellman (HJB) equation. It is a partial differential equation satisfied for all time-capacity pairs (t,C) by the maximum revenue-to-go function. Of course we do not know a priori that the expected revenue-to-go function $W(t, C)$ is differentiable, which was implicitly assumed in the preceding derivation. We will show however that there is a solution to the HJB equation, that is piecewise differentiable ([37]).

At this point we remind that $I_i(t,C) \leq I_j(t,C)$, for $i < j$. If now, i is the smallest index m for which $I_m(t,C) = 1$, we define:

$$\gamma(t,C) = \sum_{m=1}^{M} \lambda_m(t) \cdot I_m(t,C) = \sum_{m=i}^{M} \lambda_m(t) \tag{4.4}$$

$$\beta(t,C) = \sum_{m=1}^{M} \lambda_m(t) \cdot f_m \cdot I_m(t,C) = \sum_{m=i}^{M} \lambda_m(t) \cdot f_m \tag{4.5}$$

We see that:

- $\gamma(t,C)$ is the reservation rate of the vessel capacity, or the expected rate of capacity reservation. It is equal to the sum of the arrival rates of the container classes that we accept while in the current policy area.
- $\beta(t,C)$ is the revenue accumulation rate, or the expected rate of incoming revenue. It is equal to the sum of the revenue accumulation rates of the container classes that we accept for transportation while in the current policy area.
- i is the lowest freight rate container class accepted for transportation in the current policy area.

Within each policy area, i.e. area where we accept only selected classes of goods, $\gamma(t,C)$ and $\beta(t,C)$ are known coefficients. As a result of the definitions of $\gamma(t,C)$ and $\beta(t,C)$, within each policy area, equation 4.3 becomes:

$$\frac{\partial W(t,C)}{\partial t} + \gamma(t,C) \cdot \frac{\partial W(t,C)}{\partial C} = \beta(t,C) \tag{4.6}$$

We repeat the boundary conditions of the DP formulation. These equations are the boundary conditions of equation 4.6 too.

$$W(t, C=0) = 0.0\,, \quad \forall\ t \in \{0,T\} \tag{4.7}$$

$$W(t=0, C) = 0.0\,, \quad \forall\ C \in \{0,K\} \tag{4.8}$$

If we are able to find a solution that satisfies the equation 4.6, at each one of the M different policy areas, and it also satisfies the boundary conditions 4.7 and 4.8, then we have found a solution to the original equation 4.2, at the limit where the number of the time intervals N into which we divide the time T, becomes $N \longrightarrow \infty$.

We have M policy areas. Within each policy area i, $\gamma(t,C)$ and $\beta(t,C)$ are known functions of t, but both $\gamma(t,C)$ and $\beta(t,C)$ are different sums from the one policy area to the next. We know the values of $\gamma(t,C)$ and $\beta(t,C)$, at each of the M policy areas. What we do not know are the boundaries of the different policy areas. Nevertheless, finding the boundaries of the policy areas, i.e. finding the classes of containers that we accept for each given combination t and C is the original problem. In the following section, we will give the solution for a relatively simple version of the problem.

4.2 Solution of the HJB equation for constant arrival rates

In this section we will give a solution to a special case of the two ports, M classes of containers problem.

We assume that the arrival rates of the different classes of customers are constant through out the booking process, and they are not a function of time. The arrival rate of each class of customers is different, in general, than the arrival rates of any other container class.

$$\lambda_i(t) = \lambda_i \ , \quad i = 1, \ldots, M \tag{4.9}$$

Despite the fact that the arrival rates are not functions of time, $\gamma(t, C)$ and $\beta(t, C)$ continue being function of time. By that I mean, that although, $\gamma(t, C)$ and $\beta(t, C)$ are constant within each policy area, they are different from the one policy area to the next. The policy area is a function of the combination (t, C). For different (t, C) combinations we can be in different policy areas. Therefore γ and β can be different for different combinations of (t, C).

We will find a solution to the differential equation 4.6 that will also satisfy the boundary conditions of equation 4.6 as they are described by equations 4.7 and 4.8. We will prove that a solution of the differential equation 4.6, subject to the boundary conditions of equation 4.6 is the solution of the following linear programming program.

$$\begin{aligned} \mathbf{z}(t, C) = \max \ \mathbf{f}^T\mathbf{x} & \\ \mathbf{A}\mathbf{x} \ \leq \ \mathbf{b} & \\ \mathbf{x} \ \geq \ \mathbf{0} & \end{aligned} \tag{4.10}$$

where:

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_M \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_M \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} C \\ \lambda_1 \cdot t \\ \lambda_2 \cdot t \\ \vdots \\ \lambda_M \cdot t \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

We also give the formulation of the dual of the linear problem 4.10:

$$\begin{aligned} \mathbf{v}(t,C) = \min\ & \mathbf{b}^T\mathbf{y} \\ & \mathbf{y}^T\mathbf{A} \geq \mathbf{f} \end{aligned} \tag{4.11}$$

with

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{M+1} \end{bmatrix}$$

The following Theorem 8 proves that the solution $\mathbf{z}(t,C)$ of the LP 4.10 (or the solution $\mathbf{v}(t,C)$ of the LP 4.11) satisfies the differential equation 4.6 of the linearized $W(t,C)$ function. Since $\mathbf{z}(t,C)$ and $W(t,C)$ have the same boundary conditions, the combination of the two, means that $\mathbf{z}(t,C) = W(t,C)$.

Corollary 9 proves that for the variables x_m for which the optimal solution is such that $x_m^* > 0$, all the respective classes m are accepted for transportation (i.e. $I_m(t,C) = 1$) under an optimal booking policy. For the variables of the LP 4.10 that are $x_m^* = 0$, corollary 9 shows that $I_m(t,C) = 0$, under the optimal booking policy.

$$x_m^* > 0 \iff \quad I_m(t,C) = 1 \tag{4.12}$$

$$x^*_m = 0 \Longleftrightarrow \quad I_m(t, C) = 0$$

From the above introduction it is obvious that the LP 4.10 is of importance for the development of our theory. Therefore, we will examine it more closely.

Input and Output of the LP 4.10 The input of the LP model is the remaining capacity, the remaining time until departure and the expected demand from the different classes of containers. The product of the arrival rate for the different classes of containers times the remaining time until departure, gives the expected demand from the different classes of containers. In other words, we only need an estimate of the expected demand for the M different classes of containers and the information on the remaining unbooked vessel capacity.

The output of the LP is the optimal solution x_m, $m = 1, \ldots, M$. It is relatively easy to see that the solution is such that $x_m = 0$, $m = 1, \ldots, i-1$ and $x_m > 0$ $m = i, \ldots, M$. Furthermore $x_m = \lambda_m \cdot t$, $m = i+1, \ldots, M$, whereas $0 < x_i < \lambda_i \cdot t$.

In the framework of the static LP, the above means that we accept no capacity bookings for containers belonging to classes $1, \ldots, i-1$, whereas we accept some reservations from class i containers, and all the reservations from classes $i+1, \ldots, M$.

If we had one more unit of capacity, this unit of capacity would be booked to a container from class i. Therefore, the class of containers i, is called the marginal class of containers. In a more compact form we have:

$$x^*_j = \frac{\partial \mathbf{z}(t, C)}{\partial f_j} \quad := \quad \begin{cases} 0 & j < i \\ C - \sum_{j=i+1}^{M} \lambda_j t & j = i \\ \lambda_j \cdot t & j > i \end{cases} \tag{4.13}$$

When we have more demand than the demand we can satisfy, the capacity constraint is active. In other words we have a marginal class of containers. When the marginal class of containers is the class i, the extra revenue from an additional unit of capacity would be equal to f_i.

$$\frac{\partial \mathbf{z}(t,C)}{\partial C} = f_i \tag{4.14}$$

When the capacity constraint is not active, i.e. when we have more capacity than the expected demand, the marginal cost is : $\frac{\partial \mathbf{z}(t,C)}{\partial C} = 0$. In this case we make the assumption that the marginal class of containers is the $0th$ class of containers with freight rate $f_0 = 0$. Therefore:

$$\frac{\partial \mathbf{z}(t,C)}{\partial C} = f_0 = 0 \tag{4.15}$$

Before we continue with Theorem 8 and Corollary 9, we will prove Lemma 7, which will be used at the proof of the Theorem 8.

Lemma 7 gives the differential equation that describes the maximum value of the objective function of the Linear Program 4.10. Lemma 7 also gives the boundary conditions of the maximum value of the LP, for reasons of completeness.

Lemma 7 $\mathbf{z}(t,C)$ *is the maximum value, and* $x^*_m(t,C)$, $m = 1,\ldots,M$, *is the corresponding optimal solution of the Linear Program 4.10.* $\mathbf{v}(t,C)$ *is the minimum value, and* $y^*_m(t,C)$, $m = 1,\ldots,M+1$, *is the corresponding optimal solution of the Linear Program 4.11. The governing differential equation for both* $\mathbf{z}(t,C)$ *and* $\mathbf{v}(t,C)$ *is the equation*

$$\sum_{j=1}^{M} f_j \cdot \frac{\partial \mathbf{z}(t,C)}{\partial f_j} - C \cdot \frac{\partial \mathbf{z}(t,C)}{\partial C} - t \cdot \frac{\partial \mathbf{z}(t,C)}{\partial t} = 0 \tag{4.16}$$

with the following boundary conditions:

$$\mathbf{z}(t, C=0) = 0 \quad and \quad \mathbf{z}(t=0, C) = 0 \tag{4.17}$$

Proof: The boundary conditions 4.17, can be easily verified from the LP formulation 4.10, if we substitute $t = 0$ or $C = 0$. From the optimality property of linear programming we get that $\mathbf{z}(t,C) = \mathbf{v}(t,C)$. Therefore:

$$\sum_{j=1}^{M} f_j x_j^* = C y_1^* + \sum_{j=1}^{M} \lambda_j t y_{j+1}^* \tag{4.18}$$

where all the x_j^*, and y_j^* are the optimal solutions to the primal and the dual linear program respectively.

$$\begin{aligned} x_j^* &= \frac{\partial \mathbf{z}(t,C)}{\partial f_j}, \quad j = 1,\ldots,M \\ y_1^* &= \frac{\partial \mathbf{z}(t,C)}{\partial C}, \\ \text{and} \quad y_j^* &= \frac{\partial \mathbf{z}(t,C)}{\partial bj}, \quad j = 2,\ldots,M+1 \end{aligned} \tag{4.19}$$

By substituting equations 4.19 into equation 4.18, we get:

$$\sum_{j=1}^{M} f_j \cdot \frac{\partial \mathbf{z}(t,C)}{\partial f_j} = C \cdot \frac{\partial \mathbf{z}(t,C)}{\partial C} + t \cdot \sum_{j=1}^{M} \lambda_j \cdot \frac{\partial \mathbf{z}(t,C)}{\partial b_{j+1}} \tag{4.20}$$

We know that the $2,\ldots,M+1$ constraints of the primal LP, are functions of time. More precisely, $b_j = \lambda_{j-1} \cdot t$, $j = 2,\ldots,M+1$. Neither the first constraint, nor any other parameter of the primal LP are functions of time. When we want to derive the partial derivative of the optimal value of the primal, with reference to time, we get:

$$\frac{\partial \mathbf{z}(t,C)}{\partial t} = \sum_{j=1}^{M} \frac{\partial \mathbf{z}(t,C)}{\partial b_{j+1}} \cdot \frac{\partial b_{j+1}}{\partial t} \tag{4.21}$$

We have that $\frac{\partial b_{j+1}}{\partial t} = \lambda_j$, and the partial derivative of z(t,C), with reference to time becomes:

$$\frac{\partial \mathbf{z}(t,C)}{\partial t} = \sum_{j=1}^{M} \lambda_j \cdot \frac{\partial \mathbf{z}(t,C)}{\partial b_{j+1}} \tag{4.22}$$

If we substitute equation 4.22 into equation 4.20, we can easily get equation 4.16. Equation 4.16 is the governing equation of the optimal value of $\mathbf{z}(t,C)$. For given values of the freight rates $f_j, j = 1, \ldots, M$, $\mathbf{z}(t, C)$ is a function of the remaining time t until departure, and the remaining vessel capacity C. Since $\mathbf{z}(t, C) = \mathbf{v}(t, C)$, the lemma is automatically true for $\mathbf{v}(t, C)$ too. **Q.E.D.**

In the following Theorem 8 we use the results from lemmas 6 and 7 to suggest that the solutions of the dynamic programming and the linear programming are equal when and only when the two following happen: 1) When the linearized dynamic programming suggests the rejection of a class of customers, the linear programming does not offer any capacity to this same class of customers. 2) When the linearized dynamic programming suggests the acceptance of a class of customers, the linear programming offers some capacity to this same class of customers.

Theorem 8 *$W(t, C)$ is the solution of the linearized version of the Dynamic Program given by equation 4.6, when the boundary conditions are given by equations 4.7 and 4.8. $I_m(t, C)$ is the control variable of the Dynamic Programming under optimal policy, that suggests the acceptance ($I_m(t, C) = 1$) or the rejection ($I_m(t, C) = 0$) of a container that belongs to class m, as a function of the remaining time t and the remaining capacity C. $\mathbf{z}(t, C)$ is the maximum value, and $x^*_m(t, C)$, $m = 1, \ldots, M$ is the corresponding optimal solution of the Linear Program 4.10. We prove the following:*

$$\{W(t, C) = \mathbf{z}(t, C)\} \iff \left\{ \begin{array}{l} I_m(t, C) = 0, \quad x^*_m = 0, \quad \forall\ m = 1, \ldots, i-1 \\ \textit{and} \\ I_m(t, C) = 1, \quad x^*_m > 0, \quad \forall\ m = i, \ldots, M \end{array} \right\} \tag{4.23}$$

or

$$\{W(t,C) = z(t,C)\} \iff \left\{ \begin{array}{lcl} \beta(t,C) = \sum_{j=i}^{M} \lambda_j f_j & & x_m^* = 0, \ \ \forall\, m = 1,\ldots,i-1 \\ & and & \\ \gamma(t,C) = \sum_{j=i}^{M} \lambda_j & & x_m^* > 0, \ \ \forall\, m = i,\ldots,M \end{array} \right\} \tag{4.24}$$

The proof of the above theorem 8, is given in the appendix B.

An alternative, and more useful expression of the above Theorem 8, is the following. Corollary 9 says that the optimal policy suggested by the linear programming is optimal policy for the linearized dynamic programming and the maximum expected revenue of the linearized dynamic programming is equal to the maximum value of the objective function of the linear program 4.10.

Corollary 9 (Equivalence Theorem) *Let $z(t,C)$ be the maximum value of the objective function and let $x_m^*(t,C)$, $m = 1,\ldots,M$, be the optimal solution of the LP 4.10. Let also $W(t,C)$ be the linearized maximum expected revenue from the differential equation 4.6, and $I_m(t,C)$, $m = 1,\ldots,M$, is the optimal policy suggested for the linearized DP. The following holds:*

$$\left\{ \begin{array}{l} x_m^* = 0, \ \ \forall\ m = 1,\ldots,i-1 \\ x_m^* > 0, \ \ \forall\ m = i,\ldots,M \end{array} \right\} \iff \tag{4.25}$$

$$\left\{ W(t,C) = z(t,C) \ \ and \ \ \begin{array}{l} I_m(t,C) = 0, \ \ \forall\ m = 1,\ldots,i-1 \\ I_m(t,C) = 1, \ \ \forall\ m = i,\ldots,M \end{array} \right\}$$

Implications of Corollary 9 Corollary 9 shows that the solution of the linearized HJB equation 4.6, is given by the optimal value of the parametric LP 4.10. Therefore the expected revenue of the DP model under optimal policy, is given by the objective function of the linear programming. Furthermore this LP formulation gives the optimal policy of the linearized dynamic programming.

Corollary 9 shows that there is no need to solve the (linearized) DP formulation of the two ports, M classes of containers booking problem, in order to find the optimal booking policy. Instead, we can solve the linear program 4.10, and the solution of the LP will give the optimal policy.

If the optimal solution of the LP for the variable x_m is $x_m = 0$, then the optimal policy for the linearized version of the DP is to refuse bookings from the containers that belong to class m ($I_m(t, C) = 0$). If the optimal solution for the x_m variable of the LP is such that $x_m > 0$, then it is optimal to accept the class m of containers for transportation ($I_m(t, C) = 1$).

We have to remind that the solution $W(t, C)$ of the differential equation 4.3, is a linearized form of the $W(t, C)$, which is defined as the cost functional of the recursive dynamic programming equation 4.2. The agreement between the optimal solution offered by the LP 4.10, and the optimal solution offered by the original recursive equation 4.2, depends on the agreement of the derivative $\frac{\partial z(t,C)}{\partial C}$, and the first difference $W(t, C) - W(t, C-1)$. $z(t, C)$ is the maximum value of the objective function of the LP 4.10, and $W(t, C)$ is the solution of the original recursive equation 4.2. The agreement between the two solutions becomes greater for larger values of the remaining capacity of the vessel.

4.2.1 A Solution for the Two Ports, M Classes of Containers Problem with Constant Arrival Rates

In this subsection we present two tables. Table 4.1 shows the optimal value of the LP 4.10 and its dual 4.11 as a function of the remaining time and remaining capacity (t, C) combinations, at all the M distinct policy areas. For each policy area, Table 4.1 also includes the optimal solutions $\mathbf{x}$ and $\mathbf{y}$ of the primal and the dual linear programs, respectively.

By comparing the optimal solutions $\mathbf{x}$ and $\mathbf{y}$ at the different policy areas it is easy to prove[1] that they actually are the optimal solutions of the primal and the dual problems respectively.

For the policy area where $\frac{C}{\sum_{i=m}^{M}\lambda_i} \leq t \leq \frac{C}{\sum_{i=m+1}^{M}\lambda_i}$, the solution of the LP 4.10 is such that $x_1 = \ldots = x_{m-1} = 0$ and $x_m > 0, \ldots, x_M > 0$. According to Theorem 9, the optimal policy in the above policy area, is to accept for transportation only the containers that belong to classes $m, \ldots, M$, whereas the containers that belong to classes $1, \ldots, m-1$ should be rejected. Alternatively:

$$\gamma(t, C) = \sum_{i=m}^{M} \lambda_i \quad \text{and} \quad \beta(t, C) = \sum_{i=m}^{M} f_i \lambda_i \tag{4.26}$$

The optimal policies applied at the different policy areas, are given at Table 4.2.

If we have a customer with more than one containers, then we do not have the luxury of accepting only the containers that we are allowed to accept under the current optimal policy. In most of the cases the vessel operator has to either accept all the containers, or he is offered no container at all. The solution of some of the possible scenaria of offers consisting of many containers is rather straight forward [2]. More complicated scenaria of

[1] From the optimality property of linear programming, if $\mathbf{x}$ and $\mathbf{y}$ are feasible solutions to the primal and the dual LP's respectively, and we also have $\mathbf{z}(t, C) = \mathbf{f}^T\mathbf{x} = \mathbf{b}^T\mathbf{y} = \mathbf{v}(t, C)$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions to the respective LP's. ([38]). (The optimality of the solutions could have also been proven with the Complementary Slackness Theorem ([38], or [39]))

For the (t, C) combinations for which we have:

$$\frac{C}{\sum_{i=m}^{M}\lambda_i} \leq t \leq \frac{C}{\sum_{i=m+1}^{M}\lambda_i},$$

the solutions $\mathbf{x}$ and $\mathbf{y}$ are such that

$\mathbf{z}(t, C) = \mathbf{f}^T\mathbf{x} = \mathbf{b}^T\mathbf{y} = \mathbf{v}(t, C) = f_m C + \sum_{i=m+1}^{M}(f_i - f_m)\lambda_i t$

Since $\mathbf{z}(t, C) = \mathbf{v}(t, C)$, in the above case, we conclude that the solutions presented here are the optimal solutions to the respective L.P.'s.

[2] If we are at the Policy Area which is defined by the inequalities $\frac{C}{\sum_{i=m}^{M}\lambda_i} \leq t \leq \frac{C}{\sum_{i=m+1}^{M}\lambda_i}$ and a customer arrives with an offer for p containers that belong to class m, we accept (if we have the option) α units of Goods belonging to Class m, where $t = \frac{C-\alpha}{\sum_{i=m+1}^{M}\lambda_i}$. If we have to accept all of the containers or none, then we solve two variations of the linear programming model. The first time, we solve the linear

Range of t	z(t,C) = v(t,C)	**x**	**y**
$0 \leq t \leq \frac{C}{\sum_{i=1}^{M} \lambda_i}$	$\sum_{i=1}^{M} f_i \lambda_i t$	$x_1 = \lambda_1 t$ $\vdots$ $x_M = \lambda_M t$	$y_1 = 0$ $y_2 = f_1$ $\vdots$ $y_{M+1} = f_M$
$\frac{C}{\sum_{i=1}^{M} \lambda_i} \leq t$ $t \leq \frac{C}{\sum_{i=2}^{M} \lambda_i}$	$f_1 C +$ $\sum_{i=2}^{M} (f_i - f_1) \lambda_i t$	$x_1 = C -$ $\sum_{i=2}^{M} \lambda_i t$ $x_2 = \lambda_2 t$ $\vdots$ $x_M = \lambda_M t$	$y_1 = f_1$ $y_2 = 0$ $y_3 = f_2 - f_1$ $\vdots$ $y_{M+1} = f_M - f_1$
$\frac{C}{\sum_{i=m}^{M} \lambda_i} \leq t$ $t \leq \frac{C}{\sum_{i=m+1}^{M} \lambda_i}$	$f_m C +$ $\sum_{i=m+1}^{M} (f_i - f_m) \lambda_i t$	$x_1 = 0$ $\vdots$ $x_{m-1} = 0$ $x_m = C -$ $\sum_{i=m+1}^{M} \lambda_i t$ $x_{m+1} = \lambda_{m+1} t$ $\vdots$ $x_M = \lambda_M t$	$y_1 = f_m$ $y_2 = 0$ $\vdots$ $y_{m+1} = 0$ $y_{m+2} = f_{m+1} - f_m$ $\vdots$ $y_{M+1} = f_M - f_m$
$\frac{C}{\lambda_M} \leq t \leq \infty$	$f_M C$	$x_1 = 0$ $\vdots$ $x_{M-1} = 0$ $x_M = C$	$y_1 = C$ $y_2 = 0$ $\vdots$ $y_{M+1} = 0$

Table 4.1: Solution of the Parametric LP's.

Policy Area	$W(t,C) = \mathbf{z}(t,C)$	$\mathbf{x}$	Optimal Policy
$0 \le t \le \frac{C}{\sum_{i=1}^{M} \lambda_i}$	$\sum_{i=1}^{M} f_i \lambda_i t$	$x_1 = \lambda_1 t$ $\vdots$ $x_M = \lambda_M t$	$I_1 = 1$ $\vdots$ $I_M = 1$
$\frac{C}{\sum_{i=1}^{M} \lambda_i} \le t$ $t \le \frac{C}{\sum_{i=2}^{M} \lambda_i}$	$f_1 C +$ $\sum_{i=2}^{M} (f_i - f_1) \lambda_i t$	$x_1 = C -$ $\sum_{i=2}^{M} \lambda_i t$ $x_2 = \lambda_2 t$ $\vdots$ $x_M = \lambda_M t$	$I_1 = 1$ $I_2 = 1$ $\vdots$ $I_M = 1$
$\frac{C}{\sum_{i=m}^{M} \lambda_i} \le t$ $t \le \frac{C}{\sum_{i=m+1}^{M} \lambda_i}$	$f_m C +$ $\sum_{i=m+1}^{M} (f_i - f_m) \lambda_i t$	$x_1 = 0$ $\vdots$ $x_{m-1} = 0$ $x_m = C -$ $\sum_{i=m+1}^{M} \lambda_i t$ $x_{m+1} = \lambda_{m+1} t$ $\vdots$ $x_M = \lambda_M t$	$I_1 = 0$ $\vdots$ $I_{m-1} = 0$ $I_m = 1$ $I_{m+1} = 1$ $\vdots$ $I_M = 1$
$\frac{C}{\lambda_M} \le t \le \infty$	$f_M C$	$x_1 = 0$ $\vdots$ $x_{M-1} = 0$ $x_M = C$	$I_1 = 0$ $\vdots$ $I_{M-1} = 0$ $I_M = 1$

Table 4.2: Optimal Solution for the LP, and Optimal Policy for the DP

multiple container orders are examined at chapter 10.

4.2.2 Contribution

Dynamic programming gives an accurate and realistic formulation of the reservation problem for ocean transportation without many restrictive assumptions. The only restrictive assumption we have used so far is the assumption that the arrival rates for all the classes of containers are constant until departure. This assumption will be relaxed in the following chapters of this thesis.

The input of the DP model is the remaining capacity C, the remaining time t until departure, and the estimate for the transportation demand, in the form of the arrival rates (λ's) for the different classes of containers (see figure 4-2).

The output of the DP formulation is the optimal policy for the combination (t, C) and the maximum expected revenue until departure, given that an optimal policy is applied. The derivation of the solution of the DP for the particular (t, C) combination, also includes the derivation of the optimal solution for all (t_1, C_1) combinations with $t_1 < t$ and $C_1 < C$.

If the estimates for the different λ's remain the same through out the booking process, then the operator has to run the DP model only once, and then follow the optimal policy suggested for the different (t_1, C_1) combinations.

In general, linear programming is a method that gives fast solutions (relatively to the number of the variables and constraints involved). If we tried to describe a dynamic process like the booking process, with LP, one would think that the linear programming model would be inaccurate, and the solution suggested by the LP would be unreliable.

The input of the linear programming model described by formula 4.10, is a condensed form of the input of the dynamic programming. The dynamic programming formulation needs the arrival rates for all classes of containers for all t's, whereas the linear program-

programming model under the scenario that we accept all the p containers. We add up the revenue from this customer and the expected optimal revenue from the remaining capacity. We compare this amount to the expected revenues from the other run under the scenario that we do not give any capacity to the customer.

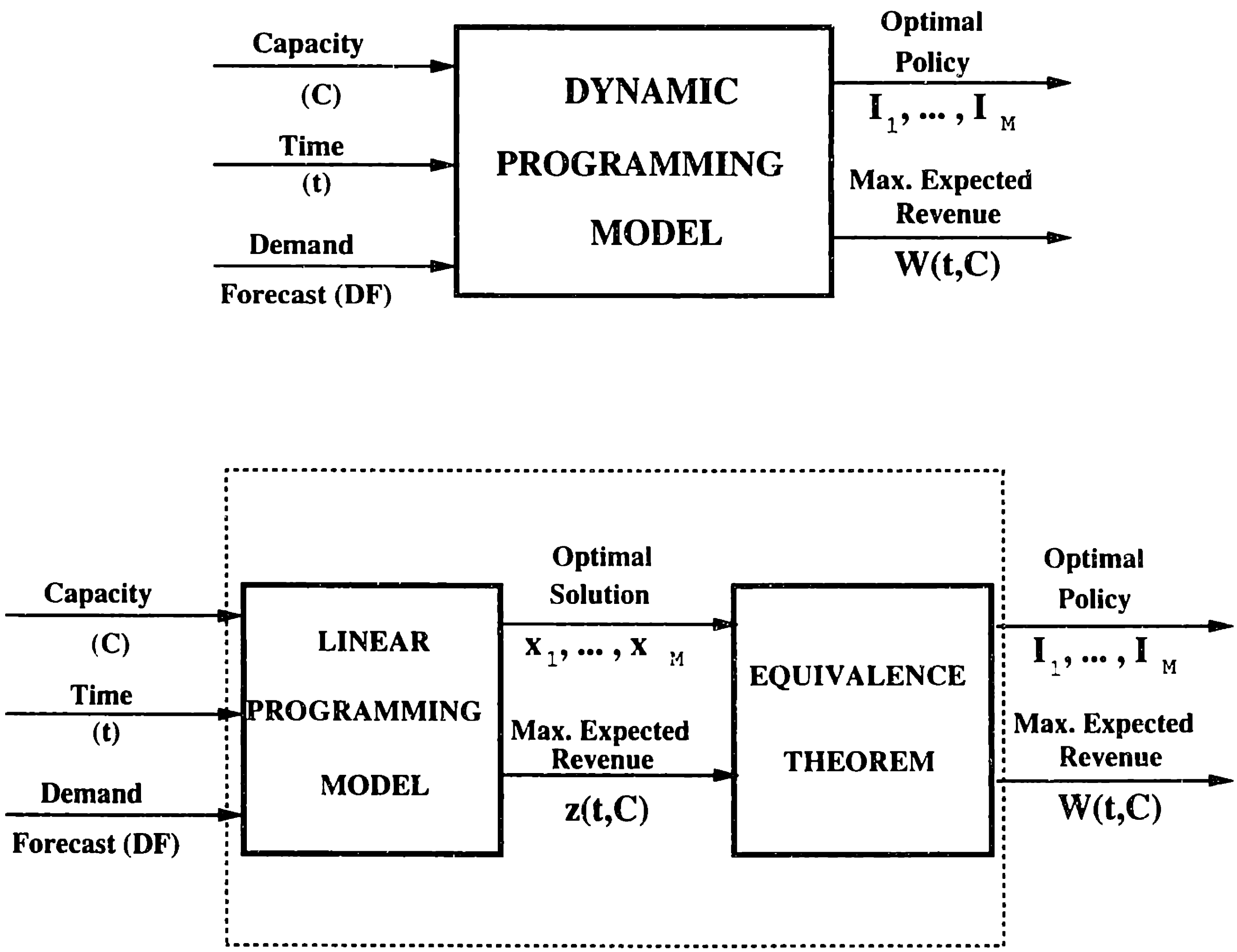


Figure 4-2: Equivalence Theorem: Comparison of the DP and the LP models

ming model just needs the expected value of the total arrival or demand from each class of containers. The demand for each class is equal to the integral of the arrival rates from the current time t until the departure of the vessel.

The output of the LP model is the optimal LP solution, which has the form $x_m = 0,\ m = 1, \ldots, i-1$ and $x_m > 0,\ m = i, \ldots, M$, and the maximum value $\mathbf{z}(t, C)$ of the objective function of the LP. Thanks to the *Equivalence Theorem*, we know that, the optimal policy of the DP with the same input, is such that $I_m = 0,\ m = 1, \ldots, i-1$ and $I_m = 1,\ m = i, \ldots, M$. Additionally, the maximum expected revenue of the DP model is $W(t, C) = \mathbf{z}(t, C)$. (We have to remember that the DP model and the LP models are equivalent for values of the capacity such that $C >> 1$).

With the help of the *Equivalence Theorem*, we can combine the best features of dynamic programming and linear programming. Dynamic programming offers modeling accuracy and reliable solutions. The solution of an LP is in general easy, whereas, the solution of a DP can be tedious. We bypass this disadvantage of dynamic programming with the introduction of the *Equivalence Theorem*. The *Equivalence Theorem*, suggests that when given the same input, the linear program 4.10 gives the same output as the dynamic programming model given by the formulas 4.1 and 4.2. The solution of the LP, is not as computationally intensive as the solution of the DP.

For the simple two ports, M classes of containers problem model that we examined in this chapter, we have found the analytical solution of the linearized expected revenues function $W(t, C)$. The analytical solution and the optimal policy of the linearized DP are exhibited in Table 4.2. For the simple example of the two ports problem, with M classes of containers, and constant arrival rates, the solution of the booking problem is given in Table 4.2.

Figures The following figures 4-3, 4-4, 4-5, and 4-6 show the agreement between the solution from the LP and the solution from the original DP equations 4.1 and 4.2, for different values of the remaining capacity. The capacity is $C = 2$, $C = 20$, $C = 100$, and

$C = 1,000$ at the four figures respectively.

The first observation that we make is that for strong demand (upper right corner) and weak demand (lower left corner), the results from the two methods are identical. The discrepancy that we observe occurs at the intermediate cases of not very strong demand.

As expected, the discrepancy between the two methods decreases as the remaining capacity increases. For remaining capacity $C = 1,000$, the results from the two different methods cannot be distinguished easily (see figure 4-6).

A further observation is that the solution given by the linear programming, gives consistently higher or equal values than the dynamic programming model. That happens because, although the two models are basically the same, the LP model is a relaxed form of the DP. The variables of the DP have to be either 0 or 1, whereas the variables of the LP do not even have to be integers. Since the LP has fewer constraints than the DP, we expect the objective function of the LP to be greater than the values of the cost functional of the DP.

4.3 Summary

In this chapter we present a discrete time DP model for the two ports yield management problem, where the different classes of customers arrive with a constant rate. From the discrete time DP model, we derive the continuous time DP model. We assume the expected revenues $W(t, C)$ to be a (piecewise) linear function of the capacity, and we derive the differential equation that governs the maximum expected revenues of the booking process. We prove that the solution to this differential equation is given by the maximum value of the objective function of a linear programming formulation. We also show that the optimal solution of the linear programming gives the optimal policy for the linearized version of DP model. We present graphs that show the agreement between the DP model and its linear programming approximation. The agreement of the two methods is greater for larger values of the capacity.

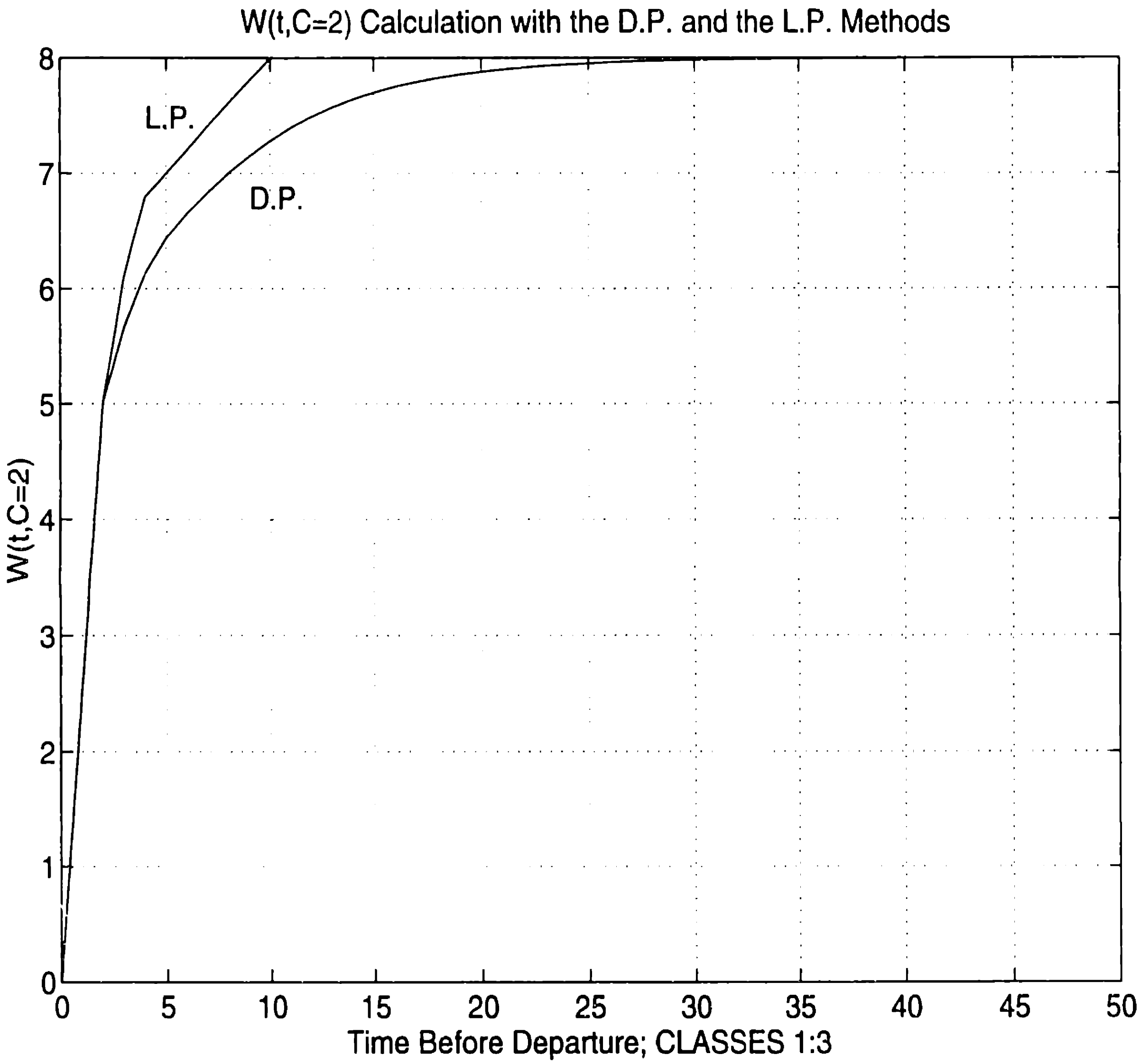


Figure 4-3: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 2, and Constant Arrival Rates ($\lambda(t) = \lambda$).

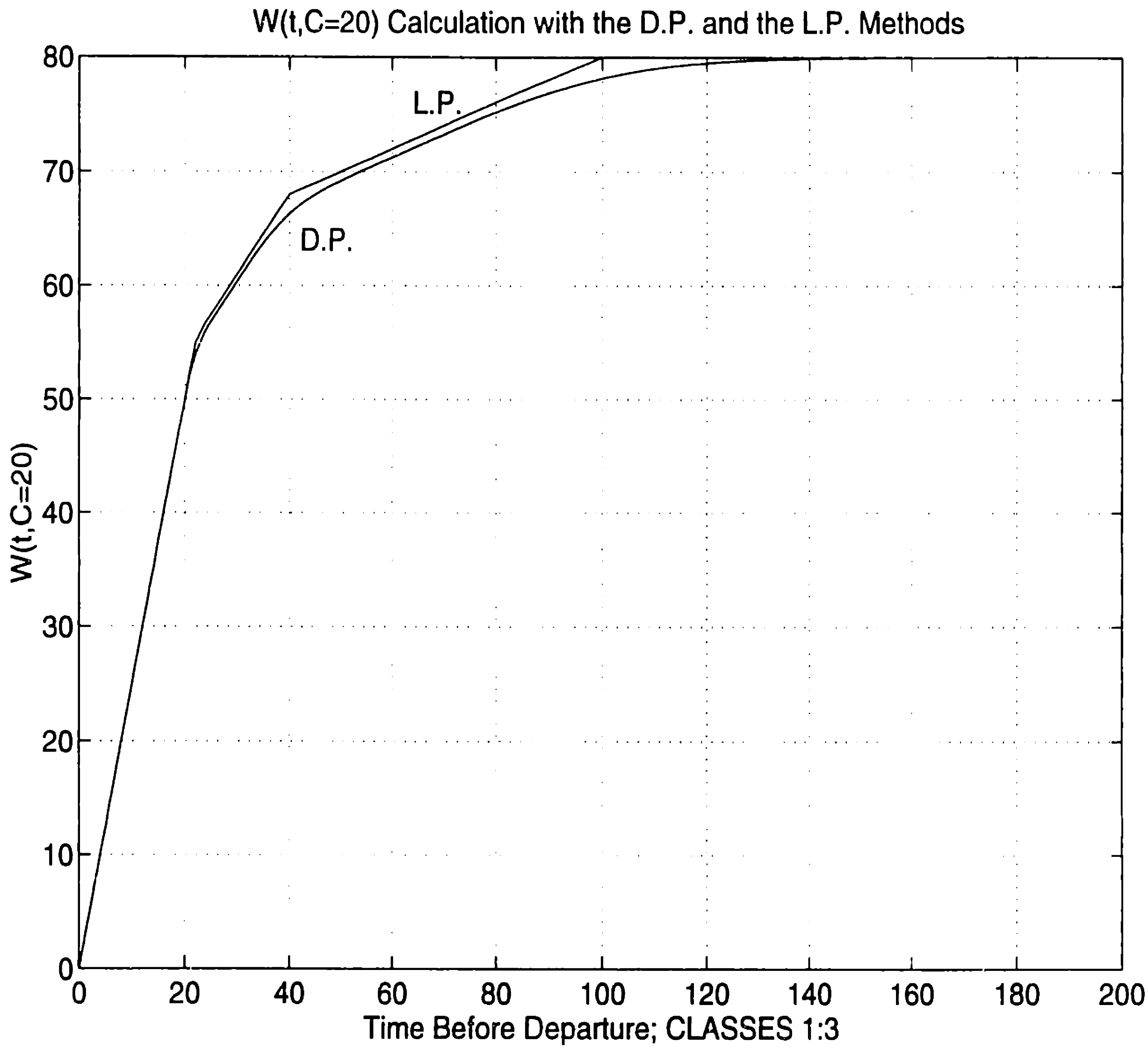


Figure 4-4: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 20, and Constant Arrival Rates ($\lambda(t) = \lambda$).

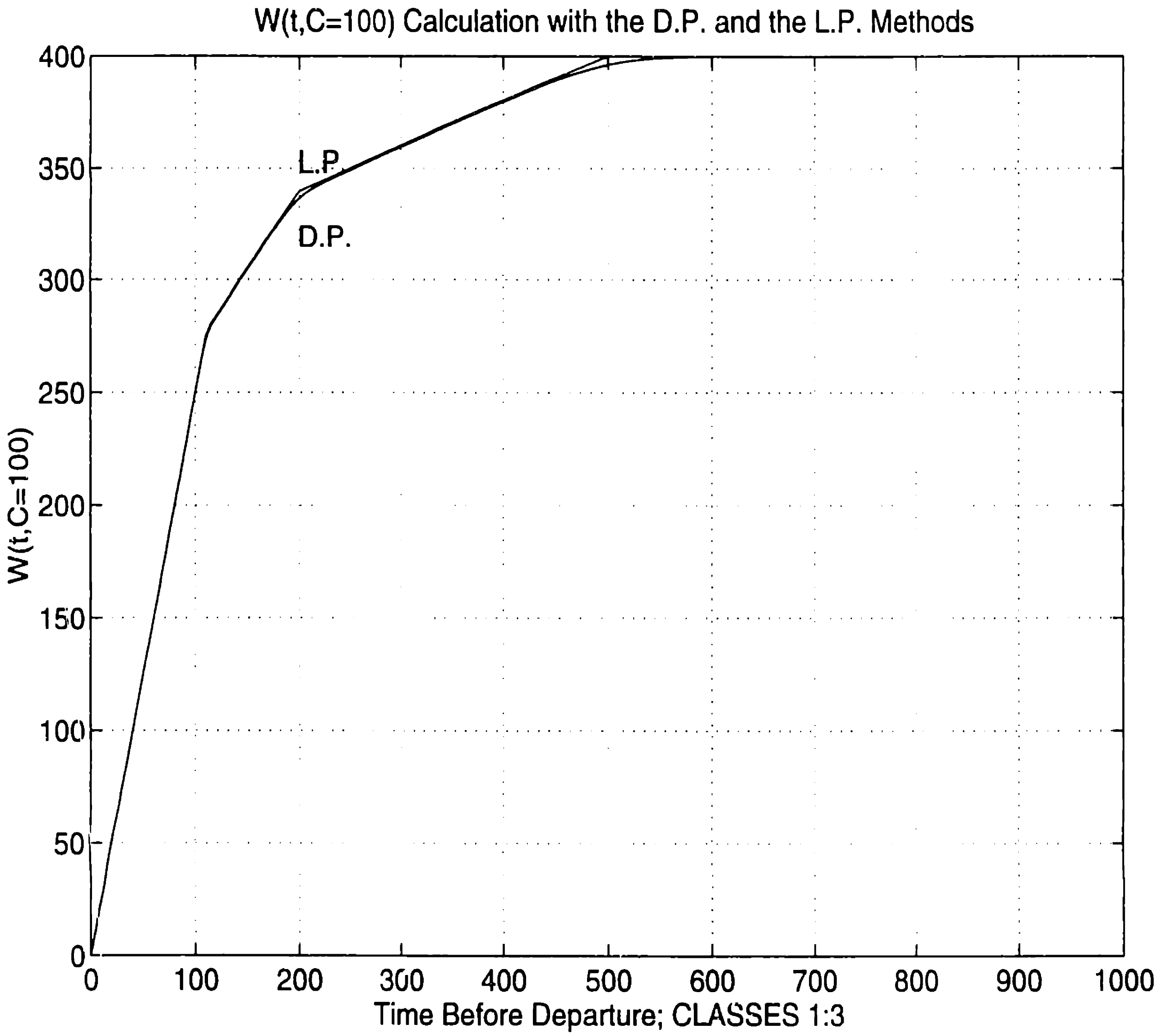


Figure 4-5: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 100, and Constant Arrival Rates ($\lambda(t) = \lambda$).

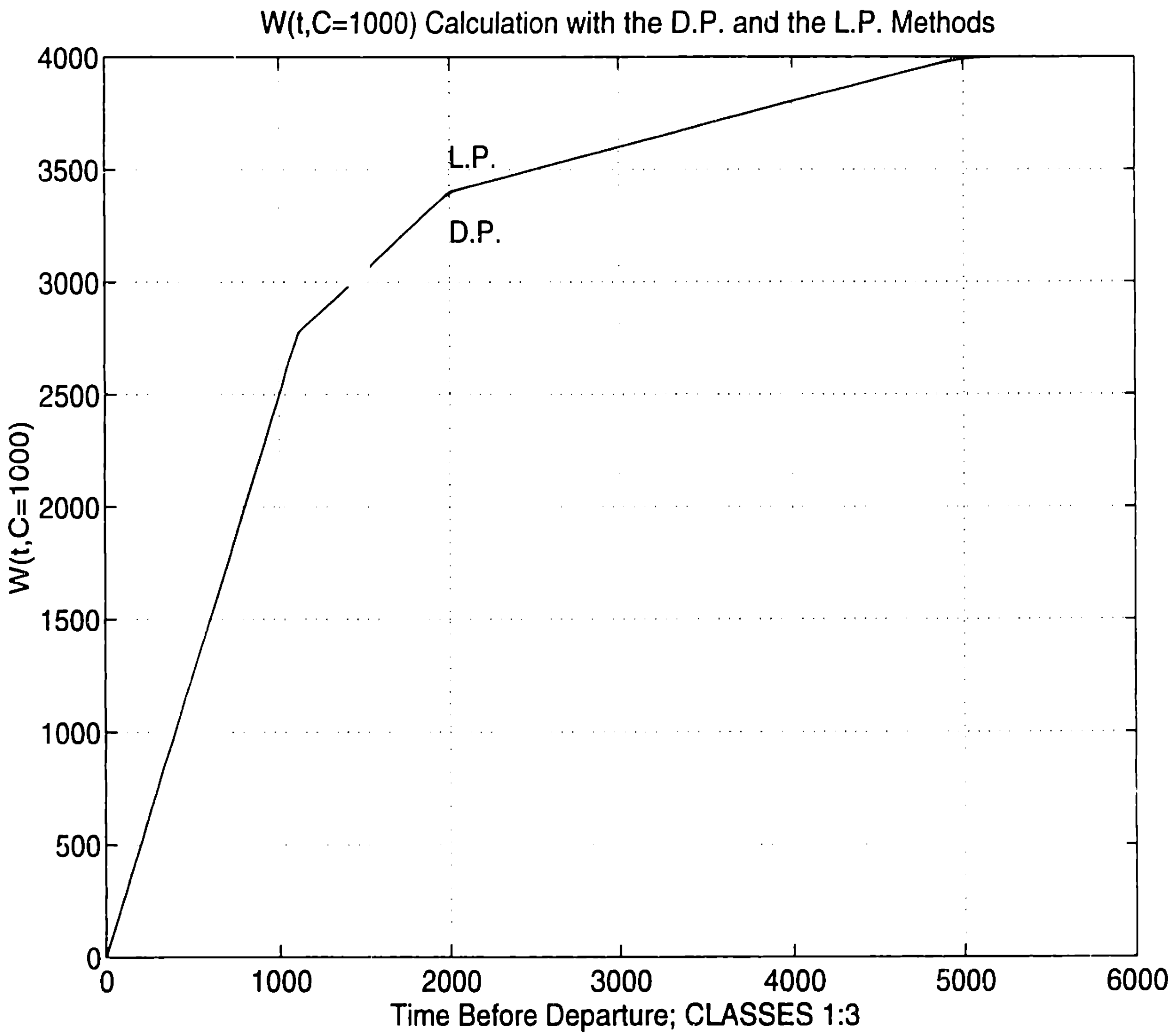


Figure 4-6: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 1000, and Constant Arrival Rates ($\lambda(t) = \lambda$).

Chapter 5

Continuous Model for N Legs, M Classes of Goods

In the previous chapter we examined the two ports, M classes of containers ocean yield management (OYM) problem. The method that we presented incorporates the modeling accuracy of dynamic programming modeling, and the facility of solution that an LP can offer. Furthermore, we gave an analytical solution for the two ports, M classes of containers problem, in a tabular form. As a result, we get the optimal policy suggested by the dynamic programming model from Table 4.2 and there is no need to solve the LP every time there is a container arrival. Table 4.2, gives the boundaries of all M policy areas, and for any combination (t, C), we know what policy area we are in, and which is the optimal policy that we should apply.

Despite the superiority of the method that we presented at the previous chapter over other methods that address similar forms of the yield management problem, the model we developed at the previous chapter is both restrictive and incomplete.

It is restrictive, because it includes the assumption that the arrival rates for all the classes of containers are constant and the same from the announcement of the itinerary, until the departure of the vessel. We know that the arrival rates of any class of containers is a function of time. That means that for a particular class of containers, we would expect the rate of customer arrivals to vary through the booking period of the vessel. Depending on the class of the containers, the rate of arrival can be greater away from the departure, or closer to it. Low value containers tend to book capacity earlier, whereas high value containers, tend to be last minute shoppers. As a result, the special case of constant arrival rates, is not realistic enough to cover the case of customers with varying arrival patterns.

The model that we presented at the previous chapter, is also incomplete. We developed a model for the simplest case of a vessel that serves an itinerary of just two ports. Nevertheless, containerships usually serve a number of several ports. When there are no explicit bilateral agreements that restrict the operations of the vessels, the containers load cargo at any port of their itinerary, to transfer it at any other port of the route they serve. The booking process for a containership that operates at a multi-port route is more complicated than the booking process that involves an itinerary of just two ports.

In the two ports problem we only have to decide whether we should assign a container slot to the low value customer who asks for it now, or just wait for a high value customer who might arrive later. In the multi-port problem we have more alternatives. Instead of giving the container slot to the low value customer who asks for it now, we could give it to a high value customer with the same itinerary, or we could assign it to an other high or low value customer who has a different port of origin and/or destination but an overlapping part of the itinerary with the low value customer who asks for the container slot now. The booking decisions that we will make at one port will influence the earning potential of the vessel at several of the ports in the itinerary of the vessel.

5.1 Definitions

Before we continue further, we will give some definitions that will be used for at the description of the network served by the vessel and the optimal policies within each policy area.

We have the set $\mathbf{S} = \{ODC_1, \ldots, ODC_F\}$ of all the possible origin-destination and container class combinations (ODC Combination). I create N sets $\mathbf{S_i}$, one for each legs of the network. $\mathbf{S_i}$ includes all those ODC combinations that have to use leg i, for their transportation.

$$\mathbf{S_i} = \{ODC_{Si,1}, \ldots, ODC_{Si,m}\}, \quad i = 1, \ldots, N \tag{5.1}$$

We brake each set $\mathbf{S_i}$ further, into two mutually exclusive and collectively exhaustive sets:

$$\begin{aligned} \mathbf{NS_i} &= \{ODC_{NSi,1}, \ldots, ODC_{NSi,m}\}, \quad i = 1, \ldots, N \\ \mathbf{Y_i} &= \{ODC_{Yi,1}, \ldots, ODC_{Yi,m}\}, \quad i = 1, \ldots, N \end{aligned}$$

- $\mathbf{NS_i}$ is the set that includes the ODC combinations that would use leg i, but they are denied transportation under the current optimal policy. In the context of linear programming, one ODC that would use leg i, belongs to $\mathbf{NS_i}$, if the optimal solution for this ODC is such that $x_{ODC} = 0$. In the context of dynamic programming a ODC combination belongs to $\mathbf{NS_i}$, if $I_{ODC} = 0$.

- $\mathbf{Y_i}$ is the set of all the ODC that are accepted for transportation, and leg i belongs to the path they use for their transportation. In the context of linear programming, one ODC that uses leg i, belongs to $\mathbf{Y_i}$, if the optimal solution for this ODC is such that $x_{ODC} > 0$. In the context of dynamic programming a ODC combination that uses leg$i$, belongs to $\mathbf{Y_i}$, if $I_{ODC} = 1$.

We brake each set $\mathbf{Y_i}$ further, into two mutually exclusive and collectively exhaustive sets $\mathbf{MS_i}$ and $\mathbf{AS_i}$. Those sets are defined only relatively to the linear programming modeling of the multi-port, multi-class ocean yield management (OYM) problem. The definition of the sets $\mathbf{MS_i}$ and $\mathbf{AS_i}$ is meaningless in the case of the dynamic programming modeling.

$$\mathbf{MS_i} = \{ODC_{MSi,1}, \ldots, ODC_{MSi,m}\}, \quad i = 1, \ldots, N$$

$$\mathbf{AS_i} = \{ODC_{ASi,1}, \ldots, ODC_{ASi,m}\}, \quad i = 1, \ldots, N$$

- $\mathbf{MS_i}$ is the set that includes these ODC combinations that are accepted for transportation over the i_{th} leg, but are not accepted fully. In other words, the vessel operator does not accept all the expected demand from this ODC for transportation. An ODC belongs to $\mathbf{MS_i}$ if there exists at least a leg of the trip at which, if we increase the capacity by one unit, we increase the number of containers from this ODC by one unit.

- $\mathbf{AS_i}$ includes all the ODC combinations that we accept for transportation over the i_{th} leg and do not belong to $\mathbf{MS_i}$. We accept all the available offers (which is equal to the expected demand of the respective ODC's) from elements of $\mathbf{AS_i}$. We would accept more offers from the ODC's that belong to $\mathbf{AS_i}$ if the demand was greater. In order to do that, we would displace elements of the set $\bigcup_{m=1}^{N} \mathbf{MS_i}$.

In conclusion:

$$\mathbf{S_i} = \mathbf{NS_i} \cup \mathbf{Y_i}$$
$$\mathbf{Y_i} = \mathbf{AS_i} \cup \mathbf{MS_i}$$

I further define:

$$\mathbf{N} = \bigcup_{i=1}^{N} \mathbf{NS_i}$$
$$\mathbf{Y} = \bigcup_{i=1}^{N} \mathbf{Y_i}$$
$$\mathbf{A} = \bigcup_{i=1}^{N} \mathbf{AS_i}$$
$$\mathbf{M} = \bigcup_{i=1}^{N} \mathbf{MS_i} \qquad (5.2)$$

- $\mathbf{N}$ is the set of all the ODC that are not accepted for transportation.

- **Y** is the set of all the ODC that are admitted in total or partially for transportation.
- In the context of linear programming modeling we get:

  A is the set of all the ODC that are offered space without any restriction.

  M is the set of all the ODC that are marginal. The set **M** has at most N elements.

5.2 DP Model and Boundary Conditions

The dynamic programming (DP) formulation of the ocean yield management problem (OYM) changes as follows, for the case of the $N+1$ ports, or N legs trip, with M classes of goods, from each port. The assumption of M classes of goods is general enough and there is no need to introduce a different number of container classes at each port. The freight rate of each container, depends not only on the class at which the container belongs, but also the itinerary of the container.

The operator serves all $N+1$ ports. The initial port of the itinerary is port 1, and the last port is port $N+1$. The operator accepts container offers for all possible combinations of origin and destination and for all classes of containers. We define the path from any port k to any other port s, where $k < s$, to be path OD_i. We further assume that the container that needs to be transported from port k to port s, belongs to the g class of containers. The scenario that we have just described of the container that belongs to class g and needs to be transported from port k to port s, is the origin-destination combination (ODC), OD_i, g. In general, we have $N \cdot (N+1)$ OD_i paths. If we multiply the number of paths, by the total number of the classes of containers M, we have defined the total number of the potential origin-destination combinations.

At the one leg (two ports) problem the vessel sails from the first port to go to the second port which is the final port of the trip. For the two ports problem, the remaining time before departure, is the remaining time from the present time, until the departure of the vessel from the port of origin. The remaining time until departure is the same for all OD combinations. When we have an itinerary with more than one legs (three or more ports), the remaining time until the departure from the second port is greater than the remaining time before the departure from the first port, and so on. Similarly, the remaining available time for booking of containers originating from the second port is greater than the remaining available time for bookings of containers originating from the first port and so on.

We therefore see that in the case of the N legs problem, the remaining time before departure, and therefore the remaining booking period for each port, cannot be expressed by one single number and one might say that we would have to express the stage of our process as a vector of different "remaining times". Nevertheless, the departures from the different ports are scheduled at fixed time intervals. The departure from the last port from which we accept reservations, (i.e. port N out of $N+1$ Ports) is scheduled for t_{kN} time units after the departure from port k.

We assume that currently the vessel accepts reservations from all the ports of origin, (i.e. the next port the vessel calls is port 1), and the remaining time before departure from port N is t. It is a restrictive assumption to consider that the port to be visited next is the 1_{st} out of the remaining N ports of the itinerary. When we know the departure time from one port and the time intervals between the departures from each consequent port, we know the remaining time before departure from each port. Therefore, the time before departure from port k is $t - t_{kN}$, where t_{kN} is the travel time between ports k and N.

The above show that in order to know the departure time for all ODC's, from each port, we only have to know the departure time from port 1. The remaining time until departure for the ODC, that belongs to class g and wants transportation capacity on the path OD_i, is $t_{OD_{i,g}}$. The time $t_{OD_{i,g}}$ is equal to:

$$t_{OD_{i,g}} = t - t_{kN} \tag{5.3}$$

The remaining time until departure from the port of origin for the itinerary OD_i is $t_{OD_i,g}$. When we know the time t_{kN}, all the information is included in the time t.

The state of the DP model that we will present, is the number of the remaining container slots that are still available at each leg of the trip, and have not been assigned yet to any shipper. The capacity at the different legs of the trip is $C_1, \ldots, C_M$. Therefore, the state of the system is described by the vector $\mathbf{C}$. If we accept a container of class m to travel on the path OD_i, (i.e. $I_{OD_i,m}(t, \mathbf{C}) = 1$), and the vector of capacity before the acceptance is $\mathbf{C}$, it becomes $\mathbf{C}_{OD_i,m}$ afterwards.

$$\mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}, \quad \text{and} \quad \mathbf{C}_{I_{OD_i,m}} = \begin{bmatrix} C_1 \\ \vdots \\ C_{k-1} \\ C_k - I_{OD_i,m} \\ \vdots \\ C_{s-1} - I_{OD_i,m} \\ C_s \\ \vdots \\ C_N \end{bmatrix},$$

where C_r is the capacity available at the rth leg of the trip.

We generalize the discrete model that we introduced at chapter 3 and we exhibited at figure 3-1. The basic feature of the dynamic programming model that we introduced there is that at each discrete time t, we have the arrival of at most one container that belongs to one of the $N \cdot (N+1) \cdot M$ ODC combinations. The probability of arrival for a container that belongs to the OD_i, g ODC combination, is equal to $\lambda_{OD_i,g}(t)$. In the general case, the probability of arrival for each ODC is a function of time. The probability of simultaneous arrival of more than one containers is equal to zero.

The cost functional is the following:

$$V_{OD_i,m}(t,\mathbf{C}) = \max_{I\in\{0,1\}} [I \cdot f_{OD_i,m} + W(t,\mathbf{C}_I)] \quad \forall\, m \in \{1, M\} \text{ and } \forall\, OD_i\text{'s} \tag{5.4}$$

where

$$W(t,\mathbf{C}) = \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} [\lambda_{OD_i,g}(t-1) \cdot V_{OD_i,g}(t,\mathbf{C})] + \left[1 - \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t-1) \right] \cdot W(t-1,\mathbf{C}) \tag{5.5}$$

$V_{OD_i,m}(t,\mathbf{C})$ is the maximum expected revenue at time t, when the vector of the available container slots at each of the N legs of the trip is $\mathbf{C}$. At time t we have the arrival of container that belongs to class m. The container needs transportation on the OD_i itinerary or Origin-Destination Pair. k is the port of origin of the itinerary and s the port of destination for the itinerary OD_i.

$W(t,\mathbf{C})$ is the maximum expected revenue at time t with $\mathbf{C}$ being the vector of the available container slots at the N legs of the trip. $W(t,\mathbf{C}))$ is conditional on the event that we have no arrival of any customer at time t.

$\lambda_{OD_i,g}(t)$ is the rate of arrival (or the probability of arrival) at time t, for a class g customer, who wants to travel on the OD_i itinerary.

I is the control variable when we have the arrival of a class m customer who asks for transportation on the OD_i itinerary. The optimal control variable at stage t and state $\mathbf{C}$, is $I_{OD_i,m}(t,\mathbf{C})$. i.e. $V_{OD_i,m}(t,\mathbf{C}) = I_{OD_i,m}(t,\mathbf{C}) \cdot f_{OD_i,m} + W(t,\mathbf{C}_{I_{OD_i,m}})$.

where $I_{OD_i,m}(t,\mathbf{C}) \in \{0,1\}$. When we have no customer arrival, $I_{OD_i,m}(t,\mathbf{C}) = 0$ by default.

$f_{OD_i,m}$ is the freight rate for containers that belong to class m and need to be transported on the OD_i itinerary.

The boundary conditions of the above cost functional are:

$$V_{OD_i,m}(t,\mathbf{C}) = 0.0\,, \quad \forall\ t > 0 \text{ and } m \in \{1,\ldots,M\}$$

$$W(t,\mathbf{C}=\mathbf{0}) = 0.0\,, \quad \forall\ t > 0 \tag{5.6}$$

$$W(t=0,\mathbf{C}) = 0.0\,, \quad \forall\ \mathbf{C} \geq [0\ldots0]^T \tag{5.7}$$

The convention we have made here is that $t = 0$ when the vessel departs from the one before last port of the itinerary (i.e. port N). After the vessel departs from port N, and sails to the $N+1$ port, which is the final port of the trip, there is no further earning potential for the vessel and naturally $W(t = 0, \mathbf{C}) = 0.0$.

5.3 The governing equation of the multi-leg, multi-product OYM problem

We have considered the N legs, M classes of containers model to be a discrete time model. The model was assuming that we had possible container arrivals at discrete points in time. We have explained the benefits of the continuous models in the previous chapter. Besides, we were able to find a solution for the linearized $W(t, C)$. This solution was the optimal objective function of an LP. In the current chapter, we will try to repeat the method of the previous chapter and find a similar LP that would be the solution to the HJB equation.

We will divide the remaining time t into T time intervals. When the number of time intervals $T \longrightarrow \infty$, then the duration of each time interval $\Delta t = \frac{t}{T} \longrightarrow 0$. Our assumptions are modified in order to include the event that at the beginning (or expiration) of each time interval Δt we have a possible arrival of a container that wants to travel the path OD_i (i.e. from Port k to Port s), and belongs to class m, with probability $\Delta p_{OD_i,m}(t) = \Delta t \cdot \lambda_{OD_i,m}(t)$.

At the limit $\Delta t = \frac{t}{T} \longrightarrow 0$ the time is considered to be a continuous variable, and we will consider that at each time interval Δt we have a possible arrival of Class m good that wants to travel on the OD_i Origin-Destination Pair i.e. from Port k to Port s, with probability $\Delta p_{OD_i,m}(t) = \Delta t \cdot \lambda_{OD_i,m}(t)$.

If we treat t and C_r $r = 1, \ldots, N$ as continuous variables, the cost functional is the following:

$$\begin{aligned} V_{OD_i,m}(t, \mathbf{C}) &= \max_{I \in \{0,1\}} [I \cdot f_{OD_i,m} + W(t, \mathbf{C}_I)] \\ & \forall\, m \in \{1, M\} \text{ and } \forall\, OD_i\text{'s} \end{aligned} \tag{5.8}$$

whereas the formula for $W(t, \mathbf{C})$ becomes:

$$\begin{aligned} W(t, \mathbf{C}) &= \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t - \Delta t) \cdot V_{OD_i,g}(t, \mathbf{C}) \\ &+ \left[1 - \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t - \Delta t) \right] \cdot W(t - \Delta t, \mathbf{C}) \end{aligned} \tag{5.9}$$

The Boundary Conditions as described from equations 5.6 and 5.7 still hold.

The following lemma 10 gives the differential equation that describes the linearized form of the maximum expected revenue, as it is given by the dynamic programming formulation. The differential equation given by the lemma is the Hamilton-Jacobi-Bellman equation of control theory.

Lemma 10 *$W(t, \mathbf{C})$ is the maximum expected revenue of the continuous time model under optimal policy . The optimal policy is such that $I_i(t, \mathbf{C}) = 0$, $\forall i \in \mathbf{N}$, and $I_i(t, \mathbf{C}) = 1$, $\forall\, i \in \mathbf{Y} = \bigcup_{j=1}^{N} \mathbf{Y_j}$. We assume that the partial derivatives of $W(t, \mathbf{C})$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t, \mathbf{C})$ is the equation:*

$$\frac{\partial W(t,\mathbf{C})}{\partial t} + \sum_{j=1}^{N} \frac{\partial W(t,\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} = \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i \qquad (5.10)$$

The solution of the lemma 10, is given in appendix C.

Equation 5.10 is the so called Hamilton-Jacobi-Bellman (HJB) equation. It is a partial differential equation satisfied for all time-capacity pairs (t,C) by the maximum revenue-to-go function. If we can find a solution to the this equation 5.10, we will have found the (linearized) optimal solution to the original DP formulation given by formulas 5.4 and 5.5.

We define $\gamma_j(t, \mathbf{C})$ and $\beta(t, \mathbf{C})$ as follows:

$$\gamma_j(t,\mathbf{C}) = \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \; \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t,\mathbf{C}) \qquad (5.11)$$

$$\beta(t,\mathbf{C}) = \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \; \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} \cdot I_{OD_i,g}(t,\mathbf{C}) \qquad (5.12)$$

- $\gamma_j(t, C)$ is the reservation rate of the vessel capacity for the leg j of the network, or the expected rate of capacity reservation at the leg j. In the summation, we include only the arrival rates of all the ODC combinations that we accept in the current policy area, and among them only those ODC's that include the leg j in their paths. In other words, we include the rates of arrival for the classes of containers that we accept under the current policy and which include the leg j, in their paths OD_i.

The number of γ's is equal to the number of the legs of the network, and each γ_j is equal to the current reservation rate of leg j.

- $\beta(t, \mathbf{C})$ is the revenue accumulation rate, or the expected rate of incoming revenue. It is equal to the sum of the revenue accumulation rates of all the ODC's, or all the container classes from all paths OD_i that we accept for transportation while in the current policy area.

(At chapter 3, we proved that $I_g(t, C) \leq I_r(t, C)$, if $g < r$. It is easy to prove that at the path OD_i, we have that $I_{OD_i,g}(t, \mathbf{C}) \leq I_{OD_i,r}(t, \mathbf{C})$, if $g < r$.)

As a result, within each policy area, equation 5.10 becomes:

$$\frac{\partial W(t, \mathbf{C})}{\partial t} + \sum_{\text{all legs } j} \gamma_j(t, \mathbf{C}) \frac{\partial W(t, \mathbf{C})}{\partial C_j} = \beta(t, \mathbf{C}) \tag{5.13}$$

The solution of the above differential equation has to satisfy the boundary conditions as described from equations 5.6 and 5.7.

If we are able to find a solution that satisfies the equation, at the different policy areas, and it satisfies the boundary conditions, then we have found a solution to the equation 5.13.

We have $M \cdot N \cdot (N-1)$ policy areas. At each policy area, the different γ_j's and the β are constant. They differ though from the one policy area to the next. If we knew what the γ_j's and β are at each policy area, we would be able to find the value of $W(t, \mathbf{C})$ for all t and $\mathbf{C}$'s. Nevertheless, finding the γ_j's and β, or finding the optimal policy for the given t and $\mathbf{C}$, constitutes our original problem.

5.4 Solution for the N legs, M classes problem with time varying arrival rates

In section 4.2 we proved the equivalence between the dynamic programming formulation of the two ports, M classes of containers yield management problem and the linear programming formulation 4.10, when the arrival rates of the different classes of containers are constant through out the booking process.

In the current chapter, we consider the more complicated N legs, M classes of containers with variable arrival rates yield management problem. In the following we will prove the equivalence between the N legs, M classes of containers with variable arrival rates ($\lambda_{OD_{i,g}}(t)$) dynamic model, and a linear programming formulation.

We repeat the defining differential equation 5.13. It defines $W(t, \mathbf{C})$, which is the HJB equation of control theory.

$$\frac{\partial W(t,\mathbf{C})}{\partial t} + \sum_{\text{all legs } j} \gamma_j(t,\mathbf{C}) \frac{\partial W(t,\mathbf{C})}{\partial C_j} = \beta(t,\mathbf{C}) \tag{5.14}$$

The policy defined variables γ_j , $j = 1, \ldots, M$, and β, are given by equations 5.11, and 5.12, respectively.

Instead of repeating the definitions 5.11, and 5.12, we will use the definitions that we introduced at the previous section for the description of γ_j , $j = 1, \ldots, M$, and β.

$$\gamma_j(t,\mathbf{C}) = \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \tag{5.15}$$

$$\beta(t,\mathbf{C}) = \sum_{i \in \mathbf{Y}} f_i \lambda_i(t) \tag{5.16}$$

The boundary conditions of the differential equation 5.14, are given by equations 5.6

and 5.7. For the sake of completeness, we repeat the boundary conditions here:

$$W(t, \mathbf{C} = \mathbf{0}) = 0.0\,, \quad \forall\ t > 0 \tag{5.17}$$

$$W(t = 0, \mathbf{C}) = 0.0\,, \quad \forall\ \mathbf{C} \geq [0 \ldots 0]^T \tag{5.18}$$

We will prove that a solution of the differential equation 5.14, subject to the boundary conditions given by the equations 5.17 and 5.18 is the solution of the following linear programming program 5.19.

$$\begin{aligned} \mathbf{z}(t, \mathbf{C}) = \max\ \ \mathbf{f^T x} & \\ \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned} \tag{5.19}$$

The form of the constraints, as they are given by the inequalities

$$\mathbf{Ax} \leq \mathbf{b}$$

is the following:

$$\begin{aligned} \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \ \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} x_{OD_i,g} &\leq C_j, \ j = 1, \ldots, N \\ x_{OD_i,g} &\leq \int_{t=0}^{t} \lambda_{OD_i,g}(t) dt, \ \forall\ OD_i, g \in \mathbf{S} \end{aligned} \tag{5.20}$$

We should keep in mind that:

$$\int_{t=0}^{t} \lambda_{OD_i,g}(t) dt = \text{expected demand from the } ODC \text{ combination } OD_i, g \tag{5.21}$$

The first of the constraints 5.20, suggests that all ODC (origin, destination, class) combi-

LINEAR PROGRAMMING FOR THE MULTI PORT - MULTI CLASS YIELD MANAGEMENT PROBLEM

OBJECTIVE	:	**revenue maximization**
CONSTRAINTS	:	
total number of containers accepted to use leg j	=<	**remaining capacity of vessel at leg j**
number of accepted containers from each ODC combination	=<	**expected demand from this ODC combination**

Figure 5-1: LP model for the N legs, M classes, no cancelations OYM problem

nations that are accepted over each one of the N legs of the network, have to satisfy the capacity constraint of each leg.

The second of the constraints 5.20, suggests that no origin, destination and class combination (ODC) can be accepted at a level greater than its expected demand. The expected demand is defined as the expected demand from the current time t, until the departure of the vessel from the port of departure of the particular ODC. For notational simplicity we have considered the integral of the expected demand to extend to the time of departure of the vessel from the port N. We assume that the arrival rate between the departure from the port of origin and the port N, is $\lambda_{OD_{i,g}}(t) = 0$.

Figure 5-1 presents an alternative form of the linear programming model that we use to describe the N legs, M classes of containers yield management problem with time varying arrival patterns.

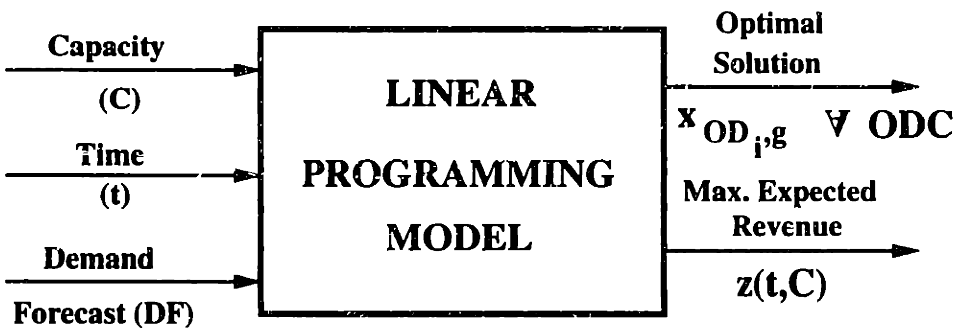


Figure 5-2: Input and output of the linear program for the multi port, multi class OYM problem

We solve the linear programming problem and we find an optimal solution $x^*_{OD_i,g}$ $\forall$ OD_i's and $\forall$ Classes g. Among the ODC's that use the leg j we have that:

$$
\begin{aligned}
OD_i, g \in \mathbf{NS_j} &\iff x^*_{OD_i,g} = 0 \\
OD_i, g \in \mathbf{AS_j} &\iff x^*_{OD_i,g} = \int_{t=0}^{t} \lambda_{OD_i,g}(t)dt \\
OD_i, g \in \mathbf{MS_j} &\iff 0 < x^*_{OD_i,g} < \int_{t=0}^{t} \lambda_{OD_i,g}(t)dt,
\end{aligned}
$$

When $OD_i, g \in \mathbf{MS_j}$, then:

$$
f_{OD_i,g} = \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}} \frac{\partial z(t, \mathbf{C})}{\partial C_j}
$$

Figure 5-2 gives an illustration of the input and the output of the linear program for the case of the multi-port, multi class ocean yield management (OYM) problem.

Before we continue with the statement and proof of Theorem 12 and the Corollary 13, we will give Lemma 11, which will be used at the proof of the Theorem 12. Here, I will give only the statement of the lemma. The proof is given in appendix C.

Lemma 11 gives the differential equation that describes the maximum value of the objective function of the Linear Program 5.19. Lemma 11 also gives the boundary conditions of the maximum value of the LP, for reasons of completeness.

Lemma 11 *$\mathbf{z}(t, \mathbf{C})$ is the maximum value, and $x^{*}_{OD_{i},g}(t, \mathbf{C})$, $\forall$ OD_i's and $\forall$ g, is the corresponding optimal solution of the Linear Program 5.19. The governing differential equation for $\mathbf{z}(t, \mathbf{C})$ is the equation*

$$\frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial t} + \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} = \sum_{i \in \mathbf{Y}} f_i \lambda_i(t) \tag{5.22}$$

with the following boundary conditions:

$$\mathbf{z}(t, \mathbf{C} = \mathbf{0}) = 0 \quad \textit{and} \quad \mathbf{z}(t = 0, \mathbf{C}) = 0 \tag{5.23}$$

The proof of the above lemma is given in appendix C.

In the following Theorem 12 we use the results from lemmas 10 and 11 to suggest that the solutions of the dynamic programming and the linear programming are equal when and only when the two following happen: 1) When the linearized dynamic programming suggests the rejection of a class of customers, the linear programming does not offer any capacity to this same class of customers. 2) When the linearized dynamic programming suggests the acceptance of a class of customers, the linear programming offers some capacity to this same class of customers.

Theorem 12 *$W(t, \mathbf{C})$ is the solution of the linearized version of the Dynamic Program given by equation 5.14, when the boundary conditions are given by equations 5.17 and 5.18. $I_{OD_i,g}(t, \mathbf{C})$ is the control variable for the class g containers that travel on the OD_i path, given by the Dynamic Programming under optimal policy. The acceptance ($I_{OD_i,g}(t, \mathbf{C}) = 1$)*

or the rejection ($I_{OD_i,g}(t, \mathbf{C}) = 0$) of a container that belongs to the OD_i, g ODC combination is as a function of the remaining time t and the remaining capacity vector $\mathbf{C}$. $z(t, \mathbf{C})$ is the maximum value, and $x^*_{OD_i,g}(t, \mathbf{C})$, $\forall$ OD_i's and $\forall$ g, is the corresponding optimal solution of the Linear Program 5.19. We prove the following:

$$\{W(t,\mathbf{C}) = z(t,\mathbf{C})\} \iff \left\{ \begin{array}{l} I_{OD_i,g}(t,\mathbf{C}) = 0, \quad x^*_{OD_i,g} = 0, \quad \forall\ OD_i, g \in \mathbf{N} \\ \textit{and} \\ I_{OD_i,g}(t,\mathbf{C}) = 1, \quad x^*_{OD_i,g} > 0, \quad \forall\ OD_i, g \in \mathbf{Y} \end{array} \right\} \tag{5.24}$$

or:

$$\{W(t,\mathbf{C}) = z(t,\mathbf{C})\} \iff$$

$$\left\{ \begin{array}{lcl} \beta(t,\mathbf{C}) = \sum_{OD_i,g \in \mathbf{Y}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} & & x^*_{OD_i,g} = 0, \quad \forall\, OD_i, g \in \mathbf{N}, \\ & \textit{and} & \\ \gamma(t,\mathbf{C}) = \sum_{OD_i,g \in \mathbf{Y}} \lambda_{OD_i,g}(t) & & x^*_{OD_i,g} > 0, \quad \forall\, OD_i, g \in \mathbf{Y} \end{array} \right\} \tag{5.25}$$

Proof: The right hand side of the two expressions 5.24 and 5.25 are equivalent, because

$$\left. \begin{array}{l} I_{OD_i,g}(t,\mathbf{C}) = 0, \quad \forall\ OD_i, g \in \mathbf{N} \\ \text{and} \\ I_{OD_i,g}(t,\mathbf{C}) = 1, \quad \forall\ OD_i, g \in \mathbf{Y} \end{array} \right\} \iff \left\{ \begin{array}{l} \beta(t,\mathbf{C}) = \sum_{OD_i,g \in \mathbf{Y}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} \\ \text{and} \\ \gamma(t,\mathbf{C}) = \sum_{OD_i,g \in \mathbf{Y}} \lambda_{OD_i,g}(t) \end{array} \right. \tag{5.26}$$

Therefore we do not have to do a separate proof for each one of them. As a matter of fact, we will use the two equivalent expressions of 5.26, interchangeably.

($\Longrightarrow$)

We assume that $W(t,\mathbf{C}) = \mathbf{z}(t,\mathbf{C})$, and we will prove that the second part of formulas 5.24 and 5.25 are true, by contradiction.

From the assumption $W(t,\mathbf{C}) = \mathbf{z}(t,\mathbf{C})$, we get that:

$$\begin{aligned} \frac{\partial W(t,\mathbf{C})}{\partial t} &= \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} \\ \frac{\partial W(t,\mathbf{C})}{\partial C_j} &= \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j}, \ j = 1,\ldots,N \end{aligned} \tag{5.27}$$

Let $\mathbf{Y}^{\bullet}$ be the set of the ODC (OD_i, g) combinations that we accept for transportation (i.e. $I_{OD_i,g}(t,\mathbf{C})$=1), under the dynamic programming formulation.
Let $\mathbf{Y}$ be the set of the ODC (OD_i, g) combinations that we accept for transportation (i.e. $x^{\bullet}_{OD_i,g}(t,\mathbf{C}) > 0$), under the linear programming formulation.
We further assume that $\mathbf{Y}^{\bullet} \neq \mathbf{Y}$ $(\mathbf{Y}^{\bullet}_{\mathbf{j}} \neq \mathbf{Y}_{\mathbf{j}}, \ j = 1,\ldots,N)$.
From the above assumption about the optimal policy for the linearized $W(t,\mathbf{C})$, we get that:

$$\begin{aligned} \gamma_j(t,\mathbf{C}) &= \sum_{i \in \mathbf{Y}^{\bullet}_{\mathbf{j}}} \lambda_i(t) \\ \beta(t,\mathbf{C}) &= \sum_{i \in \mathbf{Y}^{\bullet}} f_i \lambda_i(t) \end{aligned} \tag{5.28}$$

If we substitute the above assumption 5.28 in the HJB equation 5.14, we get

$$\frac{\partial W(t,\mathbf{C})}{\partial t} = \sum_{i \in \mathbf{Y}^{\bullet}} f_i \lambda_i(t) - \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y}^{\bullet}_{\mathbf{j}}} \lambda_i(t) \right\} \cdot \frac{\partial W(t,\mathbf{C})}{\partial C_j} \tag{5.29}$$

From lemma 11, we get that the governing equation of the linear programming 5.19

$$\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} = \sum_{i \in \mathbf{Y}} f_i \lambda_i(t) - \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y}_{\mathbf{j}}} \lambda_i(t) \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} \tag{5.30}$$

From the combination of the two above equations 5.29, and 5.30, and equations 5.27, which are direct results of the assumption $W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C})$, we get:

$$\sum_{i \in \mathbf{Y}^*} f_i \lambda_i(t) - \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y}_j^*} \lambda_i(t) \right\} \cdot \frac{\partial W(t, \mathbf{C})}{\partial C_j} = \sum_{i \in \mathbf{Y}} f_i \lambda_i(t) - \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y}_j} \lambda_i(t) \right\} \frac{\partial W(t, \mathbf{C})}{\partial C_j} \tag{5.31}$$

If we assume that $\mathbf{Y}^* \neq \mathbf{Y}$, (and also that $\mathbf{Y}_j^* \neq \mathbf{Y}_j \;\; j = 1, \ldots, N$), equation 5.31 does not hold. We therefore have to accept that $\mathbf{Y}^* = \mathbf{Y}$ and $\mathbf{Y}_j^* = \mathbf{Y}_j \;\; j = 1, \ldots, N$.

($\Longleftarrow$) If we make the assumptions $\mathbf{Y}^* = \mathbf{Y}$ and $\mathbf{Y}_j^* = \mathbf{Y}_j \;\; j = 1, \ldots, N$, with the help of equations 5.28, we can easily prove that equations 5.14 and 5.22 are the same differential equation. Both, the solution $W(t, \mathbf{C})$ of the linearized DP and the solution $\mathbf{z}(t, \mathbf{C})$ of the LP satisfy the same boundary conditions. i.e $W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C}) = 0$, when either $t = 0$ or when $\mathbf{C} = \mathbf{0}$. That means that equations 5.14, and 5.22 satisfy the requirements of the Cauchy-Kowalevski Theorem [40, p. 74]. Therefore the solution of the two equations 5.14, and 5.22 is unique. Consequently, we get that $W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C})$. **Q.E.D.**

An alternative, and more useful expression of the above Theorem 12, is the following. Corollary 13 says that the optimal policy suggested by the linear programming is optimal policy for the linearized dynamic programming and the maximum expected revenue of the linearized dynamic programming is equal to the maximum value of the objective function of the linear program 5.19.

Corollary 13 (Equivalence Theorem) *Let $\mathbf{z}(t, \mathbf{C})$ be the maximum value of the objective function and let $x^*_{\wedge D_{i,g}}(t, \mathbf{C})$, $m \in \mathbf{S}$, be the optimal solution of the LP 5.19. Let also $W(t, \mathbf{C})$ be the linearized maximum expected revenue from the differential equation 5.14, and $I_{OD_{i,g}}(t, \mathbf{C})$, $m \in \mathbf{S}$, is the optimal policy suggested for the linearized DP. The following holds:*

$$\left\{\begin{array}{l} x^*_{OD_i,g} = 0, \quad \forall \; OD_i, g \in \mathbf{N} \\ x^*_{OD_i,g} > 0, \quad \forall \; OD_i, g \in \mathbf{Y} \end{array}\right\} \Longleftrightarrow \tag{5.32}$$

$$\left\{ W(t,C) = z(t,C) \quad and \quad \begin{array}{l} I_{OD_i,g}(t,\mathbf{C}) = 0, \quad \forall \; OD_i, g \in \mathbf{N} \\ I_{OD_i,g}(t,\mathbf{C}) = 1, \quad \forall \; OD_i, g \in \mathbf{Y} \end{array} \right\}$$

Implications of Corollary 13 Corollary 13 shows that the solution of the linearized HJB equation, as it is given at lemma 10, is given by the optimal value of the LP 5.19. Therefore the expected revenue of the network dynamic programming model under optimal policy, is given by the objective function of the linear programming. Furthermore this LP formulation gives the optimal policy of the linearized dynamic programming.

We therefore see that there is no need to solve the multi-leg dynamic programming model in order to find the optimal booking policy of the linearized dynamic programming. We see that the method that we suggest here is an easy way to get results that have the accuracy that can be offered only by dynamic programming. Furthermore, the method suggested here bypasses the "*curse of dimensionality*" that a DP model for a real life multi-leg network application would encounter.

Corollary 13 shows that there is no need to solve the (linearized) DP formulation of the N legs itinerary, M classes of containers booking problem, in order to find the optimal booking policy. Instead, we can solve the linear program 5.19, and the solution of the LP will give the optimal policy.

If the optimal solution of the LP for the variable $x_{OD_i,g}$ is $x_{OD_i,g} = 0$, then the optimal policy for the linearized version of the DP is to refuse bookings from the containers that want to travel on the OD_i path, and belong to class g ($I_{OD_i,g}(t, \mathbf{C} = 0)$). If the optimal solution for the $x_{OD_i,g}$ variable of the LP is such that $x_{OD_i,g} > 0$, then it is optimal to accept for transportation the containers that want to travel on the OD_i path, and belong to class g ($I_{OD_{i,g}}(t, \mathbf{C}) = 1$).

We have to remind that the solution $W(t, \mathbf{C})$ of the differential equation of lemma 10, is a linearized form of the $W(t, \mathbf{C})$, which is defined as the cost functional of the recursive dynamic programming equation 5.5.

The agreement between the optimal solution offered by the LP 5.19, and the optimal solution offered by the original recursive equation 5.5, depends on the agreement of the product $(\mathbf{C} - \mathbf{C}_{OD_{i,m}}) \cdot \nabla \mathbf{z}(t, \mathbf{C})$, and the first difference $W(t, \mathbf{C}) - W(t, \mathbf{C}_{OD_{i,m}})$.

$\mathbf{z}(t, \mathbf{C})$ is the maximum value of the objective function of the LP 5.19, and $W(t, \mathbf{C})$ is the solution of the original recursive equation 5.5. The agreement between the two solutions becomes greater for larger values of the remaining capacity of the vessel.

5.4.1 A Solution for the Two Ports, M Classes of Goods Problem with Variable Arrival Rates

We have presented the general way we can find the optimal policy for a booking problem on a multi-leg trip. The presentation of the solution of the two ports problem can be given in an analytic form.

In order to find the solution and the Policy Areas to the two ports problem when the rates of arrival for the different classes of containers, we would follow the process that we followed at subsection 4.2.1. In a way similar to that of subsection 4.2.1, it is easy to show that the solutions presented at the following Table 5.1 are optimal to the respective Primal and Dual Problems. That can be proven with the help of the Optimality Property of Linear Programming, or with the Complementary Slackness Theorem ([38], or [39]). The Primal is given at formulation 5.19. We repeat it here:

$$
\begin{aligned}
\mathbf{z}(t, \mathbf{C}) = \max \quad & \mathbf{f}^{\mathbf{T}} \mathbf{x} \\
\mathbf{A}\mathbf{x} & \leq \mathbf{b} \\
\mathbf{x} & \geq \mathbf{0}
\end{aligned}
\tag{5.33}
$$

The dual of the above LP has as follows:

$$\mathbf{v}(t, C) = \min \ \mathbf{b}^T\mathbf{y}$$
$$\mathbf{y}^T\mathbf{A} \ \geq \ \mathbf{f} \qquad (5.34)$$

All the variables have been explained before.

By comparing the optimal solutions $\mathbf{x}$ and $\mathbf{y}$ at the different policy areas it is easy to prove that they actually are the optimal solutions of the primal and the dual problems respectively.

For the policy area where $\sum_{i=m+1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right] \leq C \leq \sum_{i=m}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$, the solution of the LP 5.19 is such that $x_1 = \ldots = x_{m-1} = 0$ and $x_m > 0, \ldots, x_M > 0$. According to Theorem 12, the optimal policy in the above policy area, is to accept for transportation only the containers that belong to classes $m, \ldots, M$, whereas the containers that belong to classes $1, \ldots, m-1$ should be rejected.

In Table 5.1 we summarize the parametric solution to the linear programming. The optimal policies applied at the different policy areas, are given at Table 5.2.

We have presented a solution to the differential equation, for the Two Ports Problem, with Variable Arrival Rates.

5.5 Discussion

In this chapter we have shown the equivalence between the Optimal Policy derived from the Linearized form of the D.P. and the linear programming model described by 5.33. In other words, if the L.P. suggests that we should accept some quantity from a particular ODC combination (i.e. $x_{OD_i,m} > 0$), then the optimal decision at the Linearized D.P. is to accept the Container with the specific ODC combination for transportation (i.e. $I_{OD_i,m} = 1$), and vice versa. When the optimal solution of the LP is such that $x_{OD_i,m} = 0$, then the optimal reservation policy is $I_{OD_i,m} = 0$.

The above result is important in the sense that we can have the detailed and accurate

Ranges of t and C	$\mathbf{z}$(t,C) = $\mathbf{v}$(t,C)	$\mathbf{x}$	$\mathbf{y}$
$\int_{\xi=0}^{t}\left[\sum_{i=1}^{M}\lambda_i\right]d\xi \le C$	$\sum_{i=1}^{M} f_i \int_{\xi=0}^{t}\lambda_i d\xi$	$x_1 = \int_{\xi=0}^{t}\lambda_1 d\xi$ $\vdots$ $x_M = \int_{\xi=0}^{t}\lambda_M d\xi$	$y_1 = 0$ $y_2 = f_1$ $\vdots$ $y_{M+1} = f_M$
$\sum_{i=2}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right] \le C$ $C \le \sum_{i=1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$	$f_1 C - \sum_{i=2}^{M}$ $(f_i - f_1)\int_{\xi=0}^{t}\lambda_i d\xi$	$x_1 = C -$ $\sum_{i=2}^{M}\int_{\xi=0}^{t}\lambda_i d\xi$ $x_2 = \int_{\xi=0}^{t}\lambda_2 d\xi$ $\vdots$ $x_M = \int_{\xi=0}^{t}\lambda_M d\xi$	$y_1 = f_1$ $y_2 = 0$ $y_3 = f_2 - f_1$ $\vdots$ $y_{M+1} = f_M - f_1$
$\sum_{i=m+1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right] \le C$ $C \le \sum_{i=m}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$	$f_m C + \sum_{i=m+1}^{M}$ $(f_i - f_m)\int_{\xi=0}^{t}\lambda_i d\xi$	$x_1 = 0$ $\vdots$ $x_{m-1} = 0$ $x_m = C -$ $\sum_{i=m+1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$ $x_{m+1} = \int_{\xi=0}^{t}\lambda_{m+1} d\xi$ $\vdots$ $x_M = \int_{\xi=0}^{t}\lambda_M d\xi$	$y_1 = f_m$ $y_2 = 0$ $\vdots$ $y_{m+1} = 0$ $y_{m+2} = f_{m+1} - f_m$ $\vdots$ $y_{M+1} = f_M - f_m$
$C \le \int_{\xi=0}^{t}\lambda_M d\xi$	$f_M C$	$x_1 = 0$ $\vdots$ $x_{M-1} = 0$ $x_M = C$	$y_1 = C$ $y_2 = 0$ $\vdots$ $y_{M+1} = 0$

Table 5.1: Solution of the Parametric LP's, for Variable Arrival Rates

Policy Area	W(t,C) = z(t,C)	x	Optimal Policy
$\int_{\xi=0}^{t}\left[\sum_{i=1}^{M}\lambda_i\right]d\xi \leq C$	$\sum_{i=1}^{M} f_i \int_{\xi=0}^{t}\lambda_i d\xi$	$x_1 = \int_{\xi=0}^{t}\lambda_1 d\xi$ $\vdots$ $x_M = \int_{\xi=0}^{t}\lambda_M d\xi$	$I_1 = 1$ $\vdots$ $I_M = 1$
$\sum_{i=2}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right] \leq C$ $C \leq \sum_{i=1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$	$f_1 C + \sum_{i=2}^{M}$ $(f_i - f_1)\int_{\xi=0}^{t}\lambda_i d\xi$	$x_1 = C-$ $\sum_{i=2}^{M}\int_{\xi=0}^{t}\lambda_i d\xi$ $x_2 = \int_{\xi=0}^{t}\lambda_2 d\xi$ $\vdots$ $x_M = \int_{\xi=0}^{t}\lambda_M d\xi$	$I_1 = 1$ $I_2 = 1$ $\vdots$ $I_M = 1$
$\sum_{i=m+1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right] \leq C$ $C \leq \sum_{i=m}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$	$f_m C + \sum_{i=m+1}^{M}$ $(f_i - f_m)\int_{\xi=0}^{t}\lambda_i d\xi$	$x_1 = 0$ $\vdots$ $x_{m-1} = 0$ $x_m = C-$ $\sum_{i=m+1}^{M}\left[\int_{\xi=0}^{t}\lambda_i d\xi\right]$ $x_{m+1} = \int_{\xi=0}^{t}\lambda_{m+1} d\xi$ $\vdots$ $x_M = \int_{\xi=0}^{t}\lambda_M d\xi$	$I_1 = 0$ $\vdots$ $I_{m-1} = 0$ $I_m = 1$ $I_{m+1} = 1$ $\vdots$ $I_M = 1$
$C \leq \int_{\xi=0}^{t}\lambda_M d\xi$	$f_M C$	$x_1 = 0$ $\vdots$ $x_{M-1} = 0$ $x_M = C$	$I_1 = 0$ $\vdots$ $I_{M-1} = 0$ $I_M = 1$

Table 5.2: Optimal Solution for the LP, and Optimal Policy for the DP, for Variable Arrival Rates

formulation of a D.P. formulation combined with the relatively few computations necessary for the solution of a Linear Program.

Furthermore, we do not have to solve the LP every time we get a reservation request, in order to make a decision. The optimal solution of the linearized DP depends on the variables of the LP that are simply greater than zero. Every time we solve the LP we should do a sensitivity analysis of the range of the right hand side of the constraints (i.e. the $\mathbf{b}$ of equation 5.19) over which every $x_{OD_{i,m}}$ stays either positive or zero. So we will know the range of t's and $\mathbf{C}$ for which our policy of accepting or rejecting a ODC combination remains the same. Once we are out of this region, we run the LP again.

In the attached graphs I give comparative $W(t, \mathbf{C})$ curves that have been derived both with the L.P. and the D.P. (non-linearized) method.

5.5.1 Contribution

In the current chapter we have expanded the dynamic programming formulation that we introduced in chapter 3. We also gave a solution to the linearized differential equation of this expansion. In order to find the solution of the linearized booking problem we used a methodology similar to the methodology that we employed in chapter 4 for the solution of the linearized version of the two port problem.

The dynamic programming model that describes the booking process and defines the optimal policy is easy to use only for small applications of the yield management problem. When the number of the legs of the problem increases, the computation time needed for the solution of the problem increases by a factor of C, where C is the capacity of the vessel. The result is that the number of computations needed for the solution of the DP increases exponentially with the number of ports. This is the problem of the *curse of dimensionality* as Bellman (the father of dynamic programming) had defined it.

In this chapter we give the solution to the linearized form of the dynamic programming in the form of the objective function of a linear programming problem. The theory of this chapter shows and the attached graphs verify that the prediction of the expected revenue done by the linear program is a satisfactory approximation of the expected revenue as it is given by the dynamic programming model, even for small values of the capacity of the vessel. For large values of the vessel capacity the two different methods give values that are asymptotically the same.

The output of the dynamic programming includes both the expected revenue from the booking under optimal policy, as well as the optimal policy for all combinations $(t_1, \mathbf{C}_1)$ of remaining time t and remaining capacity $\mathbf{C}$ with $t_1 < t$ and $\mathbf{C}_1 < \mathbf{C}$. On the other hand, the output of the linear program gives the approximation to the expect revenues of the trip and the optimal solution of the LP. We show that the optimal solution of the LP gives the optimal booking policy to the linearized version of the dynamic programming model.

We state the important findings of this chapter at what we call the *Equivalence Theorem*. The Equivalence Theorem suggests that the optimal solution of the linear program gives the optimal solution of the linearized dynamic program. Additionally, the expected revenue that the linearized D.P. gives, is equal to the objective function of the L.P.

The Equivalence Theorem shows that in order to find the optimal booking policy of the yield management model and the revenue potential, we do not have to solve the D.P. Instead, we only have to solve the linear program 5.19 that we presented earlier in this chapter. This L.P. gives both the optimal policy and the expected revenues for the yield management problem. We should mention that even for relatively small applications, for instance a D.P. model of the three port yield management problem, it can take tens of hours to solve. On the other hand, it only takes less than ten seconds on the same computer, to solve the equivalent L.P. that gives the same optimal policy and maximum expected revenue with the D.P.

Additionally, the L.P. model is a much easier model to use. Although the input of the L.P. is the same as the input of the D.P., it is in a condensed form.

In order to use the D.P. for practical applications, we would have to input into the model the probabilities of arrival of the different classes of containers at each point in time. If we use the L.P., we can have almost the same results, by using as the input of the L.P. the expected values of the demand from the different classes of containers. It is much easier for the analysts of a shipping company to forecast the expected demand rather than the expected demand as a function of time.

The Equivalence Theorem gives an easy solution to the multi-leg yield management problem. In the case of the one leg problem the equivalence Theorem gives an analytical solution. This solution is given in Table 5.2.

Figures The following figures 5-3, 5-4, and figure 5-5 verify the theory that has been developed in this chapter. In the first three graphs we present the two ports model with variable arrival rates for three classes of containers. The capacity is $C = 20$, $C = 100$, and $C = 1,000$ at the three figures respectively.

In figures 5-6 through figure 5-10, we present the three ports problem for three classes of customers (only one class of customers at each of the three possible paths of the network). In figures 5-6 and 5-7, the container arrival rates are variable, whereas at the remaining figures the arrival rates are constant. The agreement between the results from the linear programming and the dynamic programming is greater when the capacity of the vessel is larger.

We again observe that for strong demand (upper right corner) and weak demand (lower left corner), the results from the two methods are identical. The discrepancy is greater at the intermediate cases of not very strong demand. As expected, the discrepancy between the two methods decreases as the remaining capacity increases. For reasons that we have explained at the previous chapter 4, the linear program gives consistently higher or equal values than the dynamic programming model.

5.6 Summary

In this chapter we presented a generalization for the dynamic programming formulation of the multi-port yield management problem that we presented and solved in the previous two Chapters. Initially we presented a discrete time DP model for the multi-port yield management problem. We assumed that the rate of arrival for the different classes of customers is variable. From the discrete time DP model, we derive the continuous time DP model. We work again with the methodology that we introduced in the previous chapter. We linearized the expected revenues $W(t, C)$ as a function of the capacity, and we derived the Hamilton-Jacobi-Bellman (HJB) differential equation that governs the booking process.

We observed and proved that the solution to the HJB equation, i.e. the maximum expected revenue from the reservations and the optimal booking policy, is given by a linear program. We also gave the analytical solution to the two ports booking problem.

The graphs that we present at the end of the chapter show the agreement between the theory and the computational results. The graphs include descriptions for the two ports model for variable container arrival rates, and the three ports model for both constant and variable container arrival rates. The agreement between the solution given by the linear programming model and the solution offered by the dynamic programming model becomes better as the capacity of the vessel increases.

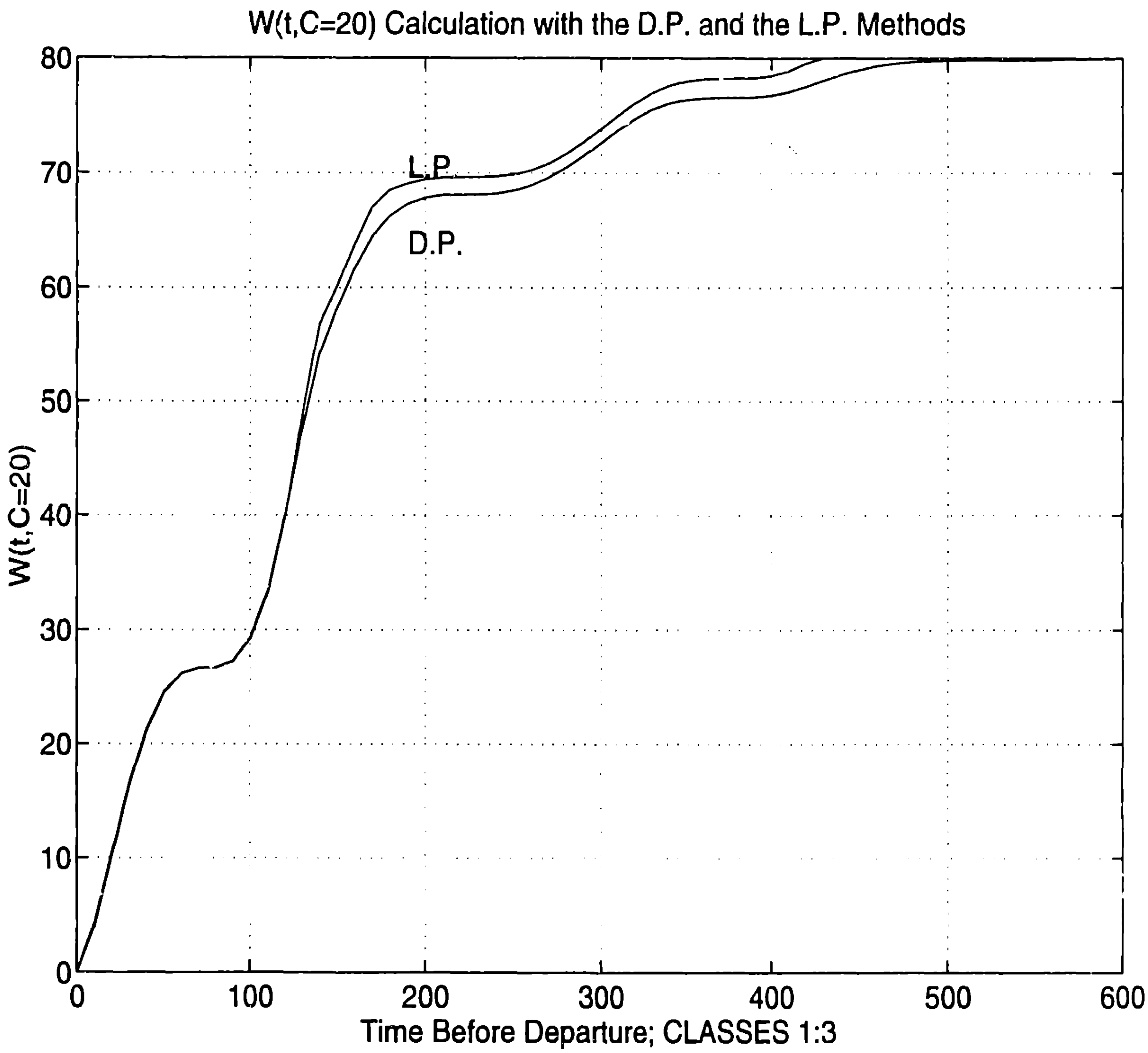


Figure 5-3: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 20, and Time Variable Arrival Rates $\left(\lambda_i(t) = c_i \cdot \log(t+1) \cdot \left[\cos\left(\frac{8\pi t}{600}\right) + 1\right]\right)$

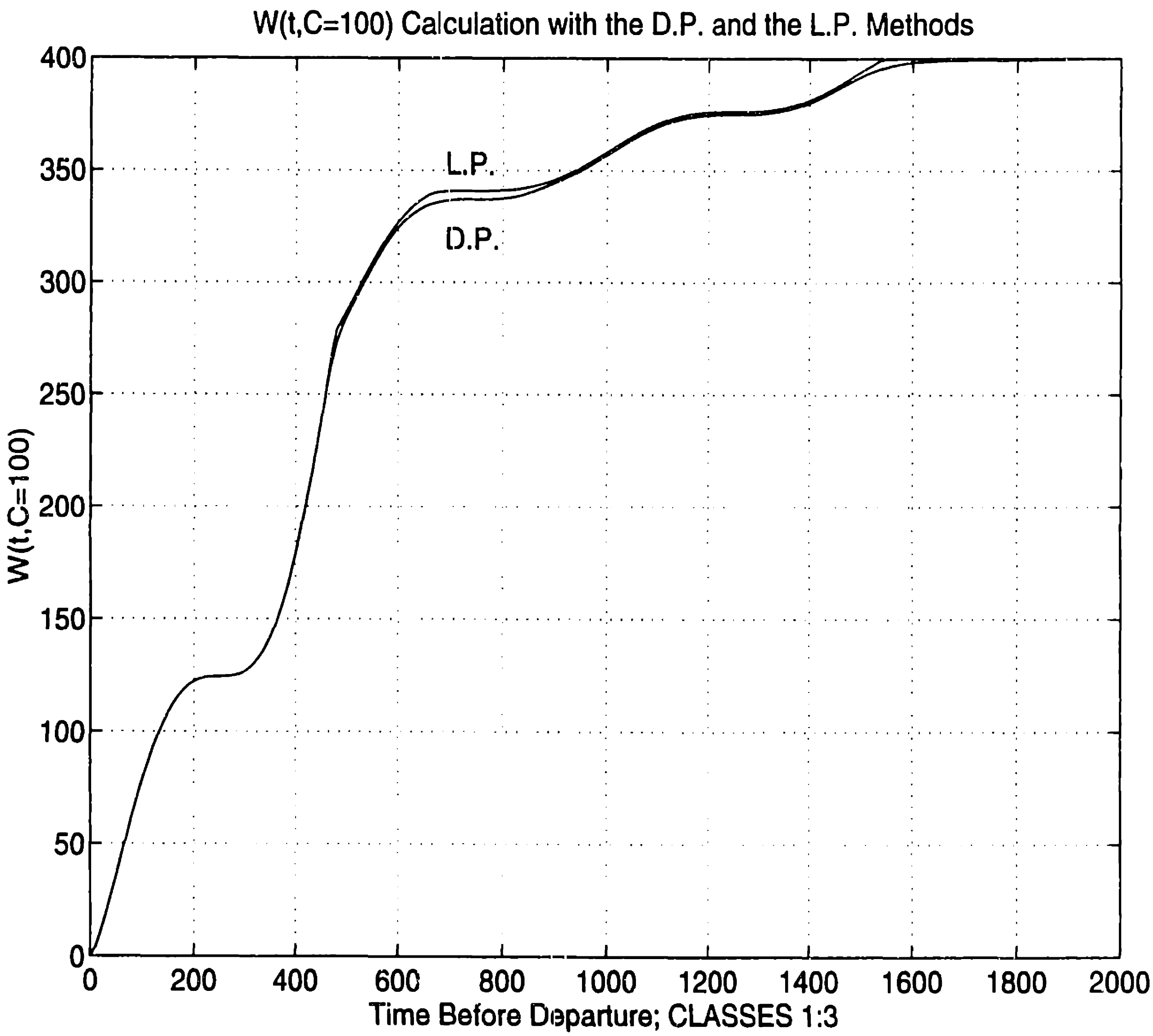


Figure 5-4: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 100, and Time Variable Arrival Rates $\left(\lambda_i(t) = c_i \cdot \log(t+1) \cdot \left[\cos\left(\frac{8\pi t}{2000}\right) + 1\right]\right)$

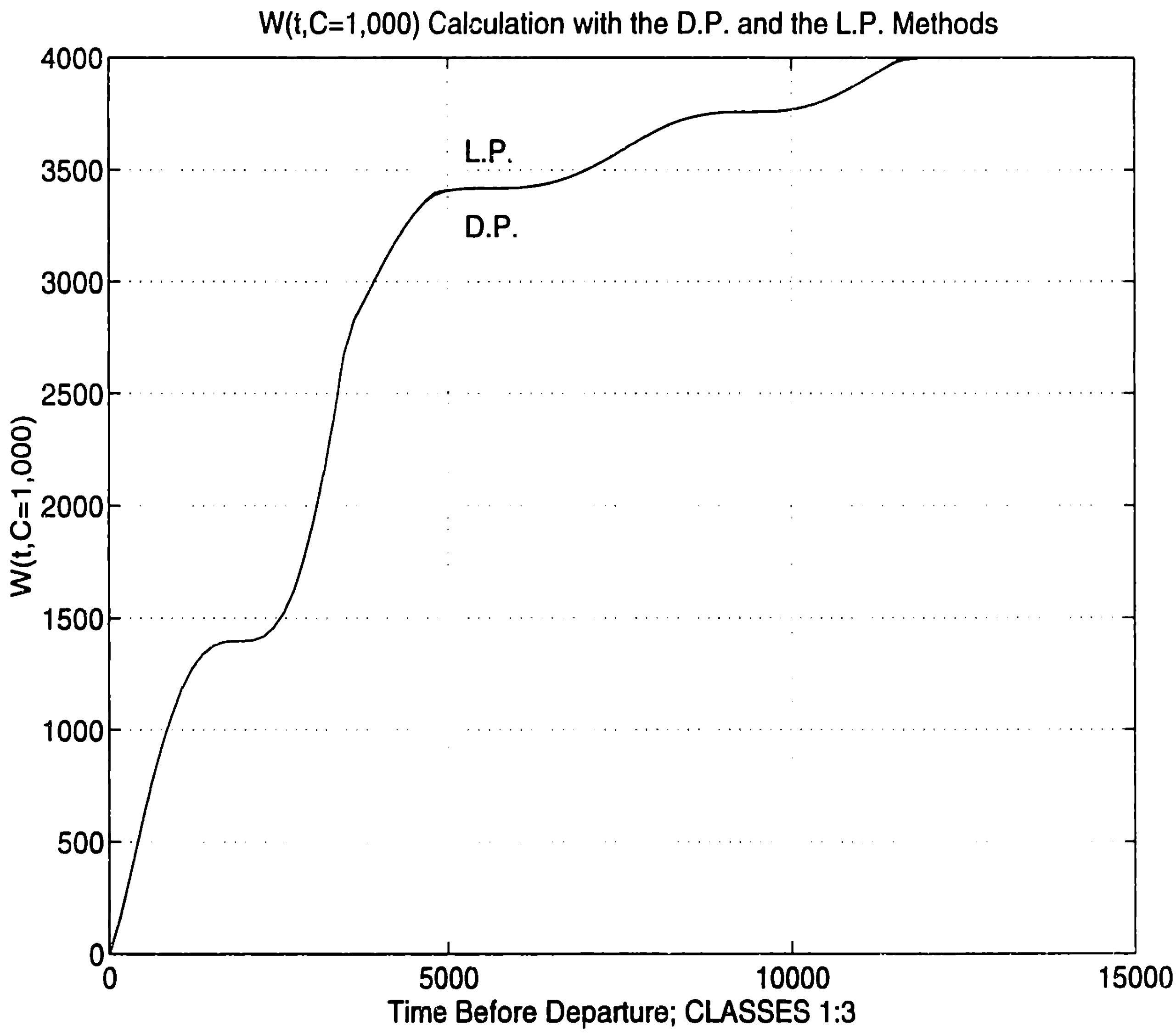


Figure 5-5: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 1,000, and Time Variable Arrival Rates $\left(\lambda_i(t) = c_i \cdot \log(t+1) \cdot \left[\cos\left(\frac{8\pi t}{15{,}000}\right) + 1\right]\right)$

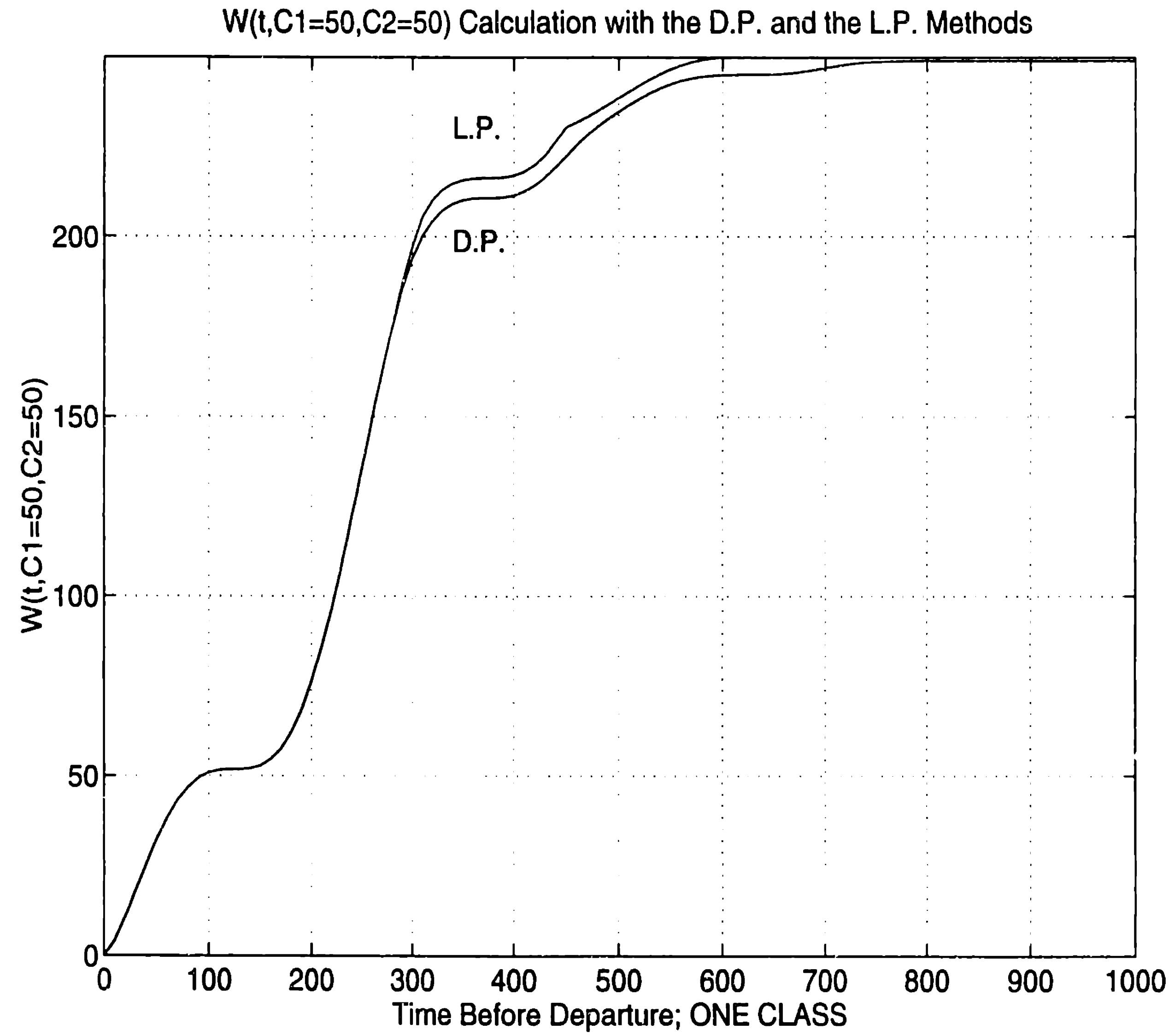


Figure 5-6: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Three Ports Problem with CAPACITY1 = 50, CAPACITY2 = 50, and Time Variable Arrival Rates $\left(\lambda_i(t) = c_i \cdot \log(t+1) \cdot \left[\cos\left(\frac{8\pi t}{1{,}000}\right) + 1\right]\right)$

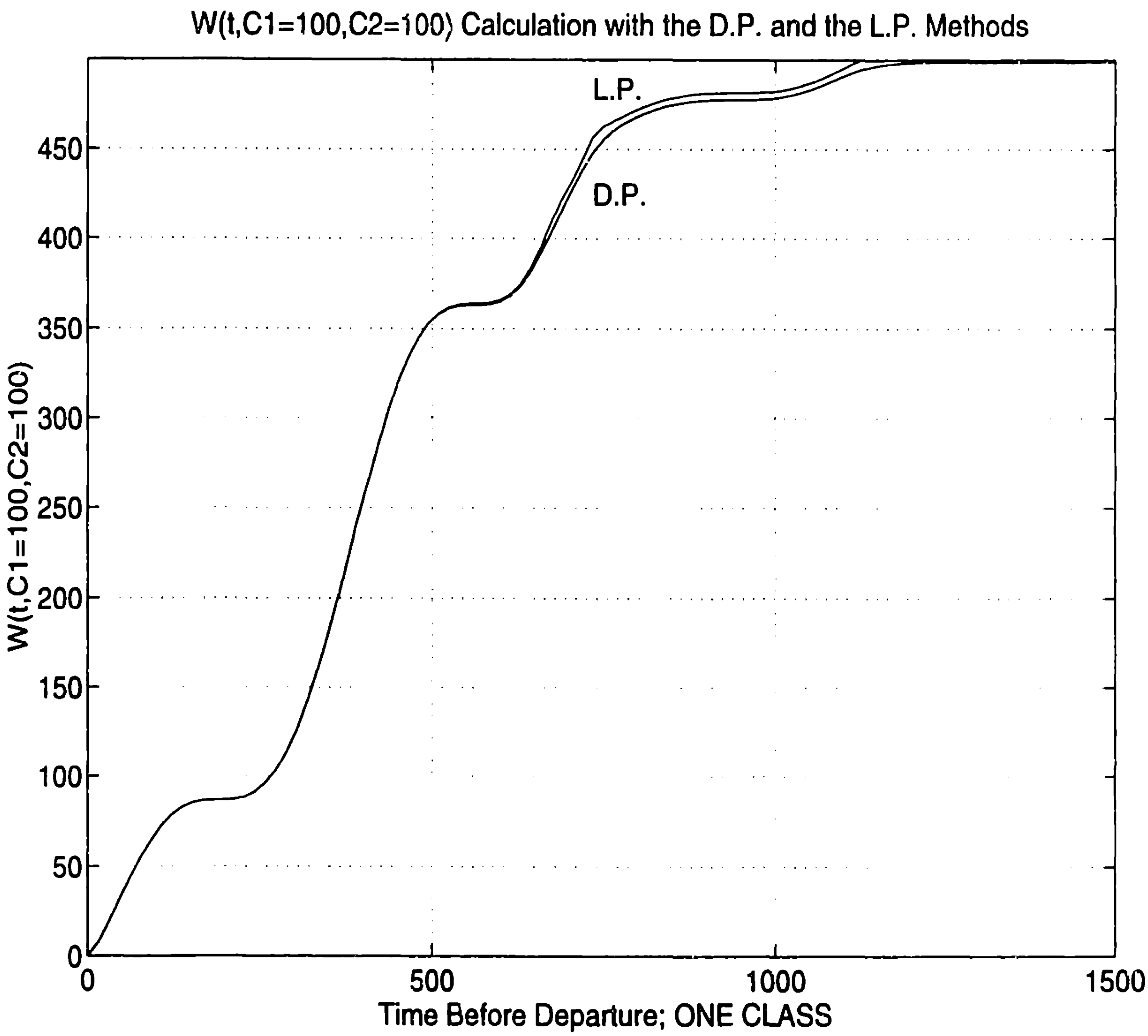


Figure 5-7: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Three Ports Problem with CAPACITY1 = 100, CAPACITY2 = 100, and Time Variable Arrival Rates $\left(\lambda_i(t) = c_i \cdot \log(t+1) \cdot \left[\cos\left(\frac{8\pi t}{1{,}500}\right) + 1\right]\right)$

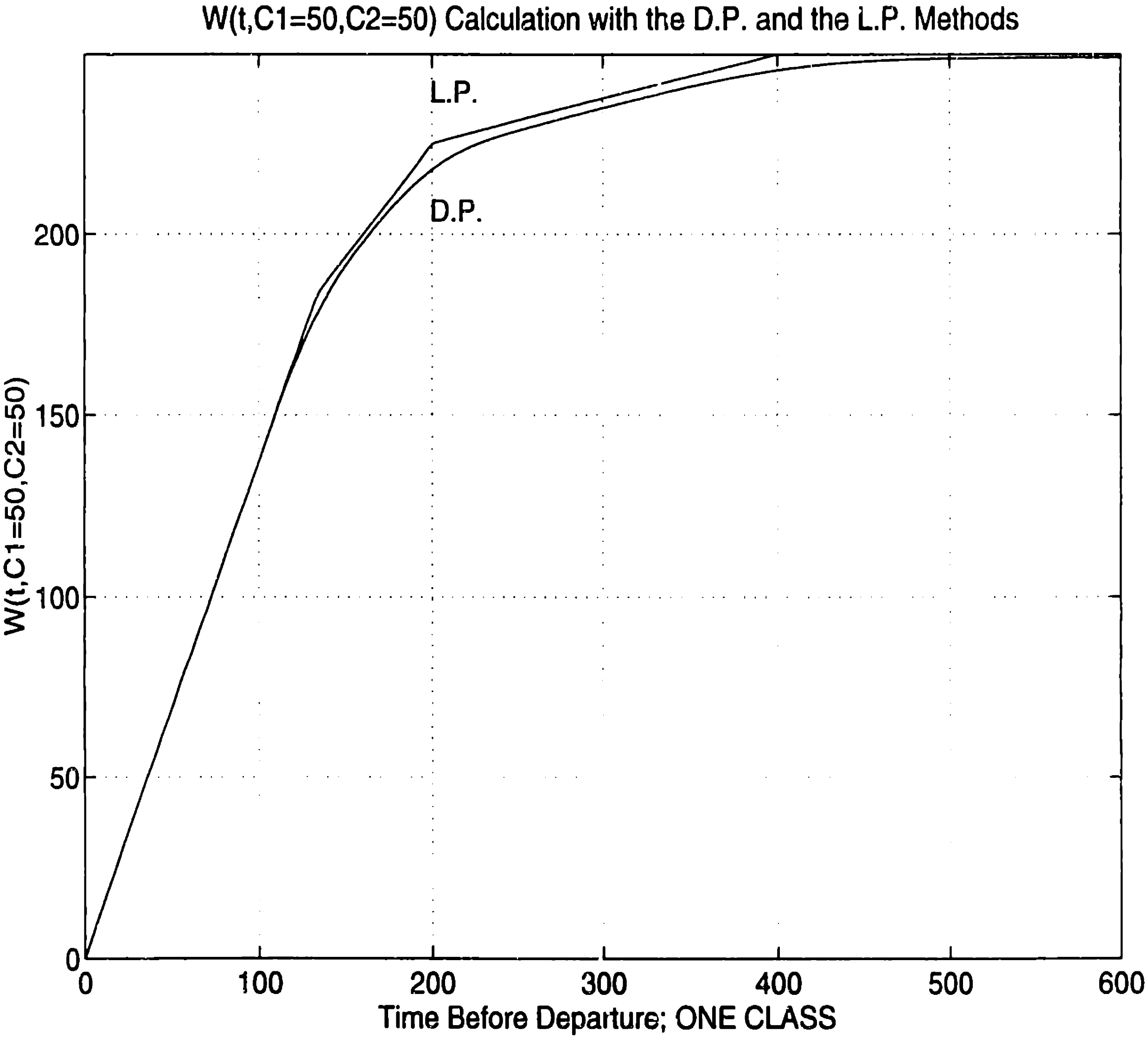


Figure 5-8: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Three Ports Problem with CAPACITY1 = 50, CAPACITY2 = 50, and Constant Arrival Rates ($\lambda(t) = \lambda$).

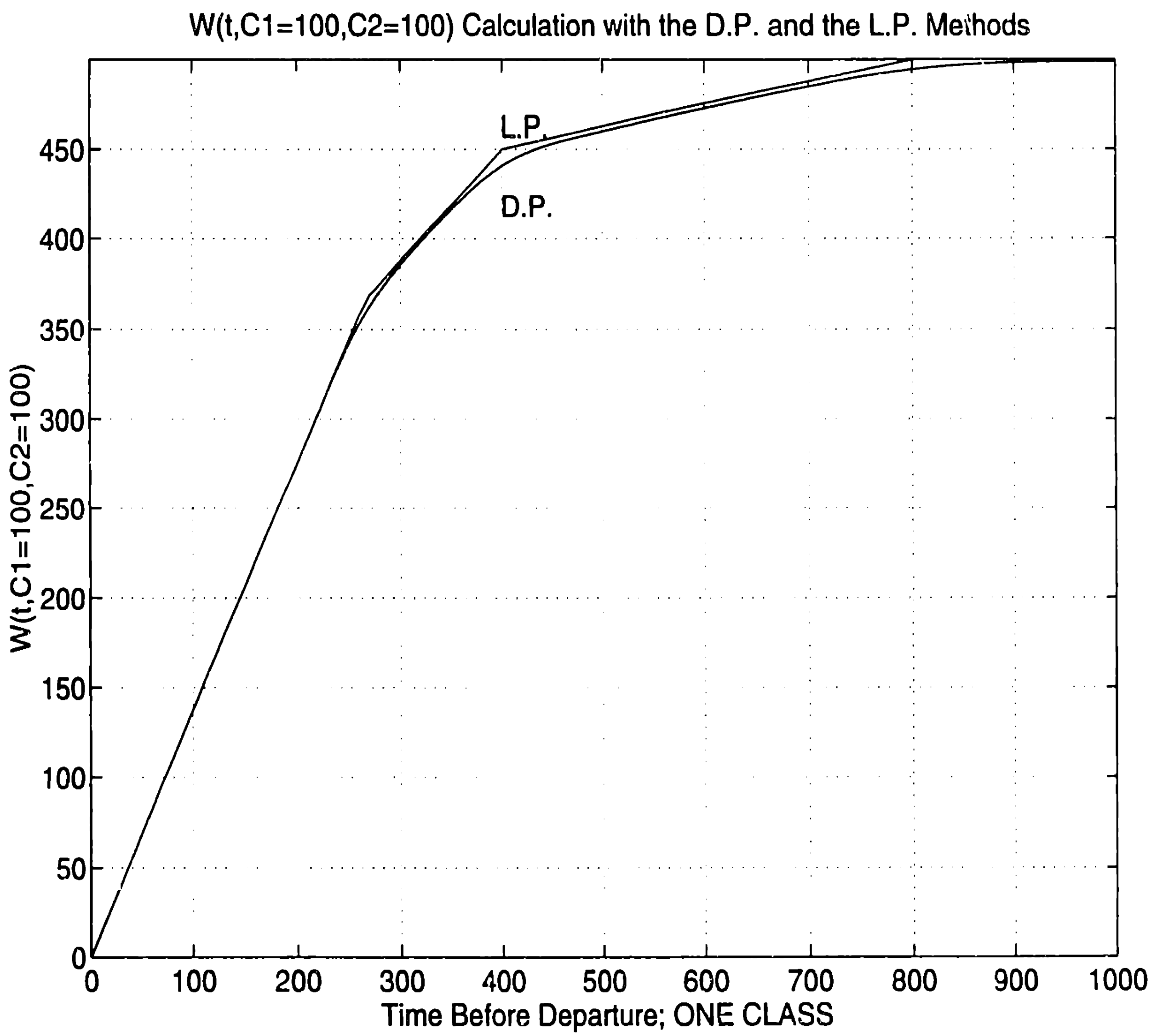


Figure 5-9: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Three Ports Problem with CAPACITY1 = 100, CAPACITY2 = 100, and Constant Arrival Rates ($\lambda(t) = \lambda$).

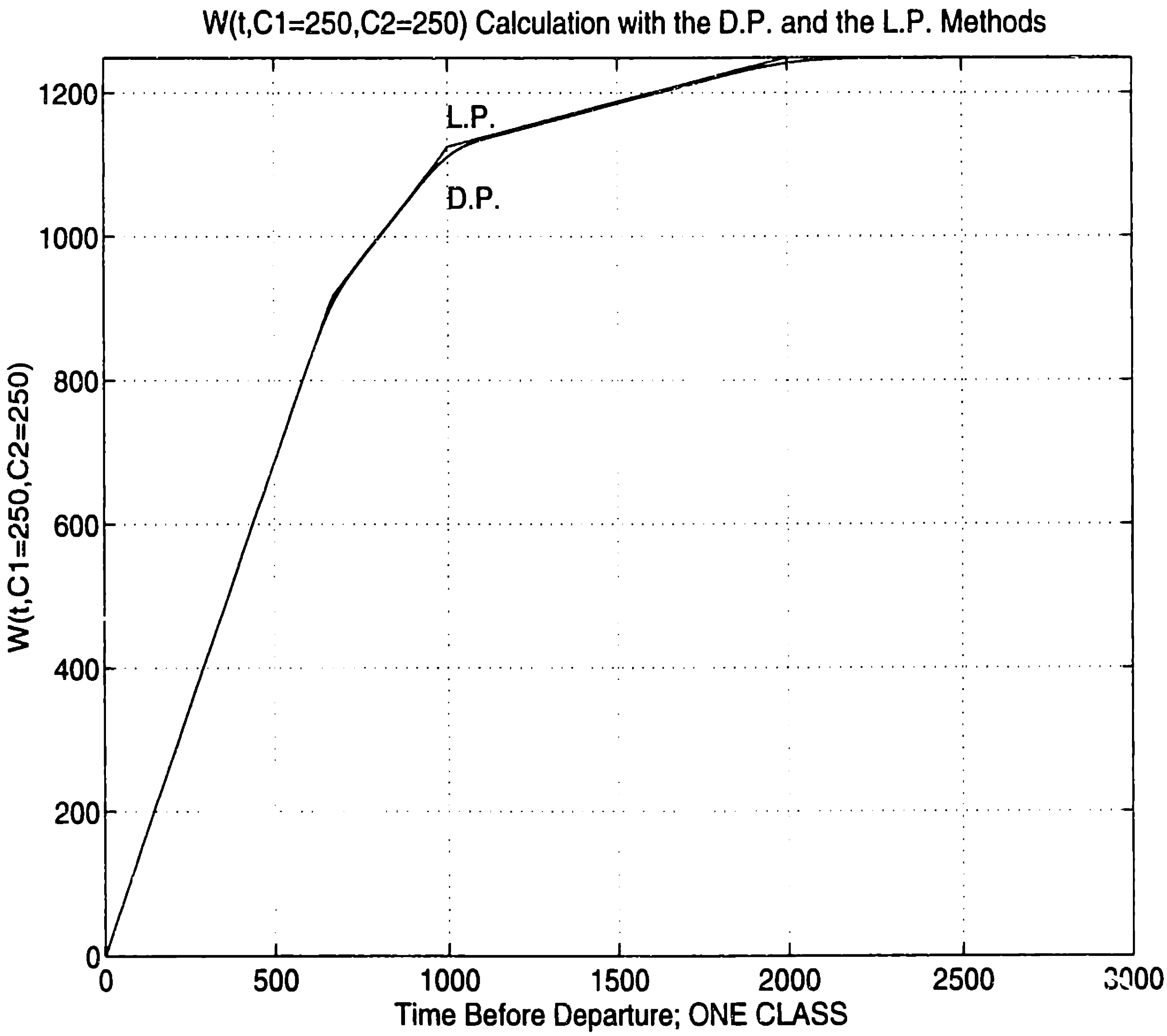


Figure 5-10: Comparison of the Linear Programming Method relatively to the Dynamic Programming Method for the Three Ports Problem with CAPACITY1 = 250, CAPACITY2 = 250, and Constant Arrival Rates ($\lambda(t) = \lambda$).

Chapter 6

Multi-leg, multi-class yield management problem with cancellations

6.1 Introduction

It is not uncommon for a customer who has reserved space with a containership to cancel his order. Many customers book space in anticipation of transportation demand that they will have. When this anticipation is not realized, they cancel the order at a minimal cost to them.

The cost of cancellations has two sources. The first source is the costs the vessel operator incurs in anticipation of the order that eventually gets canceled. For instance, the operator might send at his cost, containers to the customer for them to be filled with the goods to be transported. If the customer changes his mind after the vessel operator has send the containers, the operator incurs the cost of both sending and repositioning the containers. This phenomenon is mostly encountered with shippers of cargo of low value.

The second source of cost is the cost of opportunity to the operator. When the operator considers the vessel capacity to be reserved and he turns away customers, he does not exploit the revenue generating ability of the vessel capacity to the full extend. A hidden cost of the cancellations is that customers who want to use the services of the vessel cannot do so and therefore the quality of service offered by the operator decreases.

It is a fact that the customers who reserve capacity far in advance of the departure tend to cancel their orders more easily than the customers who book capacity close to the departure of the vessel. That can be explained by the fact that the longer the time span between the reservation of the capacity and the sailing, the more probable it is for the shipper to be subject to an unfavorable or unexpected event that might require him to cancel the order. The low freight rate customers usually reserve capacity before the high freight rate customers do so, and it is the cancellations from low freight rate containers that disrupt the booking process. They prevent the operator from accepting the high freight rate customers, and that increases further the cost of opportunity of the operator.

Cancellations become more frequent in times of strong capacity demand. Many shippers are afraid that the shortage of transportation capacity will prevent them from transporting their products if they do not book capacity early. Therefore, many shippers tend to book transportation capacity even before they are certain about their real transportation needs. Such an approach can only increase cancellations.

In conclusion, the cancellations usually spring from low freight rate customers who prevent high freight rate customers to have access to the capacity of the vessel. This phenomenon is intensified during times of high demand for capacity, with detrimental effects to the profitability potential of the vessel. It is therefore advisable to the operator, while accepting reservations, to take into consideration the tendency for cancellations exhibited by several classes of customers.

6.2 DP formulation and boundary conditions

In this section we present the dynamic programming formulation of the ocean yield management problem (OYM) with cancellations, for the case of the $N+1$ ports, or N legs trip when there are M classes of containers that can use the services of the vessel.

As in the model of the previous chapter, the stage of the cost functional is the remaining time units t until departure from the N_{th} port. The state of the dynamic programming is the number of the remaining container slots that are still available at time t, and have not been assigned yet to any shipper.

The operator accepts container offers for all possible combinations of origin and destination and for all classes of containers. The OD_i pair corresponds to the port of origin k, and the port of destination s. We further assume that the container that needs to be transported from port k to port s, belongs to the g class of containers. The scenario that we have just described of the container that belongs to class g and needs to be transported from port k to port s, is the origin-destination combination (ODC), OD_i, g. In general, we have $N \cdot (N+1)$ OD_i paths.

During the booking process, the vessel operator experiences bookings cancellations from some of the containers that have reserved capacity with the vessel. In that case the owners of the containers get back the freight rate that they have already paid, and the cancellation takes place at a small or no cost for the canceling shipper.

When we accept a Container of Class m to travel on itinerary OD_i, (i.e. $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C}) = 1$), and the vector of Capacity before the acceptance is $\mathbf{C}$, it becomes $\mathbf{C}_{I_{O_{i},m}}$ afterwards. If we have a cancellation from a Container of Class m, that had reserved capacity for the itinerary OD_i, and the vector of Capacity before the acceptance is $\mathbf{C}$, it becomes $\mathbf{C}_{OD_i,m}$ afterwards, with:

$$\mathbf{C} = [C_1 C_2 \dots C_N]^T$$

and

$$\mathbf{C}_{I_{OD_i,m}} = \begin{bmatrix} C_1 \\ \vdots \\ C_{k-1} \\ C_k - I_{OD_i,m} \\ \vdots \\ C_{s-1} - I_{OD_i,m} \\ C_s \\ \vdots \\ C_N \end{bmatrix}, \text{ and } \mathbf{C}_{OD_i,m} = \begin{bmatrix} C_1 \\ \vdots \\ C_{k-1} \\ C_k + 1 \\ \vdots \\ C_{s-1} + 1 \\ C_s \\ \vdots \\ C_N \end{bmatrix},$$

where C_r is the capacity available at the rth leg of the trip, before the Booking or the cancellation respectively.

Let $A_{OD_i,m}$ be the total number of the containers that belong to freight rate class m, and have booked transportation capacity with the vessel on the origin-destination pair OD_i. $\mathbf{A}$ is the vector of all the $A_{OD_i,m}$'s. In other words it is the vector of the reservations up to time t, classified according to the origin, destination and class combination to which they belong.

When we have an additional reservation from an ODC, for instance OD_i, m, ($I_{OD_i,m}(t, \mathbf{A}, \mathbf{C}) = 1$), the inventory $A_{OD_i,m}$ increases by $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$. At the vector level we symbolize the new reservations vector as: $\mathbf{A}_{I_{OD_i,m}}$.

When we have a cancellation from a container that belongs to ODC OD_i, m, then $A_{OD_i,m}$ decreases by one unit. The symbol for the modified reservations vector is: $\mathbf{A}_{OD_i,m}$.

We assume that at the beginning (or the end) of each time interval Δt we can have:

- a possible arrival of class m container that wants to travel on the OD_i origin destination path i.e. from port k to port s, with probability $\Delta p_{OD_i,m}(t) = \Delta t \cdot \lambda_{OD_i,m}(t)$,

- a possible cancellation of the booking for a container that belongs to any class and had booked for the itinerary OD_i. Each container has a probability of $\Delta t \cdot \mu_{OD_i,m}(t)$

of canceling its order.

- no arrival or cancellation
- we can have no more than one arrival of one container or one cancellation of one container.

The cost functional is the following:

$$V_{OD_i,m}(t,\mathbf{A},\mathbf{C}) = \max_{I\in\{0,1\}}\left[I\cdot f_{OD_i,m} + W(t,\mathbf{A}_{I_{OD_i,m}},\mathbf{C}_{I_{OD_i,m}})\right] \quad \forall\, m\in\{1,M\} \text{ and } OD_i\text{'s} \tag{6.1}$$

The cancellation functional is the following:

$$F_{OD_i,m}(t,\mathbf{A},\mathbf{C}) = -f_{OD_i,m} + W(t,\mathbf{A}_{OD_i,m},\mathbf{C}_{OD_i,m}) \quad \forall\, m\in\{1,M\} \text{ and } OD_i\text{'s} \tag{6.2}$$

and:

$$\begin{aligned}
W(t+\Delta t,\mathbf{A},\mathbf{C}) &= \\
&= \Delta t \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \lambda_{OD_i,g}(t)\cdot V_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\
&+ \Delta t \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \mu_{OD_i,g}(t)\cdot A_{OD_i,g}\cdot F_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\
&+ \left[1-\Delta t \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \{\mu_{OD_i,g}(t)\cdot A_{OD_i,g} + \lambda_{OD_i,g}(t)\}\right]\cdot W(t,\mathbf{A},\mathbf{C})
\end{aligned} \tag{6.3}$$

$V_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$ is the maximum expected revenue at time t, given that we have the arrival of a container that belongs to class m, and which wants to be transported on the OD_i origin destination itinerary. The vector of containers that have made their reservations is $\mathbf{A}$. The vector of container slots available at each of the N legs of the trip is $\mathbf{C}$. The OD_i origin destination Path corresponds to the path with port of origin the port k, and port of destination the port s.

$F_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$ is the maximum Expected Revenue at time t, given that we have a cancellation from a class m container, that was to be transported on the OD_i path. The vector of the number of the containers that have made reservations for each itinerary and class is $\mathbf{A}$. The vector of container slots available at each of the N legs of the trip is $\mathbf{C}$.

$W(t, \mathbf{A}, \mathbf{C})$ is the maximum expected revenue under optimal policy at time t, given that we have no arrivals or cancellations from any container at time t. The vector of the number of the containers that have made reservations for each itinerary and class is $\mathbf{A}$. The vector of container slots available at each of the N legs of the trip is $\mathbf{C}$.

$\lambda_{OD_i,m}(t)$ is the rate of arrival (or the probability of arrival) at time t, for a class m container, that wants to be transported on the OD_i itinerary.

$\mu_{OD_i,m}(t)$ is the rate of cancellation (or the probability of cancellation) at time t, for a class m container, that had a booking for transportation on the OD_i itinerary. We consider the cancellation process to be a memoryless process. Each container that has already reserved capacity, has a probability of canceling that depends only on the class of the container and the itinerary. The probability of cancellation does not depend on the elapsed time since the reservation. Nevertheless, the probability of cancellation is greater when the reservation is done further away from the day of departure. In order to cover the general case, we have assumed the probability of cancellation to also be a function of time. The reason the cancellation process was assumed to be memoryless was because the shippers cancel their orders when something unexpected happens that causes them to change their business plans. There is no way to predict these "unexpected" events. They are independent from one another and they are assumed to be the product of a memoryless process. These events, though influence all the containers that belong to the same class and travel on the same itinerary. For instance, the cancellations from fruit exporters depend on "unexpected" weather changes, that cause them to send their products earlier, later or even in different quantities than originally planned.

$I_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$ is the control variable at time t, when we have an arrival of a class m container that asks for transportation capacity on the OD_i itinerary. The state of the model is $\mathbf{A}, \mathbf{C}$. If we decide to offer the space to the container, we have $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C}) = 1$. Otherwise, $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C}) = 0$.

$f_{OD_i,m}$ is the freight rate for containers that belong to class m, and are transported on the OD_i itinerary.

The boundary conditions are:

$$\begin{aligned} V_{OD_i,m}(t, \mathbf{A}, \mathbf{C} = \mathbf{0}) &= 0.0\,, \quad \forall\ t > 0 \text{ and } m \in \{1, M\} \\ F_{OD_i,m}(t, \mathbf{A} = \mathbf{0}, \mathbf{C}) &= 0.0\,, \quad \forall\ t > 0 \text{ and } m \in \{1, M\} \\ W(t = 0, \mathbf{A}, \mathbf{C}) &= 0.0\,, \quad \forall\ \mathbf{A}, \mathbf{C} \geq [0 \ldots 0]^T \end{aligned} \tag{6.4}$$

6.3 Derivation and solution of the HJB equation

We have considered the N legs, M classes of containers model to be a discrete time model. The model was assuming that we had possible container arrivals at discrete points in time.

For the network problem without reservation cancellations, we were able to find a solution for the linearized $W(t, C)$. This solution was the optimal objective function of an LP. In the current chapter, we will try to repeat the method of the previous chapter and find a similar LP that would be the solution to the HJB equation of the network model that includes cancellations.

The remaining time t is divided into T time intervals and the duration of each time interval is such that $\Delta t = \frac{t}{T}$. When the number of time intervals $T \longrightarrow \infty$, then the duration of each time interval $\Delta t \longrightarrow 0$.

We consider that at each time interval Δt we have a possible arrival of class m good that wants to travel on the OD_i Origin-Destination Pair i.e. from port k to port s, with probability $\Delta p_{OD_i,m}(t) = \Delta t \cdot \lambda_{OD_i,m}(t)$, or a possible cancellation of the reservation from a container that belongs to the same ODC combination. The cancellation probability during the time interval Δt is equal to $\Delta c_{OD_i,m}(t) = \Delta t \cdot \mu_{OD_i,m}(t)$.

The Model is described by the equations 6.1, 6.2 and 6.3 and the boundary conditions are described by equations 6.4.

If we treat both t and C_r, $r = 1, \ldots, N$ as continuous variables, we can get the differential equation that governs the booking process. In other words, the following lemma ?? gives the differential equation that describes the linearized form of the maximum expected revenue, as it is given by the dynamic programming formulation. The differential equation given by the lemma is the Hamilton-Jacobi-Bellman equation of control theory.

Lemma 14 *$W(t,\mathbf{A},\mathbf{C})$ is the maximum expected revenue of the continuous time model under optimal policy . The optimal policy is such that $I_i(t,\mathbf{A},\mathbf{C}) = 0$, $\forall i \in \mathbf{N}$, and $I_i(t,\mathbf{A},\mathbf{C}) = 1$, $\forall\ i \in \mathbf{Y} = \bigcup_{j=1}^{N} \mathbf{Y_j}$. We assume that the partial derivatives of $W(t,\mathbf{A},\mathbf{C})$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t,\mathbf{A},\mathbf{C})$ is the equation:*

$$\begin{aligned} \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{i\in\mathbf{Y}} \lambda_i(t)\cdot f_i - \sum_{i\in\mathbf{S}} \mu_i(t)\cdot A_i\cdot f_i \\ &+ \sum_{i\in\mathbf{Y}} \lambda_i(t)\cdot\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_i} - \sum_{i\in\mathbf{S}} \mu_i(t)\cdot A_i\cdot\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_i} \\ &- \sum_{j=1}^{N}\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j}\cdot\left\{\sum_{i\in\mathbf{Y_j}}\lambda_i(t) - \sum_{i\in\mathbf{S_j}}\mu_i(t)\cdot A_i\right\} \end{aligned} \tag{6.5}$$

The proof of the above lemma 14, is given in appendix D.

6.3.1 Solution of the HJB equation

In the previous section we gave the dynamic programming formulation, when we make the approximation that the number of containers is a continuous number instead of integer. The formula is equation 6.5, and the boundary conditions of this differential equation are given by equation 6.4.

We will prove the equivalence of solutions between the Dynamic Programming formulation and a Linear Programming formulation. First we will describe the LP formulation and the idea behind it, and then we will prove the equivalence of the solutions.

Let us assume that we have only one Class of containers, X and no Capacity Constraint, and that we accept all cargo offers. The rate of arrival for the Class X is $\lambda(t)$, and the Cancellation Rate is $\mu(t)\cdot X$. Therefore, the governing equation of X is:

$$\frac{\partial X}{\partial t} = \lambda(t) - \mu(t)\cdot X \tag{6.6}$$

and the Boundary Condition is: $X(t=0) = A$.
This is the first order linear differential equation, expressed in it's Canonical form [41, p. 1129]. The general solution of the differential equation 6.6 is:

$$X(t) \;=\; e^{-\int_{\nu=0}^{t}\mu(\nu)d\nu} \cdot \left[A + \int_{\xi=0}^{t} \lambda(\xi) e^{\int_{\nu=0}^{\xi}\mu(\nu)d\nu} d\xi\right] \qquad (6.7)$$

The above equation gives the number of containers we would have after t time units, if the original number of containers is A , the rate of container arrival is $\lambda(t)$ and the rate of cancellation for each container that has booked capacity is $\mu(t)$.
When the original inventory is zero, i.e when $A = 0$, equation 6.7 becomes:

$$X(t) \;=\; e^{-\int_{\nu=0}^{t}\mu(\nu)d\nu} \cdot \left[\int_{\xi=0}^{t} \lambda(\xi) e^{\int_{\nu=0}^{\xi}\mu(\nu)d\nu} d\xi\right] \qquad (6.8)$$

The Linear Programming formulation that follows, is based on the above idea.

$$\begin{aligned} \mathbf{z}(t, \mathbf{A}, \mathbf{C}) = \max \ \mathbf{f}^T \cdot (\mathbf{x} \ &- \ \mathbf{A_c}) \\ \mathbf{Bx} \ &\leq \ \mathbf{b} \\ \mathbf{x} \ &\leq \ \mathbf{L} \\ \mathbf{x} \ &\geq \ \mathbf{0} \end{aligned} \qquad (6.9)$$

where $\mathbf{A_c}$, is a vector with elements $A_{\mathbf{c}OD_i,g}$, such that :

$$A_{\mathbf{c}OD_i,g} = A_{OD_i,g} \cdot \left(1 - e^{-\int_{\nu=0}^{t}\mu_{OD_i,g}(\nu)d\nu}\right) \qquad (6.10)$$

$A_{\mathbf{c}OD_i,g}$ is the number of the expected cancellations from the ODC combination OD_i, g, between the current time t, and the departure of the vessel from the port of origin of the itinerary OD_i. Consequently $\mathbf{A_c}$ is the vector of the expected cancellations from the different ODC combinations, between the time t, and the departure of the vessel from the port of origin of each ODC.

The form of

$$\mathbf{Bx} \leq \mathbf{b}$$

is the following:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} x_{OD_i,g} \leq C_j + \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} A_{COD_i,g}, \; j = 1, \ldots, N \tag{6.11}$$

The sum at the right hand side of the formula 6.11 is equal to the space that will be freed at the leg j of the network, as a result of the cancellations of bookings from containers that would use the j leg of the trip. The extra capacity is equal to the sum of the expected cancellations from the containers that have already booked and belong to the ODC's that use the leg j.

The constraint 6.11 suggests that the number of the containers that will book to use the leg j for their transportation needs have to be fewer than the remaining capacity on this leg, augmented by the expected cancellations from containers that have already booked to use the leg j.

A more detailed form of the constraints

$$\mathbf{x} \leq \mathbf{L}$$

is

$$x_{OD_i,g} \leq L_{OD_i,g}(t), \; \forall \; OD_i, g \in \mathbf{S} \tag{6.12}$$

$L_{OD_i,g}(t)$ is the expected demand from containers that belong to the ODC OD_i, g between the time t, and the departure of the vessel from the port of origin of the itinerary OD_i.

The expected demand $L_{OD_i,g}(t)$ is the expected demand of the OD_i, g containers that will book capacity with the vessel and will not cancel their bookings. In mathematical terms, the expected demand $L_{OD_i,g}(t)$, is given by the following equation, which is similar to the equation 6.7.

$$L_{OD_i,g}(t) = e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu} \cdot \left[\int_{\xi=0}^{t} \lambda(\xi) e^{\int_{\nu=0}^{\xi} \mu_{OD_i,g}(\nu)d\nu} d\xi\right], \ \forall \ OD_i, g \ \in \ \mathbf{S} \quad (6.13)$$

The above constraint suggests that no origin, destination and class (ODC) combination can be accepted at a level greater than its expected demand (with the expected cancellations taken into consideration).

Figure 6-1 presents an alternative form of the linear programming model that we use to describe the N legs, M classes of containers yield management problem with reservations cancellations.

We solve the Linear Programming Problem and we find a solution $x^*_{OD_i,g}$, $\forall$ OD_i's and $\forall$ Class g. We assume that the ODC combination OD_i, g uses the leg j for transportation. If $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) > 0$, then $OD_i, g \in \mathbf{Y_j}$. If $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) = 0$, then $OD_i, g \in \mathbf{N_j}$.

Lemma 15 gives the differential equation that describes the maximum value of the objective function of the Linear Program 6.9. For reasons of completeness, lemma 15 also gives the boundary conditions of the maximum value of the LP.

Linear programming for the multi port
multi class yield management problem
with reservations cancelations

OBJECTIVE	**:**	**maximization of** **reservations revenues − cancelations refunds**
CONSTRAINTS	**:**	
total number of containers accepted to use leg j	**=<**	**remaining capacity at leg j + expected cancelations at leg j**
number of accepted containers from each ODC combination	**=<**	**expected demand from this ODC combination**

Figure 6-1: LP model for the multi-leg, multi class OYM problem, with cancellations

Lemma 15 $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ *is the maximum value, and* $x^*_{OD_i, g}(t, \mathbf{A}, \mathbf{C})$, $\forall\ OD_i$*'s and* $\forall\ g$, *is the corresponding optimal solution of the Linear Program 6.9. The governing differential equation for* $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ *is the equation*

$$\begin{aligned}
\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot f_i \\
&+ \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} \\
&- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) - \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \qquad (6.14)
\end{aligned}$$

with the following boundary condition:

$$\mathbf{z}(t = 0, \mathbf{A}, \mathbf{C}) = 0 \qquad (6.15)$$

The proof of the above lemma is given in appendix D.

In the following Theorem 16 we use the results from lemmas 6.9 and **??** to suggest that the solutions of the dynamic programming and the linear programming are equal when and only when the two following happen: 1) When the linearized dynamic programming suggests the rejection of a class of customers, the linear programming does not offer any capacity to this same class of customers. 2) When the linearized dynamic programming suggests the acceptance of a class of customers, the linear programming offers some capacity to this same class of customers.

Theorem 16, is similar to Theorems 8 and 12. Theorem 16 is a generalization of Theorem 12, that includes cancellations.

Theorem 16 $W(t, \mathbf{A}, \mathbf{C})$ *is the solution of the linearized version of the Dynamic Program given by equation 6.5, when the boundary conditions are given by equations 6.4.*

$I_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$ is the control variable for the class g containers that travel on the OD_i path, given by the Dynamic Programming under optimal policy. The acceptance ($I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) = 1$) or the rejection ($I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) = 0$) of a container that belongs to the OD_i, g ODC combination is as a function of the remaining time t, the reservations vector $\mathbf{A}$, and the remaining capacity vector $\mathbf{C}$. $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ is the maximum value, and $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$, $\forall\ OD_i$'s and $\forall\ g$, is the corresponding optimal solution of the Linear Program 6.14. We prove the following:

$$\{W(t, \mathbf{A}, \mathbf{C}) = \mathbf{z}(t, \mathbf{A}, \mathbf{C})\} \iff \left\{ \begin{array}{l} I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) = 0, \quad x^*_{OD_i,g} = 0, \quad \forall\ OD_i, g \in \mathbf{N} \\ \textit{and} \\ I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) = 1, \quad x^*_{OD_i,g} > 0, \quad \forall\ OD_i, g \in \mathbf{Y} \end{array} \right\} \tag{6.16}$$

Proof:

($\Longrightarrow$)

We assume that $W(t, \mathbf{A}, \mathbf{C}) = \mathbf{z}(t, \mathbf{A}, \mathbf{C})$. We will prove that the second part of formula 6.16 is true by contradiction.

From the assumption $W(t, \mathbf{A}, \mathbf{C}) = \mathbf{z}(t, \mathbf{A}, \mathbf{C})$, we get that:

$$\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial t} = \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} \tag{6.17}$$

$$\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} = \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j}, \ j = 1, \dots, N \tag{6.18}$$

$$-\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_i} = \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i}, \ i \in \mathbf{S} \tag{6.19}$$

Let $\mathbf{Y}^*$ be the set of the ODC (OD_i, g) combinations that we accept for transportation (i.e. $I^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$=1), under the dynamic programming formulation.

Let $\mathbf{Y}$ be the set of the ODC (OD_i, g) combinations that we accept in whole or in part for transportation, (i.e. $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) > 0$), under the linear programming formulation.

We further assume that $\mathbf{Y}^* \neq \mathbf{Y}$ ($\mathbf{Y}_\mathbf{j}^* \neq \mathbf{Y}_\mathbf{j}$, $j = 1, \ldots, N$).

From lemma 14, lemma 15, and equation 6.17, we see that the right hand side of the equations 6.5 and 6.14 are equal. With the help of equations 6.18 and 6.19, after a few manipulations we get:

$$\sum_{i \in \mathbf{Y}^*} \lambda_i(t) \cdot f_i + \sum_{i \in \mathbf{Y}^*} \lambda_i(t) \cdot \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_i} - \sum_{j=1}^{N} \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y}_\mathbf{j}^*} \lambda_i(t) \right\} =$$
$$= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i + \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_i} - \sum_{j=1}^{N} \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y}_\mathbf{j}} \lambda_i(t) \right\} \tag{6.20}$$

Equation 6.20 suggests that $\mathbf{Y}^* = \mathbf{Y}$ (and $\mathbf{Y}_\mathbf{j}^* = \mathbf{Y}_\mathbf{j}$, $j = 1, \ldots, N$).

($\Longleftarrow$) If we make the assumptions $\mathbf{Y}^* = \mathbf{Y}$ and $\mathbf{Y}_\mathbf{j}^* = \mathbf{Y}_\mathbf{j}$ $j = 1, \ldots, N$, equations 6.5 and 6.14 of the lemmas 14 and 15, respectively are the same differential equation. Both, the solution $W(t, \mathbf{A}, \mathbf{C})$ of the linearized DP and the solution $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ of the LP satisfy the same boundary conditions for $t = 0$ and $\mathbf{C} = \mathbf{0}$ (equation 6.4). That means that equations 6.5, and 6.14 satisfy the requirements of the Cauchy-Kowalevski Theorem [40, p. 74]. Therefore the solution of the two equations 6.5, and 6.14 is unique. As a result, $W(t, \mathbf{A}, \mathbf{C}) = \mathbf{z}(t, \mathbf{A}, \mathbf{C})$. **Q.E.D.**

An alternative, and more useful expression of the above Theorem 16, is the following. Corollary 17 says that the optimal policy suggested by the linear programming is optimal policy for the linearized dynamic programming and the maximum expected revenue of the linearized dynamic programming is equal to the maximum value of the objective function of the linear program 6.9.

Corollary 17 (Equivalence Theorem) *Let $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ be the maximum value of the objective function and let $x^*_{OD_{i,g}}(t, \mathbf{A}, \mathbf{C})$, $m \in \mathbf{S}$, be the optimal solution of the LP 6.14. Let* *also* $W(t, \mathbf{A}, \mathbf{C})$ *be the linearized maximum expected revenue from the differential equation 6.5, and* $I_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$, $m \in \mathbf{S}$, *is the optimal policy suggested for the linearized DP. The following holds:*

$$\left\{ \begin{array}{ll} x^*_{OD_i,g} = 0, & \forall \ OD_i, g \in \mathbf{N} \\ x^*_{OD_i,g} > 0, & \forall \ OD_i, g \in \mathbf{Y} \end{array} \right\} \Longleftrightarrow \tag{6.21}$$

$$\left\{ W(t, \mathbf{A}, \mathbf{C}) = \mathbf{z}(t, \mathbf{A}, \mathbf{C}) \quad and \quad \begin{array}{ll} I_{OD_i,g}(t, \mathbf{C}) = 0, & \forall \ OD_i, g \in \mathbf{N} \\ I_{OD_i,g}(t, \mathbf{C}) = 1, & \forall \ OD_i, g \in \mathbf{Y} \end{array} \right\}$$

6.4 Contribution

The contribution of the current chapter is the extension of the results of the previous chapter for the model that incorporates booking cancellations. The model of cancellations is a more complicated dynamic programming model than the D.P. models we presented in the previous chapter. We presented a linear programming model that for large values of the capacity gives the same results as the D.P. model with cancellations.

The linear programming model that corresponds to the D.P. model with cancellations is much easier to solve. and it only needs a condensed version of the data needed for the dynamic programming model. On the other hand, the methodology that we developed combines the best features of both D.P. and L.P. modeling. The model has the modeling accuracy of a dynamic programming model. Because of the fact that the booking criterion is a linear program, our method has the ability to solve large applications that are often encountered in practice. The L.P. needs only the expected value of the demand and cancellations and not the probabilities of a customer arrival or cancellation at every point in time. That makes the L.P. model and the methodology that we developed a much easier method to use in our yield management application.

Figures In the following figures 6-2, through figure 6-5, we present the numerical results of a two ports model with tree classes of containers. Both the rates of arrivals of the containers, as well as the rates of cancellation for all three classes of containers are constant.

Figure 6-2, shows that for very small values of the vessel capacity ($C = 4$) and for high cancellation rates, the fit between the DP and the LP models is, as expected, poor. In the following figure 6-3, we see that for greater vessel capacity ($C = 40$), and with the same high cancellation rate, the fit between the DP and the LP models very good. In this model the cancellation rate is so high that we cannot have enough containers to approach the limit of the vessel capacity. At figure 6-4, where the cancellation rate (or outflow) is smaller than the arrival rates (inflow), the curve from the LP predicts reasonably well the curve derived from the dynamic programming. The same happens at figure 6-5, where we examine the maximum expected revenue curve when we have an inventory from all three classes of containers.

When the demand is strong, as well as when the demand is weak, the results from both the LP and the DP are identical. The discrepancy is greater at the intermediate cases. As expected, the discrepancy between the two methods decreases as the remaining capacity increases.

6.5 Summary

Cancellations are costly because they cause under-utilization of the capacity of the vessel. In this chapter we study the influence of cancellations on the revenue potential of the vessel and the booking policy of the vessel operator. We incorporate the cancellations into the dynamic programming model. From the discrete time DP model, we derive the continuous time DP model. We follow the methodology we worked with at the previous chapters. We prove that the solution to the linearized version of the linearized DP is the solution of a linear programming model. Both the arrival rates of containers and the cancellation rates of orders are variable in the general case.

We first present a graph that shows the limits of the application of the linear theory, when the capacity is very small and the cancellation rates are very high. Besides this graph, we present graphs that show the agreement between the DP model and the linear programming approximation to the dynamic programming model, when we are away from the extreme cases represented at the first graph.

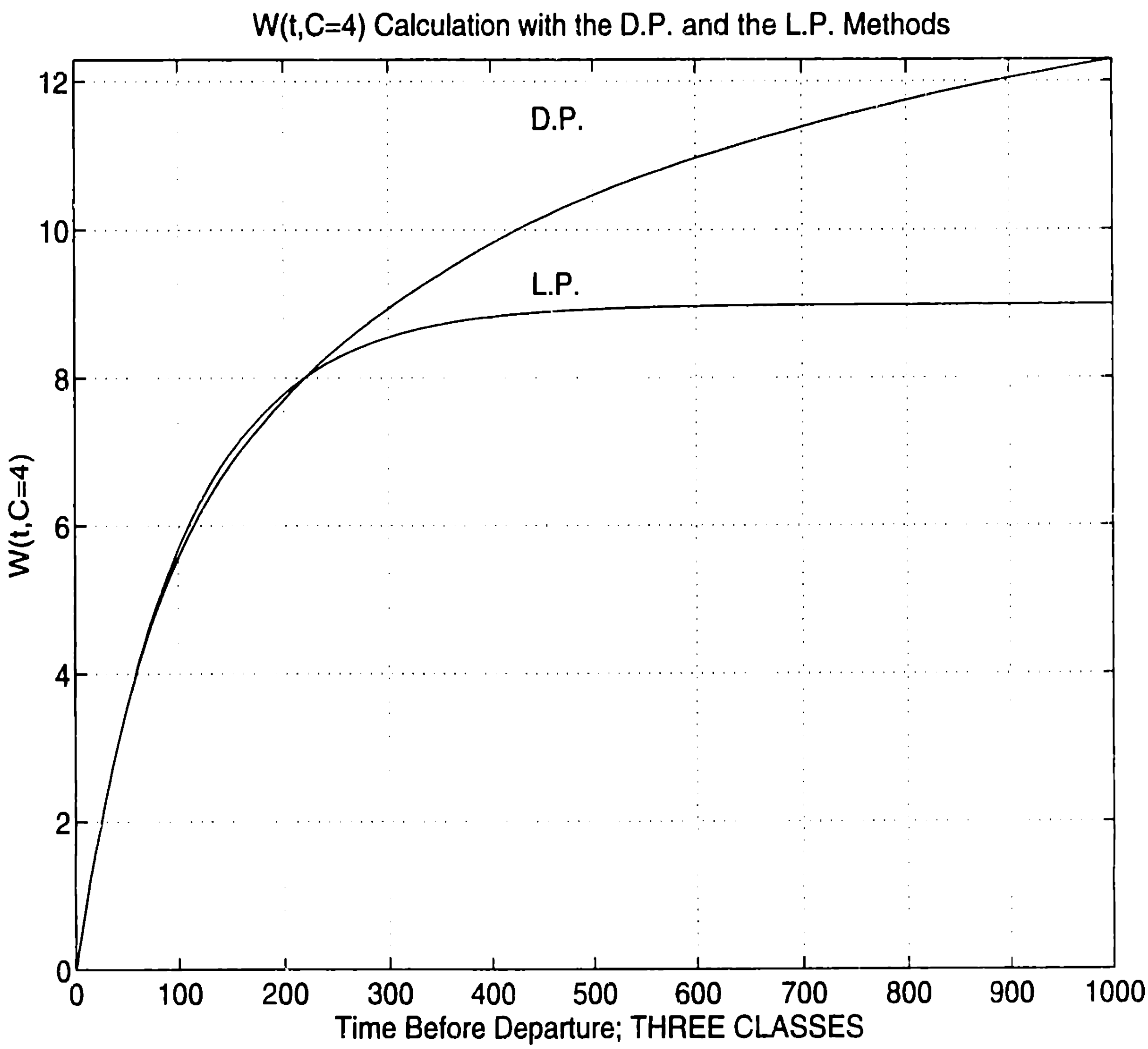


Figure 6-2: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 4, and Time Arrival Rates ($\lambda_1 = \lambda_2 = \lambda_3 = 0.01$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.01$)

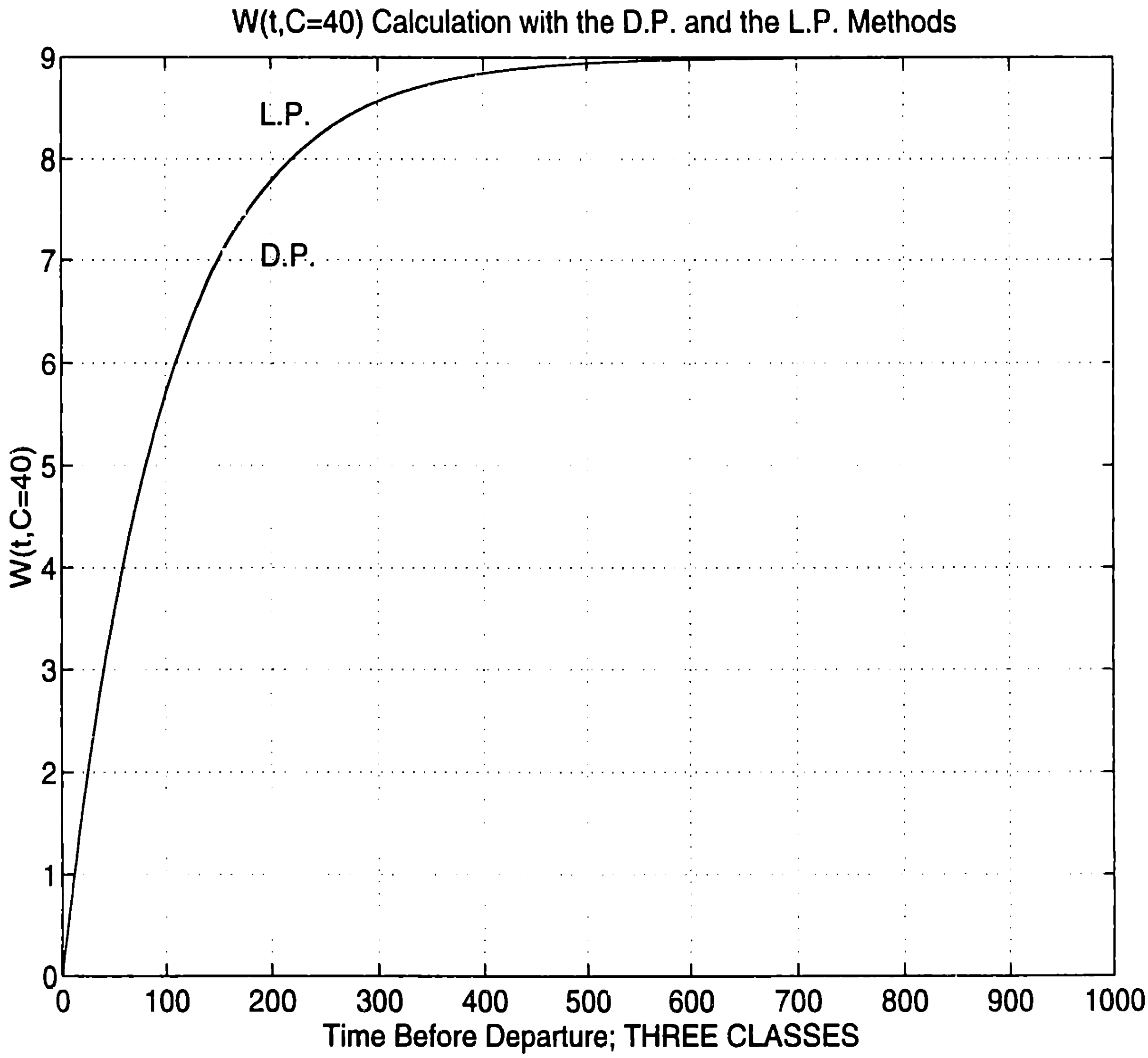


Figure 6-3: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 40, and Time Arrival Rates ($\lambda_1 = \lambda_2 = \lambda_3 = 0.01$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.01$)

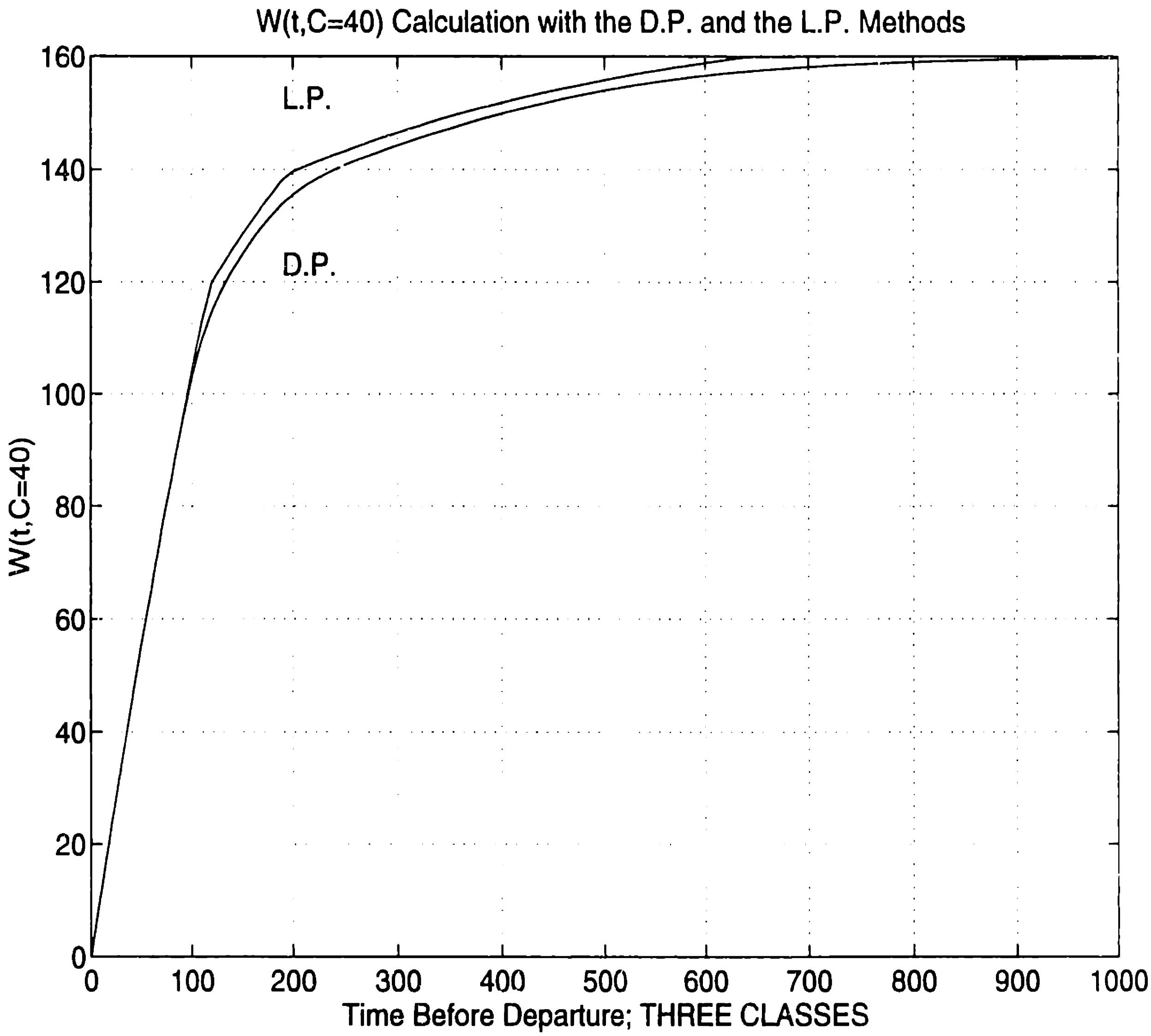


Figure 6-4: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 40, and Time Arrival Rates ($\lambda_1 = \lambda_2 = 0.135$, $\lambda_3 = 0.125$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.0025$), Containers ($A_1 = A_2 = A_3 = 0$)

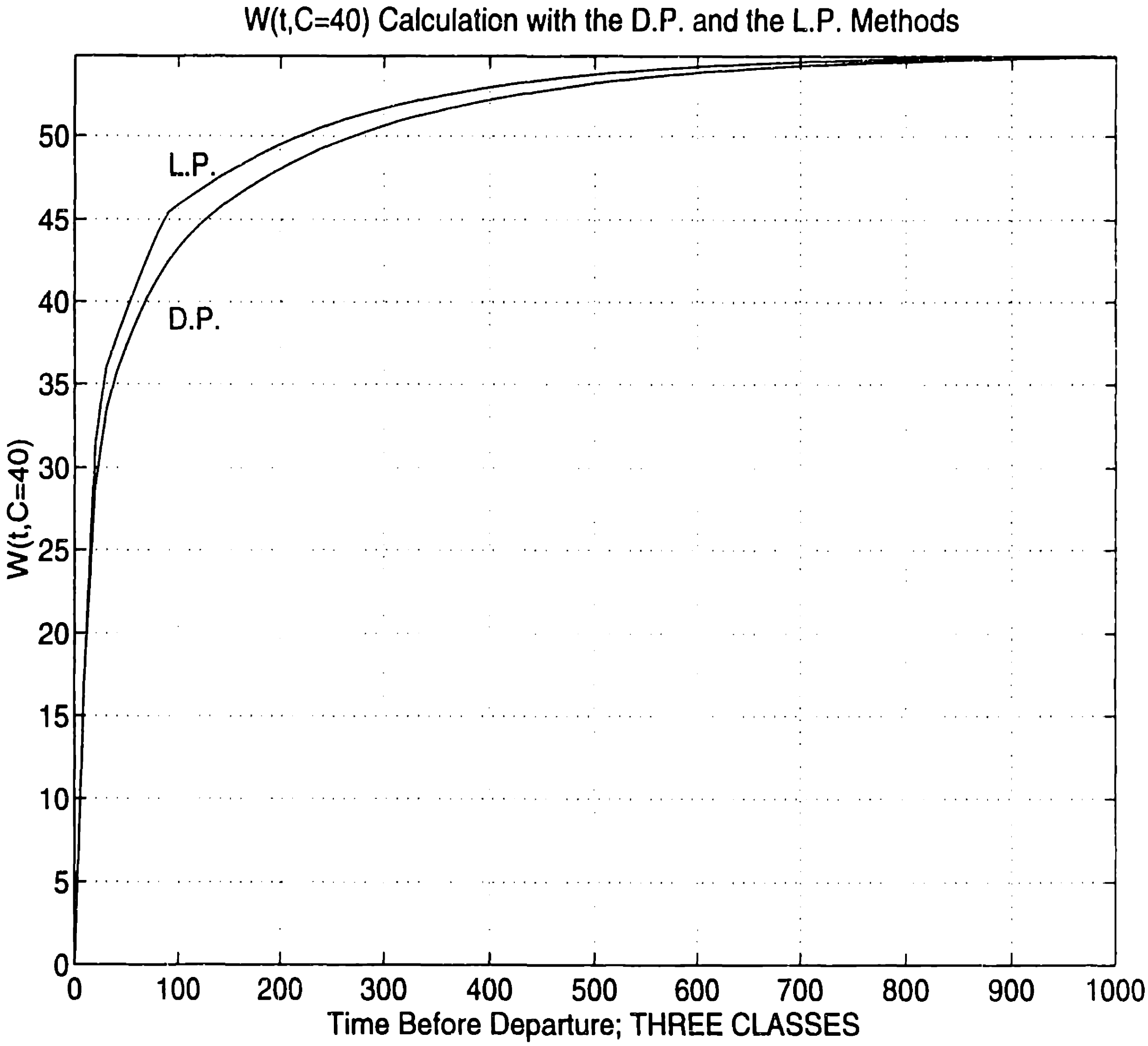


Figure 6-5: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 40, and Time Arrival Rates ($\lambda_1 = \lambda_2 = 0.25$, $\lambda_3 = 0.24$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.005$), Containers ($A_1 = A_2 = 5$, $A_3 = 10$)

Chapter 7

Multi-leg, multi-class OYM problem with cancelations and overbooking

In the previous chapter we discussed the cancelations of container space reservations. In review, we could say that the cost to the vessel operator from bookings cancelations is multi-faceted and it consists of:

- Expenses of the operator in anticipation of the order
- Cost of opportunity
- Lowered level of service

There are several ways that a vessel operator could react and prevent the phenomenon of cancelations. For instance, they can impose a penalty on all cancelations. Such a penalty would deter those customers who are not certain that they can honor their commitment with the vessel operator, from reserving capacity in anticipation of transportation needs that are not certain. Furthermore, such a penalty would help the operator recover some of the costs incured and it would be a compensation for the cost of opportunity of the reserved container slots that might remain empty because of the reservation that was later canceled.

It is rather difficult to justify to customers, and especially to the loyal ones, a policy that would require them to pay the whole or part of the freight rate shortly after the reservations. Such a practice would be necessary in order for the policy of the penalties to succeed. Nevertheless, that policy runs contrary to the current practice of the shipping companies. At the same time this practice would create some operational problems at the transactions between the vessel operator and the shippers. The reasons that have been mentioned prevent many vessel operators from imposing strict policies like the one described above.

The usual way in which the operators combat cancelations is through overbooking. They estimate that several of the customers will cancel their orders and there is no need to stop the booking process when the vessel has reached its capacity. There will always be some customers who will cancel their reservations and eventually the operators will not have lost cost of opportunity because of the cancelations. In the unlikely (and unpleasant) event when shortly before departure there are more containers than the containers that could be accommodated by the operator, some of the containers have to be left out. This is a strain to the relations of the operator and the shipper whose containers have been left out. The operator does not suffer any obvious cost (at least in most of the cases), but there is always a good will cost associated with overbooking. Usually, in cases of overbooking, the vessel operators try to secure capacity to their best customers, those who have the big accounts and those who are consistent users of the services of the company, do not cancel their reservations etc. Although, measures like that minimize the damage to the public relations of the company, there is always a cost attached to every rejection because of overbooking.

In conclusion, overbooking is the most widely used and successful method to combat cancelations of reservations. In the following figure 7-1 we see the increased expected results when we practice overbooking, over the expected revenues of an operator who does not practice overbooking techniques.

7.1 The DP model and the boundary conditions

The main difference between the dynamic programming model of the yield management model with cancelations that was described at the previous chapter 6, and the current yield management dynamic programming model with cancelations and overbooking, is the fact that the remaining capacity at the later model can take negative values. After all, negative remaining capacity or reservations in excess of the capacity of the vessel is the main aspect of an overbooking overbooking policy. A direct result of the overbooking and the negative capacity is the fact that the boundary conditions are different. In the cancelations model, when the remaining time departure becomes zero, the future expected revenues from the trip is zero. At the model with overbooking, at the time of departure, the expected revenue is not necessarily equal to zero. If the number of the reservations is greater than the number of the container slots of the vessel, the vessel operator has to refund the freight rate they collected from the rejected shippers. For every container they reject they incur a cost either directly at the form of compensation for each rejected container or indirectly at the form of an implicit cost, the cost of good will for the violation of the established working relationship between the operator and the shipper.

As in the case of the model with cancelations, we assume that at the beginning (or the end) of each of the time intervals Δt at which we separate the time until the departure of the vessel, we can have:

- a possible arrival of class m container that wants to travel on the OD_i origin destination path with probability $\Delta p_{OD_i,m}(t) = \Delta t \cdot \lambda_{OD_i,m}(t)$,
- a possible cancellation of the booking for a container that belongs to any class and had booked for the itinerary OD_i. Each container has a probability of $\Delta t \cdot \mu_{OD_i,m}(t)$ of canceling its order.
- no arrival or cancellation

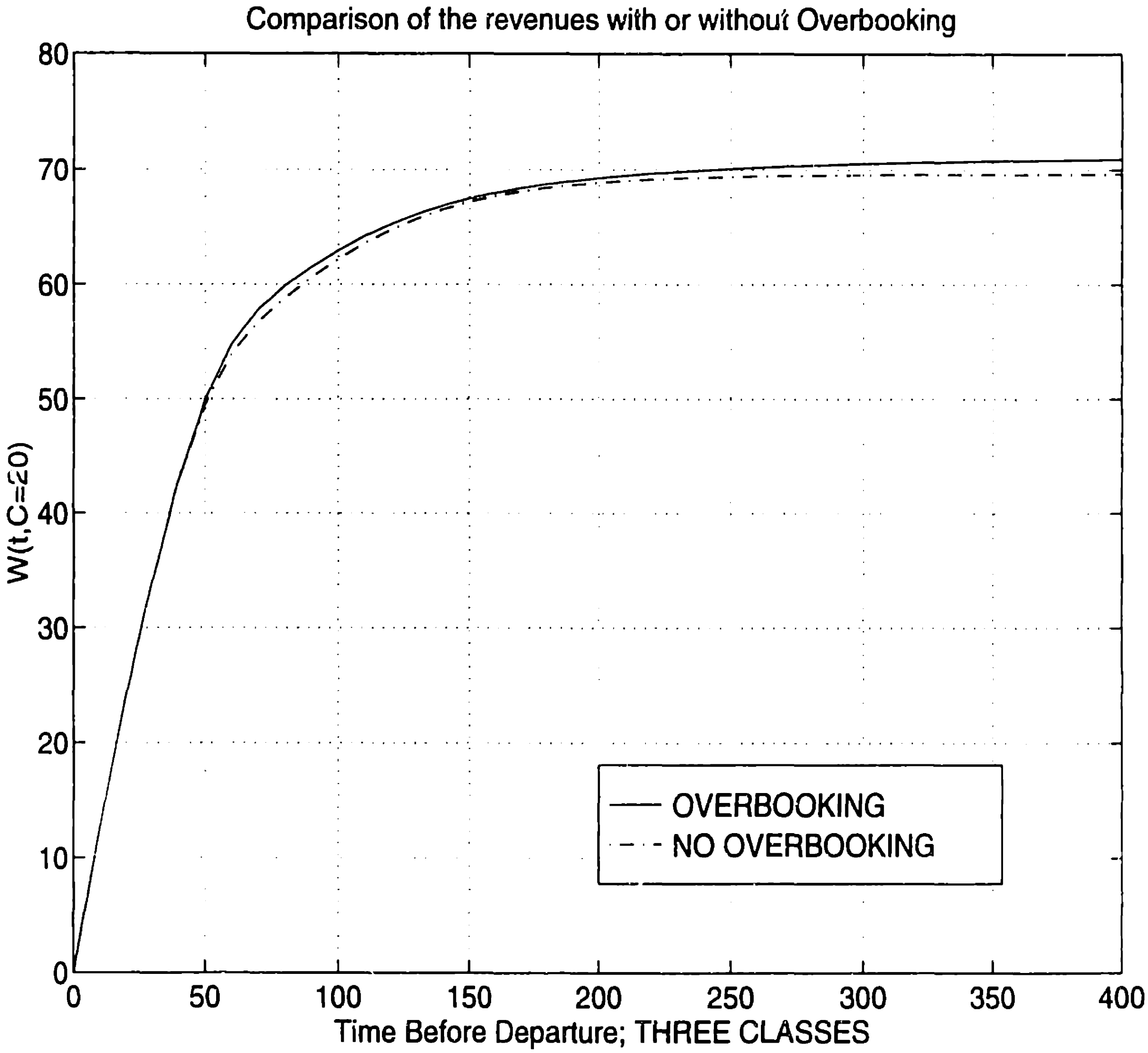


Figure 7-1: Comparison of the maximum expected revenue, with and without the policy of overbooking

- we can have no more than one arrival of one container or one cancellation of one container.

The cost functional is the following:

$$V_{OD_i,m}(t,\mathbf{A},\mathbf{C}) = \max_{I\in\{0,1\}}\left[I\cdot f_{OD_i,m} + W(t,\mathbf{A}_{I_{OD_i,m}},\mathbf{C}_{I_{OD_i,m}})\right] \quad \forall\, m \in \{1,M\} \text{ and } OD_i\text{'s} \tag{7.1}$$

The cancellation functional is the following:

$$F_{OD_i,m}(t,\mathbf{A},\mathbf{C}) = -f_{OD_i,m} + W(t,\mathbf{A}_{OD_i,m},\mathbf{C}_{OD_i,m}) \quad \forall\, m \in \{1,M\} \text{ and } OD_i\text{'s} \tag{7.2}$$

and:

$W(t,\mathbf{A},\mathbf{C})$ is the maximum expected revenue under optimal policy at time t, given that we have no arrivals or cancelations from any container at time t. The vector of the number of the containers that have made reservations for each itinerary and class is $\mathbf{A}$. The vector of container slots available at each of the N legs of the trip is $\mathbf{C}$. Because of the overbooking, the capacity at some legs of the trip can be negative.

$$\begin{aligned} W(t+\Delta t,\mathbf{A},\mathbf{C}) &= \\ &= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t)\cdot V_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\ &+ \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t)\cdot A_{OD_i,g}\cdot F_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \end{aligned}$$

$$+ \left[1 - \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \{\mu_{OD_i,g}(t) \cdot A_{OD_i,g} + \lambda_{OD_i,g}(t)\} \right] \cdot W(t, \mathbf{A}, \mathbf{C}) \tag{7.3}$$

The definition of $V_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$, $F_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$, $\lambda_{OD_i,g}(t)$, $\mu_{OD_i,g}(t)$, $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$ and $f_{OD_i,m}$, has been given in section 6.2. The Boundary Conditions are not the expressed by the following equation 7.4. Instead, equation 7.4 is an approximation, if we consider the number of containers to be continuous rather than integer numbers. At time $t = t_{1N}$, which is the time of departure from Port 1 (which the next port from which the vessel will depart), the boundary conditions are:

$$\begin{aligned} W(t = t_{1N}, \mathbf{A}, \mathbf{C}) = \max \ \mathbf{f}^T \cdot \mathbf{x} &\ - \ (\mathbf{f} + \mathbf{k})^T \cdot \mathbf{u} \\ \mathbf{D}(\mathbf{x} - \mathbf{u}) &\ \leq \ \mathbf{b} \\ \mathbf{x} &\ \leq \ \mathbf{L} \\ \mathbf{u} &\ \leq \ \mathbf{A_e} \\ \mathbf{x} &\ \geq \ \mathbf{0} \\ \mathbf{u} &\ \geq \ \mathbf{0} \end{aligned} \tag{7.4}$$

The cost to the operator for every overbooked container is the cost $k_{OD_i,g}$.

The form of $\mathbf{D}(\mathbf{x} - \mathbf{u}) \leq \mathbf{b}$, is the following:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} (x_{OD_i,g} - u_{OD_i,g}) \ \leq \ C_j, \ \ j = 1, \ldots, N \tag{7.5}$$

The above equation suggests that all the reservations that use the leg j of the itinerary and are in excess of the total available capacity at this leg, are overbookings.
The form of $\mathbf{x} \leq \mathbf{L}$ is:

$$x_{OD_i,g} \;\leq\; L_{OD_i,g}(t_{1N}),\ \forall\ OD_i, g\ \in\ \mathbf{S}$$

The above inequality suggests that the total number of reservations for each ODC combination, cannot be in excess of the demand for this particular ODC combination.
The form of $\mathbf{u} \leq \mathbf{A_e}$ is:

$$u_{OD_i,g} \;\leq\; A_{eOD_i,g}(t_{1N}),\ \forall\ OD_i, g\ \in\ \mathbf{S} \tag{7.6}$$

$\mathbf{A}$ is the vector of the current reservations from all the possible ODC's with the vessel.

$\mathbf{A_e}$ is the vector of the reservations from the vector $\mathbf{A}$ that will have remained (i.e. not have canceled their reservations) at the time of departure.
The above equation suggests that the overbookings cannot be more than the actual bookings.

The form of $A_{eOD_i,g}$, which are the elements of vector $\mathbf{A_e}$ is the following:

$$A_{eOD_i,g}(t) = A_{OD_i,g} \cdot e^{-\int_{\nu=0}^{t_{1N}} \mu_{OD_i,g}(\nu)d\nu} \tag{7.7}$$

7.2 Derivation of the HJB equation

With the exception of the Boundary Conditions, and the fact that we can have negative remaining capacity at one or more legs, (i.e. we can have overbooking) the Dynamic Programming Formulation for overbooking, is identical to the formulation of the model for Cancelations. By repeating the same process as we did in the previous chapter, with the cancelations, we arrive again at lemma 14, which we repeat here for the cancelations and overbooking model.

Lemma 18 *$W(t, \mathbf{A}, \mathbf{C})$ is the maximum expected revenue of the continuous time model under optimal policy . The optimal policy is such that $I_i(t, \mathbf{A}, \mathbf{C}) = 0$, $\forall i \in \mathbf{N}$, and $I_i(t, \mathbf{A}, \mathbf{C}) = 1$, $\forall\ i \in \mathbf{Y} = \bigcup_{j=1}^{N} \mathbf{Y_j}$. We assume that the partial derivatives of $W(t, \mathbf{A}, \mathbf{C})$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t, \mathbf{A}, \mathbf{C})$ is the equation:*

$$\begin{aligned}\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot f_i \\ &+ \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_i} - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_i} \\ &- \sum_{j=1}^{N} \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) - \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \qquad (7.8)\end{aligned}$$

The approximate boundary conditions of $W(t, \mathbf{A}, \mathbf{C})$ are described by the linear programming model 7.4

7.3 Solution of the HJB equation

In the previous section we gave the above lemma 18, which is the form of the dynamic programming model, when we make the approximation that the number of containers is a continuous number instead of an integer. The differential equation that governs the reservations with cancelations and overbooking (the HJB equation) is equation 7.8 of lemma 18. The approximate boundary conditions of the HJB equation are given by the linear programming model 7.4.

We will prove the solution of the differential equation (HJB equation) which is derived by the dynamic programming formulation 7.3, reduces to the solution of a linear programming formulation. The Linear Programming formulation that follows, is a generalization of the formulation of equation 6.9

$$
\begin{aligned}
\mathbf{z}(t,\mathbf{A},\mathbf{C}) = \max \;\; \mathbf{f}^T\cdot(\mathbf{x}-\mathbf{A_c}) &\;-\; (\mathbf{f}+\mathbf{k})^T\cdot\mathbf{u} \\
\mathbf{B}(\mathbf{x}-\mathbf{u}) &\;\leq\; \mathbf{b} \\
\mathbf{x} &\;\leq\; \mathbf{L} \\
\mathbf{u} &\;\leq\; \mathbf{A_e} \\
\mathbf{x} &\;\geq\; \mathbf{0} \\
\mathbf{u} &\;\geq\; \mathbf{0}
\end{aligned}
\tag{7.9}
$$

$\mathbf{A_c}$, is the vector of the expected cancelations from the containers that have made their reservations by the current time t.

$$
A_{\mathbf{c}OD_i,g} = A_{OD_i,g}\cdot\left(1-e^{-\int_{\nu=0}^{t}\mu_{OD_i,g}(\nu)d\nu}\right) \tag{7.10}
$$

The form of $\mathbf{u}\leq\mathbf{A_e}$ is:

$$
u_{OD_i,g} \;\;\leq\;\; A_{\mathbf{e}OD_i,g}(t),\;\; \forall\; OD_i,g\;\in\;\mathbf{S} \tag{7.11}
$$

$\mathbf{A}$ is the vector of the current reservations from all the possible ODC's with the vessel. $\mathbf{A_e}$ is the vector of the reservations from the vector $\mathbf{A}$ that will have remained (i.e. not have canceled their reservations) at the time of departure.
The above equation suggests that the overbookings cannot be more than the actual bookings.

The form of $A_{\mathbf{e}OD_i,g}$, the elements of vector $\mathbf{A_e}$ is the following:

$$
A_{\mathbf{e}OD_i,g}(t) = A_{OD_i,g}\cdot e^{-\int_{\nu=0}^{T_{OD_i}}\mu_{OD_i,g}(\nu)d\nu} \tag{7.12}
$$

where T_{OD_i} is the remaining time before departure from the port of origin of the itinerary

OD_i. The form of $\mathbf{B}(\mathbf{x}-\mathbf{u}) \leq \mathbf{b}$ is the following:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} (x_{OD_i,g} - u_{OD_i,g}) \leq C_j + \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} A_{\mathbf{c}OD_i,g}, \quad j = 1, \ldots, N \tag{7.13}$$

A more detailed form of the constraints $\mathbf{x} \leq \mathbf{L}$ is

$$x_{OD_i,g} \leq L_{OD_i,g}(t), \ \forall \ OD_i, g \in \mathbf{S} \tag{7.14}$$

$L_{OD_i,g}(t)$ is the expected demand from containers that belong to the ODC OD_i, g between the time t, and the departure of the vessel from the port of origin of the itinerary OD_i. The expected demand $L_{OD_i,g}(t)$ is the expected demand of the OD_i, g containers that will book capacity with the vessel and will not cancel their bookings. where:

$$L_{OD_i,g}(t) = e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu} \cdot \left[\int_{\xi=0}^{t} \lambda(\xi) e^{\int_{\nu=0}^{\xi} \mu_{OD_i,g}(\nu)d\nu} d\xi\right], \ \forall \ OD_i, g \in \mathbf{S} \tag{7.15}$$

The form of $A_{\mathbf{c}OD_i,g}$ is given in equation 7.12. We remind here that $\mathbf{A_c}$ is the vector of the expected cancelations from the different ODC Combinations, between the time t and the departure of the vessel from the port of Origin of this particular ODC. The Constraints say that any Origin-Destination and Class Combination (ODC) can be accepted up to the level of its expected demand (with the expected cancelations taken into consideration), and all the ODC's that use a particular leg have to satisfy the Capacity Constraint of that leg.

The boundary conditions of the LP for $t = 0$ are given by the equation of the boundary conditions for $W(t, \mathrm{A}, \mathrm{C})$, i.e. they are given by the LP 7.4.

Figure 7-2 presents an alternative form of the linear programming model that we use to describe the N legs, M classes of containers yield management problem with reservations cancelations.

Linear programming for the multi port
multi class yield management problem
with cancelations, overbookings

OBJECTIVE : max reservations revenues − cancelations refunds − overbooking refunds and penalties

CONSTRAINTS :

containers to use leg j − overbooking at leg j =< remaining capacity at leg j + expected cancelations at leg j

number of accepted containers from each ODC combination =< expected demand from this ODC combination

overbookings =< non cancelled reservations

Figure 7-2: LP model for the multi-leg, multi class OYM problem, with cancelations and overbooking

We solve the Linear Programming Problem and we find a solution $x^*_{OD_i,g}$, $\forall$ OD_i's and $\forall$ Class g. Let us assume that ODC OD_i, g uses the leg j for transportation. If $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) > 0$, then $OD_i, g \in \mathbf{Y_j}$. If $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) = 0$, then $OD_i, g \in \mathbf{N_j}$.

Lemma 19 gives the differential equation that describes the maximum value of the objective function of the Linear Program 7.9.

Lemma 19 *$\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ is the maximum value, and $x^*_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$, $\forall$ OD_i's and $\forall$ g, is the corresponding optimal solution of the Linear Program 7.9. The governing differential equation for $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ is the equation*

$$\begin{aligned}
\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot f_i \\
&+ \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} \\
&- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) - \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \qquad (7.16)
\end{aligned}$$

The boundary conditions of the LP are given by 7.4

The proof of the above lemma 19, is given in the appendix E.

In the following, we will give Theorem 20, which is similar to Theorems 8, 12 and 16. Theorem 20 is a generalization of Theorems 8, 12, and 16, for rates of arrival that are time variable (i.e. $\lambda_i(t)$), and for a (time variable) cancellation rate, and overbookings.

In the following Theorem 20 we use the results from lemmas 18 and 19 to suggest that the solutions of the dynamic programming and the linear programming are equal when and only when the two following happen: 1) When the linearized dynamic programming suggests the rejection of a class of customers, the linear programming does not offer any capacity to this same class of customers. 2) When the linearized dynamic programming suggests the acceptance of a class of customers, the linear programming offers some capacity to this same class of customers.

Theorem 20, is similar to Theorems 8, 12 and 16. Theorem 20 is a generalization of Theorem 16, that includes overbooking of the vessel capacity.

Theorem 20 *$W(t,\mathbf{A},\mathbf{C})$ is the solution of the linearized version of the Dynamic Program given by equation 7.8, when the boundary conditions are given by equation 7.4. $I_{OD_i,g}(t,\mathbf{A},\mathbf{C})$ is the control variable for the class g containers that travel on the OD_i path, given by the Dynamic Programming under optimal policy. The acceptance ($I_{OD_i,g}(t,\mathbf{A},\mathbf{C})=1$) or the rejection ($I_{OD_i,g}(t,\mathbf{A},\mathbf{C})=0$) of a container that belongs to the OD_i,g ODC combination is as a function of the remaining time t, the reservations vector $\mathbf{A}$, and the remaining capacity vector $\mathbf{C}$. $\mathbf{z}(t,\mathbf{A},\mathbf{C})$ is the maximum value, and $x^*_{OD_i,g}(t,\mathbf{A},\mathbf{C})$, $\forall\ OD_i$'s and $\forall\ g$, is the corresponding optimal solution of the Linear Program 7.16. We prove the following:*

$$\{W(t,\mathbf{A},\mathbf{C})=\mathbf{z}(t,\mathbf{A},\mathbf{C})\} \iff \left\{ \begin{array}{l} I_{OD_i,g}(t,\mathbf{A},\mathbf{C})=0, \quad x^*_{OD_i,g}=0, \quad \forall\ OD_i,g\in\mathbf{N} \\ \text{and} \\ I_{OD_i,g}(t,\mathbf{A},\mathbf{C})=1, \quad x^*_{OD_i,g}>0, \quad \forall\ OD_i,g\in\mathbf{Y} \end{array} \right\} \tag{7.17}$$

Proof: The proof of Theorem 20, is identical to the proof of theorem 16. For the proof of theorem 16, we used lemmas 14, and 15. For the proof of theorem 20, we use lemmas 18, and 19.

An alternative, and more useful expression of the above Theorem 20, is the following. Corollary 21 says that the optimal policy suggested by the linear programming is optimal policy for the linearized dynamic programming and the maximum expected revenue of the linearized dynamic programming is equal to the maximum value of the objective function of the linear program 7.9.

Corollary 21 (Equivalence Theorem) *Let $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ be the maximum value of the objective function and let $x^*_{OD_{i},g}(t, \mathbf{A}, \mathbf{C})$, $m \in \mathbf{S}$, be the optimal solution of the LP 7.16. Let also $W(t, \mathbf{A}, \mathbf{C})$ be the linearized maximum expected revenue from the differential equation 7.8, and $I_{OD_{i},g}(t, \mathbf{A}, \mathbf{C})$, $m \in \mathbf{S}$, is the optimal policy suggested for the linearized DP. The following holds:*

$$\left\{ \begin{array}{l} x^*_{OD_{i},g} = 0, \quad \forall \ OD_i, g \in \mathbf{N} \\ x^*_{OD_{i},g} > 0, \quad \forall \ OD_i, g \in \mathbf{Y} \end{array} \right\} \Longleftrightarrow \tag{7.18}$$

$$\left\{ W(t, \mathbf{A}, \mathbf{C}) = \mathbf{z}(t, \mathbf{A}, \mathbf{C}) \quad and \quad \begin{array}{l} I_{OD_{i},g}(t, \mathbf{C}) = 0, \quad \forall \ OD_i, g \in \mathbf{N} \\ I_{OD_{i},g}(t, \mathbf{C}) = 1, \quad \forall \ OD_i, g \in \mathbf{Y} \end{array} \right\}$$

The *Equivalence Theorem* shows that, the optimal policy of the DP with the same input as the linear programming model 7.9, is such that $I_{OD_{i},g} = 0$, $\forall \ x_{OD_{i},g} = 0$ and $I_{OD_{i},g} = 1$, $\forall \ x_{OD_{i},g} > 0$. Additionally, the maximum expected revenue of the DP model is $W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C})$. The input of the linear programming model, is just a condensed form of the input for the dynamic programming model.

With the help of the *Equivalence Theorem*, we can combine the best features of dynamic programming and linear programming. Dynamic programming offers modeling accuracy and reliable solutions. The solution of an LP is in general easy, whereas, the solution of a DP can be tedious. We bypass this disadvantage of dynamic programming with the introduction of the *Equivalence Theorem*. *Equivalence Theorem*, suggests that when given the same input, the linear program 7.9 gives the same output as the dynamic programming model given by the formula 7.3. The solution of the LP, is much easier than the solution of the DP.

Figures At the following figures 7-3, through figure 7-6, we present a two ports model with tree classes of containers. Both the rates of arrivals of the containers, as well as the rates of cancellation for all three classes of containers are constant.

Figure 7-3, shows that for negative values of the vessel capacity (i.e. for overbooking), the fit between the dynamic and the linear programming results is adequate even for relatively small absolute values of the remaining (negative) capacity.

very small values of the vessel capacity ($C = 4$) and for high cancellation rates, the fit between the DP and the LP models is, as expected, poor.

In the following figures 7-4 and 7-5, we see that for greater vessel capacities ($C = 10$) and ($C = 40$) respectively, and with relatively high cancellation rates, the fit between the DP and the LP models very good. In both figures, the cancellation rate is so high that we cannot have enough containers to approach the limit of the vessel capacity. At figure 7-6, where the cancellation rates (or outflow) are substantially smaller than the arrival rates (inflow), the curve from the LP predicts reasonably well the curve derived from the dynamic programming.

For strong demand (upper right corner) and weak demand (lower left corner), the results from both the LP and the DP are identical. The discrepancy is greater at the intermediate cases of not very strong demand.As expected, the discrepancy between the two methods decreases as the remaining capacity increases.

The solution given by the linear programming, gives consistently higher or equal values than the dynamic programming model. That happens because the LP model is a relaxed form of the DP.

7.4 Summary

The model presented at this chapter combines the description of the phenomenon of cancelations with the practice of overbooking. It is the same dynamic programming model as the DP model of the previous chapter that described the booking process with cancelations without overbooking. The only change that we have in this model is the different boundary conditions. This modifications allow the remaining capacity to become negative, and the payoff to become negative also when at the time of departure there is a capacity shortage. In the first part of the chapter we explain how the overbooking of the capacity contributes to the increased profitability of the shipping company. We also give a numerical example in the form of a graph that supports the above argument.

We follow the methodology that we followed at the previous chapters. We prove again that the solution to the linearized version of the linearized DP is the solution of a linear programming model. Both the arrival rates of containers and the cancellation rates of orders are variable in the general case.

If to the results of the present chapter we add the restriction that the remaining capacity cannot become negative, they are reduced to the results of the previous chapter where we do not consider overbooking.

The graphs that we present verify the theory that we developed in this chapter. They show the agreement between the DP model and the linear programming approximation to this dynamic programming model.

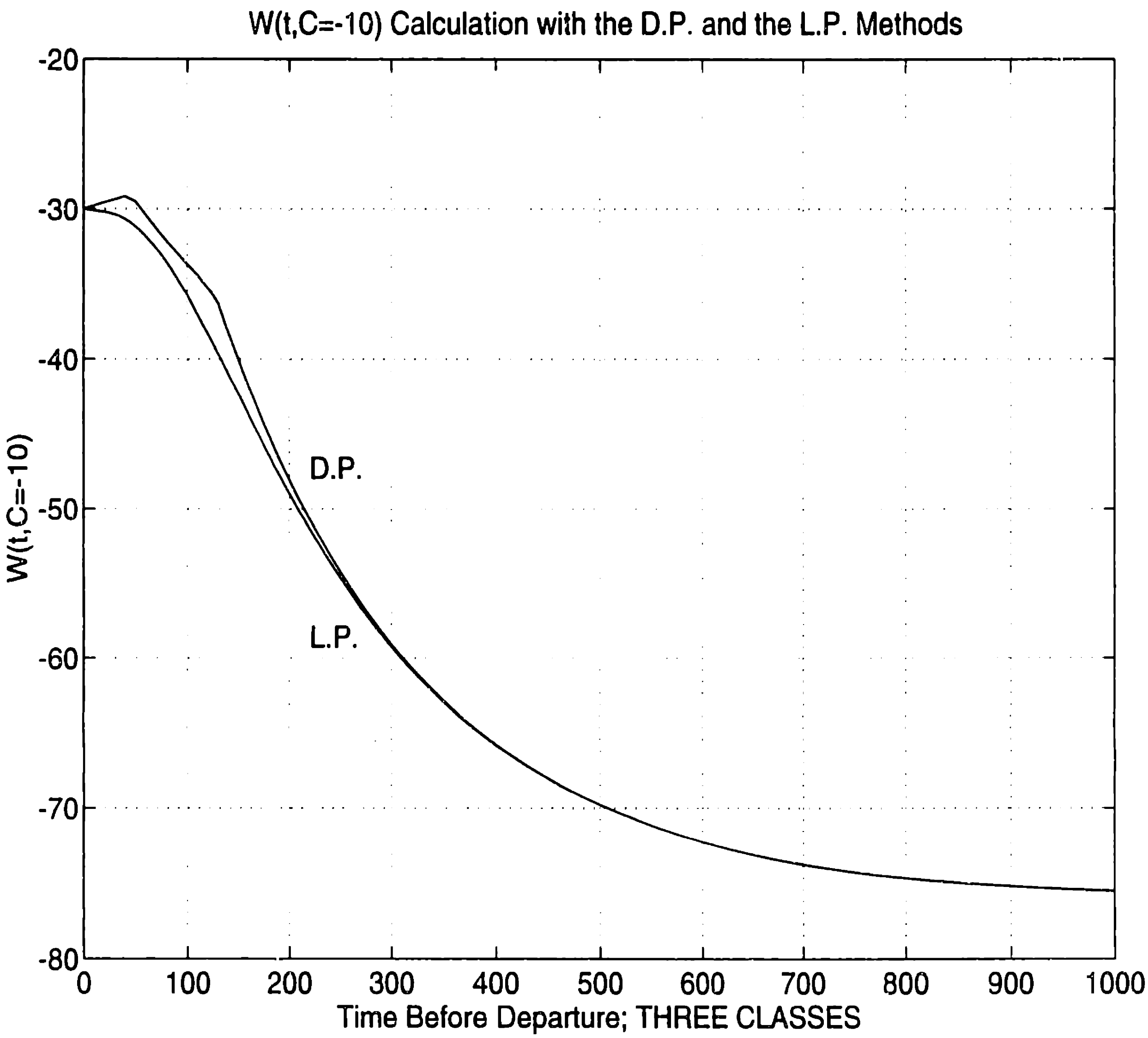


Figure 7-3: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = -10, and Time Arrival Rates ($\lambda_1 = \lambda_2 = \lambda_3 = 0.01$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.01$), Containers ($A_1 = A_2 = 5$, $A_3 = 10$)

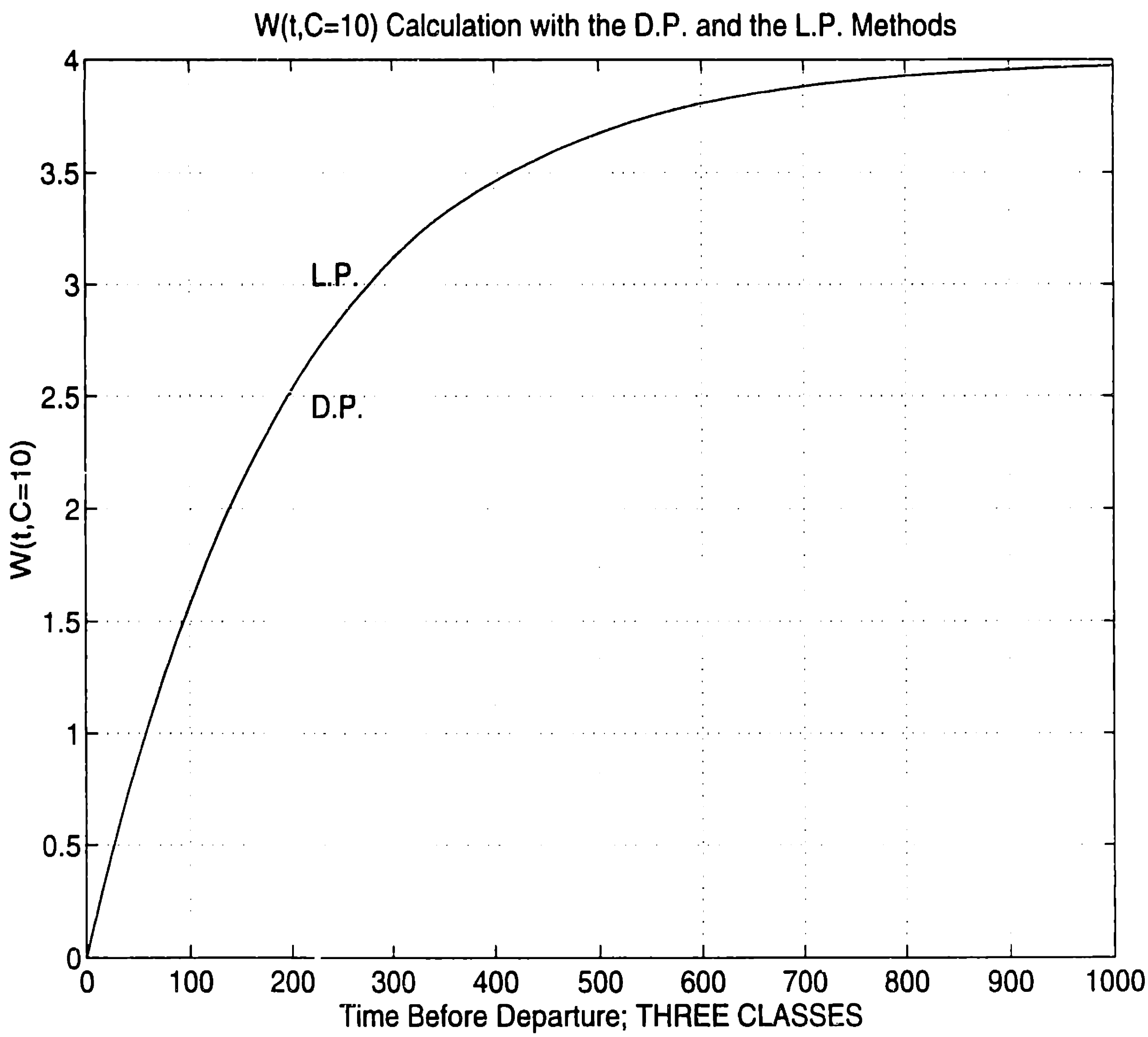


Figure 7-4: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 10, and Time Arrival Rates ($\lambda_1 = 0.01, \lambda_2 = 0.015, \lambda_3 = 0.02$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.005$), Containers ($A_1 = A_2 = 5,\ A_3 = 0$)

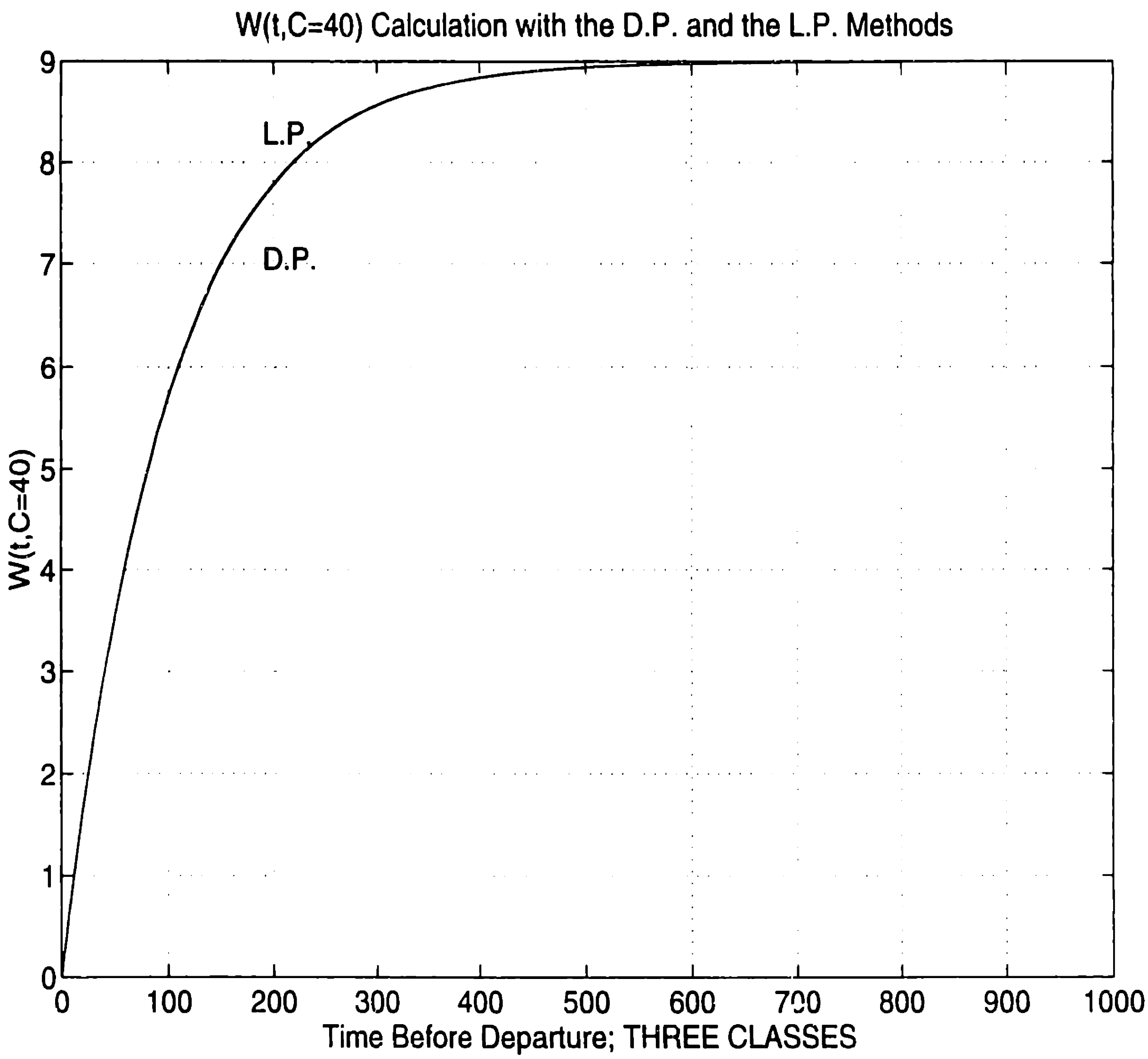


Figure 7-5: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 40, and Time Arrival Rates ($\lambda_1 = \lambda_2 = \lambda_3 = 0.01$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.01$), Containers ($A_1 = A_2 = A_3 = 0$)

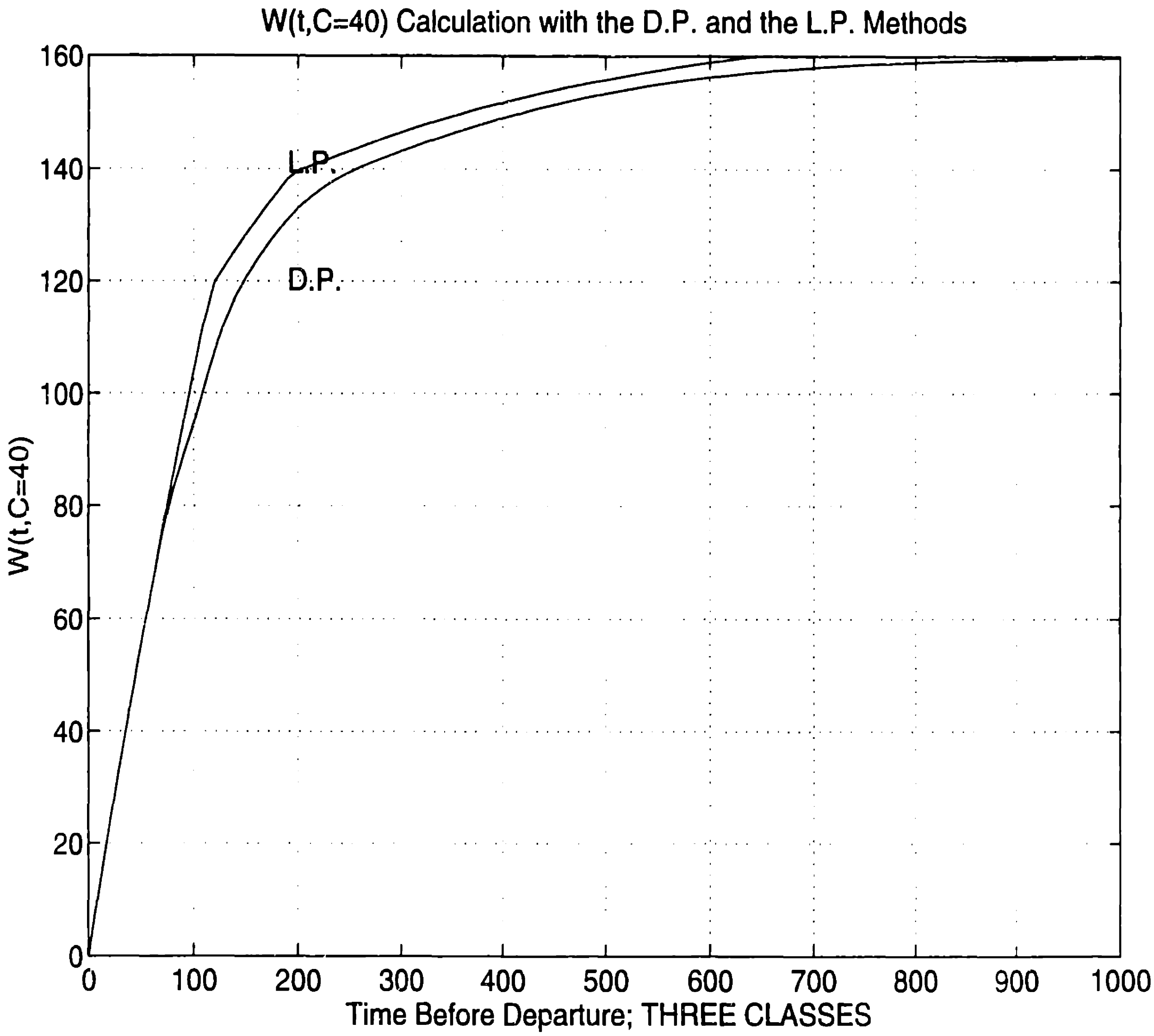


Figure 7-6: Comparison between the Linear and the Dynamic Programming Method for the Two Ports Problem with CAPACITY = 40, and Time Arrival Rates ($\lambda_1 = \lambda_2 = 0.135$, $\lambda_3 = 0.125$). Cancelation Rates ($\mu_1 = \mu_2 = \mu_3 = 0.0025$), Containers ($A_1 = A_2 = A_3 = 0$)

Chapter 8

Heuristic Linear Programming Model

8.1 Heuristic criterion for reservations

In the review of the yield management problem literature, we have introduced the assumptions of the nested model for the two port yield management problem. We recal that the main assumption of the nested model is the ordered arrival of the different classes of customers. In other words, the low revenue customers ask for transportation capacity before the high revenue customers. In the following figure 8-1, which is taken from Maragos ([23]), we can see the marginal unit revenue of the vessel. The marginal unit revenue is the extra revenue the operator makes if one extra unit of capacity is added to the current capacity of the vessel. It is a function of the current capacity of the vessel. Alternatively, the marginal revenue can be seen as the slope of the expected total revenue.

If the remaining capacity of the vessel is less than $L_3 - L_2$, the marginal expected revenue $MR(C)$, is greater than the freight rate f_2 of the second most expensive class and of course greater than the the freight rate f_1 of the lowest freight rate class. As a result, when the remaining capacity C is such that $0 < C < L_3 - L_2$, we only accept the highest freight rate class f_3. Accordingly, $L_3 - L_2$ is the protection level for class 3, at the current time t.

When the remaining capacity is such that $L_3 - L_2 < C < L_3 - L_1$, the marginal expected revenue $MR(C)$ is such that $f_3 < f_2 < MR(C) < f_1$. As a result, it is optimal for the operator to accept only class 3 and class 2 containers, and reject any arriving customers that belong to class 1. Therefore, $L_3 - L_1$, is the protection level for classes 3 and 2.

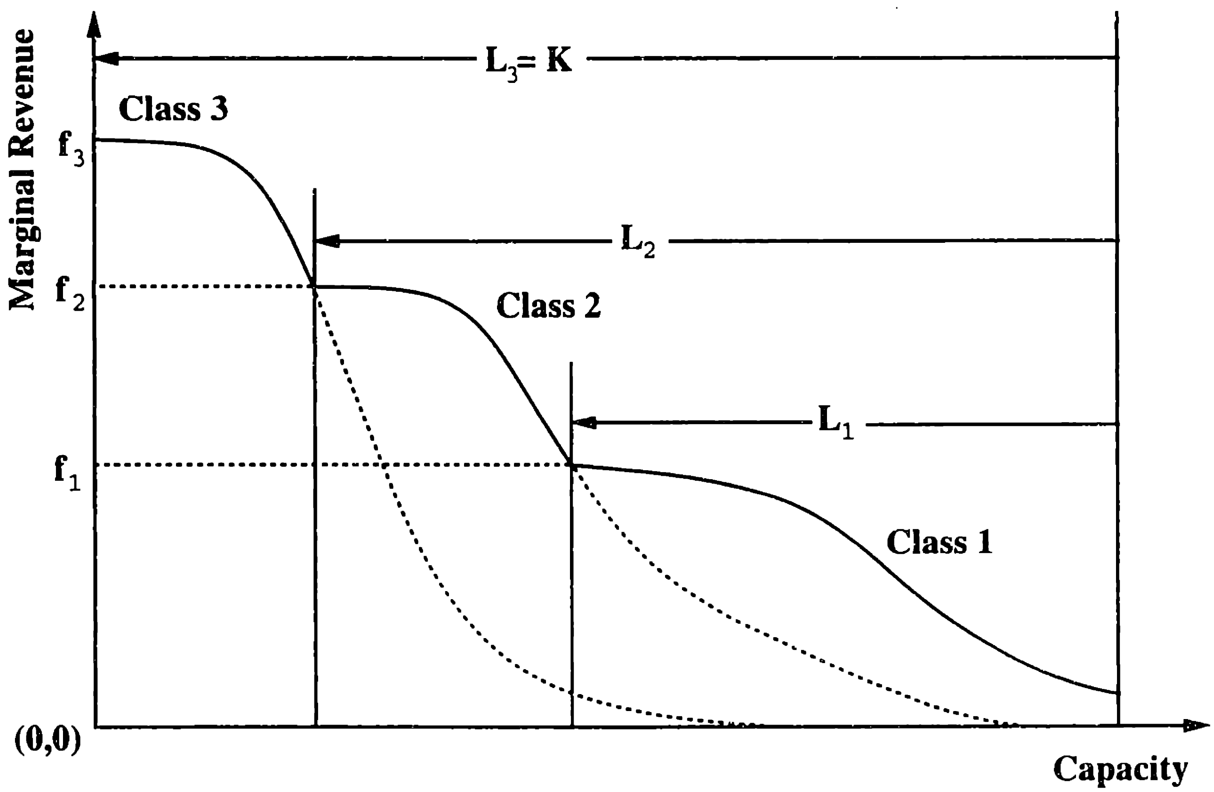


Figure 8-1: Marginal revenue and optimal capacity allocation

If we plot the marginal revenue vs. remaining capacity for the Dynamic Programming two ports model with concurrent arrivals from customers of all classes that we presented in Chapter 4, we get a curve like the smooth curve of figure 8-2. The smooth curve of figure 8-2 is the marginal expected revenue for the two ports problem, with 3 classes of containers, when the remaining time before departure is equal to $t = 100$. The freight rates for the three classes of containers are $f_1 = 2$, $f_2 = 3$, $f_3 = 4$. For the problem presented in figure 8-2, we accept containers from class 1 as long as the marginal revenue of the remaining capacity is greater than or equal to f_1. That means that at time $t = 100$, we would accept class 1 containers only if the capacity is greater than $C = 73$.

We would accept a class 2 container, if it is more profitable to give one container slot to a class 2 container rather than reserve it for class 3 containers. In this example, we would give capacity to the class 2 container, if the remaining capacity is greater than $C = 33$. Therefore, in this particular example, when the remaining time is $t = 100$, the protection level for class 3 is $C = 33$, and the joint protection level for classes 3 and 2 is $C = 73$.

If we compare the marginal revenue curves from figure 8-1 and figure 8-2, we see that the nested model gives a marginal revenue curve that is not differentiable (i.e. it is not smooth) at the points where the remaining capacity is equal to the protection level for some classes of customers. The lack of "smoothness" is due to the assumption of the ordered arrival of customers. In figure 8-2, where the arrival of all classes of customers is allowed at all times, the transition is smoother, and the curve differentiable.

The steplike function in the same graph 8-2, is the marginal expected revenue function as it is given by the linear program whose objective function is the solution of the HJB differential equation of the linearized expected revenue $W(t, C)$.

It is not a surprise that the marginal revenue, as predicted by the LP, is steplike. It could be "smoother" only if it had some of its higher order derivatives with reference to the capacity C different than zero.

The linear approximation of the marginal revenue curve suggests that the cutoff for any class of containers is the expected value of the number of containers from the higher classes of containers.

The linear approximation of the marginal revenue function is reasonably good. Actually, it is even better than it looks, if we realize that the practical role of the marginal revenue function is to compare the revenue of a particular container that needs transportation capacity against the cost of opportunity for this last unit of capacity, if we apply optimal policy from that point on. For all values of capacity the linear marginal revenue function is a good approximate criterion for the application of optimal policy, except for the areas where the capacity is in the vicinity and larger than the cutoff capacity levels suggested by the linear theory. In figure 8-2, the linear marginal revenue curve is not an optimal criterion, when the remaining capacity is $20 < C < 30$, $50 < C < 60$ and $90 < C < 100$.

The expected demand from the containers with the highest freight rate, at time $t = 100$, is equal to 20. If we examine the marginal revenue curve given by the D.P. at the area around the remaining capacity $C = 20$, we see that the marginal revenue curve, as it is given by the D.P. formulation, has a shape similar to the shape of the cumulative probability distribution function for the highest freight rate containers demand. To the right of the expected value, the marginal revenue curve is concave. For larger values it becomes flat, with values virtually equal to the price $f_2 = 3$. In this area it is almost indifferent whether we accept or reject a class 2 customer.

In the area around the expected value of the combined demands from classes 2 and 3, the curve becomes again sigmoid and so on. If, instead of having the combined expected demand of the higher classes as a cutoff value for the lower classes of containers, we had as the cutoff point a point in the flat area between the two expected values, the cutoff criterion would improve. The new (heuristic) cutoff criterion and the related marginal revenue curve (which are shown in figure 8-3), would probably lead to non optimal decisions in parts of the "flat" area. That does not make much difference though, because both the expected revenue from the last unit of capacity and the revenue from the container are practically the same.

The heuristic method that we just proposed does not really need extended "flat" areas in order to be successful. It can also be successful when the "flat" areas are shrunk, or non existent (see figure 8-4).

At the multi-leg problem, we find the cutoff points for the heuristic method, by conducting sensitivity analysis of the L.P. model. The cutoff points for the multi-leg version of the original linear programming model are the points where we have a change of basis (in simplex method terms) among the variables of the problem. The variables are the x_j^*'s which, when they are $x_j^* > 0$, from the theory that we developed, we get that $I_j^* = 1$, and when $x_j^* = 0$, we get $I_j^* = 0$.

The success of the heuristic method, both in absolute terms and relatively to the original linear marginal revenue curve, is shown at figure 8-5. Figure 8-5, shows four curves. The first is the maximum expected revenue, as it is predicted by the linear program. The second curve under it is the curve of the maximum expected revenue, calculated by the dynamic programming model. These two curves where derived from the two respective models. The two remaining curves are estimations of the maximum expected revenue curve with simulation. The first is a simulation of the attainable maximum expected revenue if we use the heuristic method. The lower curve is the curve from the simulation with the original linear program that was developed in the theory.

Because it is difficult to distinguish the curve from the calculation with dynamic programming and the curve from the simulation with the heuristic criterion we give figure 8-6, in which we have enlarged some of the areas of the figure 8-5. In figure 8-7 we plot the relative errors of the expected revenues from the two simulations relatively to the expected revenue calculated with the dynamic programming model. The relative errors are given as functions of the remaining time before departure.

In figure 8-8, we give again the same four curves for the model with the variable arrival rates that we presented at a previous chapter in figure 5-4. In figure 8-9, we give the blowup of some areas of the previous figure 8-8. Figure 8-10 gives the relative errors from the operational results of this model.

The following three figures 8-11 8-12 and 8-13, refer to a two ports, three classes of containers model with booking cancelations. The figures 8-14 8-15 and 8-16, refer to a two ports, three classes of containers model with cancelations and overbooking.

There are some groups of arrival rates for which the heuristic criterion does not work as well as the the criterion of the original linear model. One such group involves the ordered arrival rates of the nested models (see 8-17). This is due to the nature of the assumptions of the nested model, which are stringent. It is rather unlikely that in practice there is a sharp separation between the arrivals of the different classes of customers. Only if someone believes firmly in the ordering on the arrivals of the different classes, and that there is no overlap, they might want to formulate their problem as a nested model.

Nevertheless, simulations show that the error of the expected revenues from simulations using the original linear program as booking criterion, relatively to the dynamic programming, are of the order of 0.5%.

In most other circumstances, where there might exist some overlap between the arrival of the different classes, the transition from the arrival of the one class to the next, is smoother. Then, the marginal revenue curve is smoother too, and the criterion given by the modified linear programming is the appropriate criterion. Some other times, experience might show that the answer lies in the use of another modified criterion that is a hybrid of the original and the heuristic linear programming model.

Comparison of the DP and the LP formulations In the previous section, we showed that the differences between the output of the dynamic programming model, and the output of the appropriate version of the linear programming model, are not significant. If we consider that the linear programming method is much faster to run on computers, we can easily see that it is superior to the dynamic programming method. Furthermore, it is almost impossible to run the dynamic programming model for a real life application multi-port, multi-commodity problem. Our only alternative is the linear programming method.

At the same time, the input of the linear programming is much simpler and condensed than the input of the equivalent dynamic programming model. The input of the dynamic programming model includes the arrival rates at all times for all the classes of containers. The equivalent linear program needs only the expected demand for the different classes of containers. It is much easier, to find the expected demand for a variable, rather than find the whole probability distribution of the same variable. The calculation of the distribution involves both assumptions that could be erroneous and detailed estimations from scarce data. (It is difficult to estimate the arrival rate for a particular class of containers, when the time is, for instance, five days before departure). On the contrary, it is easier to calculate the expected demand of a class of containers, from aggregate data and without having to introduce assumptions about the shape of the distribution.

8.2 Summary

In the previous chapters we examined the agreement of the dynamic programming model and the linear programming model that approximates it, by examining the discrepancy between the expected revenues curves as they are given by the two methods. In addition to giving revenue potential, both models suggest an the optimal policy. In this chapter, we examine the agreement of the optimal policies as they are suggested by the two different models. We conclude that the agreement between the two different methods is satisfactory. Simulations show that the operational results when we employ the decision criterion given by the linear programming model are lower by at most 1.8% than when we employ the dynamic programming model. We observe that we can improve the linear programming criterion for optimal policy with the introduction of a heuristic decision criterion given by a modification of the linear programming formulation. The simulations that we run show that the relative errors of the simulation with the heuristic criterion become no greater than 0.2%.

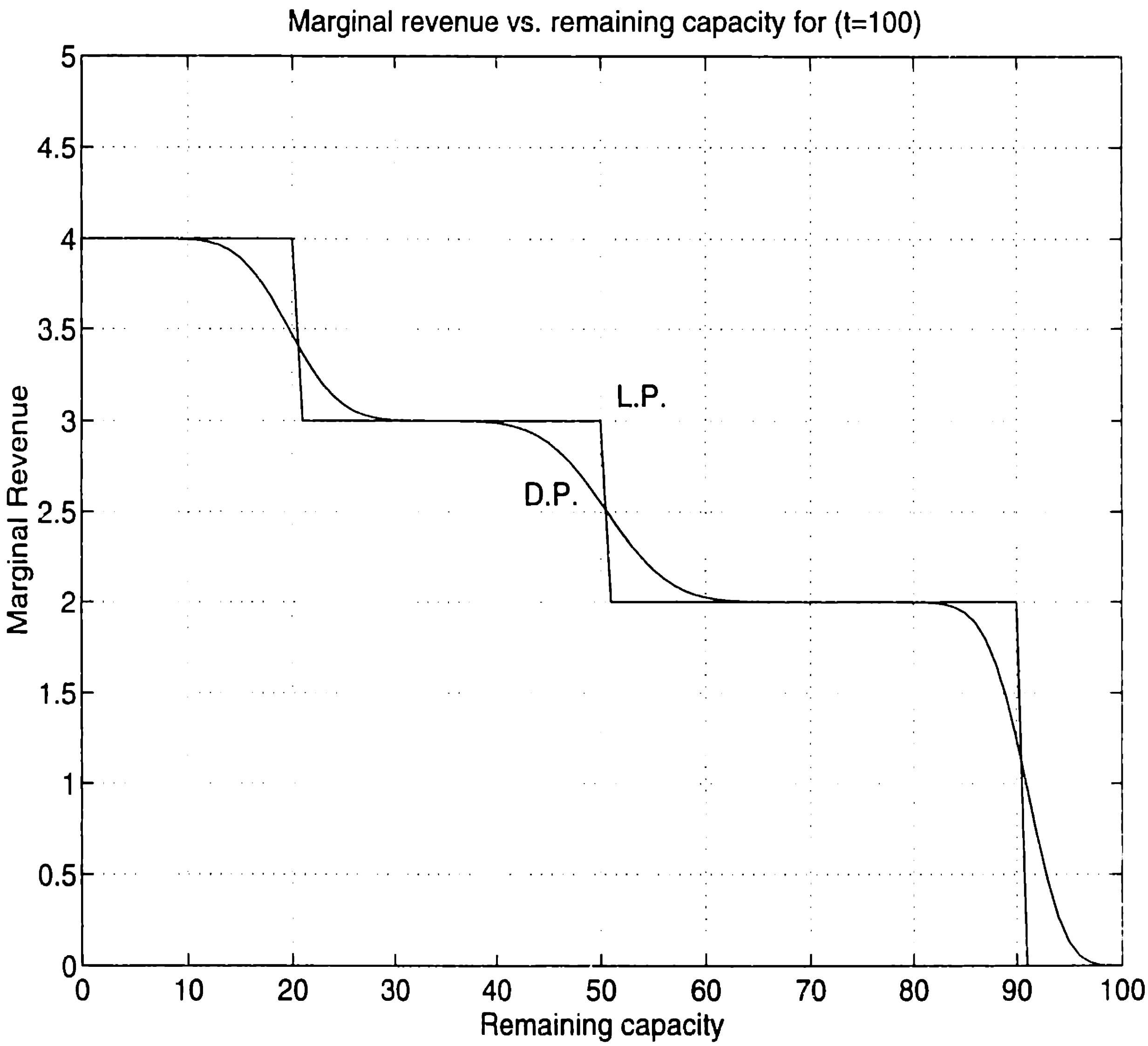


Figure 8-2: The marginal revenue ($\frac{\partial W(t,C)}{\partial C}$), given by the DP and the LP methods, for the Two Ports Problem with Constant Arrival Rates ($\lambda(t) = \lambda$), as in figure 4-5

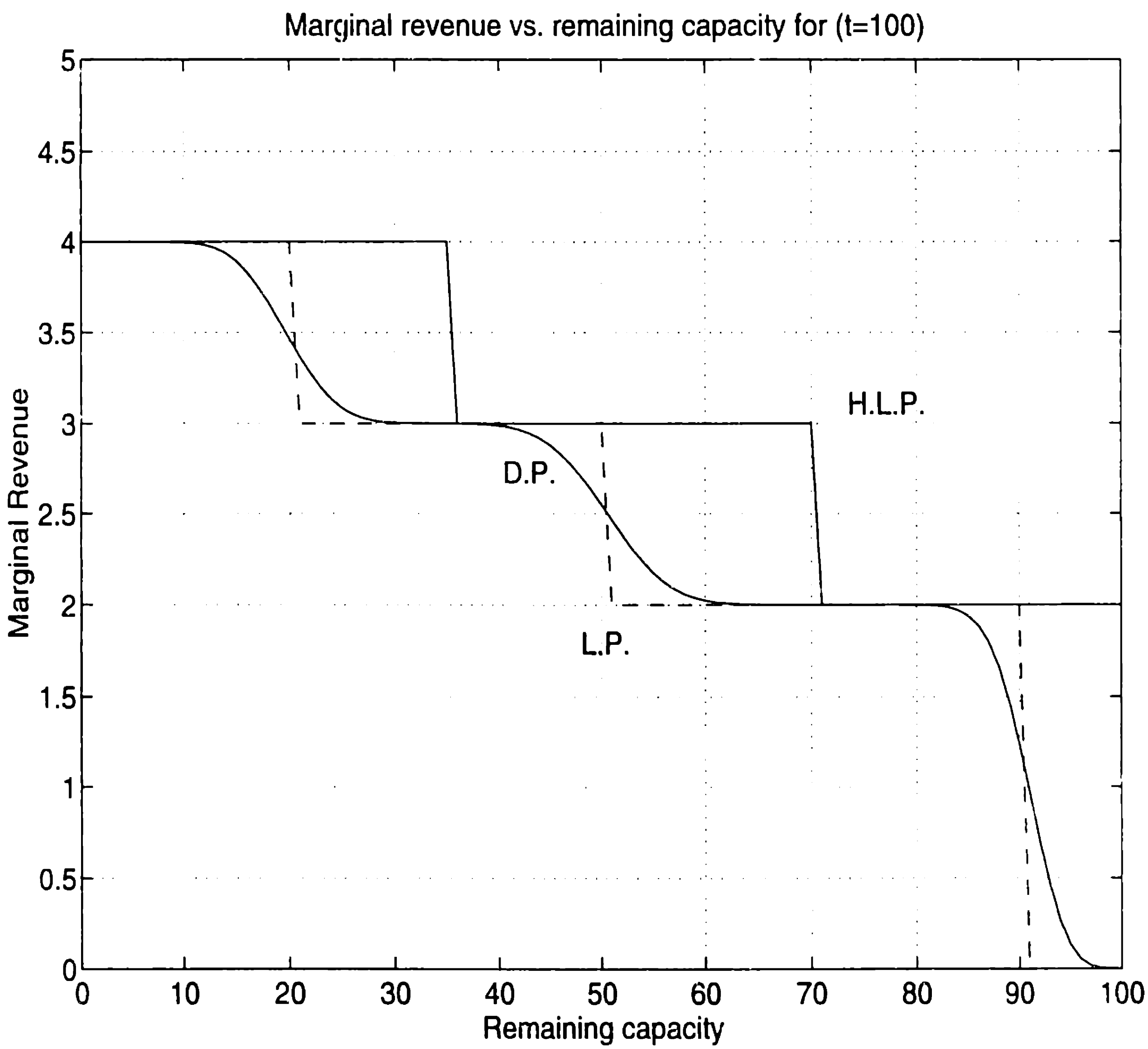


Figure 8-3: The marginal revenue ($\frac{\partial W(t,C)}{\partial C}$), given by the DP, the LP and the Heuristic LP methods, for the Two Ports Problem with Constant Arrival Rates ($\lambda(t) = \lambda$), as in figure 4-5

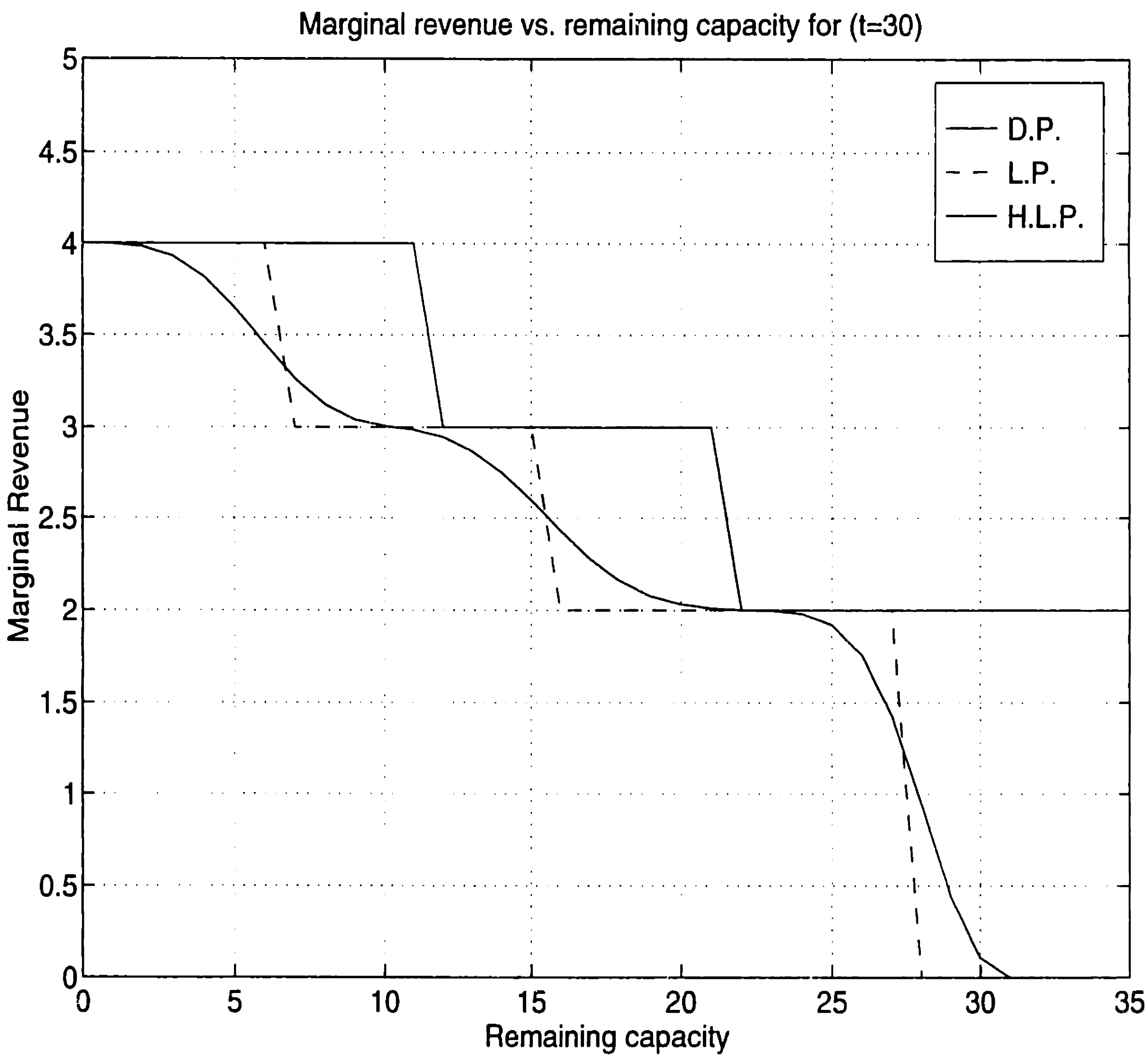


Figure 8-4: The marginal revenue ($\frac{\partial W(t,C)}{\partial C}$), given by the DP, the LP and the Heuristic LP methods, for the Two Ports Problem with Constant Arrival Rates ($\lambda(t) = \lambda$), as in figure 4-5 and the previous figure

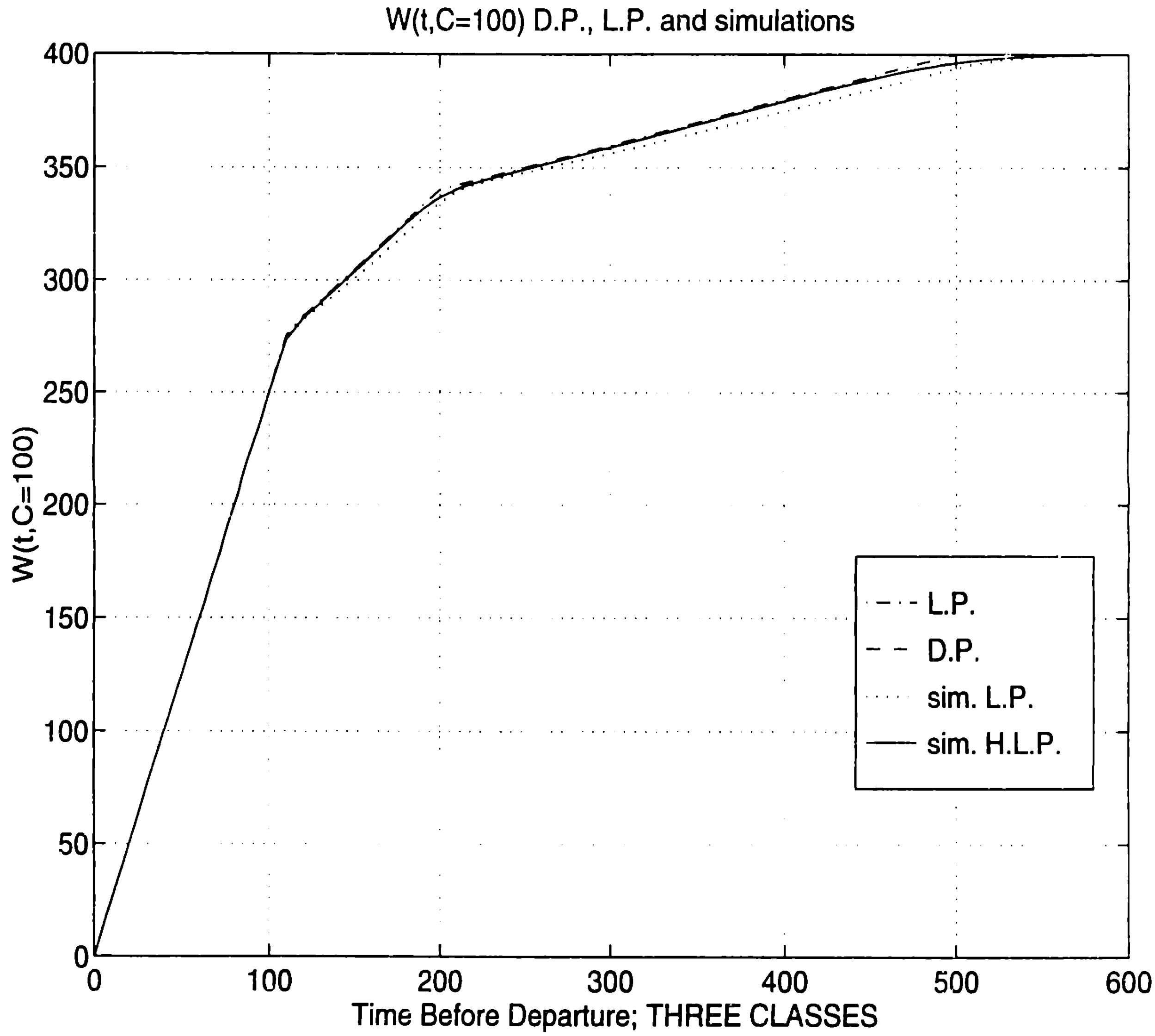


Figure 8-5: The maximum expected revenue function, given by the two theoretical curves, from L.P. and D.P. respectively, and also simulated with the use of the L.P. and the Heuristic L.P., for the model of fig. 4-5

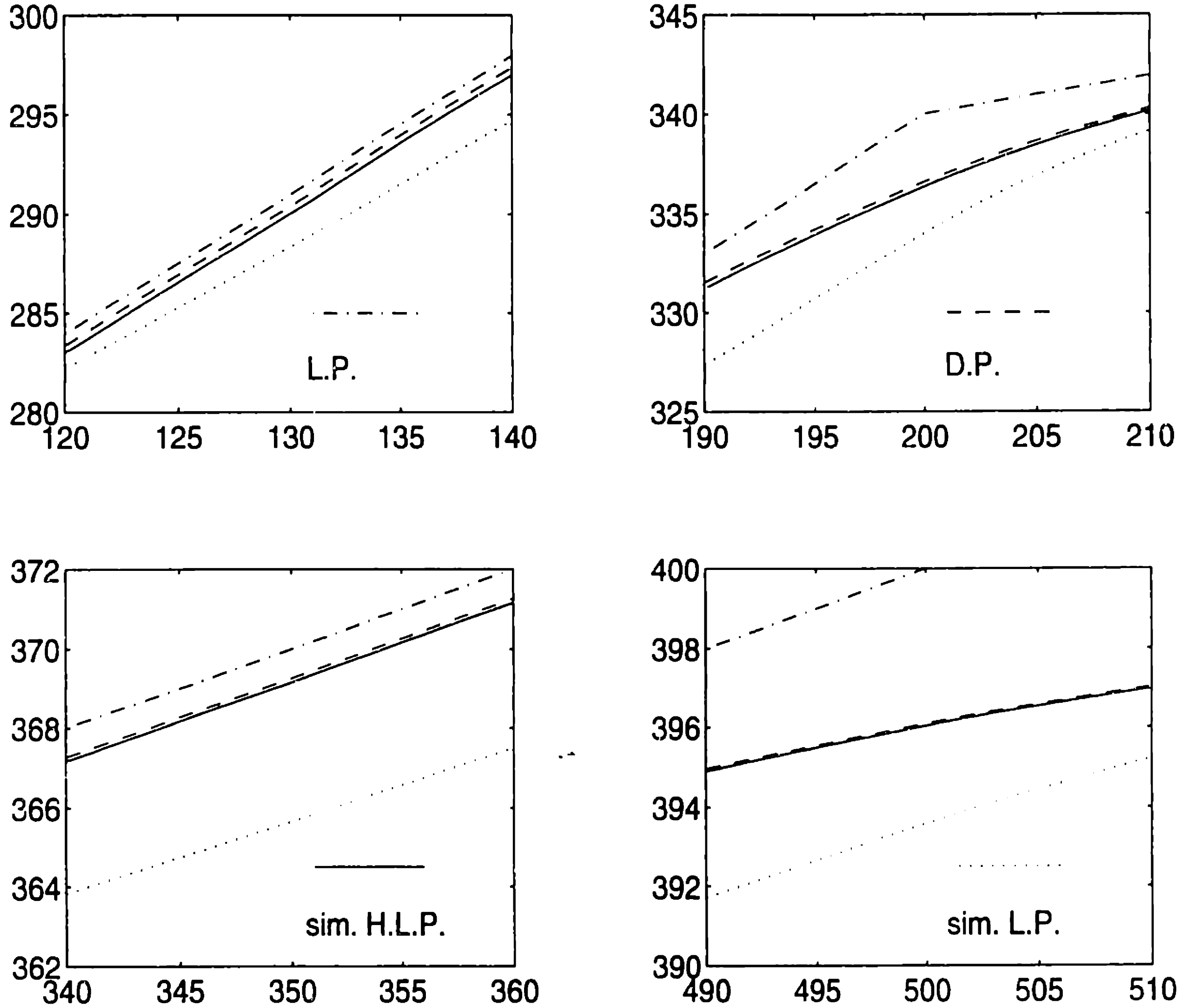


Figure 8-6: Details of the previous figure

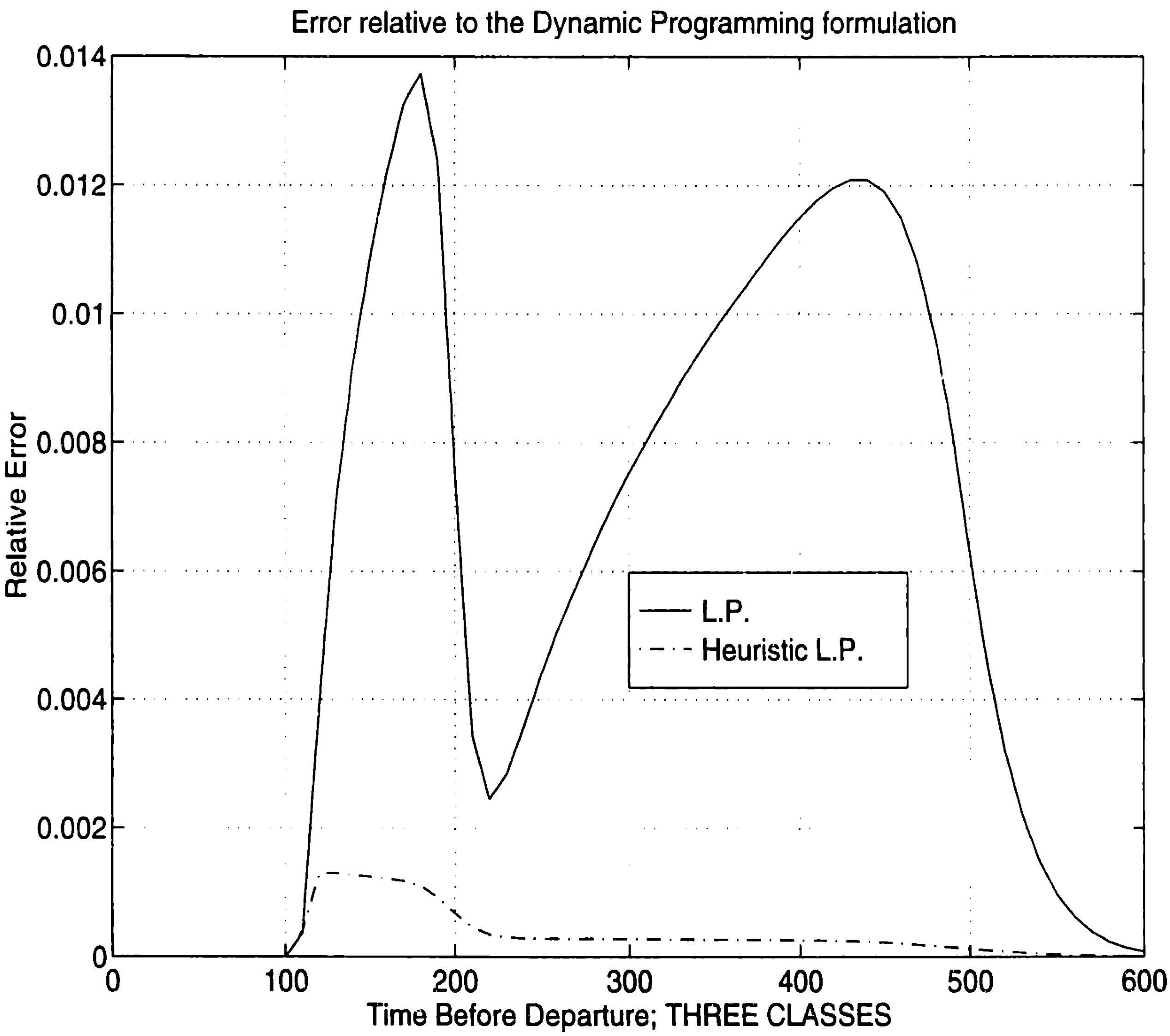


Figure 8-7: The relative error of the maximum expected revenue as it is given by the two simulations with the L.P. and the Heuristic L.P. criterion respectively, with reference to dynamic programming, for the model from figure 8-5

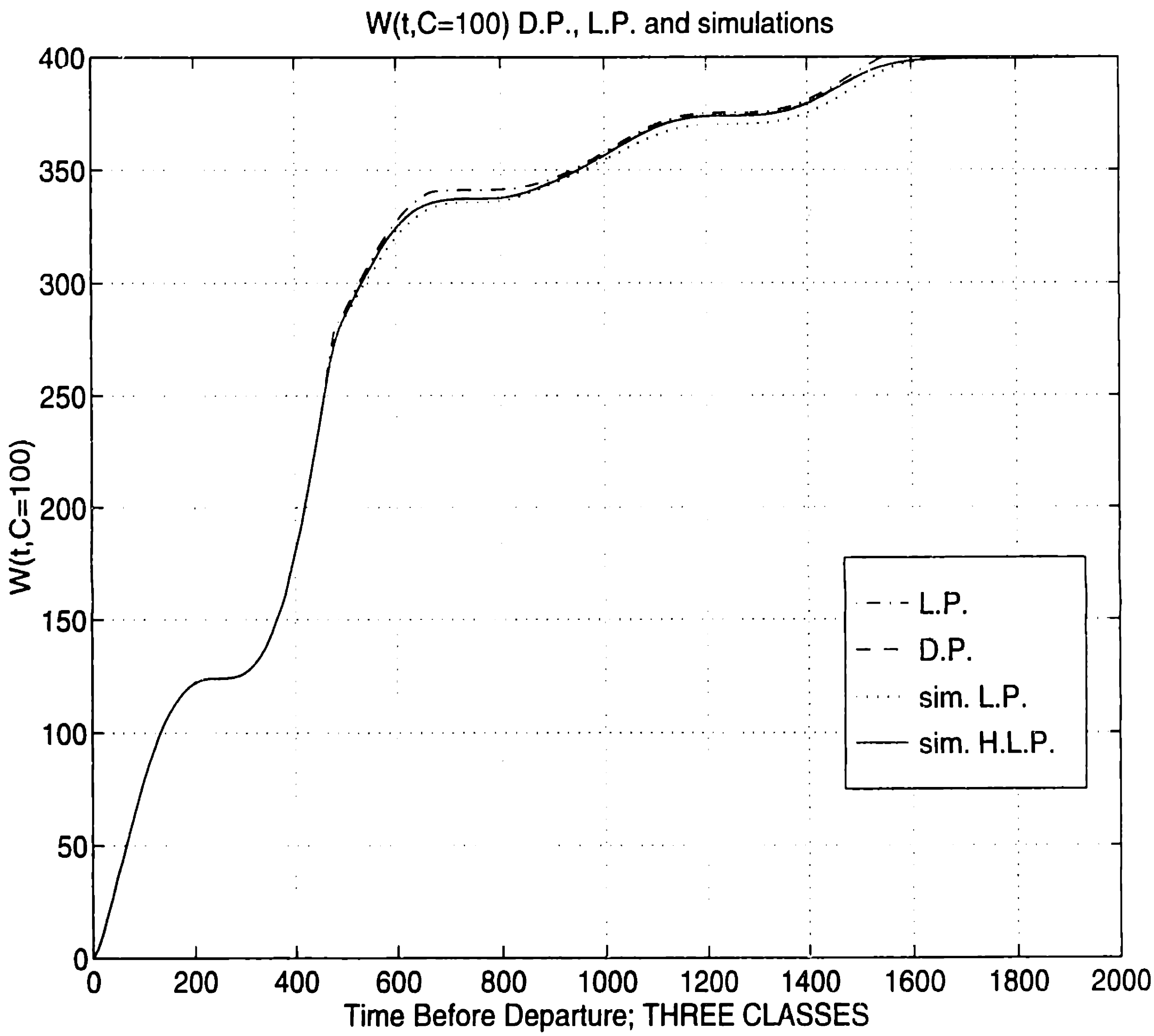


Figure 8-8: The maximum expected revenue function, given by the two theoretical curves, from L.P. and D.P. respectively, and also simulated with the use of the L.P. and the Heuristic L.P., for the model of fig. 5-4

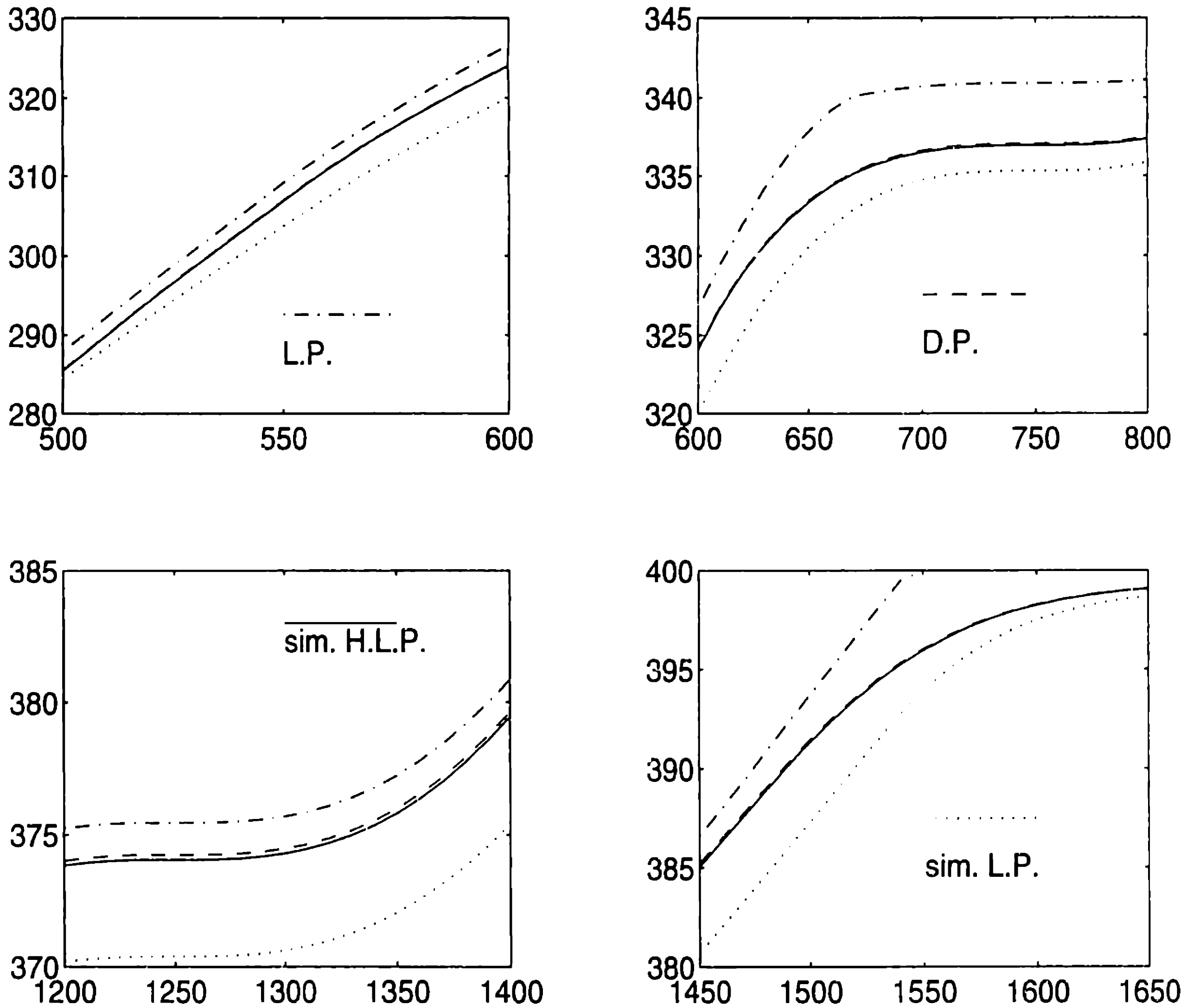


Figure 8-9: Details of the previous figure

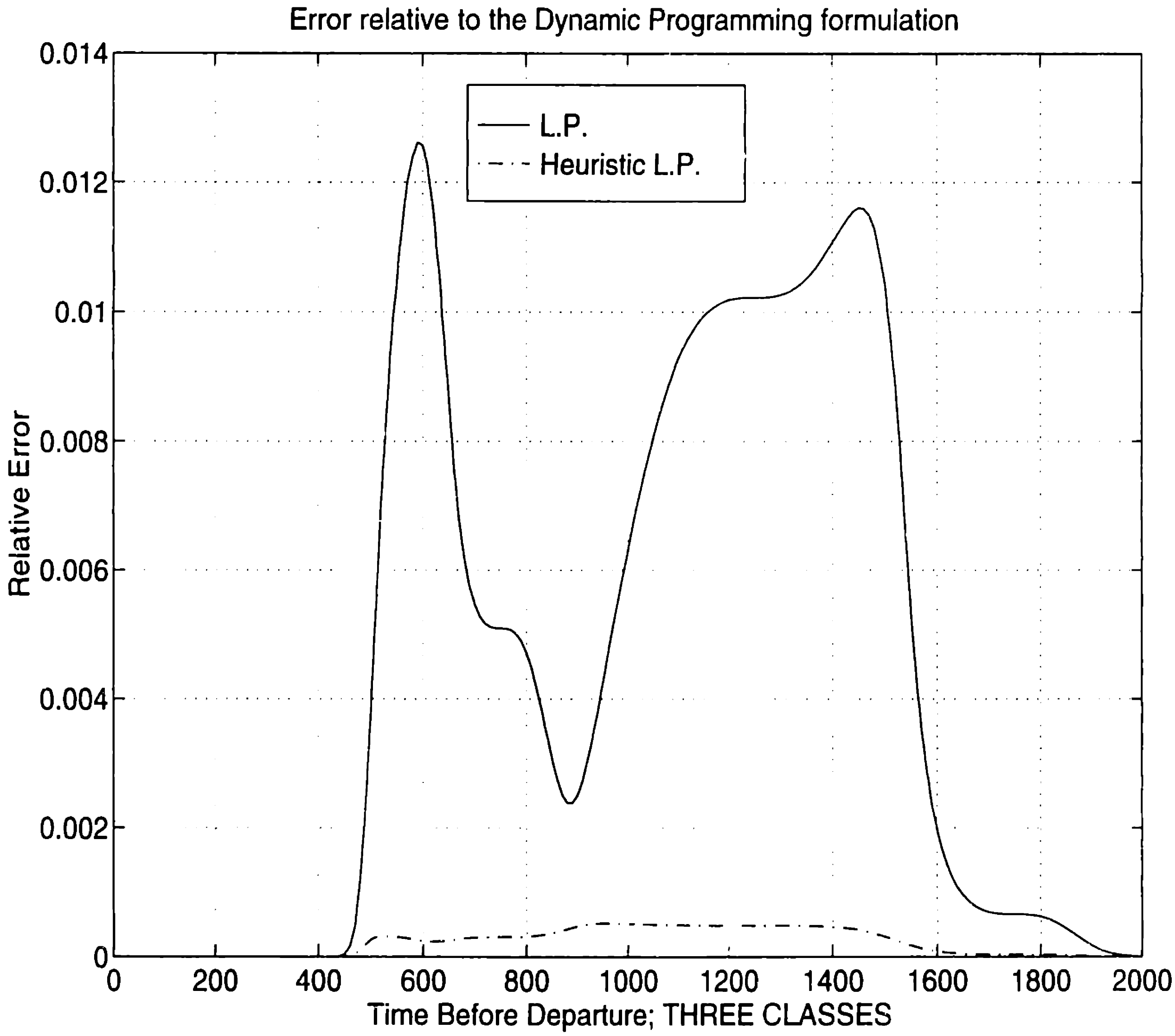


Figure 8-10: The relative error of the maximum expected revenue as it is given by the two simulations with the L.P. and the Heuristic L.P. criterion respectively, with reference to dynamic programming, for the model from figure 8-8

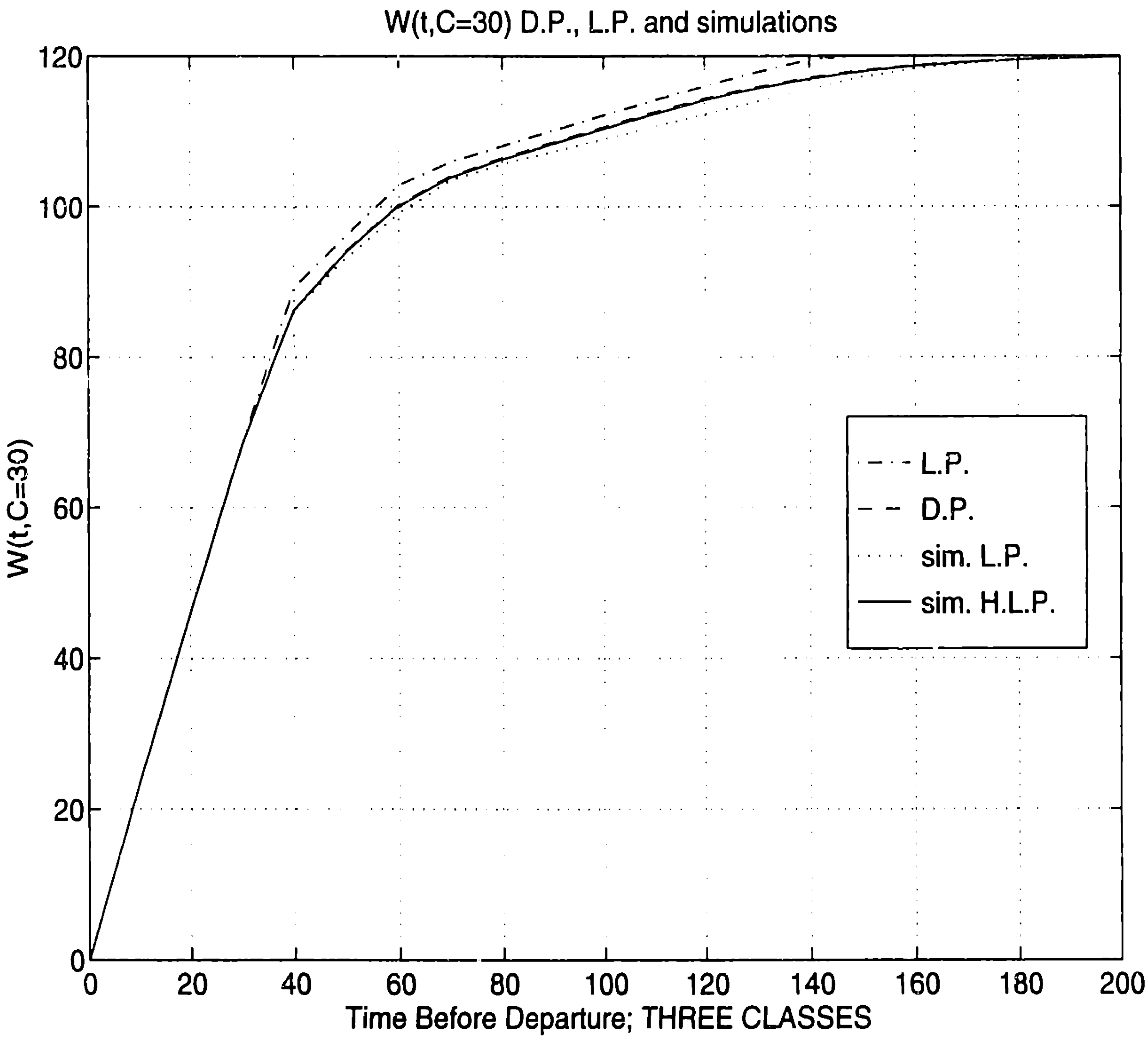


Figure 8-11: The maximum expected revenue function, given by the two theoretical curves, from L.P. and D.P. respectively, and also simulated with the use of the L.P. and the Heuristic L.P., ($\lambda_1 = 0.28, \lambda_2 = 0.27, \lambda_3 = 0.25$), ($\mu_1 = \mu_2 = \mu_3 = 0.00025$), ($A_1 = A_2 = 5,\ A_3 = 0$)

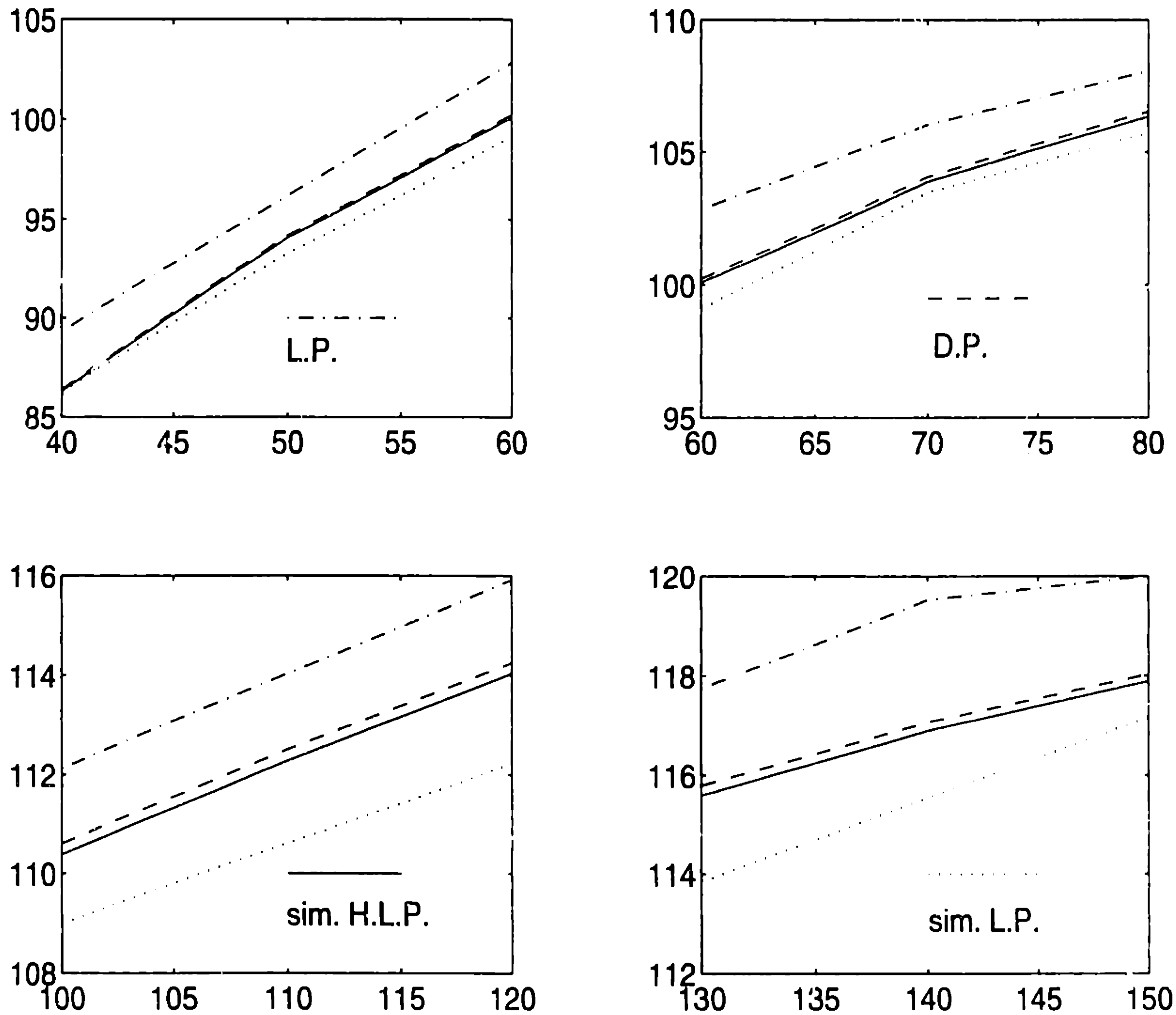


Figure 8-12: Details of the previous figure

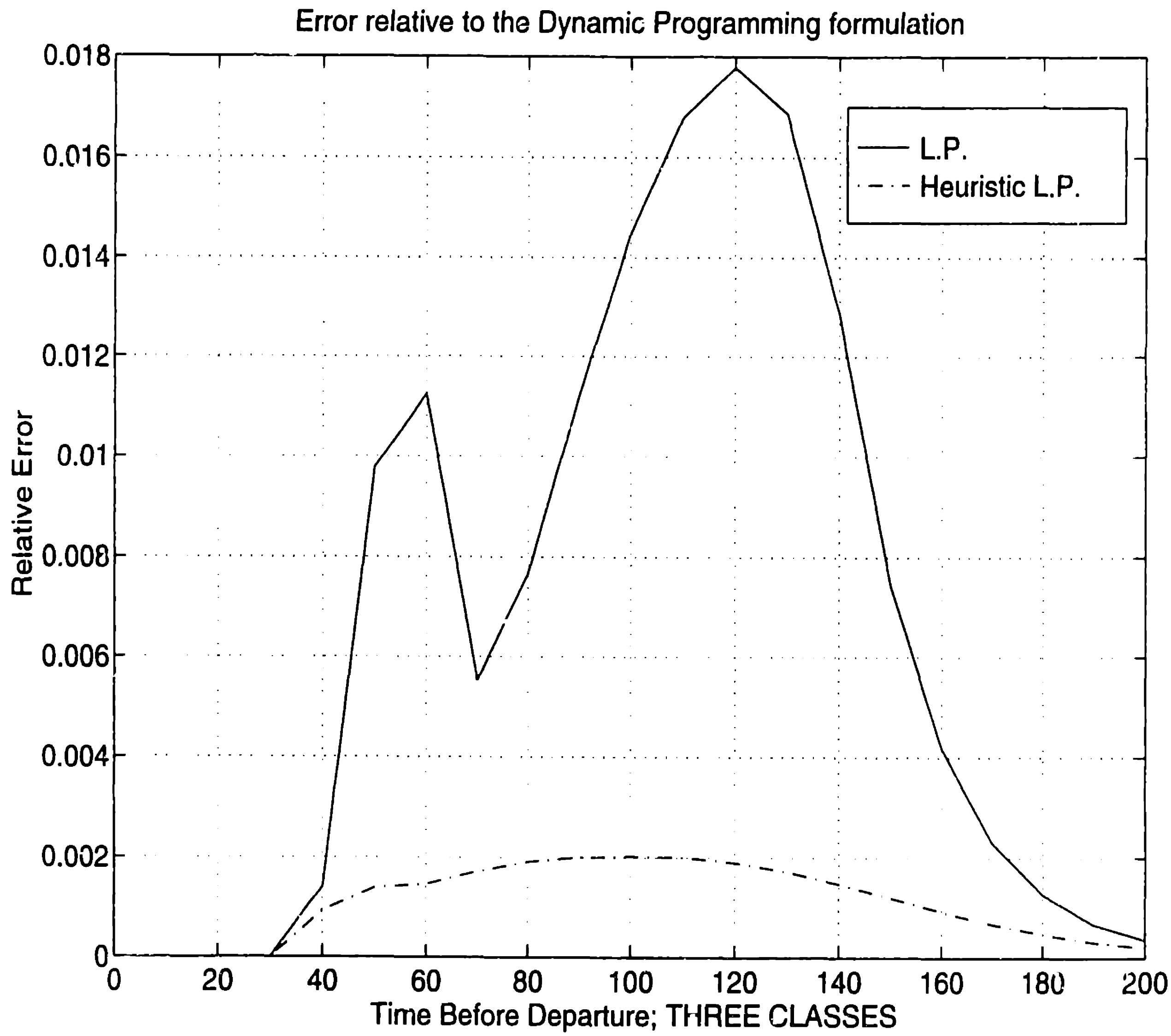


Figure 8-13: The relative error of the maximum expected revenue as it is given by the two simulations with the L.P. and the Heuristic L.P. criterion respectively, with reference to dynamic programming, for the model from figure 8-11

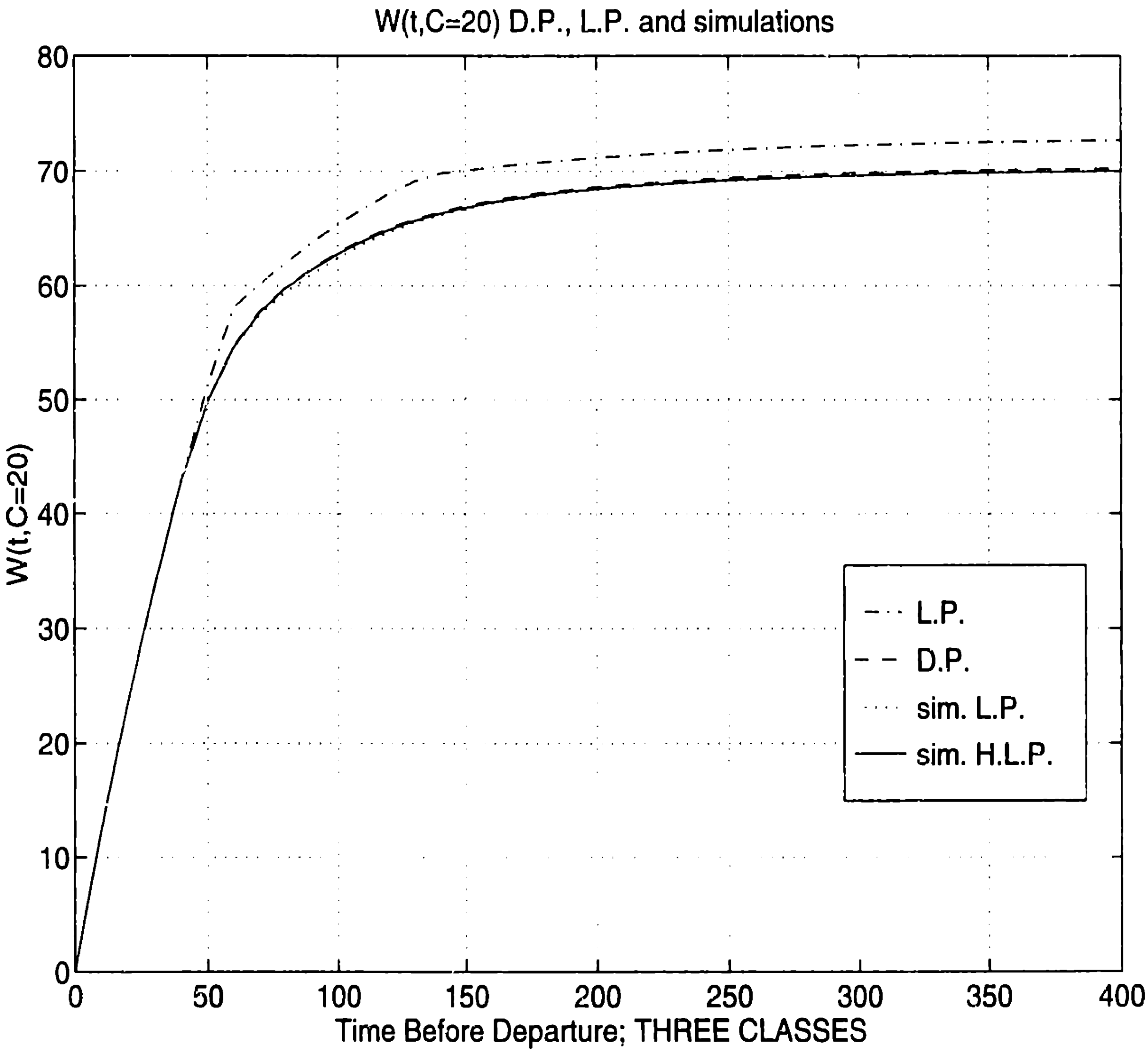


Figure 8-14: The maximum expected revenue function, given by the two theoretical curves, from L.P. and D.P. respectively, and also simulated with the use of the L.P. and the Heuristic L.P., ($\lambda_1 = 0.18, \lambda_2 = 0.14, \lambda_3 = 0.13$), ($\mu_1 = \mu_2 = \mu_3 = 0.01$), ($A_1 = A_2 = 5,\ A_3 = 0$)

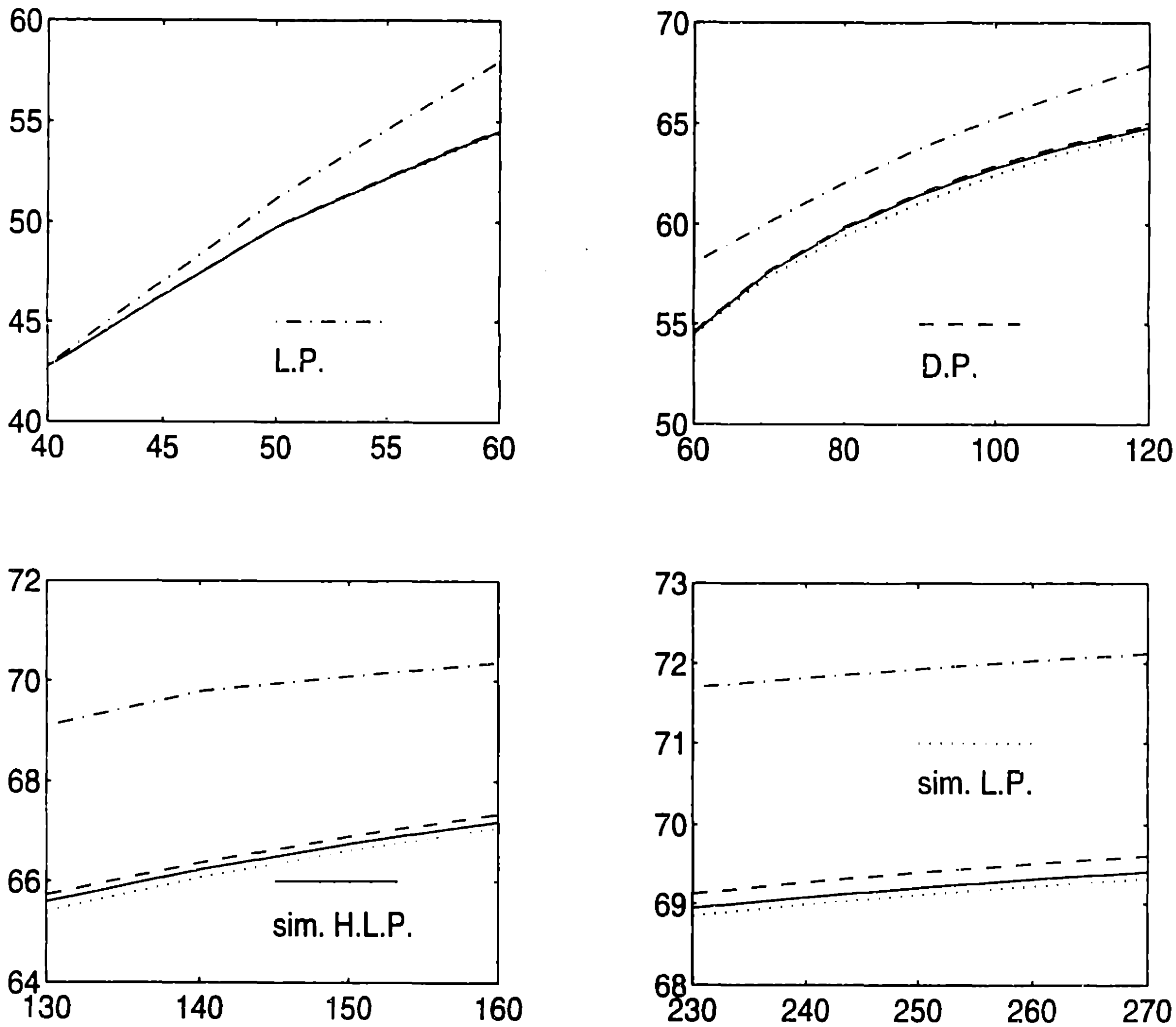


Figure 8-15: Details of the previous figure

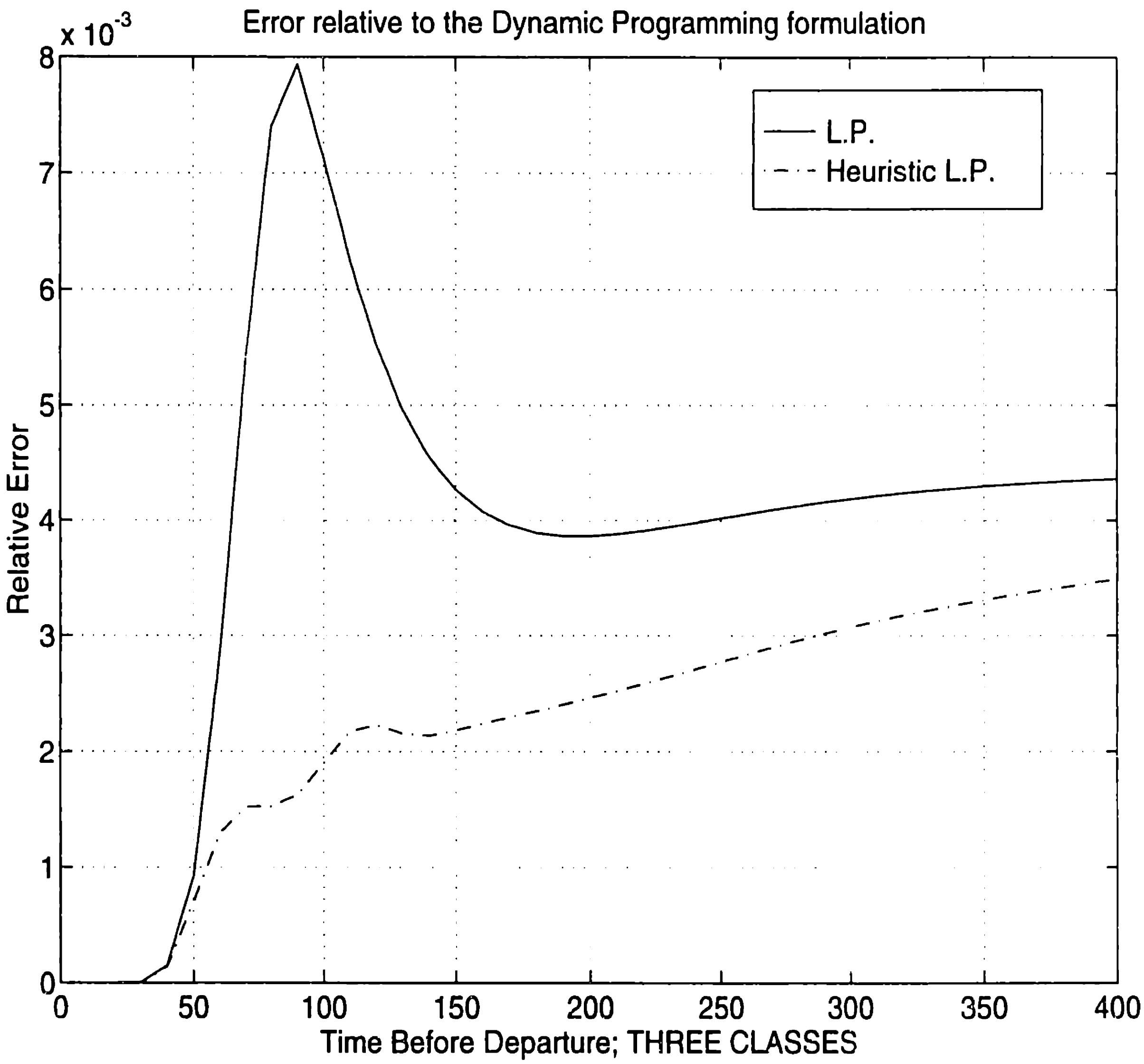


Figure 8-16: The relative error of the maximum expected revenue as it is given by the two simulations with the L.P. and the Heuristic L.P. criterion respectively, with reference to dynamic programming, for the model from figure 8-14

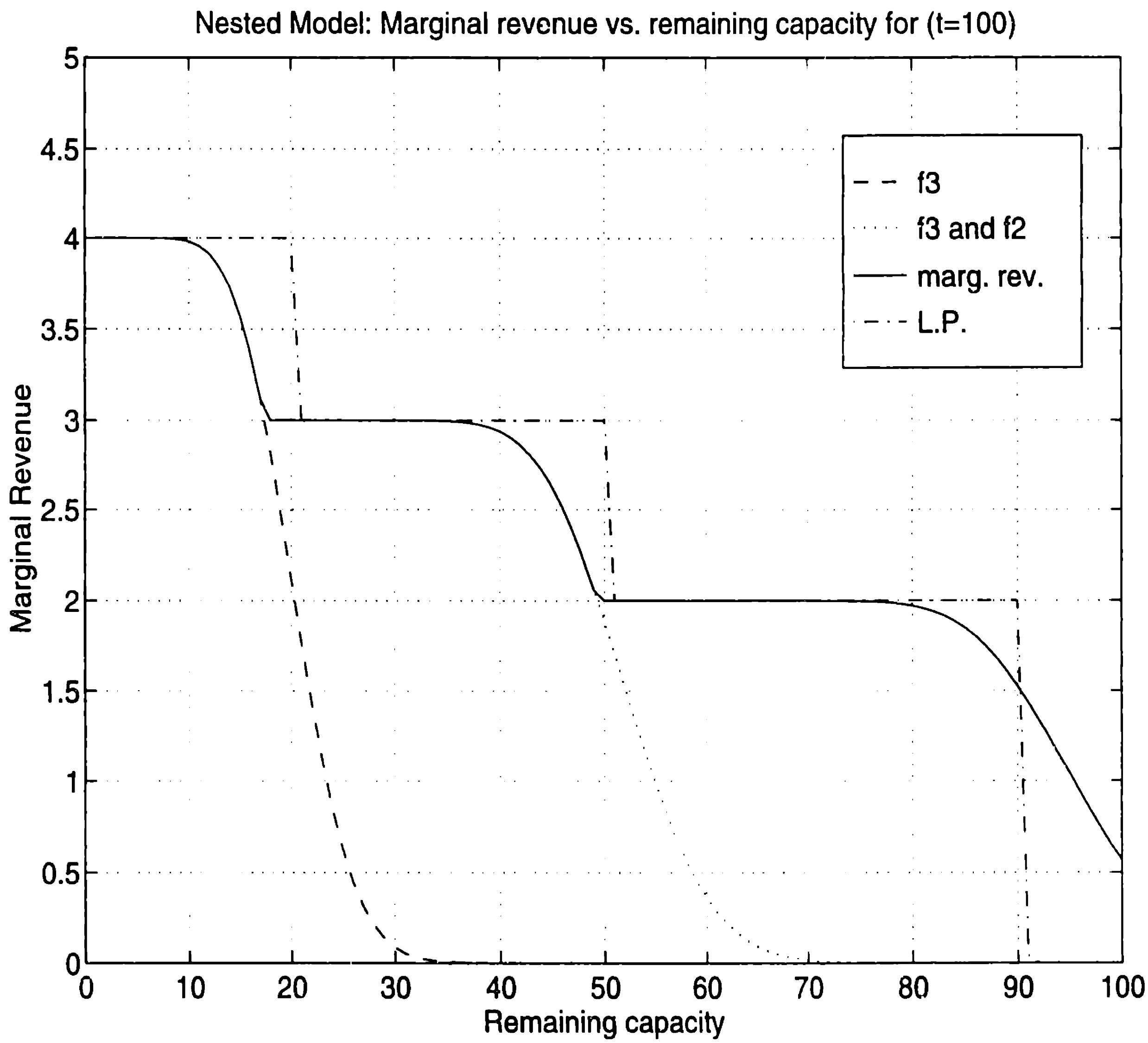


Figure 8-17: The marginal revenue ($\frac{\partial W(t,C)}{\partial C}$), given by the DP, the LP methods, for the Two Ports Problem nested model with CAPACITY = 100, and Constant Arrival Rates ($\lambda(t) = \lambda$)

Chapter 9

Yield management and optimal pricing

9.1 Introduction

The inputs of the yield management problem are the freight rates, and the estimation of the demand of the different classes of containers, the remaining capacity of the different legs of the itinerary and the confirmed reservations from the different classes of containers. The output of the model, whether it is the dynamic programming model or the equivalent linear programming model, is the optimal reservation policy for the particular combination of the input, and the expected revenues, given that we apply optimal booking policies.

We have described and solved three variations of this problem, each one being a generalization of the previous one. The three models differed based on their assumptions and they are:

- Model with time variable arrival rate
- Model with time variable arrival rate and reservations cancelations
- Model with time variable arrival rate, reservations cancelations and overbooking

The linear programming yield management models suggest an optimal booking policy, and they offer a good estimate (with less than 1% error) of the expected revenues that we find with the dynamic programming yield management models.

We therefore see that the LP yield management models can be satisfactory tools for the calculation of the expected revenue for the trip.

In a mathematical form we can conclude that the optimal value $\mathbf{z}$ of the LP model is a function of the vector $\mathbf{f}$ of the freight rates, the remaining capacity vector $\mathbf{C}$, and the confirmed reservations $\mathbf{E}$. The optimal value of the LP is also a function of the expected demand for the different classes of containers, but since the expected demand is a function of the prices, we do not have to include the demand in the parameters of the problem.

$$\mathbf{z}(t, \mathbf{E}, \mathbf{C}) = \mathbf{z}(t, \mathbf{E}, \mathbf{C}, \mathbf{f}) \tag{9.1}$$

When the operator of the vessel plans the trip, he sees that the ways to influence his expected revenues is by changing the two parameters of the problem, capacity of the vessel $\mathbf{C}$ and prices $\mathbf{f}$. It is safe to assume that when the operator has not announced the trip yet, the confirmed reservations vector $\mathbf{E} = 0$.

If we assume that the vessel that will be used on the itinerary is already decided on, the only way to increase the maximum expected revenues is through changes of the prices. We have:

$$\mathbf{z}(\mathbf{C}, \mathbf{f}) = \mathbf{z}(t, \mathbf{E}, \mathbf{C}, \mathbf{f}) \tag{9.2}$$

The vessel operator might not be able to change the rates for many reasons that range from the inflexibility of the liner conference to changes of the freight structure, to the limited interest of the liner company to changes of the freight rates.

The tool that we develop here can nevertheless be a valuable decision making tool for the management of the shipping company. Furthermore, we do not suggest dynamic pricing, because that would be infeasible due to both exogenous to the operator as well endogenous factors. Nevertheless, the operator, before the announcement of the itinerary, and knowing which vessel would serve the itinerary (i.e knowing the C), could optimize his freight rates vector **f**, so as to maximize his expected revenues **z**.

We therefore see that the yield management linear programming models that have so far been developed to assist at the operations level can also be used as pricing tools.

9.2 Two Ports, M Classes of Goods Problem

We start with a two ports example. We assume that the capacity of the vessel that operates on this route is C.

$$\begin{aligned} \mathbf{z}(C,\mathbf{f}) = \max \quad & \mathbf{f}^T\mathbf{x} \\ \mathbf{Bx} & \leq \mathbf{b} \\ \mathbf{x} & \geq \mathbf{0} \end{aligned} \tag{9.3}$$

where:

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_M \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_M \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} C \\ D_1(f_1) \\ D_2(f_2) \\ \vdots \\ D_M(f_M) \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The dimensions of $\mathbf{f} = M\text{x}1$, of $\mathbf{x} = M\text{x}1$, of $\mathbf{b} = (M+1)\text{x}1$, and of $\mathbf{B} = (M+1)\text{x}M$
In the above model we observe:

- We assume that the expected demand for each Class of Good that uses the services of the operator is a function of only the price the operator charges this Class. The prices of the competitors or other factors, like the level and the quality of service of the operator are not taken into consideration for the calculation of the demand, as they are beyond the scope of the current study.

- $\mathbf{z}$ is a function of both T and C, as well as $\mathbf{f}$. Nevertheless, for our purpose, T is the time remaining at the moment we announce the itinerary. Therefore, $D_i(T, f_i) = D_i(f_i)$. We consider $D_i(f_i)$ to be the total demand for the i_{th} Class of Good when the price is f_i. Furthermore, C is the total available capacity of the vessel before we start any cargo reservations. As a result C is considered to be one of the parameters of the problem and is not considered to be a variable.

- The above formulation of equation 9.3, is inclusive of all three LP yield management models that we developed. $D_i(T, f_i)$ can be equal to $\lambda_i t$ of the first model or equal to $\int_0^T \lambda_i(t)dt$ of the second model or to the demand of the model with the cancelations or the model with the overbookings. As far as the model with the overbooking goes, at the original stage of the itinerary announcement, when there are no containers that have booked capacity, the vector of the overbooking variables $\mathbf{u}$ (see equation 7.9) is: $\mathbf{u}=\mathbf{0}$, simply because the cost of overbooking is greater than the cost of declining a booking. In conclusion the model of equation 9.3 describes all four models that have been developed so far. These results are not confined to the case of the two Ports Problem and it is easy to see that they apply to the general case of the M Classes of Goods, N Ports Problem.

The optimization problem I want to maximize at this stage is:

$$\mathbf{R}(C) = \max_{\text{s.t. } \mathbf{f} \in \Re^M} \mathbf{z}(C, f_1, \ldots, f_M) = \max_{\text{s.t. } \mathbf{f} \in \Re^M} \mathbf{f}^T \cdot \mathbf{s}(\mathbf{f}) = \max_{\text{s.t. } \mathbf{f} \in \Re^M} \sum_{i=1}^{M} f_i \cdot s_i(f_1, \ldots, f_M) \tag{9.4}$$

We make the following observations:

- The above optimization problem is unconstrained. Of course the demand function is defined only for positive prices. We could nevertheless, define the demand function for negative prices, with a function that satisfies continuity and differentiability. That would not change the solution of the optimization problem because for negative prices the revenue would be negative and lower than any solution with non negative prices.

- The vector $\mathbf{s}$ is the vector of the optimal values of the vector $\mathbf{x}$ at the Linear Programming Model 9.3. At L.P. 9.3, we maximize the revenues when we know what the prices for each Class of Good are. At problem 9.4, we know the (maximum) expected revenues for each vector $\mathbf{f}$, and we want to find the $\mathbf{f}^*$, for which $\mathbf{R}(C) = \max\ \mathbf{z}(C, f_1, \ldots, f_M) = \mathbf{f}^{*T} \cdot \mathbf{s}$. At the optimization problem 9.4, $\mathbf{z}(C, f_1, \ldots, f_M)$, is only considered a function of $f_1, \ldots, f_M$ and we try to maximize with reference to these variables.

- In equation 9.4 we assume that the values of f are not constrained in any way. Given the appropriate demand functions, we can find, through optimization, how much each Class should be charged without demanding any particular order of the prices (i.e. $f_1 < f_2 < \ldots < f_M$). Many times though, the demand functions can describe the demand accurately for only small movements of the prices, or in an environment where the prices are ordered (i.e. $f_1 < f_2 < \ldots < f_M$). In practical therefore applications, it would be wise to have some kind of constraint for the domain of the freight rates. These constraints could refer to the ranges of the prices.

For $i \in \mathbf{A}$ we have that $s_i = D(f_i) \Longrightarrow s_i = s_i(f_i)$

For $i \in \mathbf{M}$ we have that $s_i < D(f_i) \Longrightarrow s_i = s_i(f_{i+1}, \ldots, f_M)$

Our task is to maximize $\mathbf{z}$ as a function of the prices $f_1, \ldots, f_M$. Since the problem 9.4 is unconstrained, the first order necessary condition for optimality is:

$$\nabla \mathbf{z}(f_1, \ldots, f_M) = [\frac{\partial \mathbf{z}}{\partial f_1}, \ldots, \frac{\partial \mathbf{z}}{\partial f_M}]^T = \mathbf{0} \tag{9.5}$$

(See [43], Chapter 1, on Unconstrained Optimization).

From equation 9.4, I get:

$$\frac{\partial \mathbf{z}}{\partial f_i} = s_i(f_1, \ldots, f_M) + \sum_{j=1}^{M} f_j \cdot \frac{\partial s_j(f_1, \ldots, f_M)}{\partial f_i} \tag{9.6}$$

But we have that:

$$\frac{\partial s_j}{\partial f_i} = \begin{cases} -\frac{\partial s_i}{\partial f_i} & \text{if } i \in \mathbf{A} \text{ and } j \in \mathbf{M} \\ 0 & \text{otherwise} \end{cases} \tag{9.7}$$

Therefore, for $i \in \mathbf{A}$, we get:

$$\frac{\partial \mathbf{z}}{\partial f_i} = \underbrace{s_i(f_1, \ldots, f_M)}_{=D_i(f_i)} + f_i \cdot \frac{\partial s_i(f_i)}{\partial f_i} + f_{Marg} \cdot (-\frac{\partial s_i(f_i)}{\partial f_i}) \Longrightarrow$$

$$\frac{\partial \mathbf{z}}{\partial f_i} = D_i(f_i) + (f_i - f_{Marg}) \cdot \frac{\partial s_i(f_i)}{\partial f_i}, \quad i \in \mathbf{A} \tag{9.8}$$

We also know that for $i \in \mathbf{A}$ we have:

$$\frac{\partial s_i}{\partial f_i} = \frac{\partial D_i(f_i)}{\partial f_i} \tag{9.9}$$

From equations 9.8 and 9.9, we get:

$$\frac{\partial \mathbf{z}}{\partial f_i} = D_i(f_i) + (f_i - f_{Marg}) \cdot \frac{\partial D_i(f_i)}{\partial f_i}, \quad i \in \mathbf{A} \tag{9.10}$$

For $i \in \mathbf{M} \equiv Marg$:

$$\begin{aligned} \frac{\partial \mathbf{z}}{\partial f_{Marg}} &= s_{Marg}(f_{Marg+1}, \ldots, f_M) \Longrightarrow \\ \frac{\partial \mathbf{z}}{\partial f_{Marg}} &= C - \sum_{j \in \mathbf{A}} D_j(f_j), \quad i \in \mathbf{M} \equiv Marg \end{aligned} \tag{9.11}$$

From equations 9.5 and 9.11, we see that we must have:

$$s_{Marg}(f_{Marg+1}, \ldots, f_M) = C - \sum_{j \in \mathbf{A}} D_j(f_j) = 0 \tag{9.12}$$

Therefore we do not accept any units from s_{Marg}. If we had one more unit of capacity and we did not have the right to change the prices f_i, we would accept one unit from s_{Marg}. The above result suggests that the optimal way to do Yield Management (if we do open loop control i.e. we define the control parameters at the beginning of the process and we let the process run by itself from then on) is to define the prices and which classes of goods we accept, and then to accept every customer from these classes.
The marginal class is the first class. $f_{Marg} = f_1$ and $\mathbf{M} \equiv$ class 1
We repeat the gradient of $\mathbf{z}(C, f_1, \ldots, f_M)$:

$$\nabla \mathbf{z}(f_1, \ldots, f_M) = \begin{bmatrix} C - \sum_{j \in \mathbf{A}} D_j(f_j) \\ D_2(f_2) + (f_2 - f_1) \cdot \frac{\partial D_2(f_2)}{\partial f_2} \\ \vdots \\ D_M(f_M) + (f_M - f_1) \cdot \frac{\partial D_M(f_M)}{\partial f_M} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The price vector $\mathbf{f} = (f_1, \ldots, f_M)$, which satisfies the first order necessary condition for maximization:

$$\nabla \mathbf{z}(f_1, \ldots, f_M) = \mathbf{0} \tag{9.13}$$

is a local maximum if it satisfies the condition

$$\nabla^2 \mathbf{z}(f_1, \ldots, f_M) : \text{ negative definite} \tag{9.14}$$

Of course the function $\mathbf{z}$ has to be twice differentiable. (A matrix $\mathbf{A}$ is negative definite, if $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} < 0, \ \forall \ \mathbf{x} \in \Re^n$. For more details, see [43], Chapter 1, on Unconstrained Optimization).

$$\frac{\partial^2 \mathbf{z}}{\partial f_i \partial f_j} = \begin{cases} 2 \cdot \frac{\partial D_i(f_i)}{\partial f_i} + (f_i - f_1) \cdot \frac{\partial^2 D_i(f_i)}{\partial f_i^2}, & \text{if } i = j, \text{ and } i, j \in \mathbf{A} \\ 0 & \text{otherwise} \end{cases}$$

If $\nabla^2 \mathbf{z}(f_1, \ldots, f_M)_{(i,j)}$ is negative definite for all the points (f's) that satisfy the necessary conditions for optimality, then these points are local maxima.

9.3 Network pricing problem

We continue with the model where we have M Classes of Goods, and a vessel that serves an itinerary of N ports. The vector of the vessel's available capacity is $\mathbf{C}$ with available capacity C_i at each leg i. The capacity vector $\mathbf{C}$ is the vector of the total available capacity before we start accepting any containers. More often than not, the capacity C_i is the same for all the legs of the trip.

As we have discussed in the previous section, the fact that the capacity is the original capacity and not the remaining capacity after several bookings is not restrictive at all. On the contrary, is a more appropriate and accurate formulation, since the revenue maximization through pricing is not dynamic. The pricing is done before the announcement of the itinerary and the initiation of the booking process.

For the pricing model, we do not need to use the model we developed for the cancelations or the model with cancelations and overbooking. Even if we want to use them, the substitution of $\mathbf{E} = \mathbf{0}$ and $\mathbf{E_e} = \mathbf{0}$, will simplify the model to the linear model 5.19, or the equivalent LP 9.15. We just have to keep in mind that the expected demand $\mathbf{L}$ is the expected demand from the customers who do not cancel their reservations.

$$\begin{aligned} \mathbf{z}(\mathbf{C},\mathbf{f}) = \max \ \mathbf{f}^T\mathbf{x} & \\ \mathbf{Bx} &\leq \mathbf{b} \\ \mathbf{x} &\leq \mathbf{L} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned} \tag{9.15}$$

The form of $\mathbf{Bx} \leq \mathbf{b}$ is the following:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} x_{OD_i,g} \leq C_j, \;\; j = 1, \ldots, N \tag{9.16}$$

The form of $\mathbf{x} \leq \mathbf{L}$ is the following:

$$x_{OD_i,g} \leq D_{OD_i,g}, \;\; \forall \ OD_i, g \ \in \ \mathbf{S} \tag{9.17}$$

The comments of page 220, for the two ports pricing model, are valid for the network pricing problem too.

The optimization problem I want to maximize is:

$$\mathbf{R}(\mathbf{C}) = \max_{\text{s.t. } \mathbf{f} \in \Re^n} \ \mathbf{z}(T,\mathbf{C},\mathbf{f}) = \max_{\text{s.t. } \mathbf{f} \in \Re^n} \ \mathbf{f}^T \cdot \mathbf{s} = \max_{\text{s.t. } \mathbf{f} \in \Re^n} \sum\nolimits_{i \in (\mathbf{Y} \cup \mathbf{N})} f_i \cdot s_i(\mathbf{f}) \tag{9.18}$$

Our task is to maximize $\mathbf{z}$ as a function of the price vector $\mathbf{f}$. For reasons that we have already explained, the nonlinear optimization problem 9.18 can be treated as unconstrained.

The first order necessary condition for optimality of the unconstrained problem is:

$$\nabla \mathbf{z}(\mathbf{f}) = [\ldots, \frac{\partial \mathbf{z}}{\partial f_i}, \ldots, \frac{\partial \mathbf{z}}{\partial f_j}, \ldots]^T = \mathbf{0} \tag{9.19}$$

From equation 9.18, I get:

$$\frac{\partial \mathbf{z}}{\partial f_i} = s_i(\mathbf{f}) + \sum_{j \in \mathbf{Y}} f_j \cdot \frac{\partial s_j(\mathbf{f})}{\partial f_i} \tag{9.20}$$

We introduce the following definitions:

- $\mathbf{A_{OD_r}}$ is the set that includes all the ODC that belong both to the set $\mathbf{A}$ (it includes all the ODC's from which we accept all the expected demand) and to the origin-destination pair OD_r. $\mathbf{A_{OD_r}}$ is the set that includes those and only those Classes of the OD_r pair that we fully accept.

- $\mathbf{M_{OD_r}}$ is the set that includes the one ODC that belongs to $\mathbf{M}$ and to the origin-destination pair OD_r. In other words, $\mathbf{M_{OD_r}}$ is the set that has at most one element, the marginal Class of all the Classes that belong to the OD_r pair. This element is the Class with freight rate $f_{M_{OD_r}}$.

But we have that:

$$\frac{\partial s_j(\mathbf{f})}{\partial f_i} = \begin{cases} -\frac{\partial s_i}{\partial f_i} & \text{if } i \in \mathbf{A_{OD_r}} \text{ and } j \in \mathbf{M_{OD_r}} \\ 0 & \text{otherwise} \end{cases} \tag{9.21}$$

For $i \in \mathbf{A_{OD_r}}$ we get:

$$\frac{\partial \mathbf{z}}{\partial f_i} = \underbrace{s_i(\mathbf{f})}_{=D_i(f_i)} + f_i \cdot \frac{\partial s_i(f_i)}{\partial f_i} + f_{M_{OD_r}} \cdot (-\frac{\partial s_i(f_i)}{\partial f_i}) \Longrightarrow$$

$$\frac{\partial \mathbf{z}}{\partial f_i} = D_i(f_i) + (f_i - f_{M_{OD_r}}) \cdot \frac{\partial s_i(f_i)}{\partial f_i}, \quad i \in \mathbf{A_{OD_r}} \tag{9.22}$$

We also know that for $i \in \mathbf{A_{ODr}}$ we have $s_i = D_i(f_i)$. Therefore:

$$\frac{\partial s_i}{\partial f_i} = \frac{\partial D_i(f_i)}{\partial f_i} \tag{9.23}$$

From equations 9.22 and 9.23, we get:

$$\frac{\partial z}{\partial f_i} = D_i(f_i) + (f_i - f_{M_{ODr}}) \cdot \frac{\partial D_i(f_i)}{\partial f_i}, \quad i \in \mathbf{A_{ODr}} \tag{9.24}$$

For $i \in \mathbf{M_{ODr}}$:

$$\begin{aligned} \frac{\partial z}{\partial f_{M_{ODr}}} &= s_{M_{ODr}}(\mathbf{f}) \Longrightarrow \\ \frac{\partial z}{\partial f_{M_{ODr}}} &= C - \sum_{j \in \mathbf{A_{ODr}}} D_j(f_j), \quad i \in \mathbf{M_{ODr}} \end{aligned} \tag{9.25}$$

From equations 9.5 and 9.24, we see that we must have:

$$s_{M_{ODr}}(\mathbf{f}) = C - \sum_{j \in \mathbf{A}} D_j(f_j) = 0 \tag{9.26}$$

Therefore we do not accept any units from $s_{M_{ODr}}$. If we had one more unit of capacity and we did not have the right to change the prices f_i, we would accept one unit from $s_{M_{ODr}}$. The above result suggests that the optimal way to do Yield Management (if we do open loop control i.e. we define the control parameters at the beginning of the process and we let the process run by itself from then on) is to define the prices and which classes of goods we accept, and then to accept every customer from these classes.

The price vector $\mathbf{f}$, which satisfies the first order necessary condition for maximization:

$$\nabla z(\mathbf{f}) = \mathbf{0} \tag{9.27}$$

is a local maximum if it satisfies the condition

$$\nabla^2 \mathbf{z}(\mathbf{f}) : \text{ negative definite} \qquad (9.28)$$

Of course the function $\mathbf{z}$ has to be twice differentiable.

$$\frac{\partial^2 \mathbf{z}}{\partial f_i \partial f_j} = \begin{cases} 2 \cdot \frac{\partial D_i(f_i)}{\partial f_i} + (f_i - f_{M_{OD_r}}) \cdot \frac{\partial^2 D_i(f_i)}{\partial f_i^2}, & \text{if } i = j, \text{ and } i, j \in \mathbf{A_{ODr}} \\ 0 & \text{otherwise} \end{cases}$$

If $\nabla^2 \mathbf{z}(\mathbf{f})_{(i,j)}$ is negative definite for all the points (f's) that satisfy the necessary conditions for optimality, then these points are local maxima.

9.4 Further Research

We have seen above that when we do optimal pricing with Yield Management in mind, we see that the maximum expected revenues for the vessel that operates on the given itinerary, $\mathbf{R}(\mathbf{C})$ (from equation 9.18), is a function of the vector of the total capacity that the vessel has at the different legs of the itinerary. If we consider now a company that has P vessels serving in P different itineraries, then the sum of the maximum revenues (or the sum of the rates of the maximum revenues, in case the itineraries take different time to complete) is

$$\mathbf{K} = \sum_{i=1}^{P} \mathbf{R_i}(\mathbf{C_i}) \qquad (9.29)$$

We therefore see that there is an optimal vessel allocation that would maximize the company revenues. That allocation would map each vessel with capacity vector $\mathbf{C}_\mathbf{k}^*$ to the itinerary $\mathbf{R_k}$, so as to obtain optimal revenues.

In other words:

$$\begin{aligned} \mathbf{K}^* &= \max \sum_{i=1}^{P} \mathbf{R_i}(\mathbf{X_i}) \\ & \text{s.t. } \mathbf{X_i} \in \{\mathbf{C_1}, \ldots, \mathbf{C_P}\} \text{ if } \mathbf{X_i} = \mathbf{C_j} \Longrightarrow \mathbf{X_k} \neq \mathbf{C_j}, \text{ for } i \neq k \end{aligned} \tag{9.30}$$

The natural continuation of the current research would be to focus on finding a solution to the optimization problem 9.30.

9.5 Summary

We established that the objective functions of the linear programming models presented at the previous chapters describe, to a satisfactory degree, the revenue potential of the booking process. We observe that the objective functions of the linear programming models are functions of the remaining capacity of the vessel and the level of the freight rates. If the operator of the vessel is at the planning stage of the itinerary, the L.P. can determine the revenue potential of each trip as a function of the capacity of the vessel and the freight rates that the operator will charge for his services. We can therefore see that the expected revenue under optimal policy is a function of the capacity of the vessel and the freight rates charged. Under the assumptions presented in this chapter we maximize the revenue potential of the vessel as a function of the freight rates charged by the operator. We first find the necessary conditions for optimality that the freight rates have to satisfy in the case of the two ports problem and then we derive the necessary conditions for the multi-port case.

Chapter 10

Implementation of yield management in shipping

10.1 Customers with many containers, and long term contracts

10.1.1 Customers with many containers

A major assumption of all the models on which we have worked, is that each arriving customer asks for transportation for only one container. In practice, this assumption is violated often. Many times customers arrive with more than one container. Often a customer wants transportation for several containers that belong to different classes. The difference of the scenarios mentioned above from the standard approach that we followed so far, is that when a customer gives an order for several containers, we either accept all the containers or we decline them all.

We do not have the ability to accept only the containers that belong to the classes that we accept in our current policy area. We do not have the option of accepting some of the containers just because we happen to be in a policy area that accepts these containers, and not accept them all because if we did so, we would get out of the policy area, into

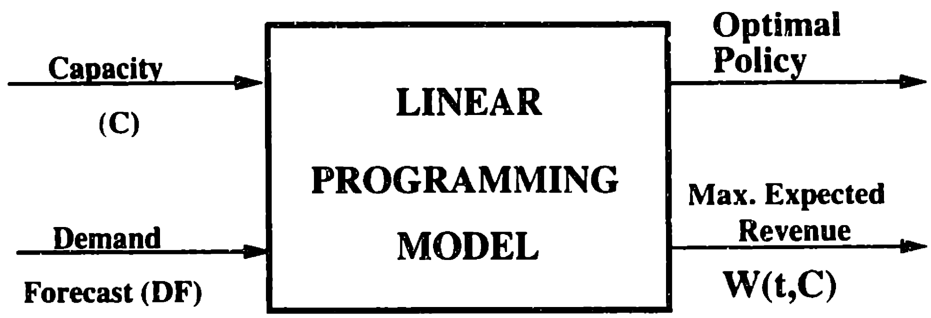


Figure 10-1: Input and output of the yield management optimization model

an other policy area, in which it is not optimal to accept these classes of containers. It is highly unlikely that a shipper would be flexible enough to accept such a booking policy on the part of the vessel operator.

The approach that we propose is the following. The different variations of the linear programming model give the expected revenue of the vessel under the assumption of optimal booking policy. The input of the optimization model is the remaining capacity of the vessel on the different legs of the network and the expected demand of the different classes of customers. The output of the model is the optimal booking policy and the expected revenues under optimal booking policy. (See Figure 10-1). The optimal booking policy is useful when we have customer arrivals with one container each. The optimal policy suggests whether to accept or not orders that consist of only one container.

When we consider the one customer/many containers problem, the optimal reservation decision will be given from the other output of the linear programming model, the expected optimal revenues. Let us assume that the vessel operator has an estimate of the expected demand for transportation with the vessel, and he knows what the remaining vessel capacity of the vessel is, when the customer with the many containers arrives. When the customer arrives, the operator has two options. He can either accept the order of the customer or refuse it.

If the operator does not accept the order, the expected revenues for the vessel is:

$$\mathbf{z}(\mathbf{C}) \tag{10.1}$$

where **C** is the vector of the remaining vessel capacity at the different legs of the network. For simplicity, we have employed the model without cancelations of reservations. The method is the same for any of the linear programming models that we presented.

Under the second scenario, the vessel operator accepts the multiple order of K_i containers from each of G classes of containers. Each class of containers has a freight rate of f_i.

If the operator accepts the order, he realizes a revenue equal to:

$$\sum_{i=1}^{G} f_i \cdot K_i \tag{10.2}$$

At the same time the available capacity for future bookings decreases to $\mathbf{C_A}$, and the revenue potential of the vessel decreases accordingly to

$$\mathbf{z}(\mathbf{C_A}) \tag{10.3}$$

When a customer with several containers wants to have them transported with the vessel of the shipping liner, the operator has two options. He can either accept the order of the customer or refuse it.

The operator should accept the multiple order, if the revenue from this order, plus the expected optimal revenue from the remaining of the capacity of the vessel, is greater than the expected maximum revenue from the current remaining capacity if we do not accept the order. In other words, the vessel operator should accept the multiple order if:

$$\sum_{i=1}^{G} f_i \cdot K_i + \mathbf{z}(\mathbf{C_A}) \geq \mathbf{z}(\mathbf{C}) \tag{10.4}$$

It should be mentioned that the containers of the multiple order do not have to originate from the same port, or have the same port as their destination. In addition, the demand forecast does not change if the operator rejects (equation 10.1), or accepts (equation 10.3), the multiple order. The forecast of the future demand does not depend on the particular decision of the operator to accept or reject a customer. The reason is that the model assumes that any customers that are turned away, will find other shipping liners to transport their containers. That means that whether we accept or reject a customer, this decision does not influence the future demand for the services of the vessel.

When all the containers offered by the customer belong to classes that we currently accept for transportation, and the number of containers per class are such that we stay in policy areas where we accept all the container classes that the customer has to offer, even after we accept the multiple order, then there is no need to apply the method we developed in this paragraph.

10.1.2 Customers with long term contracts

Let us assume now that a shipper wants to rent space on the vessel for a period of time or a specified number of trips. The shipper might want to use this space exclusively for his own transportation needs or he could sublet parts of this space to other shippers. We assume that the shipper wants to sublet $\mathbf{C_C}$ out of the total $\mathbf{C}$ vector of the capacity of the vessel ($\mathbf{C} = (C_1, \ldots, C_N)$, at the N legs of the itinerary of the vessel), for a number of S trips and for the amount of P dollars.

If we look at the problem of long term contracting, we can see that it is a more general case of the multiple orders problem. Since we are examining several trips of the vessel over an extended period of time, we should examine the impact of the contract on the revenue potential of the vessel on each one of the trips separately. We do so because, the expected demand from the different classes of containers along with the capacity of the vessel are the two parameters of the revenue potential of the vessel. Since the expected demand for transportation is subject to seasonal fluctuations, the impact of the contract on the revenue potential of the vessel is a function of time (or the particular trip).

Let us assume that the operator expects the demand for transportation for the j_{th} trip to be $\mathbf{D_j}$. Let us also assume that the capacity managed by the operator of the vessel, in case he accepts the contract with the shipper, is

$$\mathbf{C_R} = \mathbf{C} - \mathbf{C_C} \tag{10.5}$$

The signing of the contract will decrease the revenue potential of the j_{th} trip by

$$z(\mathbf{D_j}, \mathbf{C}) - z(\mathbf{D_j}, \mathbf{C_R}) \tag{10.6}$$

The above amount is the cost of opportunity for for signing the contract. It would be profitable for the vessel operator to sign the contract with the shipper, if the value of the contract is larger than the cumulative cost of opportunity from the S trips. In other words, the operator should accept the contract only if

$$P \geq \sum_{j=1}^{S} \left(z(\mathbf{D_j}, \mathbf{C}) - z(\mathbf{D_j}, \mathbf{C_R}) \right) \tag{10.7}$$

10.2 Reservation System

Although a computerized reservation system is a necessity for the successful application of yield management, it is doubtful whether more than a few shipping companies have a reservation system advanced enough to satisfy the needs of yield management practice. It is recognized that a computerized reservation system that can give at any moment the state of the booking process and the available space, or the database that provides the historical demand from which the vessel operator can make estimations for the future demand, is of major significance.

The more advanced reservation systems of the major airlines have evolved as decision support systems to automated optimization systems. The development of the reservation systems of shipping companies lags behind the development of the equivalent systems of the airlines. As a matter of fact, it is the success of the airline yield management and reservation systems that inspired the introduction of yield management in the shipping liner operations.

Due to the nature of the shipping industry and the fact that most shipping companies have hardly entered the computer age, the goal of most shipping companies is to evolve their reservation systems to become decision support tools. The major obstacle for the shipping companies that would like to establish a computerized reservation system and do scientific application of yield management is the development of a database big enough to support the application of yield management. The development of the database needs the dedication of company resources for a long time before this investment is able to give sizable returns.

The reservation systems of most shipping companies are rather simplistic. When the local office of a shipping company accepts the booking from a customer for transportation between two ports, many times they do not know the exact number of the available container slots on the legs of the trip between these two ports, let alone the expected demand for transportation capacity. Most reservation systems operate based on the judgement and experience of the users of the reservation system. In section 1.2.2, we gave an overview of the reservation system of a typical shipping company.

In summary, we should emphasize, that the customer soliciting is performed by the sales department, which is in the best position to estimate the expected demand from the different classes of customers. The reservations are accepted by the booking department, based on infrequently updated information on the remaining capacity of the legs of the trip under the jurisdiction of the local office of the shipping company. The control of the available vessel capacity and the other company equipment is performed by the equipment control department. It is through the equipment control department that the booking department is given information on the remaining capacity that has not been sold by this or an other local office, and which is available for additional reservations.

In conclusion, the current reservation system of a typical shipping company is structured in such a way that it has available all the vital information. Nevertheless, by not storing all the information in a central accessible place, it cannot exploit it to the fullest extend.

The lack of a database rich enough to give reliable estimates of the expected demand for all classes of containers and origin destination pairs, is not a reason good enough to stop a shipping company from implementing a sophisticated reservation system with the data and the tools that they have at hand. A database that gives statistical estimates of the demand, is an indispensable part of an automated yield management system. Since the shipping companies want to build a reservation system that serves only as a decision support tool at the first stages of its operation, they can do without the vast database needed to automatize the booking process. The two elements necessary for an educated decision are:

- the remaining vessel capacity at all the legs of the trip
- the expected demand from all the classes of containers, irrespectively of the port of origin and destination

The information on the remaining capacity exists within the boundaries of the company. The equipment control department has available (possibly with a time lag) all the relevant information. Information technology can make this information readily and available company wide.

The second requirement of the expected demand is a more difficult one to fulfill. When the operators are not offered the assistance of a database that can help them with the demand forecasts, they have to rely on their own experience and the indications they get from the market. For that reason at the early stages of the operation of the reservation system the performance of the system is tied to the performance of its operators.

During the first phase of the computerized reservation system, and in parallel with the application of yield management, the company can build the database required for the second phase of yield management, when the reservation system with the assistance of the database can evolve to an automated system.

The goal of most airline companies is to make their reservation systems fully automated. This is justifiable, if we consider that each reservation system accepts thousands of reservation requests every day from many small customers. The nature of the shipping business is such that the reservation system of a shipping company accepts a lower rate of requests from larger customers. As a result, the relation of a shipping company with its typical customer, is more complicated than the relation between an airline and the typical passenger. The negotiating power of the shipper of the containers might be such, that the vessel operator might want to override the recommendation of the reservation system and accept this customer. On a more positive note, the shipping company wants to offer a consistently high level of service to its best customers. The ability of the shipping company to offer transportation service to its committed customers when they need it, is a measure of the level of service the shipping company offers to its customers. This flexibility is also necessary for a long term, mutually beneficial working relationship. Wise and restrained use of the discretion of the operator to override the reservation system enhances rather than hurts the profitability of the liner company.

10.2.1 Suggestion for reservation system

In the previous chapters we have described several versions of the ocean yield management problem, ranging from the simple case of the one leg trip, to the complex version of the network of ports with customers who can cancel their reservations at no or at minimal cost, and with the shipping line practicing overbooking in defense.

Most reservation systems treat the yield management problem on the one leg level. The multi- leg problem is usually treated as a collection of several separate yield management problems.

In the current research, we have taken a global perspective of the yield management of the multi- port, multi- commodity reservation problem. We have introduced theory that can treat accurately the multi-port problem, with very few restrictive assumptions, less demanding input data than previous methods, and with limited computational intensity. All of the above suggest that the implementation of a global yield management model is not as difficult a task as it might have been otherwise.

Every yield management person within the company framework should have an understanding of the impact of their decisions concerning reservations on the revenue potential of each vessel of the company. The decision tool that would assist the yield management department to make informed decisions is the software of the reservation system. The input of the reservation system is the information on both aspects of the reservation process: How much capacity we are left with at the different legs of the itinerary of the vessel, and how much demand do we expect for the particular trip.

The need for information on all the parts of the trip, even those parts of the trip that the particular decision maker of the company is not responsible for, is necessary because the ripple effects of each booking decision are propagated across the itinerary and they affect the revenue potential of the vessel.

Description of the reservation system The shipping company accepts reservations and requests for reservations at the different ports, through the local offices of the company and various agents.

The main unit of the reservation system is the central database. It contains the data of all past bookings and realized demand, and it has the ability to analyze that data and give forecasts of the future demand. Additionally, it is the unit which collects, maintains and updates all information relevant to the current booking period. In other words, it incorporates the booking information from the all the changes in capacity availability (reservations and cancelations) on all the legs of the itinerary, and gives the status of the capacity availability at all times. It gathers the estimates of the expected demand at the different legs of the itinerary at a central point. The local offices of the shipping company, have unrestricted access to all information pertinent to the booking process.

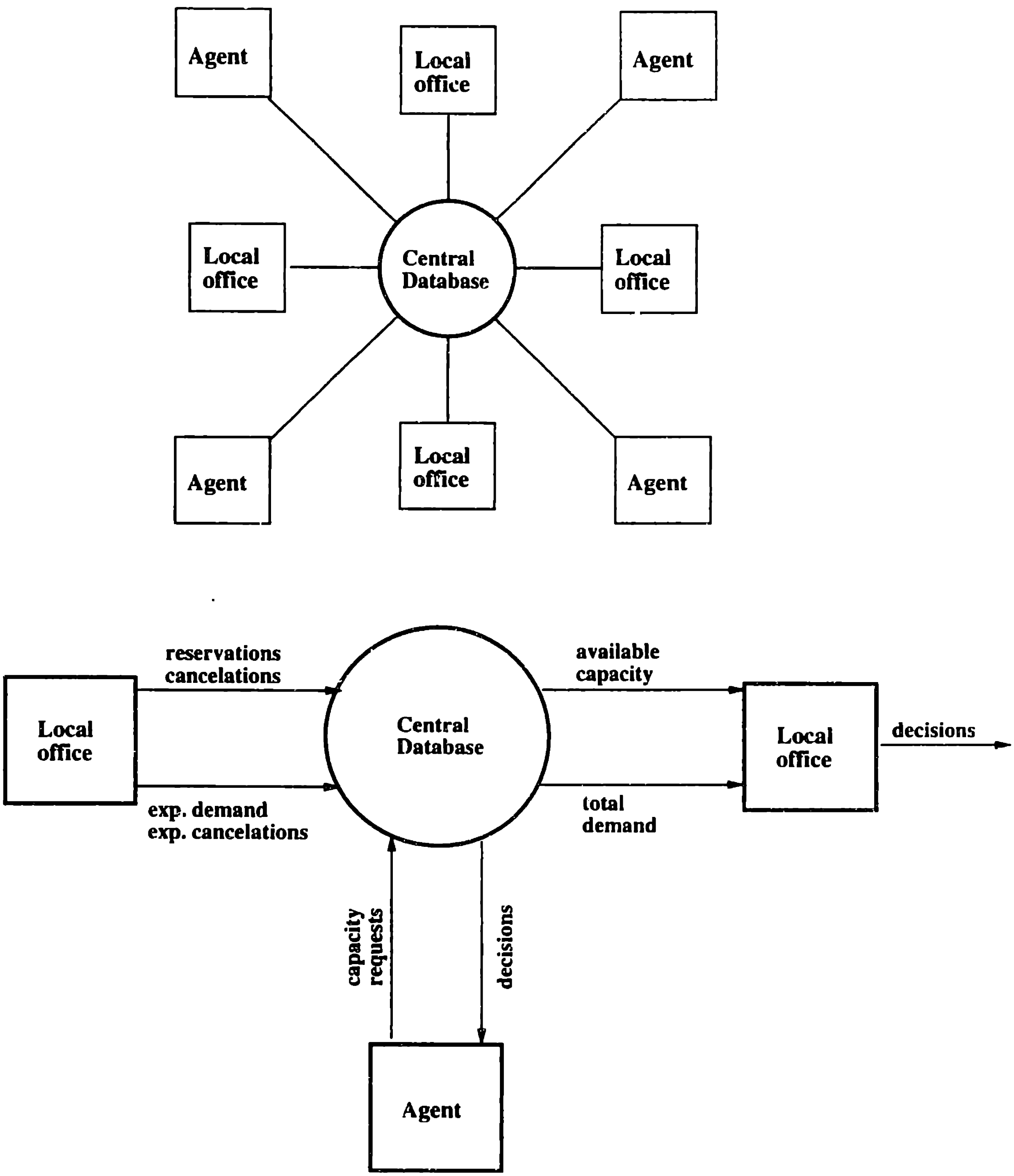


Figure 10-2: Reservation system and flow of information

The reservation acceptance process depends on whether the booking request is made directly to the local (or central) office, or through an agent. When the local office receives a booking request, they access the central data bank of the shipping company. They access the information about the available capacity at the different legs of the trip (not only the legs of the trip that concern the prospective reservation), and the expected demand for all the classes of containers and for all the possible ports of origin and destination. They process the information locally, and based on the suggestion of the reservation system, they accept or they reject the offer. After the local office has decided on whether they accept the reservation or not, they inform the central database, which updates the information concerning the status of the available capacity. Additionally, the local office informs the central database on the possible cancelations of bookings. The local process of the information gives emphasis to the fact that the suggestion of the computer can be overridden if the system operator feels that the assumptions of the computer model are weak at the present situation.

The core of the software of the decision support system is a linear programming model, which is relatively small and can be solved on a computer that would only be a small to moderate investment. At the first stage of the implementation of yield management, where the reservation system is a decision support tool, it would be advisable to make the decisions locally, based on information that is gathered at a central point. At a more advanced phase of the development of the reservation system, when the reservation process is automated, all decisions are made centrally.

The role of the local offices is not limited to the activities we described above. The local offices, having a constant interaction with the local market, are earlier and better informed about the trends in the demand and its fluctuations. As a result their opinion and expectations on the development of the demand, can be incorporated into the central database, and they become part of the information the local offices use in order to determine the optimal booking strategy.

If the bonus system of the shipping company is tied to the performance (i.e. value of reservations) of each separate local office or individual reservation officer, the company runs into the danger of having the advantages of the reservation system nullified. Each office or officer of the company will be too willing to override the recommendation of the reservation system and accept all offers from their customers, if this helps them increase their bonus. The introduction of a new bonus system, with individual pay tied to the performance of the shipping company as a whole, would re-align the incentive of the employees with the goals of the shipping company.

When a customer requests transportation capacity through an agent, the request is sent to a central point where it is processed and a decision is made. The central point updates the database on the basis of the booking decision, and the agent is informed about the status of his booking request. Local conditions and need for an existing relationship between the agents of the company and its local offices, might oblige the company to modify the reservation system and have the requests of the agents be processed through the local offices.

10.3 Data and Forecasting needs

An advanced yield management system has two components. The first is the optimization model that is used as a reservation decision tool. The second component is the database that supports the optimization model. The "optimal" suggestions of the yield management model are only as good as the input data of the optimization model. It has been reported [1] that each 10% improvement in forecast accuracy can increase the revenues of the operator by up to 4%. If we remember that the profits of the carrier are usually just a small percentage of the revenues, an increase of the revenues at the level of 4% can be a considerable increase of the profits of the company.

In the previous chapters we have introduced linear programming optimization models that for all practical purposes give results equally good as their equivalent dynamic programming models. Because of the fact that the linear programming model can address and solve easily the multi-port reservation problem, the focus should shift from the optimization method to the database that supports it.

The application of yield management in the shipping industry is still at the early stages of its development and even the leaders in yield management among the shipping companies have not developed adequate booking reservation systems. On the other hand, airline companies have developed sophisticated, and in many cases automated reservation systems. Airlines do their yield management optimization at the leg rather than the network level. That happens primarily because there had not been developed so far methods that could solve the network airline problem with a limited computer capacity, and within a reasonable accuracy level. The only method available for the solution of the network problem was the heuristic extension of the one leg model.

According to many, a reason why the airline companies have not looked for solutions to the network problem so far is the fact that the optimization model that could solve the network problem would not be able to be supported by the database of the airline company. For instance, it would be preferable for the airline company to consider the contribution of a customer who wants to travel from Boston to San Francisco, on the network level, rather than to examine the contributions of the customer at the legs Boston-Detroit and Detroit-San Francisco separately. The problem is that the airline does not have reliable statistics for all the classes of customers that want to travel from Boston to San Francisco. The data are so sparse that they make the demand distribution for each class of customers on the Boston-San Francisco itinerary to have a variance so large that limits the usefulness of the data.

On the contrary, the same airline has much more reliable statistics for the demand from Boston to the hub of the airline at Detroit. Similarly, the data for the demand for all classes of customers on the flight from Detroit to San Francisco are more reliable. This is a consequence of the hub and spoke network of the airlines. The result is that airline companies have to date focussed on yield management on the leg basis.

If we now consider the shipping liner problem with the many classes of customers and the number of itineraries that is proportional to the square of the number of the ports, we might think that the shipping companies have a more serious problem with the support of the optimization model with statistical data than the airline companies. If we examine the shipping network yield management problem more carefully, we see that the shipping companies might not have the acute problem of the airlines. The passenger of an airline who wants to fly out of Boston can fly through the hub of the airline to all the possible final destinations where the airline flies. Those destinations can easily be more than ten or twenty. On the contrary, the container that books capacity at a given port can only go to one of the other ten at the most ports that the vessel visits. At the same time, it is a feature of the nature of the shipping business that the volumes the containers transported and the capacity of the vessel itself are much larger than the equivalent numbers in the airline itineraries. The result is that the statistics for the shipping network reservation problem can be more meaningful than airline network reservation problem. Finally, we make the observation that not all the possible origin-destination and container class are plausible. In the airline industry we have both high and low fare passengers who want to fly to all possible destinations. In the case of the transpacific route of a shipping company we have refrigerated cargo (fruit) going from Seattle to Japan and Korea. In no other itinerary we can find this class of cargo. On the other hand garments come from Asia to the U.S. and few of all the possible itineraries include this class of containers.

All of the above mean that the large number of all the possible itinerary and class combinations are reduced to a smaller and potentially more manageable number. As a result, the statistics necessary for the network yield management in the shipping industry could well be more meaningful and usable than the similar statistics for the airline industry. If we want to simplify the optimization model further and make the statistical data more reliable, we could cluster in several major groups all the possible classes of containers on the basis of the value of the shipment.

In the database for the shipping liner yield management problem we want to keep track of all the capacity requests that the vessel operator had. We do not need to keep track of the timing of the capacity requests. The database should keep statistics about the cancellation rates of the different classes of customers, that can be used both for the anticipation of the potential cancelations at every trip and the creation of a list that will be used in cases of overbooking. The customers who are the most frequent to cancel their reservations would be the last to be allocated transportation capacity.

10.4 Summary

Many shipping executives doubt the usefulness of the application of yield management in their operations because they are afraid that yield management does not address the peculiarities of the shipping industry. In this chapter, we show how the linear models that we have introduced in previous chapters would be used in order to give answers to the most frequently encountered forms of capacity requests in the shipping liner industry. These forms are, the requests for transportation from customers with many containers that do not necessarily belong to the same class and do not travel on the same itinerary, and the negotiating with customers for long term transportation contracts.

In addition, we examine the place of the yield management decision tool in the framework of the reservation system of the shipping company. We suggest the form of the reservation system that would best exploit the potential of our yield management model.

We have also talked about forecasting issues. There is a consensus among researchers that the development of optimization techniques applied to the airline yield management problem has outstripped the databases on which the optimization models are based. We argue that if there existed in shipping, databases similar to the databases used for the airline yield management they would be able to support advanced network optimization models.

Chapter 11

Conclusions

11.1 Summary of Dissertation

Chapter 1

In Chapter One, the activities and the market of a Shipping Liner within the context of a Shipping Conference were described. We defined the role of a shipping company in the framework of a Shipping Conference and the market that the shipping conference serves. We showed that overcapacity is a characteristic of the quality of service that a Shipping Company offers. The "extra" capacity can be used for the service of customers who are "marginal" to the market of Liner Shipping.

Freight rates for the "marginal" customers are different than the freight rates for the customers that fully utilize the services of the Liner Service. Low freight rate customers can help the bottom line of the Liner in periods of low demand. Nevertheless, they can hurt the company in periods of high demand, if the capacity allocation is not done optimally. Yield Management can help shipping companies to utilize their capacity optimally.

We compared the Liner Shipping against the Airline business. We compared the nature of these two different industries, their reservation systems and the differences of their respective booking processes. In shipping, the freight rate of the container, is a function of the value of the cargo in the container, whereas the fares charged by the Airlines are usually functions of the remaining time from the purchase of the ticket until the day of the flight.

The differences between the two industries suggest that in order to create a decision tool for Yield Management in shipping, we should derive a model based on different assumptions than the assumptions of Airline Yield Management Models.

Chapter 2

In Chapter Two we presented the literature on Yield Management. Since there is non existent literature on Yield Management for Shipping, our literature review focused heavily on Yield Management for Airlines. In addition we presented some of the research done on Yield Management for Hotels. We concluded that the airline yield management model does not describe adequately the reservation process in the shipping industry, and the dynamic programming models of the hotel yield management are too complicated in order to have practical applications.

Chapter 3

We gave the Dynamic Programming (DP) formulation, for the Two Ports (one leg) and M Classes of Goods problem. Proof of the concavity of the optimal expected revenues function $W(t, C)$ (t and C are the remaining time and capacity respectively) We proved the monotonicity of the optimal control variable $I_m(t, C)$ (it is 0 when we do not accept a booking, and 1 when we accept it) with reference to both remaining time and remaining capacity before departure. We also gave a numerical example of the theory developed in this chapter.

The behavior of the output of the DP model was studied. That is, we studied the expected revenue function, and the reservation policies for all the combinations of remaining time and capacity (t, C). We proved the existence of policy areas. The policy areas are compact sets of the (t, C) space, with a given reservation policy, i.e. in a given policy area, we accept some classes of containers and we reject all the others.

Chapter 4

From the discrete time Dynamic Programming formulation of the previous chapter, we derived the partial differential equation, at the limit where time $t \longrightarrow 0$. This is the so called Hamilton-Jacobi-Bellman (HJB) partial differential equation that corresponds to the above DP formulation.

Finding the solution of the HJB equation means that we would have found both the Optimal expected revenues function and the optimal control variables (accept/do not accept an arriving customer) to the original stochastic DP Model. We examine the stochastic two ports, multi-commodity problem with constant commodity arrival rates. We proved that the solution to the two ports problem Yield Management HJB equation, is equal to the optimal objective function of a parametric Linear Programming (LP) Model.

In the formulation of the above L.P. each freight rate class m is represented by one variable x_m. We proved that the control variable $I_m = 1$ when and only when $x_m > 0$. Alternatively $I_m = 0$ when and only when $x_m = 0$.

The net result is that we obtained the solution of the linearized Dynamic Programming model, by solving just a Linear Programming model. The compromise is that we assume the number of the containers to be a positive real (not an integer) number. It should be remembered that the proofs in this chapter were done for constant container rate of arrivals. We gave a table with the analytical solution for the one leg, multiple freight rate classes problem with constant arrival rates.

Chapter 5

We expanded the DP formulation to describe the multi-leg multiple freight rate classes with time variable arrival rates, problem. At the limit, $t \longrightarrow 0$, we derived the Hamilton-Jacobi-Bellman (HJB) partial differential equation of the above DP. Again, we proved that the solution of the HJB equation is the optimal objective function of a parametric linear programming model. The optimal control variables I_{ODm} are equal to 0 or 1, when and only when $x_{ODm} = 0$ or $x_{ODm} > 0$ respectively (OD is a particular Origin-Destination combination or itinerary). We gave a table with the analytical solution for the one leg, multiple freight rate classes problem with time variable commodity arrival rates.

Chapter 6

Shippers have the option to cancel their bookings with liner companies at minimal if any cancellation fees. That can obstruct the optimal utilization of the containership space, and affect the revenue potential of the shipping company.

We studied the above problem, in order to optimize the capacity allocation under the assumption of the above customer behavior. We incorporated cancelations into the DP formulation. According to the model, for each reservation there is the possibility of cancellation at any time before departure. The cancellation rate for each class of containers is a function of time.

The solution to the HJB equation that we derive from the above DP, was again the objective function of a parametric LP. The derivation of the optimal control variable was similar to the derivation of the optimal control variable in the previous chapter.

Chapter 7

As a further development, we expanded the DP formulation so as to include capacity overbooking. Overbooking is the only defense that Shipping companies have when a considerable percentage of their reservations is canceled without any cost to the shippers. We derive the DP formulation that corresponds to the overbooking problem. From that formulation we obtained the respective HJB equation. Once more, the solution to the HJB equation was the objective function of a parametric LP Model. The formulation of the model for overbooking is no different than the formulation of the model for cancelations. The only difference in the D.P. model is that the capacity is allowed to become negative. The boundary conditions change too.

Chapter 8

In this chapter, we developed a heuristic tool that is a variation of the linear programming models presented at the previous chapters. This heuristic linear program gives a marginal revenue curve that becomes a booking criterion closer to the optimum than the marginal revenue curve of the linear program that was developed in the theory. Simulations showed that the maximum relative error of the Expected Revenue derived with the use of the heuristic LP's, relative to the Expected Revenue derived from the full DP Model, is less than 10^{-3}. In contrast, the L.P. that we derived in theory gives a relative error at the order of more than 1%.

Chapter 9

In Chapter Nine it was shown that the models developed so far can also be used by decision makers whose scope of revenue/profit maximization goes beyond the revenue/profit maximization through Yield Management. The models can help determine the optimal prices the operator should charge the different Classes of Customers in order to maximize the revenues/profit of the Shipping Liner.

In the previous chapters we established that the optimal expected value of a vessel trip is given by a parametric Linear Program. In fact, the optimal value of these LP's has shown to deviate no more than 1% from the expected value given by their respective DP formulation.

The objective function of the LP that represents the vessel itinerary is a function of the freight rates of the different origin-destination and class combinations that are served by the vessel. Additionally, it is a function of the capacity of the vessel. A consequence of the above observation was to consider the LP models as tools for pricing. The objective of pricing was the maximization of the shipping liner revenues. The optimal value of the LP is a function of the prices, and we maximized the optimal revenues as a function of the prices. The solution of the pricing model gives the necessary conditions for the freight rates that under optimal booking policies would maximize the revenues of the vessel.

Chapter 10

In this chapter we discussed Implementation issues. The assumption that each order consists of only one container is only infrequently true in the shipping industry. We therefore used the linear programming models developed in the previous chapters in order to incorporate optimal decision making for multiple containers orders into the yield management tool that we develop in this thesis. We also used the linear programming models in order to evaluate the revenue contribution of long term contracts.

We gave the description of a suggested reservation system for the shipping yield management problem. This reservation system suggests that the information is kept in a central database and that the decisions are made locally at the respective offices of the shipping company. We further talked about the databases of the shipping reservation systems. Although the databases of shipping companies are at a much earlier level of development than the equivalent databases of the airlines, they could eventually support the yield management optimization model in a more satisfactory way than the airline databases.

11.2 Contributions

There are several contributions made in the course of this research in terms of both theoretical contribution in the area of yield management as well as practical approaches for shipping companies to increase their revenues through the introduction of sophisticated revenue enhancing practices.

The first contribution of the current thesis is the study of the one leg dynamic programming modeling of the booking process. We concluded that the monotonicity of the booking criterion causes the creation of what we have called *Policy Areas.* The policy areas contain in a compact form all of the booking policy information that we would be given if we had optimal booking suggestions for all the time-capacity combinations of the policy area. If we are given information about the boundaries of each of the M policy areas (where M is the number of container classes), we know the optimal booking policy for all the time-capacity combinations.

The major theoretical contribution of the current work is the proof that the solution of the linearized version of the dynamic programming model is the solution of a linear programming formulation. We gave the proof first for the two ports problem and then we extended it to the multi-port case. The solution of the linearized two ports dynamic programming problem is given in an analytical form. The solutions of both the two and the multiple ports problems are valid for a variety of arrival patterns of the containers. We remind the reader that that the airline yield management literature gives solutions for the two (air)ports problem only, and only for an ordered arrival of the different classes of customers. The extension of the airline yield management problem to the network level has focussed on heuristics.

At the practical level the contribution of the methodology introduced in this thesis is that it combines the best features of both the dynamic programming and the linear programming formulations. It has the modeling accuracy of the D.P. formulation. At the same time it is a linear programming model. It can solve easily problems that involve many ports and classes of containers, and therefore it can be used for real life applications. Furthermore, the input of the L.P. is a condensed version of the data needed to be given to the D.P. model. The input of the D.P. is the probability of customer arrivals at every point in time, whereas the input of the L.P. is the expected demand for the different classes of customers. We do not have to keep a complicated record of the arrivals of the different customers in order to derive the statistics necessary for the linear programming yield management model. This simplicity of the necessary statistics simplifies the company database significantly and reduces the errors of the demand estimations. Every reduction of the statistical errors increases the revenue potential of every itinerary.

We have expanded the original dynamic program to cover the cases of cancelations and overbooking, for both the one leg and the network problem. The dynamic programming model that describes the cancelations and the overbooking is already very slow in the one leg case. On the contrary, the linear program that corresponds to the modified DP gives the solution to the optimal booking problem in a negligible fraction of the time it takes the DP formulation to derive the optimal booking policy. It is practically impossible to find a solution to the D.P. with more than one leg. On the contrary, the solution to the L.P. is always easy to find without employing computationally intensive calculations. Simulations showed that the employment of the L.P. as decision tool gives results that are not lower by more than 2% than the results given by the D.P. application.

We developed a heuristic variation of the linear programming model that we showed in the previous chapters. This heuristic L.P. gives an optimal policy criterion which approximates better the optimal policy suggested by the D.P. formulation. The performance of the heuristic is so good for the customer behavior assumed for the shipping industry that the relative error of the revenue potential between the simulation and the D.P. model is in the vicinity of 0.1%.

It has been a goal of the yield management theory and practice to combine yield management systems and pricing mechanisms in order to enhance the profitability of the carrier. We proposed a pricing model for both the one leg problem and its multi-leg extension. Since the revenue potential of the itinerary as it is given by the linear yield management models is a function of the capacity of the vessel and the level of the freight rates, it was shown how the linear programming yield management models can be used as pricing decision tools, and we gave the necessary conditions for the optimal freight rates.

The major concern of many shipping executives is that although the operational experience of the yield management practice in the airline industry can be transferred to shipping, the lessons that airlines will teach to the shipping industry are not appropriate to assure the long term profitability of the shipping companies. Most of the capacity of the vessel is sold to a few major customers and in the form of long term contracts. This thesis addresses the problem of the evaluation of both large orders and long term contracts. It proposes criteria for the acceptance or rejection of large orders and develops a method for the derivation of the "fair" price that should be paid/asked for a long term contract.

Additionally we have proposed a reservation system for the shipping company. This reservation system is a first generation advanced reservation system. It proposed to be used as a decision support tool in a decentralized decision-making, yet centrally held information system. The decentralized decision scheme was introduced in order to make the introduction of yield management in the shipping operations more easily acceptable. Furthermore, there would be a need for a centralized decision system only in case the system was designed to gradually evolve to a fully automated reservation system. The current level of development of the reservation systems and the nature of the shipping business in general make the automation of the shipping reservation systems in the foreseeable future to be doubtful.

Finally, we discussed the data requirements and forecasting needs of the shipping yield management model. The number of ports that a containership serves are more than the number of the airports an aircraft serves. Therefore the number of the potential origin-destination pairs that a vessel serves is greater. Additionally, the number of classes of containers is larger than the number of the airline customer classes. Developing statistics for all these origin-destination and class combinations would be a daunting task. The statistics would be derived from a few, on average, observations. That would compromise the accuracy of the statistics and the effectiveness of the reservation system. Nevertheless, careful observation showed that the number of the origin-destination and class combinations in shipping can be smaller than the equivalent combinations in the airline industry. We conclude that databases of the same size and level of sophistication would better serve the shipping rather than the airline problem. As a result, the network yield management model that we suggested in the current thesis could be supported by a shipping liner database that would at par with an airline database.

Appendix A

Lemma 1 *$V_m(t,n)$ and $W(t,n)$ are concave functions of n.*

Proof: We will prove the concavity of $W(t,n)$ and $V_m(t,n)$, by induction. Since V_m is a function of W, and vise versa, we will do a simultaneous proof of the concavity of the two functions.

1. From the second of the boundary conditions (equation 3.6) we get that for $\tau = 0$, $W(\tau = 0, n)$ and $V_m(\tau = 0, n)$, $m = 1, \ldots, M$ are concave.

2. We assume that $W(\tau, n)$ and $V_m(\tau, n)$, $\tau = 1, \ldots, t-1$ are concave functions of n. That is, we assume that

$$W(\tau, n+1) \geq \frac{1}{2}W(\tau, n+2) + \frac{1}{2}W(\tau, n), \quad \tau = 1, \ldots, t-1 \tag{A.1}$$

and also that

$$V_m(\tau, n+1) \geq \frac{1}{2}V_m(\tau, n+2) + \frac{1}{2}V_m(\tau, n), \quad \tau = 1, \ldots, t-1 \quad \forall\, m \in \{1, M\} \tag{A.2}$$

3. In order to prove that $W(\tau, n)$ and $V_m(\tau, n)$ are concave functions with reference to n, for $\tau = t$ and for all $m \in \{1, M\}$, we have to prove that

$$W(t, n+1) \geq \frac{1}{2}W(t, n+2) + \frac{1}{2}W(t, n) \tag{A.3}$$

and that

$$V_m(t,n+1) \geq \frac{1}{2}V_m(t,n+2) + \frac{1}{2}V_m(t,n) \; \forall \; m \in \{1, M\} \tag{A.4}$$

From the inequality A.1, and for $\tau = t - 1$, we get:

$$\begin{aligned} &\left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] W(t-1,n+1) \;\; \geq \\ \frac{1}{2}&\left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] W(t-1,n+2) \;\; + \;\; \frac{1}{2}\left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] W(t-1,n) \end{aligned} \tag{A.5}$$

From inequality A.2 and for $\tau = t - 1$, we get:

$$\sum_{i=1}^{M}[\lambda_i(t-1)V_i(t-1,n+1)] \geq \frac{1}{2}\sum_{i=1}^{M}[\lambda_i(t-1)V_i(t-1,n+2)] + \frac{1}{2}\sum_{i=1}^{M}[\lambda_i(t-1)V_i(t-1,n)] \tag{A.6}$$

If we add the left hand side terms of inequality A.5 to the left hand side terms of inequality A.6, we get :

$$\begin{aligned} &\sum_{i=1}^{M}[\lambda_i(t-1)V_i(t-1,n+1)] \;\; + \;\; \left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] W(t-1,n+1) \geq \\ \frac{1}{2}&\sum_{i=1}^{M}[\lambda_i(t-1)V_i(t-1,n+2)] \;\; + \;\; \frac{1}{2}\left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] W(t-1,n+2) + \\ &\frac{1}{2}\sum_{i=1}^{M}[\lambda_i(t-1)V_i(t-1,n)] \;\; \div \;\; \frac{1}{2}\left[1 - \sum_{i=1}^{M} \lambda_i(t-1)\right] W(t-1,n) \end{aligned} \tag{A.7}$$

With the help of the definition 3.3, we can see that inequality A.7 is equivalent to inequality A.3. Therefore, we have proven that $W(t,n)$ is a convex function.

What remains to be shown is that $V_m(t,n)$ is also a concave function of n. Before we are able to continue, we will have to extend the

I define $W(t,x)$ in the interval $(n, n+1)$ as the linear interpolation of the values $W(t,n)$

and $W(t, n+1)$. (See Figure A-1). The function continues being concave. We have:

$$W(t, n+1-i) = [i \cdot W(t,n) + (1-i) \cdot W(t, n+1)], \quad i \in [0,1] \tag{A.8}$$

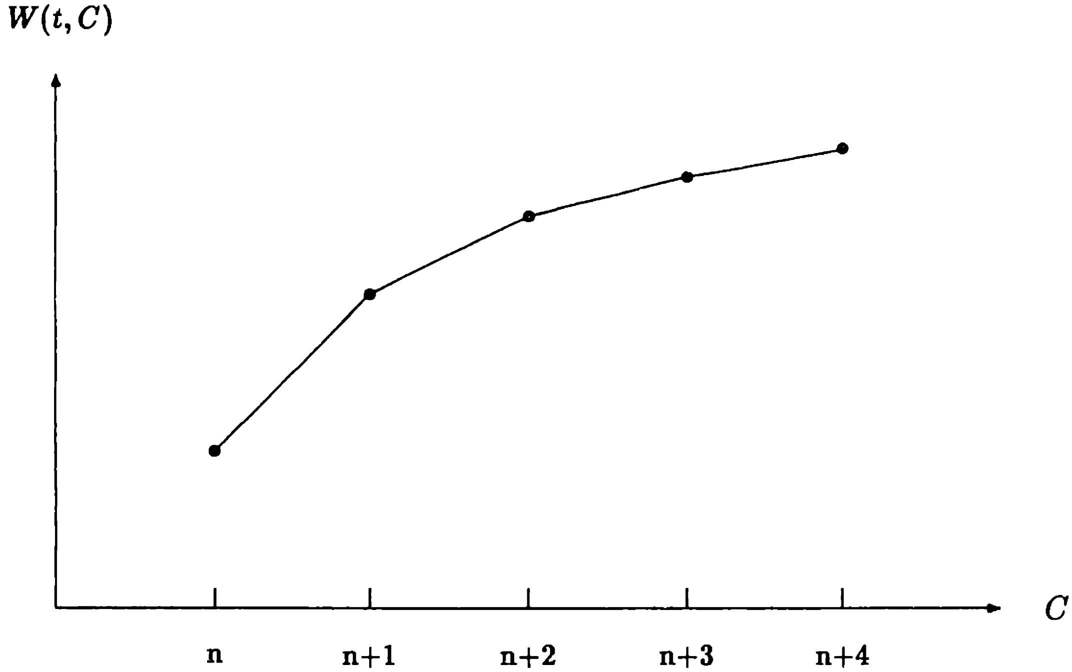


Figure A-1: Definition of $W(t,C)$ for integer and non-integer values of C

In the following we will prove that if we relax the constraint for the optimal control ($I_m(t,n) \in \{0,1\}$) and we allow $i \in [0,1]$, the value of the relaxed optimal control i^*_{m0} will be either 0 or 1. Let:

$$i^*_{mj} = \arg\max_{i \in [0,1]} V_m(t, n+j) = \arg\max_{i \in [0,1]} [i \cdot f_m + W(t, n+j-i)] \tag{A.9}$$

From equations 3.2 and A.8, we get:

$$\begin{aligned} V_m(t, n+1) &= \max_{i \in [0,1]} [i \cdot f_m + W(t, n+1-i)] \\ &= \max_{i \in [0,1]} [i \cdot f_m + i \cdot W(t,n) + (1-i) \cdot W(t, n+1)] \end{aligned}$$

$$= \max_{i \in [0,1]} [i \cdot (f_m + W(t,n)) + (1-i) \cdot W(t,n+1)] \qquad \text{(A.10)}$$

At the above equation A.10 we see that when:

$$\begin{aligned} f_m + W(t,n) \geq W(t,n+1) &\Longrightarrow \quad i^*_{m1} = 1 \\ f_m + W(t,n) < W(t,n+1) &\Longrightarrow \quad i^*_{m1} = 0 \end{aligned} \qquad \text{(A.11)}$$

i.e the optimal control variable is $i^*_{m1} = 1$. Similarly, when

$$f_m + W(t,n) < W(t,n+1)$$

the optimal control becomes $i^*_{m1} = 0$. In a similar way we can show, that even when i^*_{mj} is allowed to take any value in $[0,1]$, it takes either of the two values 0 or 1.

In short, what we have proven is, that if we relax the control variable constraint and allow $I_m(t,n)$ to get not only the integer values 0 and 1 but also to get any values at the interval $(0,1)$, then the optimal values of i will again be either 0 or 1. Therefore, even if we do not restrict $I_m(t,n)$ to get integer values it will do so anyway. That is a property of the specific way we have defined $W(t,C)$, $C \in \mathcal{R}_0^+$. It is a property ofany monotonic (increasing) and convex interpolation. What we have just proved, will help us show the concavity of $V_m(t,n)$. We have:

$$\begin{aligned} V_m(t,n+1) &= i^*_{m1} + W(t,n+1-i^*_{m1}) \\ &\geq \underbrace{\frac{1}{2} \cdot [i^*_{m0} + i^*_{m2}]}_{=0,\frac{1}{2},\text{ or } 1} + W(t,n+1-\frac{1}{2}[i^*_{m0}+i^*_{m2}]) \\ &= \frac{1}{2} \cdot [i^*_{m0} + i^*_{m2}] + W(t,\frac{1}{2}[n-i^*_{m0}] + \frac{1}{2}[n+2-i^*_{m2}]) \\ &\geq \frac{1}{2} \cdot [i^*_{m0} + i^*_{m2}] + \frac{1}{2} \cdot W(t,n-i^*_{m0}) + \frac{1}{2} \cdot W(t,n+2-i^*_{m2}) \\ &= \frac{1}{2} \cdot [i^*_{m0} + W(t,n-i^*_{m0})] + \frac{1}{2} \cdot [i^*_{m2} + W(t,n+2-i^*_{m2})] \\ &= \frac{1}{2} \cdot V_m(t,n) + \frac{1}{2} \cdot V_m(t,n+2) \end{aligned} \qquad \text{(A.12)}$$

The first inequality of A.12 comes from equation A.11, which shows that the maximum over $\{0, 1\}$, is equal to the maximum over $[0, 1]$. The second inequality of A.12 comes from the concavity of the interpolation of $W(t, C)$ for non-integer values of C. Therefore, we have concluded that:

$$V_m(t, n+1) \geq \frac{1}{2}V_m(t, n) + \frac{1}{2}V_m(t, n+2) \tag{A.13}$$

From equation A.13 we conclude that $V_m(t, n)$ is a concave function of n. That concludes the proof of Lemma 1. **Q.E.D.**

Appendix B

Lemma 6 *$W(t,C)$ is the maximum expected revenue of the continuous time model under optimal policy . The optimal policy is such that $I_m(t,C) = 0$, $\forall\ m = 1,\ldots,i-1$, and $I_m(t,C) = 1$, $\forall\ m = i,\ldots,M$. We assume that the partial derivatives of $W(t,C)$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t,C)$ is the equation:*

$$\frac{\partial W(t,C)}{\partial t} + \left[\sum_{m=i}^{M} \lambda_m(t)\right] \cdot \frac{\partial W(t,C)}{\partial C} = \sum_{m=i}^{M} \lambda_m(t) \cdot f_m \tag{B.1}$$

The boundary conditions, for $W(t,C)$ are given by equations 3.5 and 3.6.

Proof: We assume that we are in a particular policy area. Then we have:

$$V_m(t,C) = I_m(t,C) \cdot f_m + W(t, C - I_m(t,C))\ , \quad \forall\ m \in \{1, M\} \tag{B.2}$$

We remind that $I_m(t,C)$ is the optimal policy towards containers from class m while we are in the particular policy area. When $I_m(t,C) = 1$ we accept the container, if it belongs to class m. We do not accept the container of class m if $I_m(t,C) = 0$.

If we substitute equation B.2 at equation 4.2 we get:

$$W(t,C) \quad = \quad \sum_{i=1}^{M} \lambda_i(t-\Delta t) \cdot \Delta t \cdot [I_i(t-\Delta t, C) \cdot f_i + W(t, C - I_i(t-\Delta t, C))]$$

$$+ \quad \left[1 - \Delta t \cdot \sum_{i=1}^{M} \lambda_i(t - \Delta t)\right] \cdot W(t - \Delta t, C) \tag{B.3}$$

If we transfer $W(t - \Delta t, C)$ from the right to the left hand side of the above equation, devide both terms of the equation by Δt, take the limit of this equation for $\Delta t \longrightarrow 0$, and rearrange the terms at the right hand side of the equation, we get:

$$\begin{aligned} \frac{\partial W(t,C)}{\partial t} &= \sum_{i=1}^{M} \lambda_i(t) \cdot I_i(t,C) \cdot f_i \\ &+ \sum_{i=1}^{M} \lambda_i(t) \cdot [W(t, C - I_i(t,C)) - W(t,C)] \end{aligned} \tag{B.4}$$

From Taylor's formula we get:

$$\begin{aligned} W(t,C) &= W(t,C_0) + (C - C_0)\left[\frac{\partial W(t,C)}{\partial C}\right]_{(t,C_0)} + \mathcal{R}_2(t,C,C_0) \\ &\text{where:} \\ \mathcal{R}_2(t,C,C_0) &= \frac{1}{2} \cdot (C_\theta - C_0)^2 \left[\frac{\partial^2 W(t,C)}{\partial C^2}\right]_{(t,C_\theta)} \end{aligned} \tag{B.5}$$

for some $C_\theta \in (\min\{C, C_0\}, \max\{C_0, C\})$

From equation B.5, we get:

$$\begin{aligned} W(t, C - I_i(t,C)) &= W(t,C) - I_i(t,C) \cdot \left[\frac{\partial W(t,C)}{\partial C}\right]_{(t,C)} + \mathcal{R}_2(t, C - I_i(t,C), C) \\ \mathcal{R}_2(t, C - I_i(t,C), C) &= \frac{1}{2} \cdot (C_\theta - C)^2 \left[\frac{\partial^2 W(t,C)}{\partial C^2}\right]_{(t,C_\theta)} \end{aligned}$$

Assumption: We assume that all the partial derivatives of $W(t,C)$, with reference to capacity, except for the first order derivative are equal to zero. In other words we assume that

$$\mathcal{R}_2(t, C - I_i(t,C), C) \cong 0 \tag{B.6}$$

Therefore:

$$W(t, C - I_i(t,C)) \;\cong\; W(t,C) - I_i(t,C) \cdot \left[\frac{\partial W(t,C)}{\partial C}\right]_{(t,C)} \tag{B.7}$$

From equations B.4 and B.7, we get:

$$\frac{\partial W(t,C)}{\partial t} + \left[\sum_{m=1}^{M} \lambda_m(t) \cdot I_m(t,C)\right] \cdot \frac{\partial W(t,C)}{\partial C} = \sum_{m=1}^{M} \lambda_m(t) \cdot f_m \cdot I_m(t,C) \tag{B.8}$$

or:

$$\frac{\partial W(t,C)}{\partial t} + \left[\sum_{m=i}^{M} \lambda_m(t)\right] \cdot \frac{\partial W(t,C)}{\partial C} = \sum_{m=i}^{M} \lambda_m(t) \cdot f_m \tag{B.9}$$

Q.E.D.

Theorem 8 *$W(t,C)$ is the solution of the linearized version of the Dynamic Program given by equation 4.6, when the boundary conditions are given by equations 4.7 and 4.8. $I_m(t,C)$ is the control variable of the Dynamic Programming under optimal policy, that suggests the acceptance ($I_m(t,C) = 1$) or the rejection ($I_m(t,C) = 0$) of a container that belongs to class m, as a function of the remaining time t and the remaining capacity C. $z(t,C)$ is the maximum value, and $x^*_m(t,C)$, $m = 1,\ldots,M$ is the corresponding optimal solution of the Linear Program 4.10. We prove the following:*

$$\{W(t,C) = z(t,C)\} \iff \left\{ \begin{array}{l} I_m(t,C) = 0, \quad x^*_m = 0, \quad \forall\ m = 1,\ldots,i-1 \\ and \\ I_m(t,C) = 1, \quad x^*_m > 0, \quad \forall\ m = i,\ldots,M \end{array} \right\} \tag{B.10}$$

or

$$\{W(t,C) = z(t,C)\} \iff \left\{ \begin{array}{lcl} \beta(t,C) = \sum_{j=i}^{M} \lambda_j f_j & & x^*_m = 0, \quad \forall\, m = 1,\ldots,i-1 \\ & and & \\ \gamma(t,C) = \sum_{j=i}^{M} \lambda_j & & x^*_m > 0, \quad \forall\, m = i,\ldots,M \end{array} \right\}$$

(B.11)

Proof: The right hand side of the two expressions B.10 and B.11 (or 4.23 and 4.24) are equivalent, because

$$\left.\begin{array}{l} I_m(t,C)=0,\ \forall\ m=1,\ldots,i-1 \\ \text{and} \\ I_m(t,C)=1,\ \forall\ m=i,\ldots,M \end{array}\right\} \iff \left\{\begin{array}{l} \beta(t,C)=\sum_{m=i}^{M}\lambda_m f_m \\ \text{and} \\ \gamma(t,C)=\sum_{m=i}^{M}\lambda_m \end{array}\right. \tag{B.12}$$

Therefore we do not have to do a separate proof for each one of them. As a matter of fact, we will use the two equivalent expressions of B.12, interchangeably.

($\Longrightarrow$) We assume that $W(t,C) = \mathbf{z}(t,C)$. We further assume that for the particular (t,C) combination, the linearized form of the DP, suggests that the optimal policy is :

$$\begin{aligned} I_m(t,C) &= 0, \qquad \forall\ m=1,\ldots,k-1 \\ I_m(t,C) &= 1, \qquad \forall\ m=k,\ldots,M \end{aligned} \tag{B.13}$$

whereas the solution of the LP is such that:

$$\begin{aligned} x_m &= 0, \qquad \forall\ m=1,\ldots,i-1 \\ x_m &> 0, \qquad \forall\ m=i,\ldots,M \end{aligned} \tag{B.14}$$

We will prove that $i = k$.

The differential equation 4.6 is the defining equation for the linearized $W(t,C)$. We multiply both sides of the differential equation by t, we substitute $W(t,C)$ with its assumed equivalent $\mathbf{z}(t,C)$, and we move the right hand side of the equation to the left side. Then equation 4.6 becomes:

$$t \cdot \beta(t,C) - t \cdot \frac{\partial \mathbf{z}(t,C)}{\partial t} - t \cdot \gamma(t,C) \cdot \frac{\partial \mathbf{z}(t,C)}{\partial C} = 0 \tag{B.15}$$

with

$$\gamma(t,C) = \sum_{m=i}^{M} \lambda_m \qquad \beta(t,C) = \sum_{m=i}^{M} \lambda_m \cdot f_m \tag{B.16}$$

From lemma 7 we know that the governing equation for $\mathbf{z}(t,C)$ is:

$$\sum_{j=1}^{M} f_j \frac{\partial \mathbf{z}(t,C)}{\partial f_j} - t\frac{\partial \mathbf{z}(t,C)}{\partial t} - C\frac{\partial \mathbf{z}(t,C)}{\partial C} = 0 \tag{B.17}$$

From equations B.15 and B.17, we get:

$$\begin{aligned}
\sum_{j=1}^{M} f_j \frac{\partial \mathbf{z}(t,C)}{\partial f_j} - t\frac{\partial \mathbf{z}(t,C)}{\partial t} - C\frac{\partial \mathbf{z}(t,C)}{\partial C} &= t\beta(t,C) - \\
& \quad t\frac{\partial \mathbf{z}(t,C)}{\partial t} - t\gamma(t,C)\frac{\partial \mathbf{z}(t,C)}{\partial C} \Longleftrightarrow \\
\sum_{j=1}^{M} f_j \frac{\partial \mathbf{z}(t,C)}{\partial f_j} - C\frac{\partial \mathbf{z}(t,C)}{\partial C} &= t\beta(t,C) - t\gamma(t,C)\frac{\partial \mathbf{z}(t,C)}{\partial C} \Longleftrightarrow \\
\sum_{j=i+1}^{M} f_j\lambda_j t + f_i(C - \sum_{j=i+1}^{M} \lambda_j t) - C\frac{\partial \mathbf{z}(t,C)}{\partial C} &= \sum_{j=k}^{M} f_j\lambda_j t - \frac{\partial \mathbf{z}(t,C)}{\partial C}\sum_{j=k}^{M} \lambda_j t \Longleftrightarrow \\
\sum_{j=i}^{M} f_j\lambda_j t + f_i(C - \sum_{j=i}^{M} \lambda_j t) - C\frac{\partial \mathbf{z}(t,C)}{\partial C} &= \sum_{j=k}^{M} f_j\lambda_j t - \frac{\partial \mathbf{z}(t,C)}{\partial C}\sum_{j=k}^{M} \lambda_j t \Longleftrightarrow \\
f_i(C - \sum_{j=i}^{M} \lambda_j t) &= \frac{\partial \mathbf{z}(t,C)}{\partial C}(C - \sum_{j=k}^{M} \lambda_j t) + \\
& \quad (\sum_{j=k}^{M} f_j\lambda_j t - \sum_{j=i}^{M} f_j\lambda_j t) \Longleftrightarrow \\
f_i(C - \sum_{j=i}^{M} \lambda_j t) &= f_i(C - \sum_{j=k}^{M} \lambda_j t) +
\end{aligned}$$

$$
\begin{aligned}
& (\sum_{j=k}^{M} f_j \lambda_j t - \sum_{j=i}^{M} f_j \lambda_j t) \Leftrightarrow \\
i \ = & \ k
\end{aligned} \tag{B.18}
$$

We get the second equation by eliminating $t\frac{\partial \mathbf{z}(t,C)}{\partial t}$ from both sides of the first equation B.18. We go from the second equation to the third by employing equation 4.13 at the left hand side of the equation and equation 4.15 at the right hand side. We get the fourth equation by adding and subtracting $f_i \lambda_i t$ to the left hand side of the third equation. We go to the sixth equation by employing equations 4.14 and 4.15. The sixth equation is a polynomial expression of f_i. From there, we can easily go to the last equation of B.18.

($\Longleftarrow$) We assume that the optimal policy of the DP is such that

$$
\begin{aligned}
I_m(t,C) = 0, \qquad & \forall \ m = 1, \ldots, k-1 \\
I_m(t,C) = 1, \qquad & \forall \ m = k, \ldots, M
\end{aligned} \tag{B.19}
$$

and the solution of the LP given by eq. 4.10, is such that

$$
\begin{aligned}
x_m = 0, \qquad & \forall \ m = 1, \ldots, k-1 \\
x_m > 0, \qquad & \forall \ m = k, \ldots, M
\end{aligned} \tag{B.20}
$$

In other words, we assume that $i = k$. We get the first equation of B.18. From this equation and equation B.17, we conclude that:

$$
t \cdot \beta(t,C) - t \cdot \frac{\partial \mathbf{z}(t,C)}{\partial t} - t \cdot \gamma(t,C) \cdot \frac{\partial \mathbf{z}(t,C)}{\partial C} = 0 \tag{B.21}
$$

Both the linearized $W(t,C)$ and $\mathbf{z}(t,C)$ satisfy the same differential equation, given by equations 4.6 and B.21, and they satisfy the same boundary conditions for when either $t = 0$ or $C = 0$. i.e $W(t,C) = \mathbf{z}(t,C) = 0$, when either $t = 0$ or $C = 0$. That means that equations 4.6, and B.21 satisfy the requirements of the Cauchy-Kowalevski Theorem [40,

p. 74]. Therefore the solution of the two equations 4.6, and B.21 is unique. We conclude that $W(t, C) = \mathbf{z}(t, C)$ **Q.E.D.**

Appendix C

Lemma 10 *$W(t, \mathbf{C})$ is the maximum expected revenue of the continuous time model under optimal policy . The optimal policy is such that $I_i(t, \mathbf{C}) = 0$, $\forall i \in \mathbf{N}$, and $I_i(t, \mathbf{C}) = 1$, $\forall\, i \in \mathbf{Y} = \bigcup_{j=1}^{N} \mathbf{Y_j}$. We assume that the partial derivatives of $W(t, \mathbf{C})$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t, \mathbf{C})$ is the equation:*

$$\frac{\partial W(t, \mathbf{C})}{\partial t} + \sum_{j=1}^{N} \frac{\partial W(t, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} = \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i \tag{C.1}$$

Proof:

We assume that we are in a particular policy area. Each of the different OD_i, m ODC combinations, is either accepted ($I_{OD_i,m}(t, \mathbf{C}) = 1$), or not ($I_{OD_i,m}(t, \mathbf{C}) = 0$), for alll the remaining time, remaining capacity $(t, \mathbf{C})$ combinations, while we are in the particular policy area. The above can be expressed as follows:

$$\begin{aligned} V_{OD_i,m}(t, \mathbf{C}) &= I_{OD_i,m}(t, \mathbf{C}) \cdot f_{OD_i,m} + W(t, \mathbf{C}_{I_{OD_i,m}}) \\ &\qquad \forall\, m \in \{1, M\} \text{ and } \forall\, OD_i\text{'s} \end{aligned} \tag{C.2}$$

where $I_{OD_i,m}(t, \mathbf{C})$ is defined as the optimal $I_{OD_i,m}(t, \mathbf{C})$, for the particular combination of t and $\mathbf{C}$. From the combination of equations 5.9 and C.2 we get:

$$W(t,\mathbf{C}) = \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t-\Delta t) \cdot \left[I_{OD_{i,g}}(t,\mathbf{C}) \cdot f_{OD_{i,g}} + W(t,\mathbf{C}_{I_{OD_{i,g}}})\right]$$

$$+ \left[1 - \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t-\Delta t)\right] \cdot W(t-\Delta t,\mathbf{C})$$

If we transfer $W(t-\Delta t,\mathbf{C})$ from the right to the left hand side of the above equation, devide both terms of the equation by Δt, take the limit of this equation for $\Delta t \longrightarrow 0$, and rearrange the terms at the right hand side of the equation, we get:

$$\frac{\partial W(t,\mathbf{C})}{\partial t} = \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t) \cdot I_{OD_{i,g}}(t,\mathbf{C}) \cdot f_{OD_{i,g}} + \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t) \cdot \left[W(t,\mathbf{C}_{I_{OD_{i,g}}}) - W(t,\mathbf{C})\right] \tag{C.3}$$

From Taylor's formula we get:

$$W(t,\mathbf{C}) = W(t,\mathbf{C_0}) + \sum_{\text{all legs } j} (C_j - C_{j0}) \left[\frac{\partial W(t,\mathbf{C})}{\partial C_j}\right]_{(t,\mathbf{C_0})} + \mathcal{R}_2(t,\mathbf{C},\mathbf{C_0}) \tag{C.4}$$

where:

$$\mathcal{R}_2(t, \mathbf{C}, \mathbf{C_0}) = \frac{1}{2} \cdot \sum_{\text{all legs } j} (C_{j\theta} - C_{j0})^2 \left[\frac{\partial^2 W(t, \mathbf{C})}{\partial C_j^2}\right]_{(t, \mathbf{C}_\theta)} \tag{C.5}$$

for some $C_{j\theta} \in (C_j, C_{j0})$ or (C_{j0}, C_j), $j = 1, \cdots, M$

$$\text{and} \quad \mathbf{C}_\theta = \begin{bmatrix} C_{1\theta} \\ C_{2\theta} \\ \vdots \\ C_{N\theta} \end{bmatrix}$$

Assumption: We assume that the partial derivatives of $W(t, \mathbf{C})$, with reference to Capacity, higher than first degree derivatives are equal to zero. In other words we assume that

$$\mathcal{R}_2(t, \mathbf{C}, \mathbf{C_0}) \cong 0 \tag{C.6}$$

Therefore:

$$W(t, \mathbf{C}) \cong W(t, \mathbf{C_0}) + \sum_{\text{all legs } j} (C_j - C_{j0}) \left[\frac{\partial W(t, \mathbf{C})}{\partial C_j}\right]_{(t, \mathbf{C_0})} \tag{C.7}$$

If we substitute equation C.7, into equation C.3, we get:

$$W(t, \mathbf{C}_{I_{OD_{i,g}}}) = W(t, \mathbf{C}) - \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}} I_{OD_{i,g}}(t, \mathbf{C}) \left[\frac{\partial W(t, \mathbf{C})}{\partial C_j}\right]_{(t, \mathbf{C})} \tag{C.8}$$

In the above summation we only need include only the legs that belong to the path OD_i. The capacity on all the other legs that do not belong to the path OD_i, is not affected by the decision to accept or not accept a container on the path OD_i.

We remind that the path OD_i, is the path whose port of departure is port k and port of destination is port s. Therefore, in the above summation, we include only the legs $k, \ldots, s-1$.

From equations C.3 and C.8, we get:

$$\frac{\partial W(t,\mathbf{C})}{\partial t} + \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t,\mathbf{C}) \cdot \frac{\partial W(t,\mathbf{C})}{\partial C_j} = $$
$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} \cdot I_{OD_i,g}(t,\mathbf{C}) \tag{C.9}$$

At the above equation we can make the following observation. The sum of all the ODC's that we accept in the current policy area, over all the legs used by each of the ODC's, is equal to the sum of all the legs of the network, over all the ODC's that use each leg of the network.

Therefore, the triple sumation at the left hand side of the previous equation C.9, becomes:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t,\mathbf{C}) \cdot \frac{\partial W(t,\mathbf{C})}{\partial C_j} =$$
$$\sum_{\text{all legs } j} \frac{\partial W(t,\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t,\mathbf{C}) \right\} \tag{C.10}$$

If we substitute equation C.10 into equation C.9, we get:

$$\frac{\partial W(t,\mathbf{C})}{\partial t} + \sum_{\text{all legs } j} \frac{\partial W(t,\mathbf{C})}{\partial C_j} \left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t) \cdot I_{OD_{i,g}}(t,\mathbf{C}) \right\} =$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t) \cdot f_{OD_{i,g}} \cdot I_{OD_{i,g}}(t,\mathbf{C}) \tag{C.11}$$

In a more compact form, we have:

$$\frac{\partial W(t,\mathbf{C})}{\partial t} + \sum_{j=1}^{N} \frac{\partial W(t,\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} = \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i \tag{C.12}$$

Q.E.D.

Lemma 11 *$\mathbf{z}(t,\mathbf{C})$ is the maximum value, and $x^*_{OD_{i,g}}(t,\mathbf{C})$, $\forall$ OD_i's and $\forall$ g, is the corresponding optimal solution of the Linear Program 5.19. The governing differential equation for $\mathbf{z}(t,\mathbf{C})$ is the equation*

$$\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} + \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} = \sum_{i \in \mathbf{Y}} f_i \lambda_i(t) \tag{C.13}$$

with the following boundary conditions:

$$\mathbf{z}(t,\mathbf{C}=\mathbf{0}) = 0 \quad \textit{and} \quad \mathbf{z}(t=0,\mathbf{C}) = 0 \tag{C.14}$$

Proof:

At time t the value of the objective function is:

$$\mathbf{z}(t,\mathbf{C}) = \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} x^*_{OD_{i,g}}(t) f_{OD_{i,g}} \tag{C.15}$$

If we are at time $t+\Delta t$ before departure, the Expected Demand of each ODC combination is greater by $\lambda_{OD_{i,g}}(t)\Delta t$.

At leg j we have added the extra quantity (see the definition of $\mathbf{AS_j}$):

$$\sum_{i \in \mathbf{AS_j}} \lambda_i(t)\Delta t$$

The extra revenue from the extra capacity at leg j is:

$$\sum_{i \in \mathbf{AS_j}} f_i \lambda_i(t)\Delta t$$

The extra capacity that has been given to ODC Combinations that belong to the set $\mathbf{AS_j}$, has displaced an equal amount of the ODC Combination that belongs to $\mathbf{MS_j}$:

The lost Revenue is:

$$\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} \sum_{i \in \mathbf{AS_j}} \lambda_i(t)\Delta t$$

As a result, the difference of revenue between $t+\Delta t$ and t for leg j is:

$$\sum_{i \in \mathbf{AS_j}} f_i \lambda_i(t)\Delta t - \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} \sum_{i \in \mathbf{AS_j}} \lambda_i(t)\Delta t \tag{C.16}$$

The difference of revenue between $\mathbf{z}(t+\Delta t, \mathbf{C})$ and $\mathbf{z}(t, \mathbf{C})$ is:

$$\mathbf{z}(t+\Delta t, \mathbf{C}) - \mathbf{z}(t, \mathbf{C}) = \sum_{i \in \bigcup_{j=1}^{N} \mathbf{AS_j}} f_i \lambda_i(t) \Delta t - \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} \sum_{i \in \mathbf{AS_j}} \lambda_i(t) \Delta t \qquad \text{(C.17)}$$

Let

$$\begin{aligned}
A &= \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} f_i \lambda_i(t) \Delta t - \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} \sum_{i \in \mathbf{MS_j}} \lambda_i(t) \Delta t \Longleftrightarrow \\
A &= \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} f_i \lambda_i(t) \Delta t - \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} \lambda_i(t) \left\{ \sum_{\substack{\text{all Legs } j \\ \text{of path } i}} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} \right\} \Delta t \Longleftrightarrow \\
A &= \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} f_i \lambda_i(t) \Delta t - \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} \lambda_i(t) f_i \Delta t \Longleftrightarrow \\
A &= 0 \qquad \text{(C.18)}
\end{aligned}$$

At the right hand side of equation C.17, I add the equivalent of the $A = 0$ (i.e the right hand side) from the first of the equations C.18. We devide both the left and the right hand side of the resultant equation by Δt, and then we take its limit for $\Delta t \longrightarrow 0$, and we get:

$$\frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial t} + \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} \sum_{i \in (\mathbf{AS_j} + \mathbf{MS_j})} \lambda_i(t) = \sum_{i \in \bigcup_{j=1}^{N} (\mathbf{AS_j} + \mathbf{MS_j})} f_i \lambda_i(t) \qquad \text{(C.19)}$$

From the above equation C.19 and equation 5.2, we get that:

$$\frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial t} + \sum_{j=1}^{N} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} = \sum_{i \in \mathbf{Y}} f_i \lambda_i(t) \qquad \text{(C.20)}$$

Q.E.D.

A special case of the yield management problem with contanst arrival rates

For the present subsection and only for it, we will make the assumption that the time it takes to travel from the one port to the other is very small relatively to the remaining time before departure from the first port. That might be a rather weak assumption in the case of an Ocean Liner, but it can be a good assumption in other situations. For instance, an Airliner finishes an itinerary in less than 24 hours, whereas the booking period for the same itinerary lasts more than a month, in most cases.

Therefore: $t_{ODi,g} = t$

We multiply both sides fo equation 5.13 by t and we get:

$$t \cdot \frac{\partial W(t,\mathbf{C})}{\partial t} + t \cdot \sum_{\text{all legs } j} \gamma_j(t,\mathbf{C}) \frac{\partial W(t,\mathbf{C})}{\partial C_j} = t \cdot \beta(t,\mathbf{C}) \qquad \text{(C.21)}$$

with

$$\gamma_j(t,\mathbf{C}) = \sum_{i \in \mathbf{Y}} \lambda_i(t) \qquad \text{(C.22)}$$

$$\beta(t,\mathbf{C}) = \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i \qquad \text{(C.23)}$$

- $\gamma_j(t,\mathbf{C})$ is the sum of all the λ's for only those OD_i's that use leg j, and for each OD_i that includes leg j, only those classes that are accepted for transportation.

- $\beta(t,\mathbf{C})$ is the expected rate of incoming revenue for the accepted OD_i's and classes of goods that are accepted for transportation, under the current Policy.

We want to prove that if we accept Class of Goods g that uses the OD_i itinerary, an itinerary that has a given origin a given destination (i.e. $I_{OD_i,g}(t,\mathbf{C}) = 1$), for transportation under the current Optimal Policy given by the Linearized Dynamic Programming Approach, then the variable $x^*_{OD_i,g}$ of the previously described L.P. that corresponds to the same Class of Goods, is $x^*_{OD_i,g} > 0$, and vice versa.

In other words, we want to prove that $z(t, \mathbf{C}) = W(t, \mathbf{C})$ when and only when, if it is optimal to accept or not a Good for transportation under the D.P. or the L.P. formulation it is optimal to accept or not a Good for transportation under the L.P. or the D.P. formulation as well.

$$I_{OD_i,g}(t, \mathbf{C}) = 1 \iff \quad x^*_{OD_i,g} > 0 \tag{C.24}$$
$$I_{OD_i,g}(t, \mathbf{C}) = 0 \iff \quad x^*_{OD_i,g} = 0$$

We want to prove that the solution of the following L.P. and its Dual are the solutions to the differential equation C.21. The Primal is:

$$\begin{aligned} z(t, \mathbf{C}) = \max \ & \mathbf{f}^T\mathbf{x} \\ \mathbf{Ax} & \leq \mathbf{b} \\ \mathbf{x} & \geq \mathbf{0} \end{aligned} \tag{C.25}$$

and its Dual has as follows:

$$\begin{aligned} \mathbf{v}(t, \mathbf{C}) = \min \ & \mathbf{b}^T\mathbf{y} \\ \mathbf{y}^T\mathbf{A} & \geq \mathbf{f} \\ \mathbf{y} & \geq \mathbf{0} \end{aligned} \tag{C.26}$$

The form of

$$\mathbf{Ax} \leq \mathbf{b}$$

is the following:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} x_{OD_{i},g}(t) \;\leq\; C_j, \;\; j = 1, \ldots, N$$

$$x_{OD_{i},g} \;\leq\; \lambda_{OD_{i},g} \cdot t \,, \;\; \forall \;\; OD_i, g \;\in\; \mathbf{S} \tag{C.27}$$

The boundary conditions of the LP for $t = o$ are:

$$\mathbf{z}(t = 0, \mathbf{C}) = \mathbf{v}(t = 0, \mathbf{C}) = 0 \tag{C.28}$$

Proposition 1 $\mathbf{z}(t, \mathbf{C})$ *is the maximum value, and* $x^{*}_{OD_{i},g}(t, \mathbf{C})$, $\forall$ OD_i*'s and* $\forall$ g*, is the corresponding optimal solution of the Linear Program 5.19. The governing differential equation for* $\mathbf{z}(t, \mathbf{C})$ *is the equation*

$$\sum_{i=1}^{F} f_i \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial f_i} = \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_i} + t \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial t} \tag{C.29}$$

with the following boundary conditions:

$$\mathbf{z}(t, \mathbf{C} = \mathbf{0}) = 0 \quad \textit{and} \quad \mathbf{z}(t = 0, \mathbf{C}) = 0 \tag{C.30}$$

From the Optimality Property of Linear Programming, we get that $\mathbf{z}(t, \mathbf{C}) = \mathbf{v}(t, \mathbf{C})$. Therefore:

$$\sum_{i \in \mathbf{S}} f_i x_i^{*} = \sum_{i=1}^{N} C_i y_i^{*} + \sum_{i \in \mathbf{S}} \lambda_i t y_i^{*} \tag{C.31}$$

We assume that the set of all the possible OCD Combinations OD_i, g, has F elements. Then the above equation becomes:

$$\sum_{i=1}^{F} f_i x_i^{*} = \sum_{i=1}^{N} C_i y_i^{*} + t \sum_{i=N+1}^{N+F} \lambda_{i-N} \cdot y_i^{*} \tag{C.32}$$

where:

$$\begin{aligned} x_i^* &= \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_i}, \;\; i = 1,\ldots,F \\ y_i^* &= \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i}, \;\; i =,\ldots,N \text{ capacity constraints} \\ y_i^* &= \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial bi}, \;\; i = N+1,\ldots,N+F \end{aligned} \tag{C.33}$$

Therefore:

$$\sum_{i=1}^{F} f_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_i} = \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} + t \sum_{i=N+1}^{N+F} \lambda_{i-N} \cdot \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial bi} \tag{C.34}$$

But

$$\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} = \sum_{i=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial b_i} \cdot \frac{\partial b_i}{\partial t} + \sum_{i=N+1}^{N+F} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial b_i} \cdot \frac{\partial b_i}{\partial t} \tag{C.35}$$

with

$$\begin{aligned} \frac{\partial b_i}{\partial t} &= 0, \;\; i = 1,\ldots,N \\ \frac{\partial b_i}{\partial t} &= \frac{\partial(\lambda_{i-N} \cdot t_i)}{\partial t} = \lambda_{i-N}, \;\; i = N+1,\ldots,N+F \end{aligned} \tag{C.36}$$

As a result:

$$\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} = \sum_{i=N+1}^{N+F} \lambda_{i-N} \cdot \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial b_i} \tag{C.37}$$

From equations C.34 and C.37 I get:

$$\sum_{i=1}^{F} f_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_i} = \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} + t \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} \tag{C.38}$$

The above equation is the differential equation that generates the optimal value of $\mathbf{z}(t, \mathbf{C})$ as a function of the time t, the freight rates $f_i, i = 1, \ldots, F$, and the vector of the remaining Capacity $\mathbf{C}$. Q.E.D.

Proposition 2 *$W(t, \mathbf{C})$ is the solution of the linearized version of the Dynamic Program given by equation C.21, when the boundary conditions are given by equations 5.17 and 5.18. $I_{OD_i,g}(t, \mathbf{C})$ is the control variable for the class g containers that travel on the OD_i path, given by the Dynamic Programming under optimal policy. The acceptance ($I_{OD_i,g}(t, \mathbf{C}) = 1$) or the rejection ($I_{OD_i,g}(t, \mathbf{C}) = 0$) of a container that belongs to the OD_i, g ODC combination is as a function of the remaining time t and the remaining capacity vector $\mathbf{C}$. $\mathbf{z}(t, \mathbf{C})$ is the maximum value, and $x^*_{OD_i,g}(t, \mathbf{C})$, $\forall\ OD_i$'s and $\forall\ g$, is the corresponding optimal solution of the Linear Program C.25. We prove the following:*

$$\{W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C})\} \iff \left\{ \begin{array}{l} I_{OD_i,g}(t, \mathbf{C}) = 0, \quad x^*_{OD_i,g} = 0, \quad \forall\ OD_i, g \in \mathbf{N} \\ and \\ I_{OD_i,g}(t, \mathbf{C}) = 1, \quad x^*_{OD_i,g} > 0, \quad \forall\ OD_i, g \in \mathbf{Y} \end{array} \right\} \tag{C.39}$$

or:

$$\{W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C})\} \iff$$

$$\left\{ \begin{array}{lcl} \beta(t, \mathbf{C}) = \sum_{OD_i,g \in \mathbf{Y}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} & & x^*_{OD_i,g} = 0, \quad \forall\ OD_i, g \in \mathbf{N}, \\ & and & \\ \gamma(t, \mathbf{C}) = \sum_{OD_i,g \in \mathbf{Y}} \lambda_{OD_i,g}(t) & & x^*_{OD_i,g} > 0, \quad \forall\ OD_i, g \in \mathbf{Y} \end{array} \right\} \tag{C.40}$$

Proof:

($\Longrightarrow$)

We assume that $W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C})$. By substituting $\mathbf{z}(t, \mathbf{C})$ for $W(t, \mathbf{C})$ in equation C.21, we get:

$$t\beta(t,\mathbf{C}) - t\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} - t \sum_{\text{all legs } j} \gamma_j(t,\mathbf{C})\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} = 0 \qquad \text{(C.41)}$$

From equation C.38 we get:

$$\sum_{i=1}^{F} f_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_i} - t\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} - \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} = 0 \qquad \text{(C.42)}$$

From the solution of the L.P. Problem C.25 we find that out of all the x_j^*, $i = 1, \ldots, F$, some are not accepted at all ($x_j^* = 0$) and therefore $x_j^* \in \mathbf{N}$, some others are accepted partially and therefore $x_j^* \in \mathbf{M}$, and some others are accepted up to the demand that they can generate, in which case we have $x_j^* \in \mathbf{A}$.

Then we have:

$$\begin{aligned}
x_j^* = \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_j} &= 0 \quad \text{if } j \in \mathbf{NS_i} \\
x_j^* = \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_j} &= \lambda_i \cdot t \quad \text{if } j \in \mathbf{AS_i} \\
x_j^* = \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_j} &= C_j - \sum_{m \in \mathbf{AS_i}} \lambda_m t - \sum_{m \in \mathbf{MS_i} - j} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_m} \\
&= \text{capacity allocated to } j \text{ if } j \in \mathbf{MS_i} \qquad \text{(C.43)}
\end{aligned}$$

We assume that the Dynamic Programming formulation gives us $\mathbf{M}^* \neq \mathbf{M}$ and $\mathbf{A}^* \neq \mathbf{A}$. We will prove by contradiction that $\mathbf{M}^* = \mathbf{M}$ and $\mathbf{A}^* = \mathbf{A}$.

From equations C.41 and C.42 we get:

$$\begin{aligned}
t\beta(t,\mathbf{C}) - t\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} - t \sum_{\text{all legs } j} \gamma_j(t,\mathbf{C})\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} &= \qquad \text{(C.44)} \\
= \sum_{i=1}^{F} f_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_i} - t\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial t} - \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} &\Longleftrightarrow
\end{aligned}$$

$$t\beta(t,\mathbf{C}) - t \sum_{\text{all legs } j} \gamma_j(t,\mathbf{C}) \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} = \sum_{i=1}^{F} f_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_i} - \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \tag{C.45}$$

We define LHS to be the left hand side of equation C.45

$$\begin{aligned}
LHS &= t\beta(t,\mathbf{C}) - t\sum_{i=1}^{N} \gamma_i(t,\mathbf{C}) \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= t\sum_{j=1}^{F} \lambda_j f_j I_j(t,\mathbf{C}) - \sum_{i=1}^{N} t\gamma_i(t,\mathbf{C}) \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}^\bullet} t\lambda_j f_j + \sum_{j \in \mathbf{A}^\bullet} t\lambda_j f_j - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS}_i^\bullet} \lambda_j t + \sum_{j \in \mathbf{MS}_i^\bullet} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}^\bullet} \lambda_j t f_j + \sum_{j \in \mathbf{A}^\bullet} t\lambda_j f_j \\
&- \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS}_i^\bullet} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{MS}_i^\bullet} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}^\bullet} \lambda_j t f_j + \sum_{j \in \mathbf{A}^\bullet} t\lambda_j f_j \\
&- \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS}_i^\bullet} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i=1}^{N} \lambda_{MS_i} t \left\{ \sum_{\substack{\text{all Legs } j \\ \text{over which } MS_i \\ \text{extends}}} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} \right\}
\end{aligned} \tag{C.46}$$

$$\begin{aligned}
LHS &= \sum_{j \in \mathbf{M}^\bullet} \lambda_j t f_j + \sum_{j \in \mathbf{A}^\bullet} t\lambda_j f_j \\
&- \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS}_i^\bullet} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i \in \mathbf{M}^\bullet} \lambda_{MS_i} t f_{MS_i} \Longrightarrow
\end{aligned} \tag{C.47}$$

$$LHS = \sum_{j \in \mathbf{A}^\bullet} t\lambda_j f_j - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS}_i^\bullet} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \Longrightarrow$$

$$LHS = \sum_{j \in \mathbf{A}^*} t\lambda_j f_j - \sum_{i \in \mathbf{M}^*} \left\{ \sum_{j \in \mathbf{AS}_i^*} \lambda_j t \right\} f_{MS_i} \tag{C.48}$$

We have already mentioned that x_j^* is the limited capacity allocated to the class of containers with freight rate $f_j \; \forall \; j \; \in \; \mathbf{MS_i}$

$$x_j^* = \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial f_j} \tag{C.49}$$

We have:

$$\sum_{\substack{\text{all Legs } j \\ \text{over which } OD_i \\ \text{extends}}} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} = f_{MS_i} \tag{C.50}$$

because if we give one extra unit of capacity to all the legs that are used by a particular MS_i, our revenues will increase by f_{MS_i}

That can be argued as follows: If we have an increase of the capacity available at the leg i by one unit, then we accept one more unit of f_{MS_i}. Our revenues increase by f_{MS_i}. At the same time though, f_{MS_i} takes one unit of capacity from each of the legs that it uses. At each of the legs j that it uses (with the exception of leg i) it decreases the revenue by $\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j}$. Therefore we get:

$$\frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_{MS_i}} = f_{MS_i} - \sum_{\substack{\text{all Legs } j \\ \text{over which } OD_i \\ \text{extends} \\ \text{except for leg } i}} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} \Longrightarrow$$

$$f_{MS_i} = \sum_{\substack{\text{all Legs } j \\ \text{over which } OD_i \\ \text{extends}}} \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} \tag{C.51}$$

We define RHS to be the right hand side of equation C.45

$$
\begin{aligned}
RHS &= \\
&= \sum_{j=1}^{F} f_j \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_j} - \sum_{i=1}^{N} C_i \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}} f_j \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_j} + \sum_{j \in \mathbf{A}} f_j \underbrace{\frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial f_j}}_{\lambda_j t} - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{S_i}} x_j^* \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}} f_j x_j^* + \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{NS_i}} x_j^* + \sum_{j \in \mathbf{AS_i}} x_j^* + \sum_{j \in \mathbf{MS_i}} x_j^* \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}} f_j x_j^* + \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS_i}} x_j^* \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{MS_i}} x_j^* \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{M}} f_j x_j^* + \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS_i}} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i=1}^{N} x_{MS_i}^* \left\{ \sum_{\substack{\text{all Legs } j \\ \text{over which } MS_i \\ \text{extends}}} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_j} \right\} \\
&= \sum_{j \in \mathbf{M}} f_j x_j^* + \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS_i}} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i=1}^{N} x_{MS_i}^* f_{MS_i} \\
&= \sum_{j \in \mathbf{M}} f_j x_j^* + \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS_i}} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} - \sum_{i \in \mathbf{M}} x_{MS_i}^* f_{MS_i} \\
&= \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i=1}^{N} \left\{ \sum_{j \in \mathbf{AS_i}} \lambda_j t \right\} \frac{\partial \mathbf{z}(t,\mathbf{C})}{\partial C_i} \\
&= \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i \in \mathbf{M}} \left\{ \sum_{j \in \mathbf{AS_i}} \lambda_j t \right\} f_{MS_i} \qquad \text{(C.52)}
\end{aligned}
$$

From equation C.45 we know that $LHS = RHS$. Therefore

$$\sum_{j \in \mathbf{A}^*} t\lambda_j f_j - \sum_{i \in \mathbf{M}^*} \left\{ \sum_{j \in \mathbf{AS}_i^*} \lambda_j t \right\} f_{MS_i} = \sum_{j \in \mathbf{A}} f_j \lambda_j t - \sum_{i \in \mathbf{M}} \left\{ \sum_{j \in \mathbf{AS}_i} \lambda_j t \right\} f_{MS_i} \quad \text{(C.53)}$$

From the above equation we see that we have to have $\mathbf{A} = \mathbf{A}^*$, and $\mathbf{M} = \mathbf{M}^*$ for the equality to hold.

We have proven that when $W(t, \mathbf{C}) = \mathbf{z}(t, \mathbf{C}) \Longrightarrow \mathbf{M} = \mathbf{M}^*$ and $\mathbf{A} = \mathbf{A}^*$.

($\Longleftarrow$) If we have the assumptions $\mathbf{M} = \mathbf{M}^*$ and $\mathbf{A} = \mathbf{A}^*$, by following the reverse process, we get the first equation of C.45. From C.45 and equation C.42, we conclude that:

$$t\beta(t, \mathbf{C}) - t\frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial t} - t \sum_{\text{all logs } j} \gamma_j(t, \mathbf{C}) \frac{\partial \mathbf{z}(t, \mathbf{C})}{\partial C_j} = 0 \quad \text{(C.54)}$$

Both, the solution $W(t, C)$ of the linearized D.P. and the solution $\mathbf{z}(t, \mathbf{C})$ of the L.P. satisfy the same boundary conditions for $t = 0$ and $C = 0$. i.e $W(t, C) = \mathbf{z}(t, \mathbf{C}) = 0$, for either $t = 0$ or $\mathbf{C} = \mathbf{0}$. That means that equations C.21, and C.41 satisfy the requirements of the Cauchy-Kowalevski Theorem [40, p. 74]. Therefore the solution of the two equations C.21, and C.41 is unique. Therefore: $W(t, C) = \mathbf{z}(t, \mathbf{C})$ **Q.E.D.**

Appendix D

Lemma 14 *$W(t, \mathbf{A}, \mathbf{C})$ is the maximum expected revenue under optimal policy of the continuous time model. The optimal policy is such that $I_i(t, \mathbf{A}, \mathbf{C}) = 0$, $\forall\ i \in \mathbf{N}$, and $I_i(t, \mathbf{A}, \mathbf{C}) = 1$, $\forall i \in \mathbf{Y} = \bigcup_{j=1}^{N} \mathbf{Y_j}$. We assume that the partial derivatives of $W(t, \mathbf{A}, \mathbf{C})$, with reference to capacity, higher than the first derivatives, are equal to zero. The governing differential equation for $W(t, \mathbf{A}, \mathbf{C})$ is the equation:*

$$
\begin{aligned}
\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot f_i \\
&+ \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_i} - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_i} \\
&- \sum_{j=1}^{N} \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) - \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \qquad \text{(D.1)}
\end{aligned}
$$

Proof: We are in a Policy area where we have that:

$$
\begin{aligned}
V_{OD_i,m}(t, \mathbf{A}, \mathbf{C}) &= I_{OD_i,m}(t, \mathbf{A}, \mathbf{C}) \cdot f_{OD_i,m} + W(t, \mathbf{A}_{I_{OD_i,m}}, \mathbf{C}_{I_{OD_i,m}}) \\
&\quad \forall\ m \in \{1, M\} \text{ and } OD_i\text{'s} \qquad \text{(D.2)}
\end{aligned}
$$

where $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$ is defined as the optimal $I_{OD_i,m}(t, \mathbf{A}, \mathbf{C})$, for the particular combination of t, **A** and **C**. From equations 6.3 and D.2 we get:

$$
\begin{aligned}
&W(t+\Delta t,\mathbf{A},\mathbf{C}) = \\
&= \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \left[I_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \cdot f_{OD_i,g} + W(t, \mathbf{A}_{I_{OD_i,g}}, \mathbf{C}_{I_{OD_i,g}}) \right] \\
&+ \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot [-f_{OD_i,g} + W(t, \mathbf{A}_{OD_i,g}, \mathbf{C}_{OD_i,g})] \\
&+ \left[1 - \Delta t \cdot \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \{ \mu_{OD_i,g}(t) \cdot A_{OD_i,g} + \lambda_{OD_i,g}(t) \} \right] \cdot W(t,\mathbf{A},\mathbf{C}) \qquad \text{(D.3)}
\end{aligned}
$$

If we transfer $W(t, \mathbf{A}, \mathbf{C})$ from the right to the left hand side of the above equation, devide both terms of the equation by Δt, take the limit of this equation for $\Delta t \longrightarrow 0$, and rearrange the terms at the right hand side of the equation, we get:

$$
\begin{aligned}
\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} \cdot I_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\
&- \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot f_{OD_i,g} \\
&+ \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \left[W(t,\mathbf{A}_{I_{OD_i,g}},\mathbf{C}_{I_{OD_i,g}}) - W(t,\mathbf{A},\mathbf{C})\right] \\
&+ \sum_{\substack{\text{all } OD_i\\ \text{pairs}}} \sum_{\substack{\text{all Classes } g\\ \text{for each } OD_i\\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot \left[W(t,\mathbf{A}_{OD_i,g},\mathbf{C}_{OD_i,g}) - W(t,\mathbf{A},\mathbf{C})\right]
\end{aligned}
\tag{D.4}
$$

From Taylor's formula we get:

$$
\begin{aligned}
W(t,\mathbf{A}_{I_{OD_i,m}},\mathbf{C}_{I_{OD_i,m}}) &= W(t,\mathbf{A},\mathbf{C}) \\
&+ (A_j + I_{OD_i,m} - A_j)\left[\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_j}\right]_{(t,\mathbf{A},\mathbf{C})} \\
&+ \sum_{\text{all legs } j} (C_j - I_{OD_i,m} - C_j)\left[\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j}\right]_{(t,\mathbf{A},\mathbf{C})} \\
&+ \mathcal{R}_2(t,\mathbf{A}_{I_{OD_i,m}},\mathbf{A},\mathbf{C}_{I_{OD_i,m}},\mathbf{C})
\end{aligned}
\tag{D.5}
$$

where $\mathcal{R}_2(t,\mathbf{A}_{I_{OD_i,m}},\mathbf{A},\mathbf{C}_{I_{OD_i,m}},\mathbf{C})$ is the sum of the higher order terms of the Taylor summation.

Assumption: We assume that the partial derivatives of $W(t,\mathbf{A},\mathbf{C})$, with reference to Capacity, higher than first degree derivatives are equal to zero. In other words we assume that

$$\mathcal{R}_2(t, \mathbf{A}_{I_{OD_i,m}}, \mathbf{A}, \mathbf{C}_{I_{OD_i,m}}, \mathbf{C}) \cong 0 \tag{D.6}$$

Therefore:

$$\begin{aligned} W(t, \mathbf{A}_{I_{OD_i,m}}, \mathbf{C}_{I_{OD_i,m}}) &\cong W(t, \mathbf{A}, \mathbf{C}) \\ &+ I_{OD_i,m} \left[\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_j}\right]_{(t,\mathbf{A},\mathbf{C})} \\ &- I_{OD_i,m} \sum_{\substack{\text{all legs } j \\ \text{for the } OD_i \\ \text{pair}}} \left[\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j}\right]_{(t,\mathbf{A},\mathbf{C})} \end{aligned} \tag{D.7}$$

From equation D.7, we get:

$$W(t, \mathbf{A}_{I_{OD_i,m}}, \mathbf{C}_{I_{OD_i,m}}) - W(t, \mathbf{A}, \mathbf{C}) =$$

$$I_{OD_i,m} \left\{ \left[\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_i,m}}\right]_{(t,\mathbf{A},\mathbf{C})} - \sum_{\substack{\text{all legs } j \\ \text{for the } OD_i \\ \text{pair}}} \left[\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j}\right]_{(t,\mathbf{A},\mathbf{C})} \right\} \tag{D.8}$$

In a similar way, we can have that:

$$W(t, \mathbf{A}_{OD_i,m}, \mathbf{C}_{OD_i,m}) - W(t, \mathbf{A}, \mathbf{C}) =$$

$$- \left\{ \left[\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_i,m}} \right]_{(t,\mathbf{A},\mathbf{C})} - \sum_{\substack{\text{all legs } j \\ \text{for the } OD_i \\ \text{pair}}} \left[\frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \right]_{(t,\mathbf{A},\mathbf{C})} \right\} \tag{D.9}$$

In the above summation we include only the legs that belong to the path OD_i. We have to remind that the path OD_i, is the path that has port k as the port of departure and port s the port of destination.

Therefore, in the above summation, we include only the legs $k, \ldots, s-1$. We also remind here that the $I_{OD_i,g}$ is defined only for the path OD_i and the legs $k, \ldots, s$, can be either 0 or 1.

From equations D.4, D.8 and D.9, we get:

$$
\begin{aligned}
\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial t} &= \\
= & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \; \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} \cdot I_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\
- & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \; \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot f_{OD_i,g} \\
+ & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \; \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g} \cdot \left\{ \left[\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_{OD_i,g}}\right] - \sum_{\substack{\text{all legs j} \\ \text{for the } OD_i \\ \text{pair}}} \left[\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j}\right] \right\} \\
- & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \; \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot \left\{ \left[\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_{OD_i,g}}\right] - \sum_{\substack{\text{all legs j} \\ \text{for the } OD_i \\ \text{pair}}} \left[\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j}\right] \right\}
\end{aligned}
\tag{D.10}
$$

But

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) \cdot \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} =$$

$$\sum_{\text{all legs } j} \frac{\partial W(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) \right\} \tag{D.11}$$

The above equation we can be explaned with the following observation. The sum of all the ODC's that we accept in the current policy area, over all the legs used by each of the ODC's, is equal to the sum of all the legs of the network, over all the ODC's that use each leg of the network.

We also get an equation similar to D.11, for the fourth term of equation D.10.

From equations D.10 and D.11 and the equation for the fourth term of equation D.10, we get:

$$
\begin{aligned}
\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial t} &= \\
= & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot f_{OD_i,g} \cdot I_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\
- & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot f_{OD_i,g} \\
+ & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_{OD_i,g}} \cdot I_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \\
- & \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_{OD_i,g}} \\
- & \sum_{\text{all legs } j} \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot I_{OD_i,g}(t,\mathbf{A},\mathbf{C}) \right\} \\
+ & \sum_{\text{all legs } j} \frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \right\} \qquad \text{(D.12)}
\end{aligned}
$$

An alternative form of the above equation D.12, is:

$$\begin{aligned}\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{i\,\in\,\mathbf{Y}} \lambda_i(t)\cdot f_i - \sum_{i\,\in\,\mathbf{S}} \mu_i(t)\cdot A_i\cdot f_i \\ &+ \sum_{i\,\in\,\mathbf{Y}} \lambda_i(t)\cdot\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_i} - \sum_{i\,\in\,\mathbf{S}} \mu_i(t)\cdot A_i\cdot\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial A_i} \\ &- \sum_{j=1}^{N}\frac{\partial W(t,\mathbf{A},\mathbf{C})}{\partial C_j}\cdot\left\{\sum_{i\,\in\,\mathbf{Y_j}}\lambda_i(t) - \sum_{i\,\in\,\mathbf{S_j}}\mu_i(t)\cdot A_i\right\} \qquad \text{(D.13)}\end{aligned}$$

This is the final form of the Differential Equation. **Q.E.D.**

Lemma 15 $\mathbf{z}(t,\mathbf{A},\mathbf{C})$ *is the maximum value, and* $x^{*}_{OD_i,g}(t,\mathbf{A},\mathbf{C})$*,* $\forall\ OD_i$*'s and* $\forall\ g$*, is the corresponding optimal solution of the Linear Program 6.9. The governing differential equation for* $\mathbf{z}(t,\mathbf{A},\mathbf{C})$ *is the equation*

$$\begin{aligned}\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{i\,\in\,\mathbf{Y}} \lambda_i(t)\cdot f_i - \sum_{i\,\in\,\mathbf{S}} \mu_i(t)\cdot A_i\cdot f_i \\ &+ \sum_{i\,\in\,\mathbf{Y}} \lambda_i(t)\cdot\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_i} - \sum_{i\,\in\,\mathbf{S}} \mu_i(t)\cdot A_i\cdot\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_i} \\ &- \sum_{j=1}^{N}\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j}\cdot\left\{\sum_{i\,\in\,\mathbf{Y_j}}\lambda_i(t) - \sum_{i\,\in\,\mathbf{S_j}}\mu_i(t)\cdot A_i\right\} \qquad \text{(D.14)}\end{aligned}$$

with the following boundary conditions:

$$\mathbf{z}(t,\mathbf{A},\mathbf{C}) = 0 \quad \textit{and} \quad \mathbf{z}(t=0,\mathbf{A},\mathbf{C}) = 0 \qquad \text{(D.15)}$$

Proof: At time t the value of the objective function is:

$$\mathbf{z}(t,\mathbf{A},\mathbf{C}) = \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} f_{OD_{i},g} \cdot \left(x^{*}_{OD_{i},g} - A_{\mathrm{C}OD_{i},g} \right) \tag{D.16}$$

If we are at time $t + \Delta t$ before departure, the Expected Demand of each ODC combination is greater by:

$$\Delta L_{OD_{i},g}(t) = \frac{\partial L_{OD_{i},g}(t)}{\partial t} \cdot \Delta t$$

with

$$\frac{\partial L_{OD_{i},g}(t)}{\partial t} = \lambda_{OD_{i},g}(t) \cdot e^{-\int_{\nu=0}^{t} \mu_{OD_{i},g}(\nu) d\nu} \tag{D.17}$$

The additional number of the initial $A_{OD_{i},g}$ Containers from the particular ODC Combination, expected to defect in the extra Δt, is:

$$\Delta A_{\mathrm{C}OD_{i},g} = \frac{\partial A_{\mathrm{C}OD_{i},g}(t)}{\partial t} \cdot \Delta t$$

with

$$\frac{\partial A_{\mathrm{C}OD_{i},g}(t)}{\partial t} = \mu_{OD_{i},g}(t) \cdot A_{OD_{i},g} \cdot e^{-\int_{\nu=0}^{t} \mu_{OD_{i},g}(\nu) d\nu} \tag{D.18}$$

At leg j we have added the extra containers:

$$\sum_{i \in \mathbf{AS_j}} \Delta L_i(t) - \sum_{i \in \mathbf{S_j}} \Delta A_{\mathrm{C}i}(t) = \sum_{i \in \mathbf{AS_j}} \frac{\partial L_i(t)}{\partial t} \cdot \Delta t - \sum_{i \in \mathbf{S_j}} \frac{\partial A_{\mathrm{C}i}(t)}{\partial t} \cdot \Delta t \tag{D.19}$$

(see the definition of $\mathbf{AS_j}$ and $\mathbf{S_j}$).

Because of the extra containers, the additional revenue at leg j is:

$$\sum_{i \in \mathbf{AS_j}} f_i \cdot \frac{\partial L_i(t)}{\partial t} \cdot \Delta t \quad - \quad \sum_{i \in \mathbf{S_j}} f_i \cdot \frac{\partial A_{\mathrm{C}i}(t)}{\partial t} \cdot \Delta t \tag{D.20}$$

The extra Capacity that has been given to ODC Combinations that belong to the set $\mathbf{AS_j}$, has displaced an equal amount of the ODC Combination that belongs to $\mathbf{MS_j}$:
The lost Revenue at leg j is:

$$\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j}\left\{\sum_{i\in \mathbf{AS_j}}\frac{\partial L_i(t)}{\partial t}-\sum_{i\in \mathbf{S_j}}\frac{\partial A_{\mathrm{C}i}(t)}{\partial t}\right\}\cdot \Delta t \tag{D.21}$$

As a result, the difference of revenue between $t+\Delta t$ and t for leg j is:

$$\begin{aligned}\mathrm{DR}_j(t+\Delta t,t) &= \sum_{i\in \mathbf{AS_j}} f_i\cdot\frac{\partial L_i(t)}{\partial t}\cdot\Delta t-\sum_{i\in \mathbf{S_j}} f_i\cdot\frac{\partial A_{\mathrm{C}i}(t)}{\partial t}\cdot\Delta t\\ &- \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j}\left\{\sum_{i\in \mathbf{AS_j}}\frac{\partial L_i(t)}{\partial t}-\sum_{i\in \mathbf{S_j}}\frac{\partial A_{\mathrm{C}i}(t)}{\partial t}\right\}\cdot \Delta t\end{aligned} \tag{D.22}$$

The difference of revenue between $\mathbf{z}(t+\Delta t,\mathbf{A},\mathbf{C})$ and $\mathbf{z}(t,\mathbf{A},\mathbf{C})$ is:

$$\begin{aligned}&\mathbf{z}(t+\Delta t,\mathbf{A},\mathbf{C})-\mathbf{z}(t,\mathbf{A},\mathbf{C}) =\\ &= \sum_{i\in \bigcup_{j=1}^{N}\mathbf{AS_j}} f_i\cdot\frac{\partial L_i(t)}{\partial t}\cdot\Delta t-\sum_{i\in \bigcup_{j=1}^{N}\mathbf{S_j}} f_i\cdot\frac{\partial A_{\mathrm{C}i}(t)}{\partial t}\cdot\Delta t\\ &- \sum_{j=1}^{N}\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j}\left\{\sum_{i\in \mathbf{AS_j}}\frac{\partial L_i(t)}{\partial t}-\sum_{i\in \mathbf{S_j}}\frac{\partial A_{\mathrm{C}i}(t)}{\partial t}\right\}\cdot \Delta t\end{aligned} \tag{D.23}$$

Let

$$AD = \sum_{i\in \bigcup_{j=1}^{N}\mathbf{MS_j}} f_i\cdot\frac{\partial L_i(t)}{\partial t}\cdot\Delta t-\sum_{j=1}^{N}\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j}\left\{\sum_{i\in \mathbf{MS_j}}\frac{\partial L_i(t)}{\partial t}\right\}\cdot\Delta t \Longleftrightarrow$$

$$\begin{aligned} AD &= \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} f_i \cdot \frac{\partial L_i(t)}{\partial t} \cdot \Delta t - \\ &- \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} \frac{\partial L_i(t)}{\partial t} \cdot \left\{ \sum_{\substack{\text{all Legs } i \\ \text{over which } MS_j \\ \text{extends}}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_i} \right\} \Delta t \iff \\ AD &= \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} f_i \cdot \frac{\partial L_i(t)}{\partial t} \cdot \Delta t - \sum_{i \in \bigcup_{j=1}^{N} \mathbf{MS_j}} \frac{\partial L_i(t)}{\partial t} \cdot f_i \Delta t \iff \\ AD &= 0 \end{aligned} \tag{D.24}$$

At the right hand side of equation D.23, I add the equivalent of the $AD = 0$ (i.e the right hand side) from the first of the equations D.24. We devide both the left and the right hand side of the resultant equation by Δt, and then we take its limit for $\Delta t \longrightarrow 0$, and we get:

$$\begin{aligned} &\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} = \\ &= \sum_{i \in \bigcup_{j=1}^{N} (\mathbf{AS_j} + \mathbf{MS_j})} f_i \cdot \frac{\partial L_i(t)}{\partial t} - \sum_{i \in \bigcup_{j=1}^{N} \mathbf{S_j}} f_i \cdot \frac{\partial A_{ci}(t)}{\partial t} \\ &- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \in (\mathbf{AS_j} + \mathbf{MS_j})} \frac{\partial L_i(t)}{\partial t} - \sum_{i \in \mathbf{S_j}} \frac{\partial A_{ci}(t)}{\partial t} \right\} \end{aligned} \tag{D.25}$$

From the above equation D.25 and the definitions 5.2, we get that:

$$\begin{aligned} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} f_i \cdot \frac{\partial L_i(t)}{\partial t} - \sum_{i \in \mathbf{S}} f_i \cdot \frac{\partial A_{ci}(t)}{\partial t} \\ &- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{Y_j}} \frac{\partial L_i(t)}{\partial t} - \sum_{i \in \mathbf{S_j}} \frac{\partial A_{ci}(t)}{\partial t} \right\} \end{aligned} \tag{D.26}$$

We substitute $\frac{\partial L_i(t)}{\partial t}$, and $\frac{\partial A_{C_i}(t)}{\partial t}$ with their equivalents from equations D.17 and D.18. After a few manipulations (additions and subtractions) at the right hand side of the equation, we get:

$$
\begin{aligned}
\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial t} = & \sum_{i \in \mathbf{Y}} f_i \cdot \lambda_i(t) \\
- & \sum_{i \in \mathbf{Y}} f_i \cdot \lambda_i(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu)d\nu}\right) \\
- & \sum_{i \in \mathbf{S}} f_i \cdot \mu_i(t) \cdot A_i \\
+ & \sum_{i \in \mathbf{S}} f_i \cdot \mu_i(t) \cdot A_i \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu)d\nu}\right) \\
- & \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \right\} \\
+ & \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu)d\nu}\right) \right\} \\
+ & \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \\
- & \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu)d\nu}\right) \right\}
\end{aligned}
\tag{D.27}
$$

I rename the first term of the right hand side of equation D.27 to $T1$, the second to $T2$ and so on. As a result of the introduction of the new notation, we could present equation D.27, as:

$$
\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial t} = T1 - T2 + T3 + T4 - T5 + T6 + T7 - T8 \tag{D.28}
$$

Before we continue, we should find an expression for the partial derivative of the revenues, with reference to the reservations $\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_{OD_s,m}}$. We will examine one LP model with reservations vector $\mathbf{A}_{\Delta OD_s,m}$, and $\mathbf{C}$ remaining capacity, against an LP with reservations $\mathbf{A}$, and remaining capacity $\mathbf{C}$.

$\mathbf{A}$ is the vector of all the $A_{OD_i,m}$'s. If a particular $A_{OD_s,m}$ increases by $\Delta A_{OD_s,m}$, the vector $\mathbf{A}$ becomes $\mathbf{A}_{\Delta OD_s,m}$. The vector $\mathbf{A_C}$ becomes $\mathbf{A}_{\mathbf{C}\Delta OD_s,m}$. The LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ has as follows:

$$
\begin{aligned}
\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C}) = \max \ \mathbf{f}^T(\mathbf{x} &- \mathbf{A}_{\mathbf{C}\Delta OD_s,m}) \\
\mathbf{B}\mathbf{x} &\leq \mathbf{b} \\
\mathbf{x} &\leq \mathbf{L} \\
\mathbf{x} &\geq \mathbf{0}
\end{aligned}
\tag{D.29}
$$

where $\mathbf{A_C}$, is a vector whose elements are:

$$
\begin{aligned}
A_{\mathbf{C}OD_i,g} &= A_{OD_i,g} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu}\right) \ \forall \ OD_i, g \in \mathbf{S} \\
&\text{and} \\
\Delta A_{\mathbf{C}OD_s,m} &= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right)
\end{aligned}
\tag{D.30}
$$

The form of the constraints $\mathbf{x} \leq \mathbf{L}$, is given by equation 6.12. The constraints $\mathbf{Bx} \leq \mathbf{b}$ are modified as follows:

$$
\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \ \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} x_{OD_i,g} \leq C_j + \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \ \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} A_{\mathbf{C}OD_i,g}
\tag{D.31}
$$

$$
j \in \ \{\text{all legs except for the legs of the } OD_s \text{ path}\}
$$

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} x_{OD_i,g} \leq C_j + \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} A_{COD_i,g} + \Delta A_{COD_i,m}$$

$$j \in \text{the } OD_{\cdot} \text{ path} \tag{D.32}$$

From the above, we see that the LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ has increased space at the legs of the OD_s path. Due to the increased available capacity at the path OD_i, by $\Delta A_{COD_i,m}$, the earning potential of the LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ is greater than the earning potential of the $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ by:

$$\begin{aligned} \text{DRI} &= \Delta A_{COD_s,m} \cdot \sum_{\substack{\text{all Legs j} \\ \text{of path } OD_S}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \\ &= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \cdot \sum_{\substack{\text{all Legs j} \\ \text{of path } OD_S}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \end{aligned}$$

The increased reservations of the LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ will have increased cancelations. Therefore, the additional reservation of $\Delta A_{OD_i,m}$, will cause additional cancelation equal to $\Delta A_{COD_i,m}$. The additional cancelation of the reservation $\Delta A_{COD_i,m}$, requests for an additional refund to the customer OD_i, m. The revenue of the $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ decreases by the additional amount of the refund, which is:

$$\begin{aligned} \text{DRD} &= \Delta A_{COD_s,m} \cdot f_{OD_s,m} \\ &= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \cdot f_{OD_s,m} \end{aligned}$$

Therefore, the difference in revenue between the two LP's is:

$$\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C}) - \mathbf{z}(t, \mathbf{A}, \mathbf{C}) = \tag{D.33}$$

$$= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \cdot \left(\sum_{\substack{\text{all legs } j \\ \text{of path} \\ OD_s}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} - f_{OD_s,m} \right)$$

After we devide the left and the right hand side of the above equation by $\Delta A_{OD_s,m}$, and we take the limit for $\Delta A_{OD_s,m} \longrightarrow 0$, we get:

$$\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_s,m}} = \left(\sum_{\substack{\text{all legs } j \\ \text{of path} \\ OD_s}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} - f_{OD_s,m} \right) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \tag{D.34}$$

We introduce the definition:

$$f_{MS_{OD_s}} = \sum_{\substack{\text{all legs } j \\ \text{of path} \\ OD_s}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \tag{D.35}$$

Therefore:

$$\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_s,m}} = (f_{MS_{OD_s}} - f_{OD_s,m}) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \tag{D.36}$$

After we have found an expression for the derivative $\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_{OD_s,m}}$, we can resume with the analysis of the differential equation D.27. We examine the sixth term at the right hand side of the differential equation.

$$T6 = \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \right\}$$

$$= \sum_{\text{all legs } j} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \cdot$$

$$\left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot I_{OD_{i,g}}(t,\mathbf{A},\mathbf{C}) \right\}$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}}$$

$$\lambda_{OD_{i,g}}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \cdot I_{OD_{i,g}}(t,\mathbf{A},\mathbf{C})$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}}$$

$$\lambda_{OD_{i,g}}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot \left\{ \sum_{\substack{\text{all legs } j \\ \text{of path } OD_i}} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \right\} \cdot I_{OD_{i,g}}(t,\mathbf{A},\mathbf{C})$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_{i,g}}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot f_{MS_{OD_i}} \cdot I_{OD_{i,g}}(t,\mathbf{A},\mathbf{C})$$

(D.37)

We go from the second to the third of the equations D.37, by repeating the observation that, the sum of all the ODC's that we accept in the current policy area, over all the legs used by each of the ODC's, is equal to the sum of all the legs of the network, over all the

ODC's that use each leg of the network.

We go from the third of the equations D.37 to the fourth, by employing the definition D.35.

The above formula D.37 is the new expression for the sixth term of the right hand side of the differential equation D.27.

Similarly, we get:

$$
\begin{aligned}
T8 &= \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \right\} \\
&= \sum_{\text{all legs } j} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \\
&\left\{ \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \right\} \\
&= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot f_{MS_{OD_i}}
\end{aligned}
\tag{D.38}
$$

The above formula D.38 is the new expression for the eighth term of the right hand side of the differential equation D.27.

We will find the expression for the difference between $T6$ and $T2$.

$$T6 - T2 =$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu}\right) \cdot f_{MS_{OD_i}} \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$$

$$- \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu}\right) \cdot f_{OD_i,g} \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot (f_{MS_{OD_i}} - f_{OD_i,g}) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu}\right) \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_i,m}} \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C})$$

$$= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} \qquad \text{(D.39)}$$

We go from the second to the third of the equations D.39, by employing equation D.36.

$T4 - T8 =$

$$
\begin{aligned}
&= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot f_{OD_{i,g}} \\
&- \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \cdot f_{MS_{OD_i}} \\
&= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot (f_{MS_{OD_i}} - f_{OD_{i,g}}) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu}\right) \\
&= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_{i,m}}} \\
&= \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} \qquad \text{(D.40)}
\end{aligned}
$$

We go from the second to the third of the equations D.40, by employing equation D.36. From equations D.27, D.39, and D.40, we finally get:

$$
\begin{aligned}
\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot f_i \\
&+ \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_i} \\
&- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) - \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \qquad \text{(D.41)}
\end{aligned}
$$

Q.E.D.

Appendix E

Lemma 19 $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ *is the maximum value, and* $x^*_{OD_{i},g}(t, \mathbf{A}, \mathbf{C})$, $\forall$ OD_i*'s and* $\forall$ g, *is the corresponding optimal solution of the Linear Program 7.9. The governing differential equation for* $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ *is the equation*

$$\begin{aligned}
\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot f_i - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot f_i \\
&+ \sum_{i \in \mathbf{Y}} \lambda_i(t) \cdot \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_i} - \sum_{i \in \mathbf{S}} \mu_i(t) \cdot A_i \cdot \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_i} \\
&- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j} \cdot \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) - \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \qquad \text{(E.1)}
\end{aligned}$$

The boundary conditions of the LP are given by 7.4

Proof:

At time t the value of the objective function is:

$$\begin{aligned}
\mathbf{z}(t,\mathbf{A},\mathbf{C}) &= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} f_{OD_{i},g} \cdot \left(x^*_{OD_{i},g} - A_{COD_{i},g} \right) \\
&- \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \left(f_{OD_{i},g} + k_{OD_{i},g} \right) \cdot u_{OD_{i},g} \qquad \text{(E.2)}
\end{aligned}$$

If we are at time $t+\Delta t$ before departure, the Expected Demand of each ODC combination is greater by:

$$\begin{aligned} \Delta L_{OD_{i,g}}(t) &= \frac{\partial L_{OD_{i,g}}(t)}{\partial t} \cdot \Delta t \\ &\text{with} \\ \frac{\partial L_{OD_{i,g}}(t)}{\partial t} &= \lambda_{OD_{i,g}}(t) \cdot e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu} \end{aligned} \tag{E.3}$$

The additional number of the initial $A_{OD_{i,g}}$ Containers from the particular ODC Combination, expected to cancel in the extra Δt, is:

$$\begin{aligned} \Delta A_{\mathrm{c}OD_{i,g}} &= \frac{\partial A_{\mathrm{c}OD_{i,g}}(t)}{\partial t} \cdot \Delta t \\ &\text{with} \\ \frac{\partial A_{\mathrm{c}OD_{i,g}}(t)}{\partial t} &= \mu_{OD_{i,g}}(t) \cdot A_{OD_{i,g}} \cdot e^{-\int_{\nu=0}^{t} \mu_{OD_{i,g}}(\nu) d\nu} \end{aligned} \tag{E.4}$$

At leg j we have added the extra containers:

$$\sum_{i \in \mathbf{AS_j}} \Delta L_i(t) - \sum_{i \in \mathbf{S_j}} \Delta A_{\mathrm{c}i}(t) = \sum_{i \in \mathbf{AS_j}} \frac{\partial L_i(t)}{\partial t} \cdot \Delta t - \sum_{i \in \mathbf{S_j}} \frac{\partial A_{\mathrm{c}i}(t)}{\partial t} \cdot \Delta t \tag{E.5}$$

(see the definition of $\mathbf{AS_j}$ and $\mathbf{S_j}$).

Because of the extra containers, the additional revenue at leg j is:

$$\sum_{i \in \mathbf{AS_j}} f_i \cdot \frac{\partial L_i(t)}{\partial t} \cdot \Delta t \quad - \quad \sum_{i \in \mathbf{S_j}} f_i \cdot \frac{\partial A_{\mathrm{c}i}(t)}{\partial t} \cdot \Delta t \tag{E.6}$$

The extra Capacity that has been given to ODC Combinations that belong to the set $\mathbf{AS_j}$, has displaced an equal amount of the ODC Combination that belongs to $\mathbf{MS_j}$:

The lost Revenue at leg j is:

$$\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \,\in\, \mathbf{AS_j}} \frac{\partial L_i(t)}{\partial t} - \sum_{i \,\in\, \mathbf{S_j}} \frac{\partial A_{\mathbf{C}i}(t)}{\partial t} \right\} \cdot \Delta t \tag{E.7}$$

Following the same process as before in the model with the cancelations, we get again equation D.27.

$$\begin{aligned}
\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} &= \sum_{i \,\in\, \mathbf{Y}} f_i \cdot \lambda_i(t) \\
&- \sum_{i \,\in\, \mathbf{Y}} f_i \cdot \lambda_i(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \\
&- \sum_{i \,\in\, \mathbf{S}} f_i \cdot \mu_i(t) \cdot A_i \\
&+ \sum_{i \,\in\, \mathbf{S}} f_i \cdot \mu_i(t) \cdot A_i \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \\
&- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \,\in\, \mathbf{Y_j}} \lambda_i(t) \right\} \\
&+ \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \,\in\, \mathbf{Y_j}} \lambda_i(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \right\} \\
&+ \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \,\in\, \mathbf{S_j}} \mu_i(t) \cdot A_i \right\} \\
&- \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \,\in\, \mathbf{S_j}} \mu_i(t) \cdot A_i \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \right\}
\end{aligned} \tag{E.8}$$

I rename the first term of the right hand side of equation D.27 to $T1$, the second to $T2$ and so on. As a result of the introduction of the new notation, we could present equation D.27, as:

$$\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial t} = T1 - T2 + T3 + T4 - T5 + T6 + T7 - T8 \tag{E.9}$$

Before we continue, we should find an expression for the partial derivative of the revenues, with reference to the reservations $\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_s,m}}$. We will examine one LP model with reservations vector $\mathbf{A}_{\Delta OD_s,m}$, and $\mathbf{C}$ remaining capacity, against an LP with reservations $\mathbf{A}$, and remaining capacity $\mathbf{C}$.

$\mathbf{A}$ is the vector of all the $A_{OD_i,m}$'s. If a particular $A_{OD_s,m}$ increases by $\Delta A_{OD_s,m}$, the vector $\mathbf{A}$ becomes $\mathbf{A}_{\Delta OD_s,m}$. The vector $\mathbf{A_c}$ becomes $\mathbf{A_c}_{\Delta OD_s,m}$, and the vector $\mathbf{A_e}$ becomes $\mathbf{A_e}_{\Delta OD_s,m}$. The LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ has as follows:

$$\begin{aligned}
\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C}) = \max \ \mathbf{f}^T(\mathbf{x} - \mathbf{A}_{\mathbf{can}\Delta OD_s,m}) \ &- \ (\mathbf{f} + \mathbf{k})^T \cdot \mathbf{u} \\
\mathbf{B}(\mathbf{x} - \mathbf{u}) \ &\leq \ \mathbf{b} \\
\mathbf{x} \ &\leq \ \mathbf{L} \\
\mathbf{u} \ &\leq \ \mathbf{A}_{\mathbf{eff}\Delta OD_s,m} \\
\mathbf{x} \ &\geq \ \mathbf{0} \\
\mathbf{u} \ &\geq \ \mathbf{0}
\end{aligned} \tag{E.10}$$

where $\mathbf{A_c}$, is a vector whose elements are:

$$\begin{aligned}
A_{\mathbf{c}OD_i,g} \ &= \ A_{OD_i,g} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu) d\nu}\right) \ \forall \ OD_i, g \in \mathbf{S} \\
&\text{and} \\
\Delta A_{\mathbf{c}OD_s,m} \ &= \ \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu) d\nu}\right)
\end{aligned} \tag{E.11}$$

and $\mathbf{A_e}$, is a vector whose elements are:

$$A_{eOD_i,g} = A_{OD_i,g} \cdot e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu)d\nu} \;\; \forall \;\; OD_i, g \in \mathbf{S}$$

and

$$\Delta A_{eOD_s,m} = \Delta A_{OD_s,m} \cdot e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu} \tag{E.12}$$

The form of the constraints $\mathbf{x} \leq \mathbf{L}$, is given by equation 6.12. The constraints $\mathbf{B}(\mathbf{x} - \mathbf{u}) \leq \mathbf{b}$ are modified as follows:

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} (x_{OD_i,g} - u_{OD_i,g}) \leq C_j + \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} A_{cOD_i,g} \tag{E.13}$$

$$j \in \{\text{all legs except for the legs of the } OD_s \text{ path}\}$$

$$\sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} (x_{OD_i,g} - u_{OD_i,g}) \leq C_j + \sum_{\substack{\text{all } OD_i \\ \text{pairs that} \\ \text{use leg } j}} \sum_{\substack{\text{all Classes g} \\ \text{for each } OD_i \\ \text{pair}}} A_{cOD_i,g} + \Delta A_{cOD_i,m}$$

$$j \in \text{the } OD_s \text{ path} \tag{E.14}$$

From the above, we see that the LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ has increased space at the legs of the OD_s path. Due to the increased available capacity at the path OD_i, by $\Delta A_{cOD_i,m}$, the earning potential of the LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ is greater than the earning potential of the $\mathbf{z}(t, \mathbf{A}, \mathbf{C})$ by:

$$\text{DRI} = \Delta A_{cOD_s,m} \cdot \sum_{\substack{\text{all Legs j} \\ \text{of path } OD_S}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j}$$

$$= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \cdot \sum_{\substack{\text{all Legs } j \\ \text{of path } OD_S}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j}$$

The increased reservations of the LP $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ will have increased cancelations. Therefore, the additional reservation of $\Delta A_{OD_i,m}$, will cause additional cancelation equal to $\Delta A_{cOD_i,m}$. The additional cancelation of the reservation $\Delta A_{cOD_i,m}$, requests for an additional refund to the customer OD_i, m. The revenue of the $\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C})$ decreases by the additional amount of the refund, which is:

$$\begin{aligned} \text{DRD} &= \Delta A_{cOD_s,m} \cdot f_{OD_s,m} \\ &= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \cdot f_{OD_s,m} \end{aligned}$$

Therefore, the difference in revenue between the two LP's is:

$$\mathbf{z}(t, \mathbf{A}_{\Delta OD_s,m}, \mathbf{C}) - \mathbf{z}(t, \mathbf{A}, \mathbf{C}) = \tag{E.15}$$

$$= \Delta A_{OD_s,m} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \cdot \left(\sum_{\substack{\text{all legs } j \\ \text{of path} \\ OD_s}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} - f_{OD_s,m} \right)$$

After we devide the left and the right hand side of the above equation by $\Delta A_{OD_s,m}$, and we take the limit for $\Delta A_{OD_s,m} \longrightarrow 0$, we get:

$$\frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial A_{OD_o,m}} = \left(\sum_{\substack{\text{all legs } j \\ \text{of path} \\ OD_s}} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} - f_{OD_s,m} \right) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_s,g}(\nu)d\nu}\right) \tag{E.16}$$

We introduce the definition:

$$\pi_{MS_{OD_S}} = \sum_{\substack{\text{all legs } j \\ \text{of path} \\ ODs}} \frac{\partial z(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \tag{E.17}$$

$\pi_{MS_{OD_s}}$ is the marginal cost at the path ODs, for an extra unit of capacity at each leg of the path ODs. In the analysis for the model with cancelations, we had $\pi_{MS_{OD_S}} = f_{MS_{OD_S}}$ because one extra unit of capacity at the path ODs, would mean one extra unit of good belonging to the marginal Class for the path ODs. Now that we can have potential overbooking, one more unit of capacity at the path ODs, might mean that we will have one less overbooked unit that wants to travel on the path ODs. If this is the case, we have to pay

$$\pi_{MS_{OD_S}} = f_{MS_{OD_S}} + k_{MS_{OD_S}} \tag{E.18}$$

less, for the marginal overbooking unit. We remind that $f_{MS_{OD_S}}$ is the freight rate that has already been paid by the customer, and $k_{MS_{OD_S}}$ is the good will cost assigned to the particular class of customers. When we do not have overbooking, it is still

$$\pi_{MS_{OD_S}} = f_{MS_{OD_S}} \tag{E.19}$$

By working on the terms of equation E.8, and by repeating the process we used for equation D.37, we get:

$$T6 = \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{Y_j}} \lambda_i(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \right\}$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \lambda_{OD_i,g}(t) \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu) d\nu}\right) \cdot \pi_{MS_{ODi}} \cdot I_{OD_i,g}(t, \mathbf{A}, \mathbf{C}) \tag{E.20}$$

and similarly:

$$T8 = \sum_{j=1}^{N} \frac{\partial \mathbf{z}(t, \mathbf{A}, \mathbf{C})}{\partial C_j} \left\{ \sum_{i \in \mathbf{S_j}} \mu_i(t) \cdot A_i \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_i(\nu) d\nu}\right) \right\}$$

$$= \sum_{\substack{\text{all } OD_i \\ \text{pairs}}} \sum_{\substack{\text{all Classes } g \\ \text{for each } OD_i \\ \text{pair}}} \mu_{OD_i,g}(t) \cdot A_{OD_i,g} \cdot \left(1 - e^{-\int_{\nu=0}^{t} \mu_{OD_i,g}(\nu) d\nu}\right) \cdot \pi_{MS_{ODi}} \tag{E.21}$$

From here on we continue working as we did in the previous chapter, from equation D.37 onwards. The only difference here is that we substitute $\pi_{MS_{OD_S}}$ for every $f_{MS_{OD_S}}$ in equations D.37 to 6.14.

Proceeding, as in the case of the model with cancelations, we finaly get:

$$\begin{aligned}\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial t} &= \sum_{i\,\in\,\mathbf{Y}} \lambda_i(t)\cdot f_i - \sum_{i\,\in\,\mathbf{S}} \mu_i(t)\cdot A_i\cdot f_i \\ &+ \sum_{i\,\in\,\mathbf{Y}} \lambda_i(t)\cdot\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_i} - \sum_{i\,\in\,\mathbf{S}} \mu_i(t)\cdot A_i\cdot\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial A_i} \\ &- \sum_{j=1}^{N}\frac{\partial \mathbf{z}(t,\mathbf{A},\mathbf{C})}{\partial C_j}\cdot\left\{\sum_{i\,\in\,\mathbf{Y_j}}\lambda_i(t) - \sum_{i\,\in\,\mathbf{S_j}}\mu_i(t)\cdot A_i\right\} \qquad \text{(E.22)}\end{aligned}$$

Q.E.D.

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