ON A CLASS OF TEMPORALLY NON-HOMOGENEOUS MARKOV PROCESSES AND THEIR RELATIONSHIP TO INFINITE PARTICLE GASES

by

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ABSTRACT

ON A CLASS OF TEMPORALLY NON-HOMOGENEOUS MARKOV PROCESS AND THEIR RELATIONSHIP TO INFINITE PARTICLE GASES

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Submitted to the Department of Mathematics on August 22, 1966 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Consider the class of right continuous sample paths $x(t)$, $t \geq o$ with values ± 1 and assume that for each probability measure f on $E = \pm 1$ and for each $e \in E$, there exist probability measures P_f and $P_{f|e}$ for which

(1) $P_{f|e}(\cdot) = P_f(\cdot | x(o) = e)$

(2) $P_f(x(o) = e) = f(e)$

(3) $P_{f|e}(x(t+h) \in A | \mathcal{M}_t) = P_{f_t|x_t}(x(h) \in A)$ $[\text{a.e. } P_{f|e}]$,

where $\mathcal{M}_t$ is the σ-algebra generated by the events $x(s)$, $s \leq t$, A is a set of points in E, and $f_t(A) = P_f(x(t) \in A)$.

If

$$\gamma_{\pm}(u) = \frac{\partial}{\partial t} P_{f|\pm 1}(x(t) = 1)\Big|_{t=o} \quad , \; u = f(+1)$$

exists, then the functions γ_+ and γ_- will, under certain technical conditions, uniquely determine the distribution of the process $x(t)$. Such a process is a temporally non-homogeneous Markov process and will be called a *-process.

Suppose that $x(t)$, $t \geq o$ is a *-process, f its initial distribution and f_t its distribution at time t. Then it is easily shown that f_t is the (formal) solution of

$$\frac{\partial}{\partial t} f_t = B[f_t]$$

where

$$B[f](+1) = -B[f](-1) = u\gamma_+(u) + (1-u)\gamma_-(u), \quad u = f(+1).$$

B is, in general, a non-linear operator. When B is linear and bounded it is natural to think of f_t as exp (tB)f. However, when B is non-linear this cannot be done, although a replacement for exp (tB) can be found.

H. P. McKean, Jr. [3] has done this for $\gamma_\pm(u) = \pm(u-1)$. He defines a linear operator D mapping functions of one variable into functions of two variables and then extends D to functions of any finite number of variables in such a manner that the solution f_t of $\frac{\partial}{\partial t} f_t = B[f_t]$ can be expressed as

$$f_t(e) = \int_{E^\infty} f(d\xi_1)f(d\xi_2)\ldots\exp(tD)[\chi_e](\xi_1,\xi_2,\ldots)$$

where E^∞ is the infinite product of $E = \pm 1$ with itself and χ_e is the indicator function of e. McKean then shows that the operator D leads to a natural description of the *-process as the motion of a tagged particle in an infinite particle "gas" undergoing binary collisions; the motion of this tagged particle can be calculated from the formula

$$P_f[x(t_1) = e_1, x(t_2) = e_2, \ldots, x(t_n) = e_n]$$

$$= \int f(d\xi_1)f(d\xi_2)\ldots e^{t_1 D}\chi_{e_1}e^{(t_2-t_1)D}\chi_{e_2}\ldots e^{(t_n-t_{n-1})D}\chi_{e_n}.$$

This paper extends the results of McKean to those *-processes for which $\mp\gamma_\pm$ is positive on the open interval $0<u<1$ with at most algebraic roots at 0 and 1, and real analytic on the closed interval $0 \leq u \leq 1$. The equation

$$\frac{\partial}{\partial t} f_t = B[f_t]$$

is solved using a linear operator D mapping functions of one variable into functions of infinitely many variables. D, in turn, suggests by its form that the *-process can be described as the motion of a single tagged particle in an infinite particle gas. However, unlike McKean's model, collisions of arbitrarily high, but finite order are allowed.

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1. INTRODUCTION

Consider the right continuous sample paths $x(t)$, $t \geq o$ on the space $E = \pm 1$ and assume that for each probability measure f on E and $e \in E$, there exist probability measures P_f and $P_{f|e}$ for which

$$(1) \quad P_{f|e}(\cdot) = P_f(\cdot | x(o) = e)$$

$$(2) \quad P_f(x(o) = e) = f(e)$$

$$(3) \quad P_{f|e}(x(t+h) \in A | \mathcal{M}_t) = P_{f_t | x_t}(x(h) \in A) \quad [a.e. P_{f|e}];$$

where $\mathcal{M}_t$ is the σ- algebra generated by $x(s)$, $s \leq t$, A is a set of points in E, and $f_t(A) = P_f(x(t) \in A)$.

If

$$\gamma \pm (u) = \frac{\partial}{\partial t} P_{f|\pm 1}(x(t) = 1)\Big|_{t=o} \qquad , \; u = f(+1)$$

then the functions γ_+ and γ_- will, under certain technical conditions, uniquely determine the distribution of the process $x(t)$. Such a process is a temporally non-homogeneous Markov process and will be called a *-process.

Suppose that $x(t)$, $t \geq o$ is a *-process, f its initial distribution and f_t its distribution at time t. Then it is easily shown that f_t is the (formal) solution of

$$\frac{\partial}{\partial t} f_t = B[f_t]$$

where

$$B[f](+1) = -B[f](-1) = u\,\gamma_+(u) + (1-u)\gamma_-(u), \quad u = f(+1).$$

B is, in general, a non-linear operator. When B is linear and bounded it is natural to think of f_t as exp $(tB)f$. However, when B is non-linear this cannot be done, although a replacement for exp (tB) can be found, as will now be illustrated in the following example due to H. P. McKean, Jr. [3].

Let $\gamma_{\pm}(u) = \pm(u-1)$. This gives

$$B[f](\pm 1) = \pm(2u^2-3u+1), \quad u = f(+1);$$

or, to rewrite it in a more suggestive manner,

$$B[f](e_1) = \int \left[f(e_1^*)f(e_2^*) - f(e_1)f(e_2)\right] de_2 do$$

where $\int d\, e_2$ denotes the sum over $e_2 = \pm 1$ and $\int do$ denotes the sum over the two possible outcomes of the binary collision

$$(e_1, e_2) \to (e_1^*, e_2^*) = (e_1 e_2, e_2) \text{ or } (e_1, e_1 e_2).$$

This equation is very similar to Boltzmann's equation for a spatially homogeneous Maxwellian gas without exterior forces. In fact, for such a gas, if $f(\underline{V},t)$ is the distribution of molecules with velocity d $\underline{V}$ at time t and if particles with velocities $\underline{V}_1$ and $\underline{V}_2$ have velocities $\underline{V}_1^*$ and $\underline{V}_2^*$ respectively after a collision, Boltzmann's equation becomes

$$\frac{\partial}{\partial t} f(\underline{V}_1,t) = \int Q \, d\underline{V}_2 \int_{S(1)} d\underline{e} \left[f(\underline{V}_1^*,t)f(\underline{V}_2^*,t) - f(\underline{V}_1,t)f(\underline{V}_2,t)\right]$$

where $S(1)$ is the unit sphere, $\underline{e} \in S(1)$ a unit vector, and Q is a function of the scattering angle alone.

To find a solution of $\frac{\partial}{\partial t} f_t = B[f_t]$ define an operator D, mapping functions of one variable into functions of two variables, by $D[\mathcal{Q}](e_1,e_2)=\mathcal{Q}(e_1e_2)-\mathcal{Q}(e_1)$. Letting $\mathcal{Q}_1\otimes\mathcal{Q}_2$ denote the outer product $\mathcal{Q}_1(e_1,\ldots,e_a)\mathcal{Q}_2(e_{a+1},\ldots,e_{a+b})$ when $\mathcal{Q}_1=\mathcal{Q}_1(e_1,\ldots,e_a)$ and $\mathcal{Q}_2=\mathcal{Q}_2(e_1,\ldots,e_b)$, we extend D to a derivation acting on functions of any finite number of variables by requiring

$$D[\mathcal{Q}_1\otimes\mathcal{Q}_2]=\mathcal{Q}_1\otimes D[\mathcal{Q}_2]+D[\mathcal{Q}_1]\otimes\mathcal{Q}_2.$$

With this extension,

$$\int_E\left(\frac{\partial^n}{\partial s^n}f_s\right)(e)\,\mathcal{Q}(e)\,de=\int_{E^{n+1}}f_s(e_o)\ldots f_s(e_n)D^n[\mathcal{Q}]\,de_o\ldots de_n$$

for functions $\mathcal{Q}$ of one variable. Putting $s=o$, and writing f_t as a formal Taylor series in t, we get

$$\int_E f_t(e)\,\mathcal{Q}(e)\,de=\sum_{n=o}^{\infty}\frac{t^n}{n!}\int_{E^{n+1}}f\otimes\ldots\otimes f\,D^n[\mathcal{Q}]=\int_{E^{\infty}}f^{\infty}\exp(tD)[\mathcal{Q}]$$

where E^{∞} is the infinite product of $E=\pm1$ with itself, f^{∞} is the infinite outer product of $f=f_o$ with itself, and $\exp(tD)$ is the formal power series $\sum_{n=o}^{\infty}\frac{t^n}{n!}D^n$. Thus we can formally write the solution of

$$\partial/\partial t\,f_t=B[f_t]\text{ as }\exp(tD)^*[f^{\infty}].$$

McKean goes on to show that the derivation D leads to a natural description of the *-process as the motion of a tagged particle in an infinite particle gas undergoing binary collisions; the motion of this tagged particle can be calculated from the formula

$$P_f\left[x(t_1)=e_1,\ x(t_2)=e_2,\dots,x(t_n)=e_n\right]$$

$$=\int f(d\xi_1)f(d\xi_2)\dots e^{t_1 D}\chi_{e_1}e^{(t_2-t_1)D}\chi_{e_2}\dots e^{(t_n-t_{n-1})D}\chi_{e_n},$$

χ_e being the indicator function of e.

This paper extends the results of McKean to those *-processes for which $\mp\gamma_\pm$ is positive on the open interval $o<u<1$ and with at most algebraic roots at o and 1, real analytic on the closed interval $o\leq u\leq 1$. These conditions are necessary and sufficient in order that $\mp\gamma_\pm$ can be written as a sum $\mp\sum_{n=1}^{\infty} B_n^{\pm}(u)$ where

$$B_n^{\pm}(u)=\sum_{k=o}^{n} c_n^{\pm}(k)\binom{n}{k}u^k(1-u)^{n-k},\quad c_n^{\pm}(k)\geq o$$

are Bernstein polynomials and $\sum_{n\geq 1}^{\infty} n^p \max_{k\leq n,\pm} c_n^{\pm}(k)\leq p!\,L^p$ for all positive integers p. The equation $\partial/\partial t f_t=B[f_t]$ is solved using a derivation, mapping functions of one variable into functions of infinitely many variables, which is expressed in terms of the coefficients $C_n^{\pm}(k)$. This derivations, in turn, suggests that the *-process can be described as the motion of a single tagged particle in an infinite particle gas. However, unlike McKean's model, collisions of arbitrarily high, but finite order are allowed. In fact, an n-fold collision is allowed in the infinite particle gas whenever the term $B_n^{\pm}$ in the Bernstein representation of $\mp\gamma_{\pm}$ is not identically zero; consequently this representation is fundamental in the construction of the infinite particle gas. It is also true, as I will show later, that the sample paths of any finite class of particles in the infinite particle gas are independent. Finally, I calculate the holding times for a *-process and give a brief discussion of the limiting behavior of a *-process as $t \to \infty$.

This paper is arranged as follows. The second section gives a formal description of *-processes. The third section gives a formal description of the integration of the non-linear equation $\partial/\partial t\, f_t = B[f_t]$ by which the distribution $f_t(e) = P_f(x(t)=e)$ is governed. The fourth section applies the formal results of the third section to a particular class of *-processes. The fifth section constructs the *-process as the limit of the motions of a single particle in an n-particle gas as $n \to \infty$. In the sixth section, holding times are calculated for the *-process. Finally, in the seventh section the limiting behavior of a *-process as $t \to \infty$ is discussed.

2. *-PROCESSES

Suppose we are given a sample space Ω , a state space $E = \pm 1$, and a time interval $T = [o, +\infty)$. Then a temporally homogeneous Markov process on Ω , E, T consists of:

(1) for each $t \in T$ a function $x_t(w)$ mapping Ω into E,

(2) a σ- algebra $\mathcal{M}_\infty$ on Ω together with a family of sub σ-algebras $\mathcal{M}_t$, $t \in T$ such that $[x_t \in B] \in \mathcal{M}_t$ for any $t \in T$, $B \subset E$,

(3) for each $e \in E$ a probability measure P_e on $\mathcal{M}_\infty$ which satisfies:

(a) $P_e(x(o) = e) = 1$

(b) $P_e(x(t+h) \in B \mid \mathcal{M}_t) = P_{x(t)}(x(h) \in B)$ $[a.e. P_e]$.

What we shall now do is to remove the temporal homogeneity. But, rather than letting the transition mechanism vary arbitrarily with time as one would normally do, we will let it vary via the distribution of the particle. ~~Thus, the~~ transition probability functions P_e will be replaced by a family of probability measures $P_{f|e}$ where f is a probability measure on E and $e \in E$. The expression $P_{f|e}(\Lambda)$ is to be thought of as the probability that, starting with x(o) distributed according to f, the event Λ will take place, conditional on $x(o) = e$. This is accomplished by replacing (3) with

(3') for each $e \in E$ and probability measure f on E, there exists a probability measure P_f on $\mathcal{M}_\infty$ such that:

(a') $P_f(x(o)=e)=f(e)$

(b') $P_f(x(t+h)\in A/\mathcal{M}_t)=P_{f_t/x_t}(x(h)\in A)$ [a.e. P_f]

where $f_t(B)=P_f(x(t)\in B)$ is the distribution of $x(t)$ when the starting distribution is f and $P_{f|e}(\cdot)=P_f(\cdot|x(o)=e)$.

Such a process will be called a $*$-process. It is temporally homogeneous if and only if $P_{f|e}$ is independent of f, as the reader can easily check.

Defining $P_{f|e}(t;A)=P_{f|e}(x_t\in A)$, we get a formula for the probabilities of joint observations reminiscent of the case of temporally homogeneous Markov processes:

<u>THEOREM 2.1</u> If x_t is a $*$-process, then for $t_1<\ldots\leq t_n<\infty$,

$$P_{f|e}[x(t_1)\in A_1, x(t_2)\in A_2,\ldots,x(t_n)\in A_n]$$
$$=\int_{A_1} P_{f|e}(t_1;d\xi_1)\int_{A_2} P_{f_{t_1}|\xi_1}(t_2-t_1;d\xi_2)\ldots\int_{A_n} P_{f_{t_{n-1}}|\xi_{n-1}}(t_n-t_{n-1};d\xi_n)$$

<u>PROOF</u> This is immediate from 3b'.

<u>COROLLARY 2.2</u> If x_t is a $*$-process, then

$$P_{f|e}(s+t;A)=\int P_{f|e}(t;d\xi)P_{f_t/\xi}(s;A).$$

DEFINITION 2.3 Let $p(t)=f_t(+1)$ and $p_e(t)=P_{f|e}(t;+1)$ be the probability that $x(t)=+1$ and the probability that $x(t)=+1$ conditional on $x(o)=e$ respectively, ~~given that~~ $x(o)$ has the distribution f.

According to Theorem 2.1, the function $p_e(t)$ determines the distribution of the process on cylinder sets.

DEFINITION 2.4 Letting $u=f(+1)=p(o)$, define

$$(1) \quad \gamma_e(u)=\partial/\partial t \; P_{f|e}(t;+1)\Big|_{t=o}=\frac{\partial}{\partial t}P_e(t)\Big|_{t=o}$$

$$(2) \quad \gamma(u)=u\gamma_+(u)+(1-u)\gamma_-(u)$$

DEFINITION 2.5 Let B be the operator (usually non-linear) mapping distributions f into functions $B[f]$ defined by

$$B[f](+1)=-B[f](-1)=\gamma[f(+1)].$$

THEOREM 2.6 If x_t is a *-process and if $p_e(t)$ is differentiable in $t\geq o$, then

$$(1) \quad \frac{\partial}{\partial t}p(t)=\gamma\left[p(t)\right]$$

or, to put it in an equivalent form

$$\frac{\partial}{\partial t}f_t(e)=B[f_t](e)$$

$$(2) \quad \frac{\partial}{\partial t}p_e(t)=p_e(t)\gamma_+\left[p(t)\right]+\left[1-p_e(t)\right]\gamma_-\left[p(t)\right].$$

PROOF Taking the equation in Corollary 2.2 and differentiating both sides with respect to s and leeting $s=o$, we get (2). (1) follows from (2) if we notice that $p(t)=\int f(de)p_e(t)$.

Equation (1) of Theorem 2.6, which is in general nonlinear, has a unique solution bounded by o and 1 if γ satisfies a Lipschitz condition and if $\gamma(o) \geq o$ and $\gamma(1) \leq o$. Once the solution of (1) is known, equation (2) becomes a linear problem for $p_e(t)$:

$$\frac{d}{dt} p_e(t) = F(t) + G(t) p_e(t) \text{ where } F(t) = \gamma_{-}[p(t)]$$

$$\text{and } G(t) = \gamma_{+}[p(t)] - \gamma_{-}[p(t)] .$$

This equation, in turn, has a unique solution bounded by o and 1 if γ_+ and γ_- are continuous and $\mp \gamma_{\pm} \geq o$. Having uniquely determined the transition function $P_{f|e}$, we can construct the *-process by defining probabilities on the cylinder sets in the manner suggested in Theorem 2.1:

$$p\left[x(t_1)\in A_1, x(t_2)\in A_2, \ldots, x(t_n)\in A_n\right]$$

$$= \int_{A_1} P_{f|e}(t_1; d\xi_1) \int_{A_2} P_{f_{t_1}|\xi_1}(t_2-t_1; d\xi_2) \ldots \int_{A_n} P_{f_{t_{n-1}}|\xi_{n-1}}(t_n-t_{n-1}; d\xi_n)$$

where $o < t_1 < \ldots < t_n < \infty$.

Finally, one can regard a *-process as a temporally homogeneous Markov process on $E \times [o,1]$ by adjoining $p(t)$ as a new coordinate. The transition probabilities of this process are

$$P_t^*(e,u;A,B) = \begin{cases} P_{f|e}(t;A) & \text{if } p(t)\in B, f(+1) = u = p(o) \\ o & \text{otherwise} \end{cases}$$

and its generator is given by

$$(GF)(e,p(t)) = \gamma_e\left[p(t)\right]\left[F(+1,p(t)) - F(-1,p(t))\right] + \frac{\partial}{\partial t} F(e,p(t)),$$

formally at least.

3. FORMAL SOLUTION OF $\frac{\partial}{\partial t} f_t = B[f_t]$

Since B will usually be non-linear, we linearize the problem by constructing a linear operator D mapping functions φ of one variable into functions $D\varphi$ of infinitely many variables such that

$$\int B[f](d\xi)\varphi(\xi) = \int f^{\infty} D\varphi$$

where

$$\int f^{\infty}\psi = \int f(d\xi_1)f(d\xi_2)\ldots\psi(\xi_1,\xi_2,\ldots).$$

The actual choice of D is very arbitrary since there are many such operators. However, in section 4 we shall somewhat restrict the possibilities by requiring that

$$\gamma_{\pm}(u) = \int_{\{11\}^c} f^{\infty}(D\chi_{+1})(\pm 1,\ldots), u = f(+1).$$

This implies that

$$p_e(t) = \int_{\{11\}^c} f^{\infty}(e^{tD}\chi_{+1})(e,\ldots) \quad , p(o) = f(+1)$$

as will be shown in section 5.

Once D is defined on functions φ of one variable, it will be extended to a class of functions of infinitely many variables in such a manner that if φ and ψ have no common variables, a state of affairs which we indicate by writing the product $\varphi\psi$ as $\varphi\otimes\psi$, then

$D[\varphi \otimes \psi] = \varphi \otimes D\psi + \psi \otimes D\varphi$. When D is extended in this manner, it will be called a derivation for B. This extension will allow us to define $D^2\varphi, D^3\varphi, \ldots$; and, in the cases we shall eventually consider, it will be shown that not only are $D\varphi, D^2\varphi, \ldots$ in the domain of D, but that $e^{tD}\varphi = \sum_{n=o}^{\infty} \frac{t^n}{n!} D^n\varphi$ converges for sufficiently small t. A calculation which is the basic result of this section shows that if $\partial/\partial t\, f_t = B[f_t]$ and D is a derivation for B, then

$$\int \frac{\partial^p}{\partial t^p} f_t(d\xi)\varphi(\xi) = \int f_t^{\infty} D^p\varphi.$$

Thus, letting $t=o$, $f=f_o$ and writing f_t as a formal Taylor series around $t=o$, we get

$$\int f_t(d\xi)\varphi(\xi) = \int f^{\infty} e^{tD}\varphi.$$

Thus

$$f_t = (e^{tD})^*[f^{\infty}]$$

is the formal solution of $\frac{\partial}{\partial t} f_t = B[f_t]$.

The complications which follow are due to the fact that the derivation I will be using usually maps functions of one variable into functions of infinitely many variables rather than into functions of a finite number of variables. Thus, in extending D to functions of more than one variable, I need a large reserve of variables so as to ensure that $D(\varphi \otimes \psi) = \varphi \otimes D\psi + \psi \otimes D\varphi$ at each stage.

Decompose the set of positive integers I^+ into a sequence of disjoint infinite sets $I_1, I_2, \ldots$ and within each set order its elements according to their natural order, denoting the j^{th} integer in I_i by the pair ij. The elements of these sets will be used as indices of variables e_{ij} having values in E. Introduce the following definitions.

DEFINITION 3.1 If $\mathcal{Q}$ is a function whose variables have indices in I^+ and $J \subset I^+$, then $\int_J f^{\infty} \mathcal{Q}$ is to mean $\int \prod_{ij \in J} f(d\xi_{ij}) \mathcal{Q}$ whenever the integral exists.

DEFINITION 3.2 J^c denotes the complement of $J \subset I^+$

DEFINITION 3.3 Let C^i, $i \geq 1$ be the space of all functions $\mathcal{Q}$ which can be expressed as a countable sum $\sum_\alpha \mathcal{Q}_\alpha$, $\sum_\alpha \|\mathcal{Q}_\alpha\| < \infty$ where $\mathcal{Q}_\alpha$ has a finite number of variables whose indices are in $I_1 \cup \ldots \cup I_i$ and $\|\mathcal{Q}\|$ denotes the uniform norm.

DEFINITION 3.4 Let C_o^i be those functions in C^i which have a finite number of variables.

DEFINITION 3.5 Suppose that we have a family of spaces C_1^m, $C_o^m \subseteq C_1^m \subseteq C^m$ and a family of linear operators D_n mapping C_1^m into C_1^{n+1} for $n \geq m$. Then the operators D_n will be called a derivation if

$$(1) \quad D_n(\mathcal{Q} \otimes \psi) = (D_n \mathcal{Q}) \otimes \psi + \mathcal{Q} \otimes D_n \psi$$

$$(2) \quad D_n \mathcal{Q} = \sum_{i=1}^{m} \sum_{j=1}^{\infty} D_n (\mathcal{Q})_{ij}$$

where $(\mathcal{Q})_{ij}$ denotes $\mathcal{Q}$ thought of as a function of the ij-th variable alone, the other variables being held constant.

(Do you want to introduce some [illegible] relations among the D_n's. Else why a derivation?

DEFINITION 3.6 A family D_n, $n=1,2,\ldots$ of derivations will be called a derivation for the operator B if and only if

(1) for any $\varphi \in C_1^m$ with only one variable, any $n \geq m$ and any distribution f,

$$\int B[f](d\xi)\varphi(\xi) = \int f^{\infty} D_n \varphi$$

(2) $D_n\varphi = 0$ whenever φ is a constant

(3) $D_n\varphi$ depends upon the variables of φ together with new variables coming only from I_{n+1}, and $D_n\varphi$ does not depend on n in the sense that $D_n\varphi$ and $D_m\varphi$ are identical if the new variables which D_n adds to φ are renamed; especially, $\int_{I_{n+1}} f^{\infty} D_n \varphi$ is independent of $n \geq m$.

DEFINITION 3.7 For $\varphi \in C_1^m$, $D^n\varphi$ will mean $D_{n+m}D_{n+m-1}\ldots D_m\varphi$.

The following theorem and its corollary provide a formal solution of $\frac{\partial}{\partial t} f_t = B[f_t]$ through the use of derivations.

THEOREM 3.8 (formal) If D is a derivation for the operator B, then

$$\int f_t^{(n)}(d\xi)\varphi(\xi) = \int f_t^{\infty} D^n \varphi$$

for all functions φ of one variable, at least formally.

PROOF by induction on n. The theorem certainly holds for $n=1$ by the definition of a derivation. Suppose it also holds for n and put $D^n\varphi = \sum_\alpha \psi_\alpha$, where ψ_α has a finite number $\tau(\alpha)$ of variables whose indices form a set $J_\alpha \subset I_1 \cup \ldots \cup I_{n+m+1}$ and $\sum_\alpha \|\psi_\alpha\| < \infty$.

Then

$$\int f_t^{(n+1)}(d\xi)\,\mathcal{Q}(\xi)$$

$$= \frac{d}{dt} \int f_t^{(n)}(d\xi)\,\mathcal{Q}(\xi)$$

$$= \frac{d}{dt} \int f_t^{\infty}\, D^n \mathcal{Q}$$

$$= \frac{d}{dt} \sum_{\alpha} \int f_t^{\infty}\, \psi_\alpha$$

$$= \frac{d}{dt} \sum_{\alpha} \int_{J_\alpha} f_t \otimes \dots \otimes f_t\, \psi_\alpha$$

where $f_t \otimes \dots \otimes f_t$ is the $\tau(\alpha)$-fold outer product

$$= \sum_{\alpha} \sum_{p \in J_\alpha} \int \dot{f}_t(d\xi_p) \int_{J_\alpha \setminus \{p\}} f_t \otimes \dots \otimes f_t\, \psi_\alpha$$

modulo interchanges of sums and differentiations; and hence, treating $\int_{J_\alpha \setminus \{p\}} f_t \otimes \dots \otimes f_t\, \psi_\alpha$ as a function of the single variable with index p, we get upon using (1) of the definition of a derivation

$$= \sum_{\alpha} \sum_{p \in J_\alpha} \int f_t^{\infty}\, D_{m+n+1}\left\{ \int_{J_\alpha \setminus \{p\}} f_t \otimes \dots \otimes f_t\, \psi_\alpha \right\}.$$

Because of the linearity of D_{m+n+1}, one gets formally

$$= \sum_{\alpha} \sum_{p \in J_\alpha} \int f_t^{\infty} \int_{J_\alpha \setminus \{p\}} f_t \otimes \cdots \otimes f_t D_{m+n+1}(\psi_\alpha)_p$$

$$= \int f_t^{\infty} \sum_{\alpha} \sum_{p \in J_\alpha} D_{m+n+1}(\psi_\alpha)_p$$

$$= \int f_t^{\infty} \sum_{p} \sum_{\alpha} D_{m+n+1}(\psi_\alpha)_p$$

$$= \int f_t^{\infty} \sum_{p} D_{m+n+1}(\sum_{\alpha} \psi_\alpha)_p$$

$$= \int f_t^{\infty} \sum_{p} D_{m+n+1}(D^n \varphi)_p$$

$$= \int f_t^{\infty} D_{m+n+1} D^n \varphi$$

$$= \int f_t^{\infty} D^{n+1} \varphi$$

<u>COROLLARY 3.9</u> If D is a derivation for the operator B and if the derivatives $f_t^{(n)}(e)$ exist, then f_t can be formally written as $f_t(e) = \int f^{\infty} e^{tD} \chi_e$ where e^{tD} is defined as the formal Taylor series $\sum_{p=0}^{\infty} \frac{t^p}{p!} D^p$.

4. SOLUTION OF $\frac{\partial}{\partial t} f_t = B[f_t]$

In this section, we will apply the methods of the last section to the problem of solving the equation $\frac{\partial}{\partial t} f_t = B[f_t]$. However, to do this we must put some restrictions on $\gamma_\pm$.

<u>DEFINITION 4.1</u> Let H be the class of functions

$$\gamma_\pm (u) = \mp \sum_{N=1}^{\infty} B_N^\pm (u)$$

where

(1) $B_N^\pm (u) = \sum_{k=0}^{N} C_N^{\pm 1}(k) \binom{N}{k} u^k (1-u)^{N-k}$, $C_N^{\pm 1}(k) \geq 0$

(2) $\sum_{N=1}^{\infty} N^p C_N \leq p! L^p$, $p \geq 1$, $C_N = \max\limits_{k \leq N, \pm 1} C_N^{\pm 1}(k)$, $L < \infty$ fixed.

A necessary condition for a function to be in H is that it be real analytic as the following theorem demonstrates.

<u>THEOREM 4.2</u> If $F \in H$, then F has derivatives of all orders and

$$\left| \left(\frac{d}{du}\right)^p F(u) \right| \leq p!(2L)^p$$

<u>PROOF</u>

$$\left| \left(\frac{d}{du}\right)^p F(u) \right|$$

$$= \left| \left(\frac{d}{du}\right)^p \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N(k) \binom{N}{k} u^k (1-u)^{N-k} \right|$$

$$= \left| \sum_{q=0}^{p} \binom{p}{q} \sum_{N=p-q}^{\infty} \sum_{k=q}^{N-p+q} C_N(k)\binom{N}{k} k\ldots(k-q+1)u^{k-q} \right.$$

$$\left. \cdot (N-k)\ldots(N-k-p+q+1)(-1)^{p-q}(1-u)^{N-k-p+q} \right|$$

$$= \left| \sum_{q=0}^{p} \binom{p}{q} \sum_{N=p-q}^{\infty} \sum_{k=0}^{N-p} C_N(k+q)\binom{N}{k+q}(k+q)\ldots(k+1)(N-k-q)\ldots \right.$$

$$\left. \cdot (N-k-p+1)(-1)^{p-q}\, u^k(1-u)^{N-p-k} \right|$$

$$\leq \sum_{q=0}^{p} \binom{p}{q} \sum_{N=p-q}^{\infty} \frac{N!}{(N-p)!}\, C_N \sum_{k=0}^{N-p} \binom{N-p}{k} u^k(1-u)^{N-p-k}$$

$$\leq \sum_{q=0}^{p} \binom{p}{q} \sum_{N=p-q}^{\infty} N^p C_N$$

$$\leq \sum_{q=0}^{p} \binom{p}{q} p!\,L^p$$

$$= p!(2L)^p$$

Notice that the term wise differentiation is justified by the convergence of the resulting sums.

COROLLARY 4.3 If $\mp\partial_{\pm} \in H$, then γ is real analytic on $[0,1]$ and the solution $f_t(+1)$ of

$$\frac{\partial}{\partial t} f_t(+1) = B[f_t](+1) = \gamma\left[f_t(+1)\right]$$

has derivatives of all orders.

LEMMA 4.4 If F, $G \in H$, then $FG \in H$.

PROOF Let

$$F(u)=\sum_{N=1}^{\infty}\sum_{k=o}^{N} c_N(k)\binom{N}{k}u^k(1-u)^{N-k}$$

$$G(u)=\sum_{N=1}^{\infty}\sum_{k=o}^{N} d_N(k)\binom{N}{k}u^k(1-u)^{N-k}$$

then

$$F(u)G(u)=\sum_{N=1}^{\infty}\sum_{k=o}^{N} e_N(k)\binom{N}{k}u^k(1-u)^{N-k}$$

where

$$e_N(k)=\binom{N}{k}^{-1}\sum_{N_1+N_2=N}\ \sum_{\substack{k_1+k_2=k\\ k_1\leq N_1\\ k_2\leq N_2}} c_{N_1}(k_1)d_{N_2}(k_2)\binom{N_1}{k_1}\binom{N_2}{k_2}.$$

Letting $e_N = \max\limits_{k\leq n} e_N(k)$ we have

$$\sum_{N=1}^{\infty} N^p e_N$$

$$\leq \sum_{N=1}^{\infty}\sum_{k=o}^{N} N^p e_N(k)$$

$$=\sum_{N=1}^{\infty}\sum_{k=o}^{N} N^p\binom{N}{k}^{-1}\sum_{N_1+N_2=N}\ \sum_{\substack{k_1+k_2=k\\ k_1\leq N_1\\ k_2\leq N_2}} c_{N_1}(k_1)d_{N_2}(k_2)\binom{N_1}{k_1}\binom{N_2}{k_2}$$

$$= \sum_{N_1,N_2} \sum_{\substack{k_1 \leq N_1 \\ k_2 \leq N_2}} (N_1+N_2)^p c_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1+N_2}{k_1+k_2}^{-1} \binom{N_1}{k_1}\binom{N_2}{k_2}$$

$$\leq \sum_{N_1,N_2} \sum_{\substack{k_1 \leq N_1 \\ k_2 \leq N_2}} (N_1+N_2)^p c_{N_1}(k_1) d_{N_2}(k_2)$$

$$\leq \sum_{N_1,N_2} N_1 N_2 \sum_{q=0}^{p} \binom{p}{q} N_1^q \, N_2^{p-q} c_{N_1} d_{N_2}$$

$$\leq \sum_{q=0}^{p} \binom{p}{q} \left(\sum N_1^{q+1} c_{N_1}\right)\left(\sum N_2^{p-q+1} d_{N_2}\right)$$

$$\leq \sum_{q=0}^{p} \binom{p}{q} (q+1)!\, L^{q+1} (p-q+1)!\, L^{p-q+1}$$

$$\leq L^{p+2}(p+1)! \sum_{q=0}^{p} \binom{p}{q}$$

$$\leq (4L)^{p+2} p!$$

<u>LEMMA 4.5</u> If $F \in H$, then $\exp(F) \in H$

<u>PROOF</u> Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} c_N(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

Then

$$\exp\,(F(u)) = \sum_{M=0}^{\infty} \sum_{j=0}^{M} \binom{M}{j} \hat{C}_M(j) u^j (1-u)^{M-j}$$

where

$\hat{C}_M(j)$

$$= \binom{M}{j}^{-1} \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1+\dots+N_n=M} \sum_{\substack{k_1+\dots+k_n=j \\ k_1 \le N_1 \\ \vdots \\ k_n \le N_n}} \binom{N_1}{k_1} \dots \binom{N_n}{k_n} C_{N_1}(k_1) \dots C_{N_n}(k_n).$$

Thus

$$\sum_{M=0}^{\infty} M^p \hat{C}_M$$

$$\le \sum_{M=0}^{\infty} \sum_{j=0}^{M} M^p \hat{C}_M(j)$$

$$= \sum_{M=0}^{\infty} \sum_{j=0}^{M} M^p \binom{M}{j}^{-1} \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1+\dots+N_n=M} \sum_{\substack{k_1+\dots+k_n=j \\ k_1 \le N_1 \\ \vdots \\ k_n \le N_n}} \binom{N_1}{k_1} \dots \binom{N_n}{k_n}$$

$$\cdot C_{N_1}(k_1) \dots C_{N_n}(k_n)$$

$$= \sum_{N_1,\dots,N_n} \sum_{\substack{k_1 \le N_1 \\ \vdots \\ k_n \le N_n}} \sum_{n=0}^{\infty} (n!)^{-1} (N_1+\dots+N_n)^p \binom{N_1+\dots+N_n}{k_1+\dots+k_n}^{-1} \binom{N_1}{k_1} \dots \binom{N_n}{k_n}$$

$$\cdot C_{N_1}(k_1) \dots C_{N_n}(k_n)$$

$$\le \sum_{N_1,\dots,N_n} \sum_{\substack{k_1 \le N_1 \\ \vdots \\ k_n \le N_n}} \sum_{n=0}^{\infty} (n!)^{-1} (N_1+\dots+N_n)^p C_{N_1} \cdots C_{N_n}$$

$$\leqslant \sum_{N_1,\dots,N_n} \sum_{n=0}^{\infty} (n!)^{-1} N_1\cdots N_n (N_1+\dots+N_n)^p C_{N_1}\cdots C_{N_n}$$

$$= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1,\dots,N_n} \sum_{s_1+\dots+s_n=p} \frac{p!}{s_1!\dots s_n!} N_1^{\ s_1}\dots N_n^{\ s_n}$$

$$\cdot N_1\dots N_n C_{N_1}\cdots C_{N_n}$$

$$= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{s_1+\dots+s_n=p} p!L^p(s_1+1)\dots(s_n+1)$$

$$= \sum_{n=0}^{\infty} (n!)^{-1} L^p 2n(2n+1)\dots(2n+p-1)$$

$$\leqslant (2L)^p p! \sum_{n=0}^{\infty} (n!)^{-1} \binom{n+p-1}{p}$$

$$\leqslant p!(2L)^p \sum_{n=0}^{\infty} (n!)^{-1} 2^n 2^{p-1}$$

$$\leqslant p!(4L)^p e^2.$$

LEMMA 4.6 (Hausdorff [1]) If a polynomial F is positive (>0) on the open interval $(0,1)$, then it can be expressed as

$$F(u) = \sum_{m=0}^{N} a_m \chi_{N,m}(u), \quad a_m \geqslant 0$$

where $\chi_{N,m}(u) = \binom{N}{m} u^m (1-u)^{N-m}$, provided that N is sufficiently large.

PROOF If $F(o)$ or $F(1)$ equals o, then we can write $F(u)$ as $u^i(1-u)^j \hat{F}(u)$ where $\hat{F}(u)$ is positive ($>o$) on the closed interval $[o,1]$. We therefore need only prove the lemma for F positive ($>o$) on $[o,1]$. Suppose that $F(u) = \sum_{k=o}^{n} \alpha_k u^k$ is a positive polynomial on the interval $[o,1]$. Then we wish to write F in the form $F(u) = \sum_{m=o}^{N} a_m \chi_{N,m}(u)$. An easy calculation shows that

$$a_m = \sum_{k=o}^{n} \alpha_k \frac{m!(N-k)!}{N!(m-k)!} = \sum_{k=o}^{n} \alpha_k \frac{m(m-1)\ldots(m-k+1)}{N(N-1)\ldots(N-k+1)}$$

or, $a_m = F_N(\frac{m}{N})$ where

$$F_N(u) = \sum_{k=o}^{n} \alpha_k \frac{Nu(Nu-1)\ldots(Nu-k+1)}{N\ (N-1)\ldots(N-k+1)}$$

But, as N increases $\dfrac{Nu-h}{N-h}$ converges to u and hence $F_N(u)$ converges to $F(u)$ and thus for large N, $|a_m - F(\frac{m}{N})| < \varepsilon$ for $m = o, 1, \ldots, N$. Thus for N sufficiently large, $a_m \geq o$ and the theorem is proved.

LEMMA 4.7 If F is a complex valued function on the complex numbers, real on the real numbers and analytic on the closed disc $|z| \leq 1$, then for any sufficiently large real constant C, $F(z)+C \in H$.

PROOF Let F be real on the real line and analytic in the closed disc $|z| \leq 1$. Then there exists a $\delta > o$ such that $F(z) = \sum_{N=o}^{\infty} \alpha_N z^N$ for $|z| \leq 1+\delta$ where

$$|\alpha_N| = \left|\frac{F^{(N)}(o)}{N!}\right| \leq (1+\delta)^{-N} A, \text{ A a positive constant.}$$

Now let

$$b_n = \begin{cases} \alpha_N & \text{if } \alpha_N \geq o \\ o & \text{otherwise} \end{cases}, \quad a_N = \begin{cases} -\alpha_N & \text{if } \alpha_N < o \\ o & \text{otherwise} \end{cases}, \quad C = d - b_o + \sum_{N=o}^{\infty} a_N, \; d \geq o$$

Then

$$F(z)+C = d + \sum_{N=1}^{\infty} b_N \, z^N + \sum_{N=1}^{\infty} a_N \, (1-z^N).$$

But

$$1-z^N = \sum_{k=o}^{N-1} \binom{N}{k} z^k (1-z)^{N-k}, \; N \geq 1$$

and hence, letting

$$B_N(z) = \sum_{k=o}^{N} C_N(k) \binom{N}{k} z^k (1-z)^{N-k}$$

where $C_1(o) = a_1 + d$, $C_1(1) = b_1 + d$ and

$$C_N(k) = \begin{cases} a_N & \text{for } o \leq k < N \\ b_N & \text{for } k = N \end{cases}, \; N > 1$$

we get

$$F(z) + C = \sum_{N=1}^{\infty} B_N(z)$$

Thus we have represented $F+C$ as a sum of Bernstein Polynomials when $C \geq -b_o + \sum_{N=o}^{\infty} a_N$. We therefore need only show that condition (2) is satisfied. But $C_N = \max_{k \leq n} C_N(k) = |\alpha_N| \leq (1+\delta)^{-N} A$.

Hence

$$\sum_{N=o}^{\infty} N^p C_N$$

$$\leqslant A \sum_{N=0}^{\infty} N^p e^{-N \ln(1+\delta)}$$

which corresponds to

$$\int_0^{\infty} t^p e^{-t \ln(1+\delta)} dt$$

$$= \frac{p!}{[\ln(1+\delta)]^{p+1}}$$

$$\leqslant p!\, L^p$$

for a suitable L.

We can now give a sufficent conditions that F be contained in H.

LEMMA 4.8 If F is analytic on the closed disc $|z| \leqslant 1$, real on the reals and positive on $[0,1]$, then $F \in H$.

PROOF Since F is analytic on $|z| \leqslant 1$, real on the reals and positive on $[0,1]$, it can be written as

$$e^{-C}(z-z_1)(z-z_1^*)\dots(z-z_n)(z-z_n^*)e^{C+G(z)}$$

where z^* is the conjugate of z, G is analytic on $|z| \leqslant 1$ and real on the reals.

For C sufficiently large, $C + G(z) \in H$ and hence $\exp\,(C+G(z)) \in H$.

Since, according to Hausdorff's lemma,

$$(z-z_1)(z-z_1^*)\dots(z-z_n)(z-z_n^*) \in H.$$

and since H is closed under products, the proof is complete.

<u>LEMMA 4.9</u> If $F, G \in H$ and G is a polynomial, $0 \leq G(u) \leq 1$ on $(0,1)$, then $F(G(\cdot)) \in H$.

<u>PROOF</u> Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} c_N(k) \binom{N}{k} u^k (1-u)^{N-k}$$

$$G(u) = \sum_{p=0}^{M} d(p) u^p (1-u)^{M-p}$$

$$1-G(u) = \sum_{q=0}^{M} e(q) u^q (1-u)^{M-q}$$

Then

$F(G(u))$

$$= \sum_{N=0}^{\infty} \sum_{k=0}^{N} c_N(k) \binom{N}{k} \left[\sum_{p=0}^{M} d(p) u^p (1-u)^{M-p} \right]^k$$

$$\cdot \left[\sum_{q=0}^{M} e(q) u^q (1-u)^{M-q} \right]^{N-k}$$

$$= \sum_{N=1}^{\infty} \sum_{j=0}^{NM} \hat{c}_{NM}(j) \binom{NM}{j} u^j (1-u)^{MN-j}$$

where

$\hat{c}_{NM}(j)$

$$= \binom{NM}{j}^{-1} \sum_{k=o}^{N} C_N(k)\binom{N}{k} \sum_{\substack{p_1+\dots+p_k+q_1+\dots+q_{N-k}=j \\ o \leq p,q \leq M}} d(p_1)\dots d(p_k)e(q_1)\dots e(q_{N-k}) \cdot$$

Using Stirling's formula we have

$$\binom{NM}{j}^{-1}$$

$$\sim (2\pi MN)^{\frac{1}{2}} \left(\frac{j}{MN}\right)^{\frac{1}{2}} \left(1-\frac{j}{MN}\right)^{\frac{1}{2}} \left(\frac{j}{MN}\right)^{j} \left(1-\frac{j}{MN}\right)^{MN-j}$$

$$\leq 2\pi MN \left(\frac{j}{MN}\right)^{j} \left(1-\frac{j}{MN}\right)^{MN-j}$$

and thus

$$\sum_{N=1}^{\infty} (NM)^p \hat{C}_{NM}$$

$$\leq \sum_{N=1}^{\infty} (NM)^p \sum_{j=o}^{NM} \hat{C}_{NM}(j)$$

$$= \sum_{N=1}^{\infty} (NM)^p \sum_{j=o}^{NM} \sum_{k=o}^{N} C_N(k)\binom{N}{k}\binom{NM}{j}^{-1} \sum_{\substack{p_1+\dots+p_k+q_1+\dots+q_{N-k}=j \\ o \leq p,q \leq M}} d(p_1)\dots d(p_k)$$

$$\cdot \; e(q_1)\dots e(q_{N-k})$$

$$\leq \sum_{N=1}^{\infty} (NM)^p \sum_{j=o}^{NM} \sum_{k=o}^{N} C_N(k)\binom{N}{k} 2\pi MN \sum_{\substack{p_1+\dots+p_k+q_1+\dots+q_{N-k}=j \\ o \leq p,q \leq M}} d(p_1) \; d(p_k)$$

$$\cdot\ e(q_1)\cdots e(q_{N-k})\left(\frac{j}{NM}\right)^{j}\left(1-\frac{j}{NM}\right)^{NM-j}$$

$$\leq 2\pi M^{p+1}\sum_{N=1}^{\infty} N^{p+1}C_N \sum_{m=o}^{MN}\sum_{k=o}^{N}\binom{N}{k}\sum_{j=o}^{NM}\sum_{\substack{p_1+\dots+p_k+q_1+\dots+q_{N-k}=j \\ o\leq p,q\leq M}} d(p_1)\dots d(p_k)$$

$$\cdot\ e(q_1)\dots e(q_{N-k})\left(\frac{m}{NM}\right)^{j}\left(1-\frac{m}{NM}\right)^{NM-j}$$

$$= 2\pi M^{p+1}\sum_{N=1}^{\infty} N^{p+1}C_N\sum_{m=o}^{MN}\sum_{k=o}^{N}\binom{N}{k}\left[G\left(\frac{m}{NM}\right)\right]^{k}\left[1-G\left(\frac{m}{NM}\right)\right]^{N-k}$$

$$= 2\pi M^{p+2}\sum_{N=1}^{\infty} N^{p+2}C_N$$

$$\leq 2\pi M^{p+2}(p+2)!\,L^{p+2}$$

$$\leq p!\,\hat{L}^{p}$$

for suitable $\hat{L}$.

<u>THEOREM 4.10</u> $F\in H$ if and only if F is positive on the open interval $(o,1)$, with at most algebraic roots at o and 1, and real analytic on the closed interval $[o,1]$.

<u>PROOF</u> If $F\in H$, then F is certainly positive on the open interval $(o,1)$ and, by Theorem 4.2, it is real analytic on the closed interval $[o,1]$. Therefore assume that F is positive

and real analytic on the closed interval $[o,1]$; if F had roots at o or 1 we could divide through by them. Then we can find a domain D, symmetric about and containing the interval $[o,1]$ on which F is analytic. If there exists a polynomial G mapping D conformally onto a disc containing the unit disc; and if G is real on the reals with $G(o)=o$ and $G(1)=1$; then $F(G^{-1}(w))$ is analytic on the closed unit disc; real on the reals and positive on $[o,1]$. Thus, by Lemma 4.8, $F(G^{-1}(\cdot)) \in H$ and by Lemma 4.9 $F(G^{-1}(G(\cdot)))$ $= F(\cdot) \in H$. Therefore, to complete the proof we need only show that G exists.

Let $G_1(z)$ be the unique conformal mapping of D onto the disc $|z| < 2$ where $G_1(o)=o$ and $G_1'(o) > o$. Since D is symmetric about the interval $[o,1]$, $[G_1(z^*)]^*$ also maps D, conformally onto $|z| < 2$ and hence $[G_1(z^*)]^* = G_1(z)$ and G_1 is real on the real axis. Since there exists a sequence of polynomials converging uniformly to G_1 on D, let G_2 be a polynomial for which $G_2(o)=o$ and $|G_1(z)-G_2(z)| < \varepsilon$ for $z \in D$; and define $G(z)$ as $a[G_2(z) + G_2(z^*)^*]$ where $a = \frac{1}{2}[G_2(1) + G_2(1)^*]^{-1}$. Then, for ε sufficiently small, G maps D into a region which contains a disc of radius $a(2-\varepsilon)$; notice that for ε small, a is approximately equal to $[G_1(1)]^{-1}$. Furthermore, G is real on the reals, $G(o)=o$ and $G(1)=1$. We need therefore only show that G is 1-1 to complete the proof. Let C be the arc in D which is the inverse image of the circle $|z| = a(2-\varepsilon)$ under the mapping G_1. Then $G_1 \neq 0, \infty$ on C and for any w contained in the disc

$$|z| < \frac{1+a(2-\varepsilon)-1}{2} \quad \text{we have}$$

$$aG_1(z)-w = G(z)-w + h(z)$$

where $|h(z)| < \varepsilon$. But $|G(z)-w| > \varepsilon > |h(z)|$ for ε sufficiently small, $z \in C$. Thus, by Rouche's Theorem, $G(z)$ takes on the value w only once. Therefore G maps a domain $D^1 \subset D$, D^1 containing the interval $[0,1]$, conformally onto a disc containing the unit disc and thus the proof is complete.

Our present task is to construct a derivation D for B when $\mp \gamma_{\pm} \in H$ and to show that $e^{tD} = \sum_{p=0}^{\infty} \frac{t^p}{p!} D^p$ is well defined for small t. Divide the positive integers into infinite classes I_i, $i=1,2,\ldots$ as in the second section, let the pair ij represent the j^{th} integer in I_i under the natural ordering, and let $\Delta_{ij}\mathcal{Q} = \mathcal{Q}(\ldots,-e_{ij},\cdots) - \mathcal{Q}(\ldots,e_{ij},\cdots)$.

<u>DEFINITION 4.11</u> For $\mathcal{Q} \in C_0^m$ and $n \geq m$, let

$$D_n\mathcal{Q} = \sum_{N=1}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{\infty} C(e_{ij}| e_{n+1,1},\ldots,e_{n+1,N}) \, \Delta_{ij} \mathcal{Q}$$

where $c(e|e_1,\cdots,e_N) = C_N^e$ (number of $+1$'s in the set $e_1,\cdots,e_N$).

This sum clearly converges since, for $\mathcal{Q}$ having p variables,

$$\sum_{N=M}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{\infty} \| C(e_{ij} | e_{n+1,1}, \dots, e_{n+1,N}) \Delta_{ij} \mathcal{Q} \|$$

$$\leq 2\, p \|\mathcal{Q}\| \sum_{N=M}^{\infty} c_N$$

converges to zero as $M \to \infty$. In order that the operators D_n be a derivation, we must define the spaces C_1^m. To do this, define by induction $C_1^1 = C_o^1$ and $C_1^n = (D_{n-1} C_1^{n-1}) \cup C_o^n$. This, of course, pre-supposes that $D_n \mathcal{Q}$ is defined for $\mathcal{Q} \in C_1^m$, $m \leq n$. This will be shown in this chapter. However, before doing this, note that if C_1^m is defined, then the family of operators D_n are indeed a derivation for B. This is easily seen by the following theorem together with a few simple observations.

<u>THEOREM 4.12</u> If $\mathcal{Q} \in C_o^{\infty}$ has only one variable, then

$$\int B[f](d\xi)\, \mathcal{Q}(\xi) = \int f^{\infty} D_n \mathcal{Q} \,.$$

<u>PROOF</u>

$$\int f^{\infty} D_n \mathcal{Q}$$

$$= \int f^{\infty} \sum_{N=1}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{\infty} C(e_{ij} | e_{n+1,1}, \dots, e_{n+1,M}) \Delta_{ij} \mathcal{Q}$$

$$= \sum_{N=1}^{\infty} \int f^{\infty} C(e_{i_o j_o} | e_{n+1,1}, \dots, e_{n+1,N}) \Delta_{i_o j_o} \mathcal{Q}$$

$$= \sum_{N=1}^{\infty} \sum_{k=o}^{N} \int_{\Lambda_k} f^{\infty} C(e_{i_o j_o} | e_{n+1,1}, \dots, e_{n+1,N}) \Delta_{i_o j_o} \mathcal{Q}$$

where Λ_k is the set of all sequences $[e_{n+1,1},\dots,e_{n+1,N}]$ containing exactly k +1's

$$=\sum_{N=1}^{\infty}\sum_{k=0}^{N}\binom{N}{k}[f(+1)]^{k}[f(-1)]^{N-k}\int f(de_{i_0j_0})C_N^{\delta}(k)\Delta_{i_0j_0}\mathcal{Q}$$

where $\delta = e_{i_0j_0}$

$$=\sum_{N=1}^{\infty}\sum_{k=0}^{N}\binom{N}{k}u^k(1-u)^{N-k}C_N^{+1}(k)[\mathcal{Q}(-1)-\mathcal{Q}(+1)]u$$

$$+\sum_{N=1}^{\infty}\sum_{k=0}^{N}\binom{N}{k}u^k(1-u)^{N-k}C_N^{-1}(k)[\mathcal{Q}(+1)-\mathcal{Q}(-1)](1-u)$$

$$=-u\sum_{N=1}^{\infty}B_N^{+}(u)[\mathcal{Q}(+1)-\mathcal{Q}(-1)]+(1-u)\sum_{N=1}^{\infty}B_N^{-}(u)[\mathcal{Q}(+1)-\mathcal{Q}(-1)]$$

$$=[u\,\gamma_+(u)+(1-u)\gamma_-(u)][\mathcal{Q}(+1)-\mathcal{Q}(-1)]$$

$$=\gamma(u)[\mathcal{Q}(+1)-\mathcal{Q}(-1)]$$

$$=B[f](+1)\mathcal{Q}(+1)-B[f](+1)\mathcal{Q}(-1)$$

$$=B[f](+1)\mathcal{Q}(+1)+B[f](-1)\mathcal{Q}(-1)$$

$$=\int B[f](d\xi)\,\mathcal{Q}(\xi).$$

Thus the proof is complete.

We will now show that $D^n \mathcal{Q}$ (and hence C_1^n) are defined and that $e^{tD}\mathcal{Q} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \mathcal{Q}$ converges for sufficiently small t.

DEFINITION 4.13 Let $\mathcal{Q} \in C_0^1$ have variables with indices 11, 12,..., $1M \in I_1$, ie., $\mathcal{Q} = \mathcal{Q}(e_{11}, e_{12}, \ldots, e_{1M})$ and define by induction on p

$$Q_1(\mathcal{Q};N) = \sum_{j=1}^{M} C(e_{1j} | e_{21}, \ldots, e_{2N}) \Delta_{1j} \mathcal{Q}$$

$$Q_{p+1}(\mathcal{Q};N_1, \ldots, N_{p+1})$$

$$= \sum_{i=1}^{p+1} \sum_{j=1}^{N_i} C(e_{ij} | e_{p+2,1}, \ldots, e_{p+2,N_{p+1}}) \Delta_{ij} Q_p(\mathcal{Q};N_1, \ ,N_p).$$

LEMMA 4.14 If $\mathcal{Q} \in C_0^1$ has variables whose indices are 11,...,1M then

$$D^p \mathcal{Q} = \sum_{N_1, \ldots, N_p = 1}^{\infty} Q_p(\mathcal{Q};N_1, \ldots, N_p)$$

provided that the sum converges.

PROOF by induction on p. For p=1,

$$\sum_{N=1}^{\infty} Q_1(\mathcal{Q};N) = \sum_{N=1}^{\infty} \sum_{j=1}^{M} C(e_{1j} | e_{21}, \ldots, e_{2N}) \Delta_{1j} \mathcal{Q} = D_1 \mathcal{Q} = D \mathcal{Q}.$$

Now assume that the theorem holds for p. Then

$$\sum_{N_1, \ldots, N_{p+1}} Q_{p+1}(\mathcal{Q};N_1, \ldots, N_{p+1})$$

$$= \sum_{N_1,\dots,N_{p+1}} \sum_{i=1}^{p+1} \sum_{j=1}^{N_i} C(e_{ij} | e_{p+2,1},\dots,e_{p+2,N_{p+1}}) \Delta_{ij} Q_p(\varphi; N_1,\dots,N_p)$$

$$= \sum_{N_{p+1}=1}^{\infty} \sum_{i=1}^{p+1} \sum_{j=1}^{N_i} C(e_{ij} | e_{p+2,1},\dots,e_{p+2,N_{p+1}})$$

$$\cdot \Delta_{ij} \sum_{N_1,\dots,N_p=1}^{\infty} Q_p(\varphi; N_1,\dots,N_p)$$

$$= \sum_{N_{p+1}=1}^{\infty} \sum_{i=1}^{p+1} \sum_{j=1}^{N_i} C(e_{ij} | e_{p+2,1},\dots,e_{p+2,N_{p+1}})$$

$$\cdot \Delta_{ij} (D_p D_{p-1} \dots D_2 D_1 \varphi)$$

$$= D_{p+1}(D_p \dots D_2 D_1)\varphi$$

$$= D^{p+1} \varphi \; .$$

<u>LEMMA 4.15</u> If $\varphi \in C_o^1$ has M variables, then

$$\| Q_p(\varphi; N_1,\dots,N_p) \|$$

$$\le 2^p \|\varphi\| \, M C_{N_p} \cdots C_{N_1} \sum_{i_p=1}^{p} \sum_{i_{p-1}=1}^{p-1} \cdots \sum_{i_1=1}^{1} N_p^{\tau_p(i_1,\dots,i_p)} \cdots N_1^{\tau_1(i_1,\dots,i_p)}$$

where $\mathcal{T}_k(i_1,\dots,i_p)$ is the number of integers $i_1,\dots,i_p$ equal to k.

PROOF

$$\|Q_p(\mathcal{Q};N_1,\dots,N_p)\|$$

$$=\Big\|\sum_{i=1}^{p}\sum_{j=1}^{N_i} C(e_{ij}|e_{p+1,1},\dots,e_{p+1,N_p})\Delta_{ij}Q_{p-1}(\mathcal{Q};N_1,\dots,N_{p-1})\Big\|$$

$$\leq 2\sum_{i=1}^{p} C_{N_p}N_i\,\|Q_{p-1}(\mathcal{Q};N_1,\dots,N_{p-1})\|$$

$$\leq 2^{p-1}C_{N_p}C_{N_{p-1}}\dots C_{N_2}\Big(\sum_{i_p=1}^{p}N_{i_p}\Big)\Big(\sum_{i_{p-1}=1}^{p-1}N_{i_{p-1}}\Big)\dots\Big(\sum_{i_2=1}^{2}N_{i_2}\Big)\|Q_1\|$$

$$\leq 2^p\|\mathcal{Q}\|MC_{N_p}\dots C_{N_1}\sum_{i_p=1}^{p}\sum_{i_{p-1}=1}^{p-1}\dots\sum_{i_1=1}^{1}N_{i_p}\dots N_{i_1}$$

$$=2^p\|\mathcal{Q}\|MC_{N_p}\dots C_{N_1}\sum_{i_p=1}^{p}\dots\sum_{i_1=1}^{1}N_p^{\mathcal{T}_p(i_1,\dots,i_p)}\dots N_1^{\mathcal{T}_1(i_1,\dots,i_p)}\quad.$$

LEMMA 4.16 If

$$A_p=\sum_{i_p=1}^{p}\sum_{i_{p-1}=1}^{p-1}\dots\sum_{i_1=1}^{1}\mathcal{T}_p(i_1,\dots,i_p)!\,\mathcal{T}_{p-1}(i_1,\dots,i_p)!\,\mathcal{T}_1(i_1,\dots,i_p)!$$

then $A_p = 1\cdot 3\cdot 5\cdot \ldots (2p-1) \leq 2^p p!$.

PROOF by induction on p. The lemma is certainly true for $p=1$. Suppose that it also holds for p. Then

$$A_{p+1} = \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_{p+1}(i_1,\ldots,i_{p+1})! \ldots \tau_1(i_1,\ldots,i_{p+1})!$$

$$= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1,\ldots,i_{p+1})! \ldots \tau_1(i_1,\ldots,i_{p+1})!$$

$$= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1,\ldots,i_p)! \ldots \tau_{i_{p+1}+1}(i_1,\ldots,i_p)!$$

$$\cdot \left[1+\tau_{i_p}(i_1,\ldots,i_p)\right]! \, \tau_{i_{p+1}-1}(i_1,\ldots,i_p)! \ldots \tau_1(i_1,\ldots,i_p)!$$

$$= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \left[1+\tau_{i_{p+1}}(i_1,\ldots,i_p)\right] \tau_p(i_1,\ldots,i_p)!$$

$$\cdot \ldots \tau_1(i_1,\ldots,i_p)!$$

$$= (p+1) \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1,\ldots,i_p)! \ldots \tau_1(i_1,\ldots,i_p)!$$

$$+\sum_{i_{p+1}=1}^{p+1}\sum_{i_p=1}^{p}\cdots\sum_{i_1=1}^{1}\tau_{i_{p+1}}(i_1,\ldots,i_p)\tau_p(i_1,\ldots,i_p)!$$

$$\ldots\tau_1(i_1,\ldots,i_p)!$$

$$=(p+1)A_p+\sum_{i_p=1}^{p}\cdots\sum_{i_1=1}^{1}\left(\sum_{i_{p+1}=1}^{p+1}\tau_{i_{p+1}}(i_1,\ldots,i_p)\right)\tau_p(i_1,\ldots i_p)!$$

$$\ldots\tau_1(i_1,\ldots,i_p)!$$

$$=(p+1)A_p+\sum_{i_p=1}^{p}\cdots\sum_{i_1=1}^{1}p\tau_p(i_1,\ldots,i_p)!\ldots\tau_1(i_1,\ldots,i_p)!$$

$$=(p+1)A_p+pA_p$$

$$=(2p+1)A_p$$

$$=1\cdot 3\cdot 5\cdot\ \ldots\cdot(2p-1)(2p+1)$$

and the lemma is proved.

THEOREM 4.17 If $\mathcal{Q}\in C_o^1$ has a finite number M of variables, then $D^p\mathcal{Q}$ is defined and $\|D^p\mathcal{Q}\|\leq p!\ (4L)^pM\|\mathcal{Q}\|$.

PROOF We have shown that

$$D^p\mathcal{Q} = \sum_{N_1,\ldots,N_p} Q_p(\mathcal{Q}; N_1,\ldots,N_p)$$

provided that the sum converges. But, by lemma 4.14, 4.15 and 4.16 and the fact that $\sum N^p C_N \leq p!L^p$, it follows that

$$\sum_{N_1,\ldots,N_p} \| Q_p(\mathcal{Q}; N_1,\ldots,N_p) \|$$

$$\leq 2^p \|\mathcal{Q}\| M \sum_{N_1,\ldots,N_p} C_{N_p} \ldots C_{N_1}$$

$$\cdot \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} N_p^{\tau_p(i_1,\ldots,i_p)} \cdots N_1^{\tau_1(i_1,\ldots,i_p)}$$

$$= 2^p M \|\mathcal{Q}\| \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1}$$

$$\cdot \left(\sum_{N=1}^{\infty} N^{\tau_p(i_1,\ldots,i_p)} C_N\right) \cdots \left(\sum_{N=1}^{\infty} N^{\tau_1(i_1,\ldots,i_p)} C_N\right)$$

$$\leq 2^p M \|\mathcal{Q}\| \sum_{i_p=1}^{p} \sum_{i_1=1}^{1} \tau_p(i_1,\ldots,i_p)!\, L^{\tau_p(i_1,\ldots,i_p)}$$

$$\ldots \tau_1(i_1,\ldots,i_p)!\, L^{\tau_1(i_1,\ldots,i_p)}$$

$$= (2L)^p M \|\mathcal{Q}\| \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1,\ldots,i_p)! \ldots \tau_1(i_1,\ldots,i_p)!$$

$$\leq (2L)^{p} M \|\varphi\| 2^{p} p!$$

$$= p!\ (4L)^{p} M \|\varphi\|.$$

Thus $\sum_{N_1,\ldots,N_p} Q_p(\varphi; N_1,\ldots,N_p)$ converges and $D^p\varphi$ is well defined.

Furthermore,

$$\|D^p\varphi\|$$

$$= \Big\|\sum_{N_1,\ldots,N_p} Q_p(\varphi; N_1,\ldots,N_p)\Big\| \leq \sum_{N_1,\ldots,N_p} \|Q_p(\varphi; N_1,\ldots,N_p)\|$$

$$\leq p!(4L)^{p} M \|\varphi\| \quad .$$

The proof is therefore complete.

We can now define C_1^p for all p since $C_1^1 = C_o^1$ and $\varphi \in C_1^p$ if and only if $\varphi = D_{p-1}D_{p-2}\ldots D_{p-q}\psi$ for some $\psi \in C_o^{p-q}$ and $o \leq q \leq p$. D_p is clearly defined on C_1^p and maps C_1^p into C_1^{p+1} . Since we have already shown that

$$\int B[f](d\xi)\,\varphi(\xi) = \int f^{\infty} D_n\varphi$$

for functions φ of only one variable and since one can easily see that for $\varphi, \psi \in C_1^p$:

$$(1)\quad D_p(\varphi \otimes \psi) = (D_p\varphi) \otimes \psi + \varphi \otimes (D_p\psi)$$

(2) $D_p \mathcal{Q} = \sum_{i=1}^{m} \sum_{j=1}^{\infty} D_p(\mathcal{Q})_{ij}$

(3) $D_p \mathcal{Q} = 0$ whenever $\mathcal{Q}$ is a constant

(4) D_p adds variables whose indices are in I_{p+1}, it follows that the family of operators D_n are indeed a derivation for B.

<u>COROLLARY 4.18</u> The proof of Theorem 3.8 becomes valid when B and D are defined as in this section.

<u>PROOF</u> For the proof of Theorem 3.8 to succeed, we need only justify the interchange of sums and termwise differentiation. This is easily seen to be the case if

$$\sum_{N_1,\dots,N_p} (M+N_1+\dots+N_p) Q_p(\mathcal{Q};N_1,\dots,N_p) < \infty.$$

Using the bounds for $\|Q_p(\mathcal{Q};N_1,\dots,N_p)\|$ given in Lemma 4.15, and using Lemma 4.16 one can see that the sum clearly converges.

Since $e^{tD}\mathcal{Q} = \sum_{p=0}^{\infty} \frac{t^p}{p!} D^p \mathcal{Q}$ converges for $t < \frac{1}{4}L$,

we get the following theorem.

<u>THEOREM 4.19</u> The solution $f_t(+1)$ of the equation $\frac{\partial}{\partial t} f_t = B[f_t]$ is unique and is equal to $\int f^{\infty} e^{tD} \chi_{+1}$.

<u>PROOF</u> Corollary 4.3 and Theorem 3.8 state that the p^{th} derivative of $f_t(+1)$ at $t=0$ exists and is equal to

$\int f^{\infty} D^{p} \chi_{+1}$. Since, by Theorem 4.17,

$$\left| \int f^{\infty} D^{p} \chi_{+1} \right| \leqslant \| D^{p} \chi_{+1} \| \leqslant p!(4L)^{p};$$

it follows that the Taylor series for $f_t(+1)$ converges for $t < \frac{1}{4L}$. This implies that

$$f_t(+1) = \int f^{\infty} e^{tD} \chi_{+1}$$

by the definition of e^{tD} and also implies the uniqueness of the solution.

5. AN INFINITE GAS

I shall now construct a model of an infinite particle gas with velocities ± 1 in which the motion of a tagged particle is a *-process with specific $\gamma_\pm$ and in which the sample paths of any two particles are independent. This will be accomplished by constructing a gas of n like particles, each of which has velocities ± 1, and then letting $n \to \infty$. Each of the n particle gases will be a Markov jump process in which one waits an exponential holding time and then picks an index i according to the uniform distribution 1/n and lets the corresponding particle collide with one or more of the remaining particles. The effect of a collision between a single particle and a set of particles will be a change of state only for the single particle.

To be more specific, suppose, as in the last section, that $\mp\gamma_\pm \in H$. $\mathfrak{X}_n(t) = [x_1^n(t), \dots, x_n^n(t)]$ will be the Markov jump process on the n-dimensional space E^n with holding time distribution in the state $[e_1, \dots, e_n]$ equal to

$$\exp\left\{-t \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \dots, j_N}^{(n)} C(e_i \mid e_{j_1}, \dots, e_{j_N})\right\}$$

where $\sum\limits_{j_o, j_1, \dots, j_N}^{(n)}$ denotes the sum taken over all sequences $(j_o, \dots, j_N)$, $o \leq j_k \leq n$; and where $C(e \mid e_1, \dots, e_N) = C_N^e$ (number of

+1's in the set $e_1,\dots,e_N$). Starting at the state $[e_1,\dots,e_n]$, the probability that the first jump is to the state $[e_1,\dots,e_{i-1}, -e_i, e_{i+1},\dots,e_n]$ is given by

$$\frac{\sum_{N=1}^{n-1} n^{-N} \sum_{j_1,\dots,j_N \neq i}^{(n)} C(e_i|e_{j_1},\dots,e_{j_N})}{\sum_{N=1}^{n-1} n^{-N} \sum_{k,j_1,\dots,j_N}^{(n)} C(e_k|e_{j_1},\dots,e_{j_N})}.$$

The generator of the n molecule gas will therefore be

$$G_n \varphi(e_1,\dots,e_n) = \sum_{N=1}^{n-1} n^{-N} \sum_{i,j_1,\dots,j_N}^{(n)} C(e_i|e_{j_1},\dots,e_{j_N}) \Delta_i \varphi$$

where $\Delta_i \varphi = \varphi(\dots,-e_i,\dots) - \varphi(\dots,e_i,\dots)$.

We will now show that the generators G_n of the n molecule gases converge to the derivation D of the last section in the sense that if φ is any function on n dimensional space E^n with variables whose indices form a fixed set J and if f is a distribution on E, then $\int_{J^c} f^\infty G_n \varphi \to \int_{J^c} f^\infty D\varphi$ as $n \to \infty$. Such convergence will be called convergence mod J^c. It will also be shown that as $n \to \infty$, the paths of any two particles of the n-molecule gas become independent and approximate the *-process with specified γ_t. The main burden is to show that the operators G_n^p converge to $D^p\varphi$ mod J^c when φ has variables with indices from J. First we will need several lemmas and a definition.

DEFINITION 5.1 Let

$$Q_{1,n}(\mathcal{Q};N) = n^{-N} \sum_{i,j_1,\dots,j_N}^{(n)} C(e_i|e_{j_1},\dots,e_{j_N})\Delta_i \mathcal{Q}$$

and

$$Q_{p+1,n}(\mathcal{Q};N_1,\dots,N_{p+1})$$

$$= n^{-N_{p+1}} \sum_{i,j_1,\dots,j_{N_{p+1}}}^{(n)} C(e_i|e_{j_1},\dots,e_{j_{N_{p+1}}})\Delta_i Q_{p,n}(\mathcal{Q};N_1,\dots,N_p).$$

LEMMA 5.2

$$G_n^p\mathcal{Q} = \sum_{N_p=1}^{n-1} \dots \sum_{N_1=1}^{n-1} Q_{p,n}(\mathcal{Q};N_1,\dots,N_p).$$

PROOF by induction on p. The lemma is certainly true for $p=1$ by the definition of $G_n\mathcal{Q}$. Suppose that the lemma is true for p.

Then

$$G_n^{p+1}\mathcal{Q}$$

$$= G_n G_n^p \mathcal{Q}$$

$$= \sum_{N_{p+1}=1}^{n-1} n^{-N_{p+1}} \sum_{i,j_1,\dots,j_{N_{p+1}}}^{(n)} C(e_i|e_{j_1},\dots,e_{j_{N_{p+1}}})\Delta_i G_n^p\mathcal{Q}$$

$$= \sum_{N_{p+1}=1}^{n-1} n^{-N_{p+1}} \sum_{i,j_1,\ldots,j_{N_{p+1}}}^{(n)} C(e_i | e_{j_1},\ldots,e_{j_{N_{p+1}}})$$

$$\cdot \Delta_i \sum_{N_p=1}^{n-1} \cdots \sum_{N_1=1}^{n-1} Q_{p,n}(\mathcal{Q};N_1,\ldots,N_p)$$

$$= \sum_{N_{p+1}=1}^{n-1} \cdots \sum_{N_1=1}^{n-1} n^{-N_{p+1}} \sum_{i,j_1,\ldots,j_{N_{p+1}}}^{(n)} C(e_i | e_{j_1},\ldots,e_{j_{N_{p+1}}})$$

$$\cdot \Delta_i Q_{p,n}(\mathcal{Q};N_1,\ldots,N_p)$$

$$= \sum_{N_{p+1}=1}^{n-1} \cdots \sum_{N_1=1}^{n-1} Q_{p+1,n}(\mathcal{Q};N_1,\ldots,N_{p+1}).$$

<u>DEFINITION 5.3</u> $\mathcal{Q} \equiv \psi$ mod J if and only if for any probability measure f, $\int_J f^\infty \mathcal{Q} = \int_J f^\infty \psi$.

<u>LEMMA 5.4</u> If $\mathcal{Q} \in C_o^1$ has M variables whose indices form a set J, then $Q_p(\mathcal{Q};N_1,\ldots,N_p) \equiv Q_p(\mathcal{Q};N_1,\ldots,N_p)+E_{p,n}(\mathcal{Q};N_1,\ldots,N_p)$ mod J_1^c where, for large n,

$$\|E_{p,n}(\mathcal{Q};N_1,\ldots,N_p)\|$$

$$\leq \left| \frac{n(n-1)\ldots(n+1-N_1-\ldots-N_p)}{n^{N_1+\ldots+N_p}} -1 \right| \|Q_p(\mathcal{Q};N_1, \ ,N_p)\|$$

$$+ n^{-1} M \, \| \mathcal{Q} \| \quad C_{N_p} \cdots C_{N_1} (M + N_1 + \cdots + N_p)^{p+1}$$

PROOF From the definition of $Q_{p,n}(\mathcal{Q}; N_1, \ldots, N_p)$ and

$Q_p(\mathcal{Q}; N_1, \ldots, N_p)$ it follows that

$$Q_{p,n}(\mathcal{Q}; N_1; \ldots; N_p)$$

$$= n^{-N_p} \cdots n^{-N_1} \sum^{(n)} C(e_{i_p} | e_{j_{p1}}, \ldots, e_{j_{pN_p}}) \Delta_{i_p}$$

$$\cdot C(e_{i_{p-1}} | e_{j_{p-1,1}}, \ldots, e_{j_{p-1,N_{p-1}}})$$

$$\cdot \Delta_{i_{p-1}} \cdots C(e_{i_1} | e_{j_{11}}, \ldots, e_{j_{1N_1}}) \Delta_{i_1} \mathcal{Q}$$

$$= n^{-N_p} \cdots n^{-N_1} \sum^{(n)*} C(e_{i_p} | e_{j_{p1}}, \ldots, e_{j_{pN_p}}) \Delta_{i_p}$$

$$\cdot C(e_{i_{p-1}} | e_{j_{p-1,1}}, \ldots, e_{j_{p-1,N_{p-1}}}) \Delta_{i_{p-1}} \cdots C(e_{j_{11}}, \ldots, e_{j_{1N_1}}) \Delta_{i_1} \mathcal{Q}$$

$$+ n^{-N_p} \cdots n^{-N_1} \sum^{(n)**} C(e_{i_p} | e_{j_{p1}}, \ldots, e_{j_{pN_p}}) \Delta_{i_p}$$

$$\cdot\, C(e_{i_{p-1}} | e_{j_{p-1,1}}, \dots, e_{j_{p-1,N_{p-1}}}) \Delta_{i_{p-1}} \cdots C(e_{i_1} | e_{j_{11}}, \dots, e_{j_{1N_1}})$$

$$\cdot\, \Delta_{i_1} \mathcal{Q}$$

where $\sum^{(n)}$ means the sum over all indices i and j, each index ranging from o to n; $\sum^{(n)*}$ means $\sum^{(n)}$ restricted to those indices for which $j_{k\alpha} \neq j_{\ell\beta}$ for $(k,\alpha) \neq (\ell,\beta)$; and $\sum^{(n)**}$ means $\sum^{(n)}$ restricted to those indices for which there exists some $(k,\alpha) \neq (\ell,\beta)$ with $j_{k\alpha} = j_{\ell\beta}$.

The first of the last two sums is equivalent mod J^c to

$$\frac{n(n-1)\dots(n+1-N_1-N_2-\dots-N_p)}{n^{N_1+\dots+N_p}} Q_p(\mathcal{Q};N_1,\dots,N_p)$$

and the second is bounded by

$$n^{-1}M(M+N_1)\dots(M+N_1+\dots+N_{p-1})(M+N_1+\dots+N_p)^2 C_{N_p}\dots C_{N_1} \|\mathcal{Q}\|$$

$$\leq n^{-1}M \|\mathcal{Q}\| (M+N_1+\dots+N_p)^{p+1} C_{N_p}\dots C_{N_1}$$

thus proving the lemma.

LEMMA 5.5

$$\sum_{N_1,\dots,N_p} (M+N_1+\dots+N_p)^{p+1} C_{N_p}\dots C_{N_1} \leq (p+1)!(4L)^{p+1} e^{M/L} .$$

PROOF

$$\sum_{N_1,\dots,N_p} (M+N_1+\dots+N_p)^{p+1} C_{N_p}\dots C_{N_1}$$

$$\leq \sum_{k=0}^{p} \binom{p+1}{k} \left\{ \sum_{N_1,\dots,N_{p-1}} C_{N_{p-1}} \cdots C_{N_1} (M+N_1 \cdots N_{p-1})^k \right\}$$

$$\cdot \left\{ \sum_{N_p=1}^{\infty} N_p^{p+1-k} C_{N_p} \right\}$$

$$\leq \sum_{k=0}^{p+1} \frac{(p+1)!}{k!} L^{p+1-k} \sum_{N_1,\dots,N_{p-1}} C_{N_{p-1}} \cdots C_{N_1} (M+N_1+\dots+N_{p-1})^k$$

$$\leq (p+1)!\, L^{p+1} \sum_{k_1=0}^{p+1} \sum_{k_2=0}^{k_1} \cdots \sum_{k_p=0}^{k_{p-1}} \frac{L^{-k_p}}{k_p!} M^{k_p}$$

$$\leq (p+1)!\, L^{p+1} e^{M/L} B_{p+1}^{p-1}$$

where $B_p^q =$ number of ways of picking $p \geq k_1 \geq \dots \geq k_q$.

Noticing that $B_p^q = B_{p-1}^q + B_p^{q-1}$ and hence $B_p^q \leq 2^{p+q}$, we have

$$\leq (p+1)!\, (4L)^{p+1} e^{M/L}$$

which is the desired result.

LEMMA 5.6 If $\varphi \in C_0^1$ and has M variables whose indices form a set J, then

$$\lim_{n\to\infty} G_n^p \varphi = D^p \varphi \mod J^c.$$

PROOF As can be seen from the previous lemmas, we can find, for any $\varepsilon > 0$, an integer $N(\varepsilon)$ such that

$$\sum_{N_1,\dots,N_p \geq N(\varepsilon)} \| \mathcal{Q}_p(\mathcal{Q}; N_1,\dots,N_p) \| < \varepsilon/3.$$

Since

$$G_n^p \mathcal{Q} = \sum_{N_1=1}^{n-1} \dots \sum_{N_p=1}^{n-1} Q_{p,n}(\mathcal{Q}; N_1,\dots,N_p)$$

and

$$D^p \mathcal{Q} = \sum_{N_1,\dots,N_p} Q_p(\mathcal{Q}; N_1,\dots,N_p)$$

it follows that

$$G_n^p \mathcal{Q} \equiv D^p \mathcal{Q} + \sum_{N_1,\dots,N_p} E_{p,n}(\mathcal{Q}; N_1,\dots,N_p) = D^p \mathcal{Q} + E_{p,n}(\mathcal{Q}) \bmod J^c.$$

But

$$\| E_{p,m}(\mathcal{Q}) \| \leq \sum_{N_1,\dots,N_p} \| E_{p,n}(\mathcal{Q}; N_1,\dots,N_p) \|$$

$$\leq \sum_{N_1,\dots,N_p} \left| \frac{n(n-1)\dots(n+1-N_1-\dots-N_p)}{n^{N_1+\dots+N_p}} - 1 \right| \| Q_p(\mathcal{Q}; N_1,\dots,N_p) \|$$

$$+ \sum_{N_1,\dots,N_p} n^{-1} M \| \mathcal{Q} \| \, C_{N_p} \cdots C_{N_1} (M + N_1 + \dots + N_p)^{p+1}$$

$$\leq \sum_{N_1,\dots,N_p \leq N(\varepsilon)} \left| \frac{n(n-1)\dots(n+1-N_1-\dots-N_p)}{n^{N_1+\dots+N_p}} - 1 \right|$$

$$\cdot \| Q_p(\Phi; N_1,\dots,N_p) \|$$

$$+ \ \varepsilon/3 + n^{-1} M \|\Phi\| e^{M/L} (p+1)!(4L)^{p+1} .$$

Since there obviously exists an integer $N_*(\varepsilon)$ for which $n > N_*(\varepsilon)$ implies that the first and third terms are each leas than $\varepsilon/3$, we have completed the proof of the Lemma.

THEOREM 5.7 If $\Phi \in C_o^1$ has M variables whose indices form a set J, then

$$\lim_{n\to\infty} e^{tG_n}\Phi \equiv e^{tD}\Phi \text{ mod } J^c \text{ for } t < \tfrac{1}{4}L .$$

PROOF

$$e^{tG_n}\Phi = \sum_{p=o}^{\infty} \frac{t^p}{p!} G_n^p \Phi = \sum_{p=o}^{\infty} \frac{t^p}{p!} D^p\Phi + \sum_{p=o}^{\infty} \frac{t^p}{p!} E_{p,n}(\Phi) .$$

But

$$\sum_{p=o}^{\infty} \frac{t^p}{p!} \| E_{p,n}(\Phi) \|$$

$$\leq \sum_{p=o}^{\infty} \frac{t^p}{p!} \| D^p\Phi\| + n^{-1} M \|\Phi\| e^{M/L} \sum_{p=o}^{\infty} \frac{t^p}{p!} (p+1)!(4L)^{p+1}$$

(see proof of previous lemma), and since both sums converge, it follows that for any $\varepsilon > 0$, there exists an integer $p(\varepsilon)$ such that

$$\sum_{p=p(\varepsilon)}^{\infty} \frac{t^p}{p!} \| E_{p,n}(\varphi) \| < \varepsilon/2.$$

Thus since $\| E_{p,n}(\varphi)\| \to 0$ as $n \to \infty$ for each p, we can find an integer $N(\varepsilon)$ such that

$$\sum_{p=0}^{p(\varepsilon)} \frac{t^p}{p!} \|E_{p,n}(\varphi)\| < \varepsilon/2, \ n > N(\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete.

Thus we have shown, for $\varphi \in C_0^1$, that

$$e^{tG_n}\varphi \to e^{tD}\varphi \mod I_1^c \text{ for } t < \tfrac{1}{4}L.$$

Since D is a derivation,

$$D[\varphi \otimes \psi] = \varphi \otimes D\psi + \psi \otimes D\varphi = \varphi \otimes D_1\psi + \psi \otimes D_1\varphi$$

when $\varphi, \psi \in C_0^1$ have disjoint sets of variables.

Similarly, applying D to $D[\varphi \otimes \psi]$ we get

$$D^2[\varphi \otimes \psi] = D_2D_1[\varphi \otimes \psi] = D_2[\varphi \otimes D_1\psi] + D_2[\psi \otimes D_1\varphi]$$

$$= (D_2\varphi) \otimes (D_1\psi) + \varphi \otimes D_2D_1\psi + (D_2\psi) \otimes D_1\varphi + \psi \otimes D_2D_1\varphi$$

since $D_2\varphi$ and $D_1\psi$ have disjoint variables, as do φ and $D_2D_1\psi$ etc.

Thus we get the following lemma.

LEMMA 5.8 If $\varphi, \psi \in C_o^1$ have disjoint variables, then

$$D^p[\varphi \otimes \psi] = \sum_{k=o}^{p} \binom{p}{k} (D_p D_{p-1} \ldots D_{k+1} \varphi) \otimes (D_k D_{k-1} \ldots D_1 \psi) \mod I_1^c$$

where the set of indices of variables of $D_p \ldots D_{k+1} \varphi$ and $D_k \ldots D_1 \psi$ are disjoint as is indicated by the notiation.

Noticing that $D_p D_{p-1} \ldots D_{q+1} \varphi \equiv D_{p-q-1} \ldots D_1 \varphi \equiv D^{p-q} \varphi \mod I_1^c$ for $\varphi \in C_o^1$, we get the following "propagation of chaos" theorem (see M.Kac [2] for the terminology and another instance of this phenomenon).

THEOREM 5.9 If $\varphi, \psi \in C_o^1$ and the set of indices of variables of φ are disjoint from ψ, then

$$\int_{I_1^c} f^\infty e^{tD} [\varphi \otimes \psi] = \left[\int_{I_1^c} f^\infty e^{tD} \varphi\right] \otimes \left[\int_{I_1^c} f^\infty e^{tD} \psi\right] \text{ for}$$

$t < \frac{1}{4}L$.

PROOF

$$\int_{I_1^c} f^\infty e^{tD} [\varphi \otimes \psi]$$

$$= \int_{I_1^c} f^\infty \sum_{p=o}^{\infty} \frac{t^p}{p!} D^p [\varphi \otimes \psi]$$

$$= \sum_{p=o}^{\infty} \frac{t^p}{p!} \sum_{q=o}^{p} \binom{p}{q} \int_{I_1^c} f^\infty \left\{ (D_p D_{p-1} \ldots D_{q+1} \varphi) \otimes (D_q D_{q-1} \ldots D_1 \psi) \right\}$$

$$= \sum_{p=o}^{\infty} \sum_{q=p}^{p} \frac{t^p}{p!} \frac{t^q}{q!} \left(\int_{I_1^c} f^\infty D^{p-q} \varphi \right) \otimes \left(\int_{I_1^c} f^\infty D^q \psi \right)$$

$$= \left(\sum_{p=0}^{\infty} \frac{t^p}{p!} \int_{I_1^c} f^\infty D^p \varphi\right) \otimes \left(\sum_{q=0}^{\infty} \frac{t^q}{q!} \int_{I_1^c} f^\infty D^q \psi\right)$$

$$= \left(\int_{I_1^c} f^\infty e^{tD} \varphi\right) \otimes \left(\int_{I_1^c} f^\infty e^{tD} \psi\right).$$

If the initial distribution of the n-molecule gas is symmetric in the n molecules, then it is also symmetric at any later time. Especially, if the molecules are initially independent and identically distributed, then the joint distribution of the first M particles is given by

$$(1) \quad P\left[X^n(t_1) \in E_1, \ldots, X^n(t_m) \in E_m\right]$$

$$= \int f(d\xi_1) \ldots f(d\xi_n) e^{t_1 G_n} \chi_{E_1} e^{(t_2 - t_1) G_n} \chi_{E_2} \ldots e^{(t_m - t_{m-1}) G_n} \chi_{E_m}$$

where $E_k = (e_1^k, \ldots, e_M^k, E, E, \ldots, E)$. As $n \to \infty$, 1) converges to

$$\int f^\infty e^{t_1 D} \chi_{E_1} e^{(t_2 - t_1) D} \chi_{E_2} \ldots e^{(t_m - t_{m-1}) D} \chi_{E_m}.$$

This limiting distribution can be used to define a combined motion of M molecules for which:

(1) the paths of any fixed number of molecules are independent.

(2) the distribution of a tagged molecule is that of the *-process which corresponds

to $\mp \gamma_{\pm}$.

The essential elements of the proofs of (1) and (2) are contained in the following lemma. We first need a few definitions and remarks.

Let $o < t_1 < t_2$ be points in $T = [o, \infty)$ and let $a_o, a_1, a_2, b_o, b_1, b_2 \in E$. Let

$$\chi_a^i(\ldots, e_i, \ldots) = \begin{cases} 1 \text{ if } e_i = a \\ 0 \text{ otherwise} \end{cases}$$

and

$$\hat{P}_{f|a}(t;b) = \int_{\{i\}^c} f^{\infty}(e^{tD} \chi_b^i)(\ldots, \xi_i = a, \ldots)$$

where, by definition of D, the right hand expression is independent of i. Finally, note that any finite number of variables on which $\mathcal{Q}$ does not depend has no effect on the limit as $n \to \infty$ of $e^{tG_n} \mathcal{Q}$.

<u>LEMMA 5.10</u> Suppose that the n coordinates of $X^n(o)$ are independent and identically distributed with distribution f. Then

$$\lim_{n \to \infty} P\left[x_1^n(t_1) = a_1, x_1^n(t_2) = a_2, x_2^n(t_1) = b_1, x_2^n(t_2) = b_2 \,/\, x_1^n(o) = a_o, x_2^n(o) = b_o\right]$$

$$= \hat{P}_{f|a_o}(t_1;a_1)\, \hat{P}_{f_{t_1}|a_1}(t_2;a_2)\, \hat{P}_{f|b_o}(t_1;b_1)\, \hat{P}_{f_{t_1}|b_1}(t_2;b_2)\ .$$

PROOF

2) $P\left[x_1^n(t_1)=a_1, x_1^n(t_2)=a_2, x_2^n(t_1)=b_1, x_2^n(t_2)=b_2 \,\middle|\, x_1^n(o)=a_o, x_2^n(o)=b_o\right]$

$$= \sum_{\zeta_3,\ldots,\zeta_n} \int f(d\zeta_3)\ldots f(d\zeta_n) e^{t_1 G_n}$$

$$\cdot\left[\chi^1_{a_1}\otimes\chi^2_{b_1}\otimes\chi^3_{\zeta_3}\otimes\ldots\otimes\chi^n_{\zeta_n}\right](a_o,b_o,\zeta_3,\ldots,\zeta_n)\, e^{t_2 G_n}$$

$$\cdot\left[\chi^1_{a_2}\otimes\chi^2_{b_2}\right](a_1,b_1,\zeta_3,\ldots,\zeta_n)$$

$$= \sum_{\zeta_3,\ldots\zeta_n} \int_{\{1,2\}^c} f^\infty_c\, e^{t_1 D}\left[\chi^1_{a_1}\otimes\chi^2_{b_1}\otimes\chi^3_{\zeta_3}\otimes\ldots\otimes\chi^n_{\zeta_n}\right](a_o,b_o,\ldots)\, e^{t_2 G_n}$$

$$\cdot\left[\chi^1_{a_2}\otimes\chi^2_{b_2}\right](a_1,b_1,\zeta_3,\ldots,\zeta_n)+o(1)$$

$$= \int \prod_{k=1}^{n} \left\{ \int_{\{1,2\}^c} f^\infty\, e^{t_1 D}\, \chi^k_{\alpha_k}(a_o,b_o,\ldots)\right\}$$

$$\cdot\, e^{t_2 G_n}\left[\chi^1_{a_2}\otimes\chi^2_{b_2}\right](a_1,b_1,\zeta_3,\ldots,\zeta_n)$$

where $\alpha_1 = a_1,\ \alpha_2 = b_1,\ \alpha_3 = \zeta_3,\ldots,\ \alpha_n = \zeta_n$.

But

$$\int_{\{1,2\}^{c}} f^{\infty} e^{t_1 D} \chi^{k}_{\zeta_k}(a_o, b_o, \ldots) = f_{t_1}(\zeta_k)$$

since it is independent of a_o, b_o and k, and

$$\int_{\{1,2\}^{c}} f^{\infty} e^{t_1 D} \chi^{1}_{a_1}(a_o, b_o, \ldots) = P_{f|a_o}(t_1; a_1)$$

and

$$\int_{\{1,2\}^{c}} f^{\infty} e^{t_1 D} \chi^{2}_{b_1}(a_o, b_o, \ldots) = P_{f|b_o}(t_1; a_1)$$

(i.e. in evaluating $\lim_{n\to\infty} e^{tG_n}$, the a_o has no effect since $\chi^{2}_{b_1}$ does not depend on that variable). Thus 2) reduces to

$$P_{f|a_o}(t_1; a_1) P_{f|b_o}(t_1; b_1)$$

$$\cdot \int f_{t_1}(d\zeta_3) \ldots f_{t_1}(d\zeta_n) e^{t_2 G_n} [\chi^{1}_{a_2} \otimes \chi^{2}_{b_2}](a_1, b_1, \zeta_3, \ldots, \zeta_n)$$

$$\cdot + o(1)$$

and, using the same arguments on the above integral, we complete the proof.

Lemma 5.10 shows that $x^{n}_{1}(t_1)$ and $x^{n}_{1}(t_2)$ become independent of $x_2^n(t_2)$ as $n \to \infty$. The same arguments can be used to show that any finite number of particles have independent paths. Similarly, we can show that

$$\lim_{n\to\infty} P\left[x_1^n(t_1)=a_1,\cdots,x_1^n(t_m)=a_m \,\middle|\, x_1^n(o)=a_o\right]$$

$$= \hat{P}_{f|a_o}(t_1;a_1)\ \hat{P}_{f_{t_1}|a_1}(t_2;a_2)\ldots\hat{P}_{f_{t_{m-1}}|a_{m-1}}(t_m;a_m)$$

and thus that the distribution of a tagged particle converges to a*-process. Finally, $\hat{\gamma}_\pm = \gamma_\pm$ since, letting $u=f(+1)$,

$$\hat{\partial}_\pm(u)$$

$$= \frac{\partial}{\partial t}\hat{P}_{f|+1}(t;1)\Big|_{t=o}$$

$$= \frac{\partial}{\partial t}\int_{\{1\}^c} f^\infty e^{tD}\chi_{+1}(\pm 1,\ldots)\Big|_{t=o}$$

$$= \int_{\{1\}^c} f^\infty D\chi_{+1}(\pm 1,\ldots)$$

$$= \int_{\{1\}^c} f^\infty \sum_{N=1}^{\infty} C(\pm 1|e_{2,1},\ldots,e_{2,N})\left[\chi_{+1}(\mp 1)-\chi_{+1}(\pm 1)\right)$$

$$= \mp\sum_{N=1}^{\infty}\sum_{k=o}^{N} C_N^{\pm 1}(k)\ \binom{N}{k}u^k(1-u)^{N-k}$$

$$= \mp \sum_{N=1}^{\infty} B_N^{\pm}(u)$$

$$= \mp \gamma_{\pm}(u)$$

We end this section with the following theorem.

THEOREM 5.11 If $\mp \gamma_{\pm} \in H$, then the sample paths of the *-process x_t can be chosen so as to be right continuous.

PROOF It is sufficient to show that the temporally homogeneous Markov process $X(t)$ defined in section 2 can be chosen in such a manner that it has right continuous sample paths. Thus, letting $\mathcal{V}_\varepsilon(u)$ be the ε-neighborhood around the point u, we need only show that $P_t^*(e,u;e,\mathcal{V}_\varepsilon(u)) \to 1$ uniformly in e,u as $t \to o$ for any $\varepsilon > o$ (see Loeve, M: Probability Theory, P. 637).

But

$$P_t^*(e,u;e,\mathcal{V}_\varepsilon(u)) = \begin{cases} P_{f|e}(t;e) & \text{if } |u-p(t)| < \varepsilon, u=p(o)=f(+1). \\ o & \text{otherwise} \end{cases}$$

Thus a simple calculation showing that $f_t(+1) \to f(+1)$ and $P_{f|e}(t;e) \to 1$ uniformly in $u=f(+1)$ as $t \to o$ completes the proof.

6. HOLDING TIMES OF A *-PROCESS

We shall now calculate the holding times

$H_o^t(u) = P_{f|+1}(x(\zeta) = +1,\ o \leq \zeta \leq t)$, $f(+1) = u$

for a *-process x_t for which the corresponding $\mp\ \gamma_{\pm}$ are contained in H.

$$\text{Let}\quad b(t,s,u) = -\log P_{f_s|+1}(t;+1)$$

and note that

$$\lim_{h\downarrow o} \frac{b(h,s,u)}{h} = -\ \gamma_+\left[f_s(+1)\right]$$

uniformly in s,u. (see last section for the existence of this limit.)

THEOREM 6.1

$$H_o^t(u) = \exp\left[\int_o^t \gamma_+\left[f_s(+1)\right] ds\right] \quad , u = f(+1).$$

PROOF Let $t = nh + \theta$ with $o \leq \theta < h$. Then, because of the right continuity of the paths,

$H_o^t(u)$

$$= \lim_{h\downarrow o} P_{f|+1}\left[x(o) = +1,\ x(h) = +1, \ldots, x(nh) = +1\right]$$

$$= \lim_{h\downarrow o} P_{f|+1}(h;+1) P_{f_h|+1}(h;+1) P_{f_{2h}|+1}(h;+1) \ldots P_{f_{nh}|+1}(h;+1)$$

$$= \lim_{h \downarrow 0} \exp\left[- \sum_{k=0}^{n} b(h, kh, u) \right]$$

$$= \lim_{h \downarrow 0} \exp\left[- \sum_{k=0}^{n} \frac{b(h, kh, u)}{h} \cdot h \right]$$

$$= \lim_{h \downarrow 0} \exp\left[\sum_{k=0}^{n} \left[\gamma_{+}\left[f_{kh}(+1) \right] h + h o(1) \right] \right]$$

$$= \exp\left[\int_{0}^{t} \gamma_{+}\left[f_{s}(+1) \right] d_{s} \right].$$

7. LIMITING BEHAVIOR AS $t \to \infty$

If $\mp \gamma_{\pm} \in H$, then γ has derivatives of all orders and $\frac{\partial}{\partial t} p(t) = \gamma[p(t)]$. This implies that $f_t(+1)$ is either monotone increasing or decreasing in t for a given f and thus $f_\infty(+1) = \lim_{t\to\infty} f_t(+1)$ exists where f_∞ has the property that $(f_\infty)_t$ is constant as t varies. Thus

$$P_{f|e}\left[x(t_1) \in A_1, \ldots, x(t_1+t_2+\ldots+t_n) \in A_n\right]$$

$$= \int_{A_1} P_{f|e}(t_1;d\xi_1) \int_{A_2} P_{f_{t_1}|\xi_1}(t_2;d\xi_2) \ldots \int_{A_n} P_{f_{t_1+\ldots+t_{n-1}}|\xi_{n-1}}(t_n;d\xi_n)$$

$$\to \int_{A_1} f_\infty(d\xi_1) \int_{A_2} P_{f_\infty|\xi_1}(t_2;d\xi_2) \ldots \int_{A_n} P_{f_\infty|\xi_{n-1}}(t_n;d\xi_n)$$

as $t_1 \to \infty$ and the limiting behavior of such a process is temporally homogeneous and stationary. Finally, if we start with $f(+1)=u$, then the holding times, which can be quite non-Markovian for small t, approach exponential holding times as $t \to \infty$. That is,

$$H_s^t(u)$$

$$= \exp\left[\int_s^{s+t} \gamma_+(f_\theta(+1))\, d\theta\right]$$

$$= \exp\left[\int_{s}^{s+t} \gamma_{+}(f_{\infty}(+1)d\theta + to(1)\right]$$

$$\longrightarrow \exp\left[t\,\gamma_{+}\left[f_{\infty}(+1)\right]\right.$$

as $s \to \infty$.

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BIOGRAPHY

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