The singular values of the GUE (less is more)

THE SINGULAR VALUES OF THE GUE (LESS IS MORE)

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ABSTRACT. Some properties that nominally involve the eigenvalues of Gaussian Unitary Ensemble (GUE) can instead be phrased in terms of singular values. By discarding the signs of the eigenvalues, we gain access to a surprising decomposition: the singular values of the GUE are distributed as the union of the singular values of two independent ensembles of Laguerre type. This independence is remarkable given the well known phenomenon of eigenvalue repulsion.

The structure of this decomposition reveals that several existing observations about large n limits of the GUE are in fact manifestations of phenomena that are already present for finite random matrices. We relate the semicircle law to the quarter-circle law by connecting Hermite polynomials to generalized Laguerre polynomials with parameter $\pm 1/2$. Similarly, we write the absolute value of the determinant of the $n \times n$ GUE as a product n independent random variables to gain new insight into its asymptotic log-normality. The decomposition also provides a description of the distribution of the smallest singular value of the GUE, which in turn permits the study of the leading order behavior of the condition number of GUE matrices.

The study is motivated by questions involving the enumeration of orientable maps, and is related to questions involving powers of complex Ginibre matrices. The inescapable conclusion of this work is that the singular values of the GUE play an unpredictably important role that had gone unnoticed for decades even though, in hindsight, so many clues had been around.

1. Introduction

This paper highlights some surprising interrelationships between problems that involve singular values of GUE random matrices. By discarding the signs of eigenvalues, we gain access to additional structure, since despite the pairwise repulsion of its eigenvalues, the singular values of the GUE can be decomposed as the union of two independent sets. The decomposition is equivalent to a result of Jackson and Visentin [23] from enumerative combinatorics, and was previously reported by Forrester in [13, Sec. 2.2]. Our contribution is to consider the decomposition as a complete result about singular values instead of a specialized result about eigenvalues, and to note that this single decomposition underlies several diverse phenomena. From this perspective, we can translate results about asymptotically large matrices to the finite setting, and we can capitalize on the independence to describe the determinant and extreme singular values of the GUE.

Several results, that we find individually surprising, are in fact hidden facets of the same phenomenon. Our aim is to expose these surprises and the interconnections between them.

(1) It is possible to partition the singular values of the GUE into two statistically independent sets (stated in [13] in terms of eigenvalues). This stands in striking contrast, almost in contradiction with, the familiar fact that eigenvalues repel.

Key words and phrases. random matrices, GUE, anti-GUE, LUE, singular values, condition number, semicircle law, quarter-circle law.

(2) The logarithm of the absolute value of the determinant of the GUE can be written as a sum of independent random variables (speculated as impossible by Tao and Vu in [35]).
(3) Matrices of nominal half-integer size play a key role.
(4) The decomposition is equivalent to a result from enumerative combinatorics that relates the cardinalities of two classes of orientable maps on surface of positive genus ([23]).
(5) A bi-diagonal model for singular values gives all the moments of the GUE determinant.
(6) The bulk-scaling limit of the GUE behaves as a superposition of two hard edges. The first author has long since argued for the relatively obvious importance of the singular value view for Laguerre (Wishart) ensembles, and the less well known, but easy to recognize, generalized singular value view for Jacobi (MANOVA) ensembles (see the first author's course notes for course 18.337 at MIT). The importance of a singular value view for the GUE, however, is far more astonishing.

Our approach is analogous to replacing a semicircle with a pair of quarter-circles. These curves occur as famous limiting distributions. In particular, Wigner's semicircle law is the limiting distribution of the eigenvalues of the (GUE). The Marchenko-Pastur distribution similarly describes the limiting distribution for the singular values of large rectangular random matrices. In particular, Laguerre ensemble singular values satisfy the quarter-circle law. There is an obvious geometric relationship between these distributions; a semicircle is the union of two quarter-circles (Figure 1–top). The semicircle and quarter-circles also have a less obvious relationship: the semicircle is symmetric about the y-axis, and its restriction to the first quadrant is the average of two quarter-circles. This second relationship generalizes to matrices of finite size (Figure 1–right), with the quarter-circles replaced by the distributions of singular values of rectangular matrices of nominal half-integer size (Figure 1–bottom). Variations of this second relationship form the basis for this paper.

Much of random matrix theory involves the behavior of eigenvalues of asymptotically large matrices. It is not always clear how such phenomena correspond to finite matrices. In this paper, we connect the infinite to the finite by phrasing phenomena in terms of singular values. For Hermitian matrices, this amounts to considering the magnitudes of eigenvalues and discarding their signs. One might assume that discarding signs limits the scope of possible conclusions, but in practice several problems that are nominally about eigenvalues are better analyzed in terms of singular values. One could even argue that existing results about the extreme eigenvalues of Laguerre and Jacobi ensembles are elegant precisely because they are essentially about singular values.

The change of setting becomes advantageous when we observe that the singular values of the $n \times n$ GUE exhibit an unexpected decomposition: Theorem 1 shows that they are distributed identically to the union of the distinct non-zero singular values of two independent anti-GUE ensembles (an anti-GUE matrix consists of purely imaginary Gaussian entries that are independently distributed subject to skew-symmetry) one of order n, the other of order $n+1$. An equivalent result was previously observed by Forrester in [13, Sec. 2.2] where it was stated explicitly for the case that n is even. Since the eigenvalues of the GUE are readily seen to be pairwise dependent, the existence of such a decomposition is itself somewhat surprising.

The decomposition allows us to analyze several statistics of the GUE, including the physically significant gap probability, in terms of the anti-GUE. Ironically, most of the relevant facts about the anti-GUE can be found in Mehta's physically motivated text, [28, Ch. 13], where his description is asserts that such matrices have "no immediate physical interest". After a change of variables, the positive eigenvalues of the anti-GUE are seen to have distributions of Laguerre-type (Section 2), corresponding to complex matrices with a half-integral dimension and Laguerre parameter $\pm\frac{1}{2}$ (this

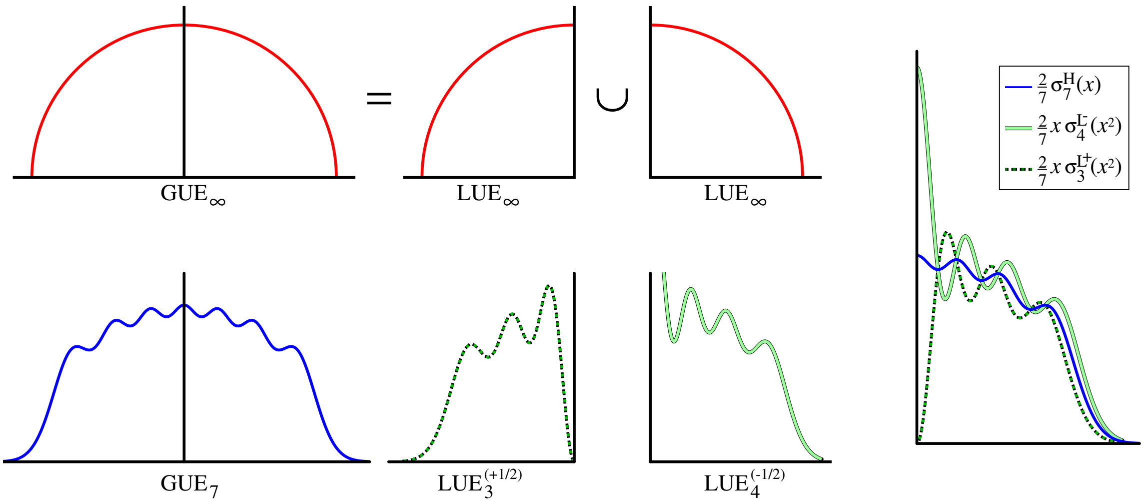

Figure 1. A semicircle describes the limiting density of the eigenvalues of the GUE. It decomposes as two quarter-circles (top), related to the limiting densities of the singular values of rectangular matrices. For finite matrices, the density of a random eigenvalue of the GUE (bottom left) is still described by a weighted average (right) of the densities of of the positive square roots of the eigenvalues from two LUEs (bottom center).

is the $\beta = 2$ case of a more general analysis presented by Dumitriu and Forrester in [8]). It is thus possible to draw conclusions about the GUE from an understanding of corresponding facts about Laguerre ensembles. Physically significant existing results about level densities, the absolute value of the determinant, the distributions of the largest singular value, and the bulk-scaling limit can all be analyzed using this framework.

As an unexpected consequence, we obtain the square of the determinant of the $n \times n$ GUE as a product of independent χ^2 random variables (Theorem 2). This is a direct analogue to the result of Goodman for Wishart matrices [18], and precisely the form that Tao and Vu speculated did not exist when discussing the log-normality of the absolute value of the determinant of the GUE in [35].

In addition to providing a common framework for understanding existing results about the GUE, the decomposition permits a study of the distribution of the smallest singular value of a matrix from the ensemble. This quantity may initially appear somewhat unnatural, but for some applications it is an appropriate analog for the smallest eigenvalue of Laguerre and Jacobi ensembles, in some ways behaving as though governed by the existence of a virtual hard-edge. The distribution of the smallest singular value is also closely related to the distribution of conditions numbers, and has implications for the analysis of numerical stability of operations involving random matrices.

The decomposition was first identified by the authors as part of an attempt to find a combinatorial derivation for a functional identity, given by Jackson and Visentin in [20], between generating series for two classes of orientable maps. Physical implications of their identity involve matrix models of 2-dimensional gravity, and are discussed in [19]. They later generalized the identity, in [23], to a stronger form that is essentially equivalent to the existence of our decomposition. Their generating series are effectively cumulant generating series for suitably scaled ensembles of matrix eigenvalues, but Jackson and Visentin appear to have been unaware of the random matrix interpretation of one of the series, possibly because its direct interpretation involves a half-integer evaluation of a parameter that nominally represents one of the dimensions of a rectangular matrix of complex Gaussians. While their work required subtle manipulation of characters of the symmetric group, we believe that the present proof is elementary and enlightening from the perspective of random matrix theory, although a combinatorial interpretation still remains elusive.

It should be noted that while the decomposition discussed here has many superficial parallels with the ideas of superposition and decimation superposition explored by Forrester and Rains ([14, 15]), the concepts are distinct, although it is not difficult to imagine a more general setting in which both their result and ours exist as special cases.

Outline. The remainder of the paper has the following structure:

- Section 2 describes the matrix ensembles we need to formulate the decomposition.
- Section 3 uses the level density of the GUE as a warm-up exercise.
- Section 4 demonstrates the decomposition. We also describes its equivalence to an identity of Jackson and Visentin, and discuses how the decomposition can be observed experimentally.
- Section 5 applies the decomposition to provide a unified explanation to existing results.
- Section 6 relates the decomposition to properties of the complex Ginibre ensemble, and draws parallels to an earlier investigation by Rains of powers of compact Lie groups [31, 32].
- Finally, in Section 7 we discuss some related questions for future work.

2. The Ensembles

Gaussian Unitary Ensembles. The Gaussian Unitary Ensemble of order n, (GUE_n), consists of $n \times n$ Hermitian matrices invariant after conjugation by any unitary matrix, and with entries that are normal, and independently distributed, subject to Hermitian symmetry. The ensemble is completely defined by specifying the variance of the diagonal entries, and we choose a normalization with diagonal entries standard normal. As a consequence, the real and imaginary parts of the off-diagonal entries are independently normal with mean 0 and variance $\frac{1}{2}$. The ensemble can be sampled as $A = \frac{1}{2}(G + G^{\mathrm{H}})$, where the real and imaginary parts of the entries of the $n \times n$ matrix G are independently standard normal, and G^{H} denotes the Hermitian conjugate of G.

Remark. It is also common to work with a normalization where real and imaginary parts of the off-diagonal entries are standard normal, as in [28, 29], or where the variance depends on n (when the primary concern is taking large-n limits). Our choice is motivated by combinatorial considerations from the map enumeration setting studied by Jackson and Visentin ([23]), and provides the property that for every partition $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$, the moment $m_\theta(n) = \mathrm{E}_{\mathrm{GUE}_n}\left[\prod_{i=1}^k \mathrm{tr}(M^{\theta_i})\right]$ is a polynomial in n with non-negative integer coefficients depending only on θ. A convenient consequence of this normalization is that $\mathrm{E}_{\mathrm{GUE}_n}\left[\det(M^{2k})\right]$ is a product of odd integers for every n and k (see Theorem 2).

An element of the GUE has real eigenvalues, so the distribution on the matrices induces a distribution on n-tuples of eigenvalues. The joint density function for this distribution on $\mathbb{R}^n$, is

$$p_n^{\mathrm{H}}(x_1, x_2, \ldots, x_n) = c_n^{\mathrm{H}} \prod_{1 \le i < j \le n} |x_i - x_j|^2 \ \exp\Big(-\frac{1}{2}\sum_{i=1}^n x_i^2\Big) \prod_{i=1}^n \mathrm{d}x_i \tag{1}$$

where c_n^{H} is such that the density defines a probability measure. A thorough discussion of the GUE is given by Mehta in [28], though with a different choice of normalization. It is convenient to consider the density as consisting of two factors: the Vandermonde squared factor, $\prod_{1\le i<j\le n}|x_i-x_j|^2$, occurs because the ensemble is unitarily invariant, while the second factor, $\exp\big(-\frac{1}{2}\sum_{i=1}^n x_i^2\big)$, is associated to the Hermite weight in the study of orthogonal polynomials (explaining the use of 'H' in our notation), and occurs because the density of a matrix M is proportional to $\exp\big(-\frac{1}{2}\operatorname{tr}(M^2)\big)$.

Remark. It is also common to consider the GUE in terms of a density on sets of eigenvalues, and thus use a density that is supported only on $x_1 \le x_2 \le \cdots \le x_n$. For the present purposes, we prefer to have a density function that is invariant under permutation of its arguments, and so we consider a density on n-tuples constructed by randomly permuting the eigenvalues. The two approaches are not substantially different, but would manifest as a factor of $n!$ if c_n^{H} were to be stated explicitly. In Section 4 we will consider an alternate density on n-tuples that induces the same density on sets.

Laguerre Unitary Ensembles. The Laguerre Unitary Ensembles (LUE) are a two-parameter family of distributions on positive definite Hermitian matrices. The parameter n corresponds to the order of the matrix, while the parameter a determines the shape of the distribution. In contrast to the GUE which traditionally found applications arising in physics, the LUE are more closely associated with statistics where the relevant matrices are often referred to as Wishart matrices. Many statistical applications of the LOE, an analogous ensemble based on real instead of complex matrix entries, can be found in [30], and much of the commentary there applies to the LUE with minor modifications. There are two related models of the LUE: one model applies when $n+a$ is a positive integer, while a second model applies when $a+1$ is a positive real number.

When $n+a$ is a positive integer, the ensemble can be sampled as $W = AA^{\mathrm{H}}$, where A is an $n\times(n+a)$ matrix of independent complex Gaussian entries. The spectrum of A is completely determined by A, and it is often more natural to work with the singular values of A than with the eigenvalues of W. By convention we will consider centered matrix entries, with equally distributed real and imaginary parts chosen such that $\mathrm{E}\left[A_{ij}\overline{A_{ij}}\right]=2$, so that the real and imaginary parts of each entry are independent standard normal. With this normalization, the joint density for the eigenvalues of the LUE on $[0,\infty)^n$, is

$$p_{n,a}^{\mathrm{L}}(x_1,x_2,\ldots,x_n) = c_{n,a}^{\mathrm{L}}\prod_{i=1}^n x_i^a \prod_{1\le i<j\le n}|x_i-x_j|^2 \exp\Big(-\frac{1}{2}\sum_{i=1}^n x_i\Big)\prod_{i=1}^n \mathrm{d}x_i. \tag{2}$$

In fact (2) continues to define a probability density for non-integral $a>-1$, and the densities are realized when A is bi-diagonal with its non-zero entries independently χ-distributed according to

$$A \sim \begin{pmatrix} \chi_{2(n+a)} & & & & \\ \chi_{2(n-1)} & \chi_{2(n+a-1)} & & & \\ & \chi_{2(n-2)} & \chi_{2(n+a-2)} & & \\ & & \ddots & \ddots & \\ & & & \chi_2 & \chi_{2(a+1)} \end{pmatrix}. \tag{3}$$

The correctness of this model for integer values of a is verified by considering the effect of applying Householder reflections to a matrix A of complex Gaussians, and can be seen to extend to non-integral a via the fact that the moments of (2) must depend polynomially on a. A complete derivation of the bi-diagonal model for Laguerre ensembles is given in a more general setting in [7]. In the present paper, we will be primarily interested in Laguerre ensembles for which $a=\pm\frac{1}{2}$ and their relationship to anti-GUE matrices of even and odd order, although ensembles corresponding to arbitrary values of a are closely related to the combinatorics in [23] that motivated the present study.

Remark. As with the GUE, moments of the LUE can be interpreted combinatorially. Taking $m = n + a$, the moments $m_\theta(n, m) = \mathrm{E}_{\mathrm{LUE}_n^{(a)}}\left[\prod_{i=1}^k \mathrm{tr}(M^{\theta_i})\right]$ are each polynomials in m and n with non-negative integer coefficients and are symmetric in m and n. These coefficients are related to the enumeration of hypermaps and associated with the generating series discussed in [23], though a direct interpretation of the combinatorial results there requires the alternate normalization $\mathrm{E}\left[A_{ij}\overline{A_{ij}}\right] = 1$.

Anti-GUE. The anti-GUE consists of anti-symmetric Hermitian matrices with independent (subject to anti-symmetric) normal entries. Such matrices were identified by Mehta as having a particularly elegant theory, with no immediate applications to physics [28, Ch. 13]. Every such matrix is of the form $M = iK$, where K is a real skew-symmetric matrix. Such a matrix is unitarily diagonalizable, so its singular values are the absolute values of its eigenvalues. Since the characteristic polynomial of K has real coefficients, its eigenvalues occur in complex conjugate pairs, and it follows that the eigenvalues of M occur in plus/minus pairs, so each non-zero singular values occurs with even multiplicity. If M is $N \times N$ for $N = 2n + r$ with $r \in \{0, 1\}$, then except on a set of measure zero, M has n distinct non-zero singular values, which we can denote by $\theta_1, \theta_2, \ldots, \theta_n$. When the imaginary parts of the entries of M are distributed as independent standard Gaussians (up to Hermitian symmetry), the joint probability density function for the distinct singular values of M (in this case also the positive eigenvalues), supported on $[0, \infty)^n$, is given by

$$p_N^{\mathrm{aG}}(\theta_1, \theta_2, \ldots, \theta_n) = c_N^{\mathrm{aG}} \prod_{j=1}^n \theta_j^{2r} \prod_{1 \le j < k \le n} \left(\theta_j^2 - \theta_k^2\right)^2 \exp\Big(-\frac{1}{2}\sum_{j=1}^n \theta_j^2\Big) \prod_{i=1}^n \mathrm{d}\theta_i, \tag{4}$$

an expression that combines the two cases described in [28, Section 3.4] or [11, Ex 1.3 q.5] after accounting for the differing choice of normalization. Key to the existence of the decomposition in Theorem 1 is that the final factor, $\exp\big(-\frac{1}{2}\sum_{j=1}^n \theta_j^2\big)$, which is common to both this density and the GUE density, is a symmetric product of even functions.

For completeness, we will outline how this density can be derived from the Laguerre density, (2), by establishing the existence of a bi-diagonal model for the singular values. This follows closely one of the approaches used by Dumitriu and Forrester in [8], where several other derivations are also presented. By applying a sequence of orthogonal Householder transformation to M, it is seen to have the same eigenvalue distribution as the tri-diagonal anti-symmetric matrix

$$\mathrm{i}\begin{pmatrix} 0 & \chi_{N-1} & & & & \\ -\chi_{N-1} & 0 & \chi_{N-2} & & & \\ & -\chi_{N-2} & 0 & \chi_{N-3} & & \\ & & -\chi_{N-3} & 0 & \ddots & \\ & & & \ddots & \ddots & \chi_1 \\ & & & & -\chi_1 & 0 \end{pmatrix},$$

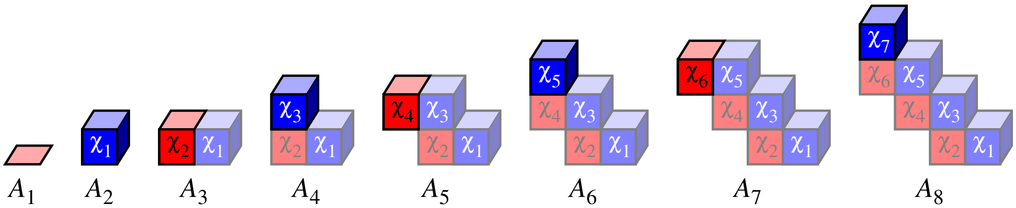

FIGURE 2. The bi-diagonal models for the positive eigenvalues of the anti-GUE form a one-parameter family, where each is obtained from the previous by adding a single non-zero matrix element.

which by simultaneously permuting rows and columns is orthogonally similar to a matrix of the form $\mathrm{i}\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$, where depending on the parity of N,

$$A_{N_{\text{odd}}} \sim \begin{pmatrix} \chi_{N-1} & \chi_{N-2} & & & \\ & \chi_{N-3} & \chi_{N-4} & & \\ & & \ddots & \ddots & \\ & & & \chi_2 & \chi_1 \end{pmatrix} \quad \text{or} \quad A_{N_{\text{even}}} \sim \begin{pmatrix} \chi_{N-1} & \chi_{N-2} & & & \\ & \chi_{N-3} & \chi_{N-4} & & \\ & & \ddots & \ddots & \\ & & & \chi_3 & \chi_2 \\ & & & & \chi_1 \end{pmatrix}.$$

Despite the differing form for even and odd N, for many purposes these bi-diagonal should be considered as comprising a one-parameter family. Their moments, for example, can be seen to linked, and to depend polynomially on N. Figure 2 emphasizes the uniformity by illustrating how each bi-diagonal matrix is obtained from one of lower order by adding a single additional non-zero matrix element. Notice that when N is odd, the matrix A_N is not square.

When N is even, the matrix $A_{N_{\text{even}}}$ is the transpose of the Laguerre form from (3), with $a = -\frac{1}{2}$. For odd values of N, the singular values of $A_{N_{\text{odd}}}$ can also be seen to be Laguerre distributed, in this case with $a = \frac{1}{2}$, by noting that

$$A_{N_{\text{odd}}} \sim \begin{pmatrix} \chi_{N-1} & \chi_{N-2} & & & \\ & \chi_{N-3} & \chi_{N-4} & & \\ & & \ddots & \ddots & \\ & & & \chi_2 & \chi_1 \end{pmatrix} \quad \text{and} \quad B_{N_{\text{odd}}} \sim \begin{pmatrix} \chi_{N} & \chi_{N-3} & & & \\ & \chi_{N-2} & \chi_{N-5} & & \\ & & \ddots & \ddots & \\ & & & \chi_5 & \chi_2 \\ & & & & \chi_3 \end{pmatrix}$$

have identically distributed singular values. Dumitriu and Forrester [8, Claim 6.5] demonstrated this equivalence by noting that $B_{N_{\text{odd}}}$ describes the distribution of the Cholesky factor of $A^{\mathrm{T}}A$. The following lemma can be used to establish the same claim while working directly with $A_{N_{\text{odd}}}$ and $B_{N_{\text{odd}}}$, potentially avoiding numerical pitfalls associated with constructing $A^{\mathrm{T}}A$.

Lemma 1. *If* $A = \begin{pmatrix} W & 0 \\ X & Y \end{pmatrix}$ *has independent entries with* $W \sim \chi_{r+s}$, $X \sim \chi_r$, *and* $Y \sim \chi_s$, *and* Q *is the reflection matrix* $Q = \dfrac{1}{\sqrt{X^2+Y^2}}\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}$, *then* $AQ = \begin{pmatrix} T & U \\ V & 0 \end{pmatrix}$ *has independent entries distributed as* $T \sim \chi_r$, $U \sim \chi_s$, *and* $V \sim \chi_{r+s}$.

Proof. This is equivalent to the more familiar fact that if W^2, X^2, and Y^2 are independent with $(W^2, X^2, Y^2) \sim (\chi^2_{r+s}, \chi^2_r, \chi^2_s)$, then X^2+Y^2, $\frac{W^2X^2}{X^2+Y^2}$, and $\frac{W^2Y^2}{X^2+Y^2}$ are also independent and distributed as $\left(X^2+Y^2, \frac{W^2X^2}{X^2+Y^2}, \frac{W^2Y^2}{X^2+Y^2}\right) \sim (\chi^2_{r+s}, \chi^2_r, \chi^2_s)$. This is established by a change of variables in appropriate joint probability density functions. □

By iteratively applying the lemma, a matrix distributed as $A_{N_{\mathrm{odd}}}$ can be orthogonally transformed into one distributed as $[B_{N_{\mathrm{odd}}} \mid 0]$ via a sequence of orthogonal matrices that act on two columns at a time. Subsequently dropping the column of zeros does not alter the singular values. In particular, the lemma gives a constructive method for sampling $B_{N_{\mathrm{odd}}}$ from a sample of $A_{N_{\mathrm{odd}}}$. Figure 3 illustrates the equivalence schematically for $N = 7$.

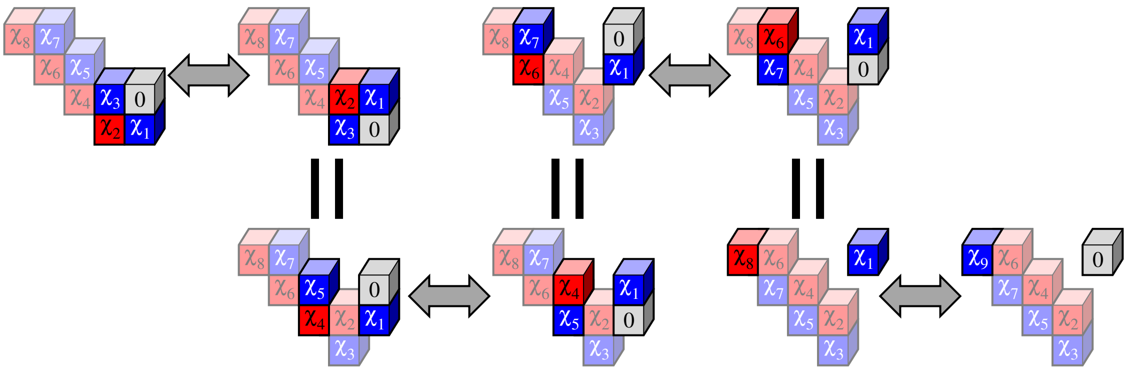

FIGURE 3. Four orthogonal transformations (gray arrows) act on two columns at a time to transform a matrix distributed as A_7 into one distributed as B_7.

Remark. Heuristically, the equivalence between the singular value distributions of $A_{N_{\mathrm{odd}}}$ and B_N can be anticipated by considering the effect of applying Householder reflections to bi-diagonalize a hypothetical complex random matrix with fractional size, namely $\frac{n-1}{2} \times \frac{n}{2}$. Beginning the process by reducing the first column and then alternating between rows and columns produces the first distribution, while starting with the first row produces the second distribution.

In both the cases of even N and odd N, the singular values of an anti-GUE matrix are the singular values of a bi-diagonal matrix of Laguerre type (Figure 4), and the probability density function (4) follows from (2) after a change of variable, taking $\theta_j^2 = x_j$ and thus $2\,\mathrm{d}\theta_j = x^{-1/2}\,\mathrm{d}x_j$, with additional factors of 2 absorbed into c_n^{aG}.

Remark. It can also be advantageous to view the equivalence between the anti-GUE and Laguerre ensembles from the opposite perspective. In particular, the relationship formalizes a sense in which the ensembles $\{\mathrm{LUE}_k^{(+1/2)}\}_{k=1}^{\infty}$ and $\{\mathrm{LUE}_k^{(-1/2)}\}_{k=1}^{\infty}$ are naturally part of a single one-parameter family. In particular, the moments of $\mathrm{LUE}_n^{(1/2)}$ and $\mathrm{LUE}_n^{(-1/2)}$ share the same polynomial dependence on n, with each evaluated at half-integers relative to the other. This matches our intuition that for the purpose of considering singular values, the dimensions of a rectangular matrix should be interchangeable, so that both $\mathrm{LUE}_3^{(+1/2)}$ and a hypothetical $\mathrm{LUE}_{3.5}^{(-1/2)}$ should involve the singular values of a nominal 3×3.5 matrix.

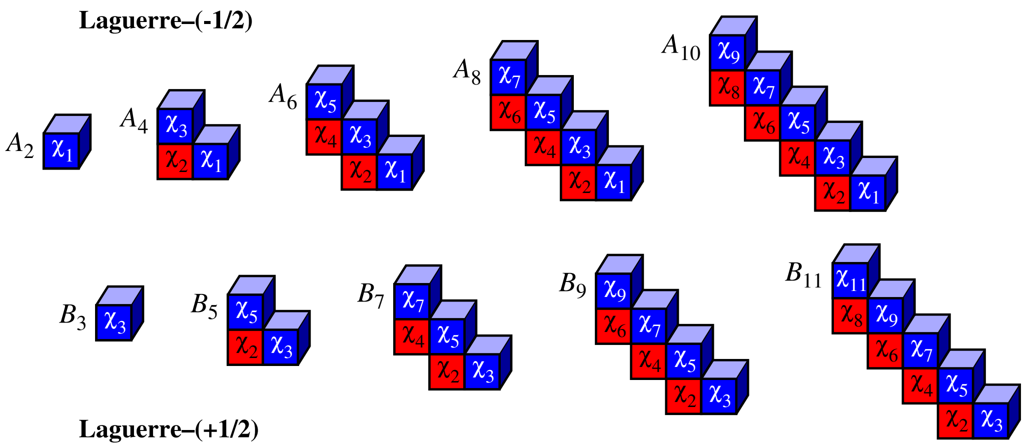

FIGURE 4. The positive eigenvalues of even and odd order anti-GUE matrices are modeled by two different families of square bi-diagonal matrices of Laguerre type.

3. Warm-up: the level densities of the GUE and the Semicircle Law

Before proceeding to the general setting, we examine more closely the motivating problem. How is the semicircle from the GUE related to the quarter-circles describing singular values of bi-diagonal matrices of Laguerre type? What is the analogous relationship for matrices of finite size? By dropping limits, and using orthogonal polynomials to represent relevant probability densities associated with finite random matrices, we see that the semicircle associated with the GUE emerges from an average of two quarter-circles.

For a distribution on n-sets, the m-point correlation function, $\sigma_n(x_1, x_2, \ldots, x_m)$ describes the induced distribution on uniformly selected subsets of size $m \le n$. By convention, $\sigma_n(x_1, x_2, \ldots, x_m)$ is not a probability distribution, but is instead normalized such that

$$\int_{\mathbb{R}^m} \sigma_n(x_1, x_2, \ldots, x_m)\, \mathrm{d}x_1 \cdots \mathrm{d}x_m = \frac{n!}{(n-m)!}.$$

Conceptually, when the underlying random process generates a single unordered n-set, it can be thought of as producing $m!\binom{n}{m} = \frac{n!}{(n-m)!}$ corresponding ordered m-tuples. We will be interested primarily in $\frac{1}{n}\sigma_n(x)$, which describes the pdf of a uniformly selected 1-set. When the distribution on the n-sets takes the form

$$p_n(x_1, x_2, \ldots, x_n) = c_n \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{i=1}^{n} w(x_i)\, \mathrm{d}x_i,$$

as with the GUE ($w(x) = \mathrm{e}^{-x^2/2}$) and the LUE ($w(x) = x^a\, \mathrm{e}^{-x/2}$), the m-point correlation function, $\sigma_n(x_1, x_2, \ldots, x_m)$, is given by an $m \times m$ determinant

$$\begin{aligned} \sigma_n(x_1, x_2, \ldots, x_m) &= c_n \frac{n!}{(n-m)!} \int \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{i=1}^{n} w(x_i)\, \mathrm{d}x_{m+1} \mathrm{d}x_{m+2} \cdots \mathrm{d}x_n \\ &= \det\left(K_2(x_i, x_j)\right)_{1 \le i,j \le m} \end{aligned} \tag{5}$$

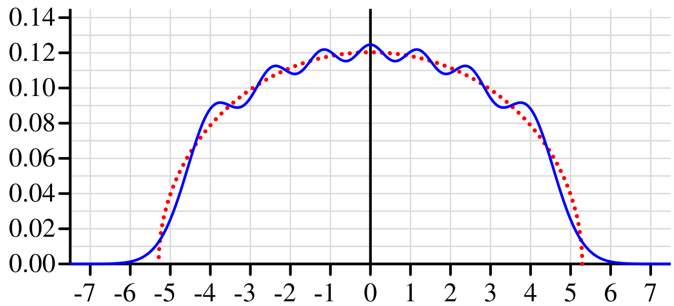

FIGURE 5. A semi-ellipse (red), with equation $y = \frac{1}{\sqrt{7}\,\pi}\sqrt{1-\frac{x^2}{4\cdot 7}}$, is approximated by the probability density function for the distributions of a uniformly selected eigenvalue of the 7×7 GUE (blue) as given by $\frac{1}{7}\sigma_7(x) = \frac{1}{7\sqrt{2\pi}}\sum_{k=0}^{6}\frac{H_k(x)^2}{k!}\mathrm{e}^{-x^2/2}$.

where $K(x,y) = \sqrt{w(x)w(y)}\sum_{j=0}^{n-1}\varphi_j(x)\varphi_j(y)$, and $\{\varphi_j(x)\colon j\ge 0\}$ are orthonormal polynomials associated with the weight $w(x)$ such that $\varphi_j(x)$ has degree j and $\int\varphi_i(x)\varphi_j(x)w(x)\,\mathrm{d}x = \delta_{i,j}$. This result is based on the fact that the Vandermonde matrix can be expanded in terms of any monic polynomials, and the resulting integrals can be evaluated column by column, and can be conceptualized as a generalization of the Cauchy-Binet formula to matrices of continuous dimension. A more complete discussion can be found, for example, in [4, Sec. 5.4] or [28, Ch. 5].

For the GUE, $w(x) = \mathrm{e}^{-x^2/2}$, and the functions φ are related to probabilists' Hermite polynomials described by the initial conditions $H_0(x) = 1$ and $H_1(x) = x$, and by the 3-term recurrence $H_{k+1}(x) = xH_k(x) - kH_{k-1}(x)$ for $k\ge 1$. It follows from the evaluation

$$\int_{-\infty}^{\infty} H_i(x)H_j(x)\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x = \delta_{i,j}\,n!\sqrt{2\pi},$$

that the level density, describing the probability density function for the distribution of a single eigenvalue selected uniformly from the eigenvalues of the order n GUE is given by

$$\frac{1}{n}\sigma_n^{\mathrm{H}}(x) = \frac{1}{n\sqrt{2\pi}}\sum_{k=0}^{n-1}\frac{H_k(x)^2}{k!}\,\mathrm{e}^{-x^2/2}. \tag{6}$$

Applying the Christoffel-Darboux formula to the sum provides the compact representation

$$\frac{1}{n}\sigma_n^{\mathrm{H}}(x) = \frac{1}{n!\sqrt{2\pi}}\big(H_n(x)^2 - H_{n-1}(x)H_{n+1}(x)\big)\,\mathrm{e}^{-x^2/2},$$

from which the eponymous semicircle law can then be recovered using asymptotic properties of Hermite polynomials, as in [28, Appendix A.9]. Figure 5 shows the level density for the 7×7 GUE as an approximation of a semicircle. The relationship to LUEs and the quarter-circle law will be observed by considering the even and odd terms of the summand in (6) separately.

For the LUE, $w(x) = x^a\,\mathrm{e}^{-x/2}$, and the relevant φ can be expressed in terms of $\{L_k^{(a)}(x)\colon k\ge 0\}$, the generalized Laguerre polynomials of parameter a, which satisfy

$$\int_0^{\infty} L_i^{(a)}(x)L_j^{(a)}(x)\,x^a\mathrm{e}^{-x}\,\mathrm{d}x = \frac{\Gamma(j+a+1)}{j!}\delta_{i,j}.$$

Although not required in the present context, it is convenient to note that the Laguerre polynomials are given explicitly by $L_n^{(a)}(x) = \sum_{i=0}^n (-1)^i \binom{n+a}{n-i} \frac{x^i}{i!}$. The weight function for the Laguerre polynomials differs by a factor of 2 in the exponential from our normalization of the LUE, and this is the source of the rescaled parameters in the subsequent formulae. When a is a half-integer, we can write $\Gamma(j+a+1)$ in terms of factorials, and obtain

$$\sigma_n^{\mathrm{L}^-}(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \frac{4^k k!^2}{(2k)!} \Big[L_k^{(-1/2)}\big(\frac{x}{2}\big)\Big]^2 \frac{1}{\sqrt{x}}\, \mathrm{e}^{-x/2}$$

$$\sigma_n^{\mathrm{L}^+}(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \frac{4^k k!^2}{(2k+1)!} \Big[L_k^{(+1/2)}\big(\frac{x}{2}\big)\Big]^2 \sqrt{x}\, \mathrm{e}^{-x/2},$$

corresponding to $a = -1/2$ and $a = +1/2$, with both functions supported on the positive real axis. The pdf for the distribution of a uniformly selected singular values is thus given by $\frac{2}{n} y \sigma_n^{\mathrm{L}^\pm}(y^2)$ (the extra factor of $2y$ is because the density is associated with an implicit differential, so the change of variable $x = y^2$ also induces the substitution $\mathrm{d}x = 2y\,\mathrm{d}y$). By the Marchenko-Pastur law, both $\frac{2}{n} y \sigma_n^{\mathrm{L}^-}(y^2)$ and $\frac{2}{n} y \sigma_n^{\mathrm{L}^+}(y^2)$ converge to quarter-ellipses as $n \to \infty$.

To see the relationship between the semicircle law and the quarter-circle law, we note that the left-right symmetry of $\sigma_n^{\mathrm{H}}(x)$ is a consequence of the fact that the matrix entries of the GUE are distributed symmetrically about the origin, so that the density of a matrix is identical to the density of its negation. The semicircle law is thus an example of a property that nominally involves eigenvalues, but can instead be analyzed in terms of singular values: it is sufficient to show a relationship between σ_n^{H} and $\sigma_n^{\mathrm{L}^\pm}$ for positive arguments.

Hermite polynomials, $H_k(x)$, are either even or odd polynomials, according to the parity of k. As a consequence, $H_{2m}(\sqrt{y})$ and $\frac{1}{\sqrt{y}} H_{2m+1}(\sqrt{y})$ are both monic polynomials of degree m in y. Applying the change of variables $y = x^2$ to the orthogonality relationship for Hermite polynomials, and using the fact that $H_i(x)H_j(x)$ is an even polynomial whenever i and j have the same parity, we see that on restricting to the positive x-axis

$$\int_0^\infty H_{2m+r}\left(\sqrt{y}\right) H_{2l+r}\left(\sqrt{y}\right)\, y^{-\frac{1}{2}} \mathrm{e}^{-\frac{y}{2}}\, \mathrm{d}y = \delta_{l,m}\, (2m+r)! \sqrt{2\pi}.$$

So the monic polynomials $\{H_{2m}(\sqrt{y}) \colon m \geq 0\}$ are orthogonal on $(0,\infty)$ relative to $y^{-\frac{1}{2}} \mathrm{e}^{-\frac{y}{2}}$, and similarly $\{\frac{1}{\sqrt{y}} H_{2m+1}(\sqrt{y}) \colon m \geq 0\}$ are orthogonal on $(0,\infty)$ relative to $y^{\frac{1}{2}} \mathrm{e}^{-\frac{y}{2}}$. As a consequence we recover the classical fact (see for example [34, Sec. 5.6]) that Hermite polynomials can also be expressed in terms of the generalized Laguerre polynomials. In particular, even ($r = 0$) and odd ($r = 1$) Hermite polynomials are given by the expression

$$H_{2n+r}(x) = x^r (-2)^n\, n!\, L_n^{(r-1/2)}\big(\frac{x^2}{2}\big) \quad \text{for } r \in \{0,1\}, \tag{7}$$

Substituting (7) into (6) we find

$$\begin{aligned}
\sigma_n^{\mathrm{H}}(x) &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n_1} \frac{4^k k!^2}{(2k)!} \Big[L_k^{(-1/2)}\big(\frac{x^2}{2}\big)\Big]^2 \mathrm{e}^{-x^2/2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n_2} \frac{4^k k!^2}{(2k+1)!} \Big[L_k^{(+1/2)}\big(\frac{x^2}{2}\big)\Big]^2 x^2\, \mathrm{e}^{-x^2/2} \\
&= x\sigma_{n_1}^{\mathrm{L}^-}(x^2) + x\sigma_{n_2}^{\mathrm{L}^+}(x^2)
\end{aligned}$$

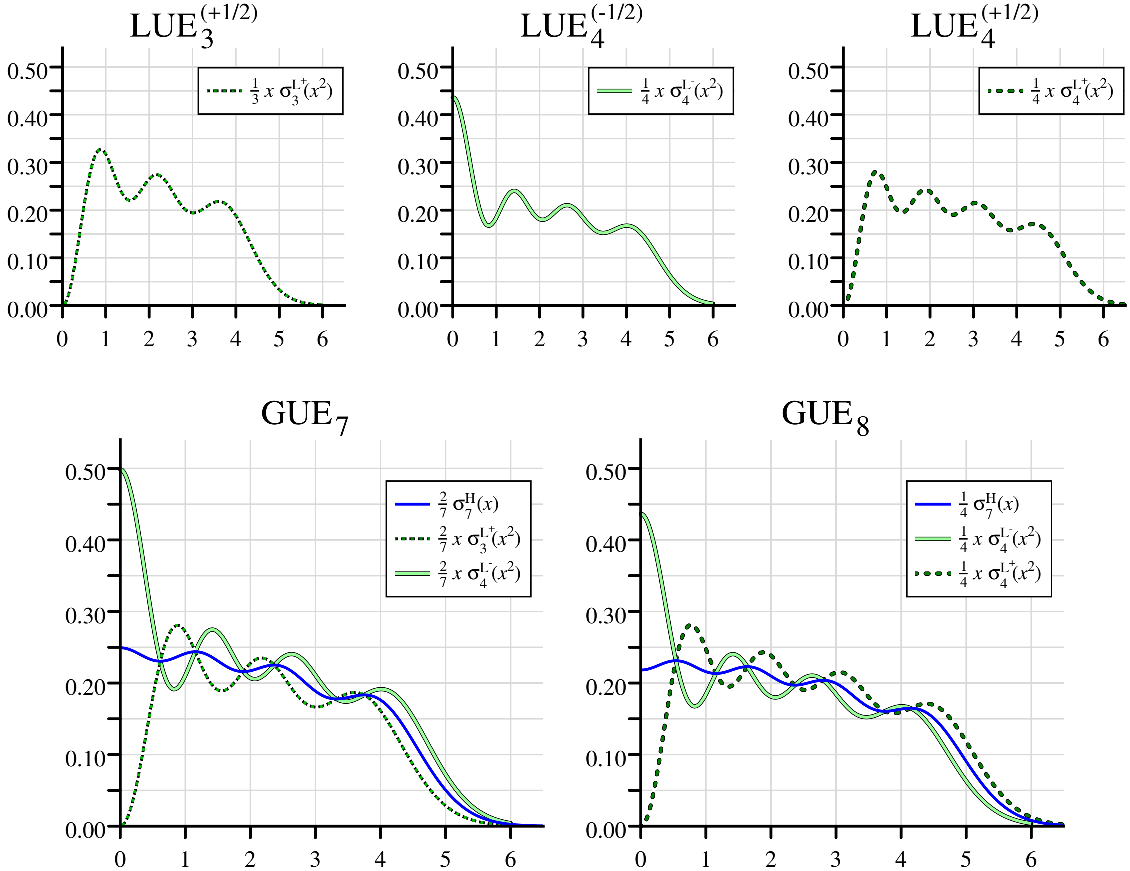

FIGURE 6. The top three plots show pdfs for distributions of the positive square root of an eigenvalue selected uniformly at random from various Laguerre ensembles. The bottom two plots give the distributions for uniformly selected singular values of GUEs (blue) as a weighted average of the Laguerre pdfs (green). For odd n, the weighting is unequal. For example, $\frac{2}{7}\sigma_7^{\mathrm{H}}(x) = \frac{1}{2}\left(\frac{4}{7}x\sigma_3^{\mathrm{L}^+}(x^2) + \frac{4}{7}x\sigma_4^{\mathrm{L}^-}(x^2)\right)$, but it is $\frac{2}{3}x\sigma_3^{\mathrm{L}^+}(x^2)$ and $\frac{2}{4}x\sigma_4^{\mathrm{L}^-}(x^2)$ that are probability densities. When n is even the weighting is equal, for example, $\frac{2}{8}\sigma_8^{\mathrm{H}}(x) = \frac{1}{2}\left(\frac{2}{4}x\sigma_4^{\mathrm{L}^+}(x^2) + \frac{2}{4}x\sigma_4^{\mathrm{L}^-}(x^2)\right)$.

where $n_1 = \left\lceil \frac{n}{2} \right\rceil$ and $n_2 = \left\lfloor \frac{n}{2} \right\rfloor$. An immediate consequence is that the restriction of the semicircle law to the positive quadrant is manifestly a weighted average of two copies of the quarter-circle law. Since $\frac{2}{n_1}x\sigma_{n_1}^{L^-}(x^2)$ and $\frac{2}{n_2}x\sigma_{n_2}^{L^+}(x^2)$ describe the pdfs of uniformly chosen singular values of bi-diagonal Laguerre matrices, we conclude that a single singular value of a GUE matrix has the same distribution as a single singular value selected from the direct sum of two LUE matrices, or via the equivalence of LUE and anti-GUE matrices two anti-GUE matrices of consecutive sizes.

Remark. We see that the 1-point correlation function for the singular values of the GUE is a sum of two 1-point correlation functions for Laguerre singular values, and conclude that picking a random singular value from the GUE is equivalent to picking one from a mixture of two appropriate LUEs. It is notable that each LUE distribution is a component of two consecutive GUE distributions. Figure 6 presents the level densities for the 7×7 and 8×8 GUE's as weighted averages of LUE singular value level densities. Notice that the $\mathrm{LUE}_4^{(+1/2)}$ distribution is a component of both mixtures.

Concretely, we conclude that a uniformly random singular value of the $n \times n$ GUE has the same distribution as uniformly random singular value from the union of an $n \times n$ anti-GUE and an $(n+1) \times (n+1)$ anti-GUE. In fact, it is possible to continue in this direction, as Forrester did in [13], to effectively show the analogous property for all m-point correlation functions, and conclude that the singular values of the GUE are a mixture of the singular values of two independent ensembles. In the next section, we take a different approach, and derive the same conclusion without appealing to the theory of orthogonal polynomials.

4. Main Result - A Decomposition of the Singular Values of the GUE

The main result of the paper equates the joint probability density functions for two distributions, the singular values of the $n \times n$ GUE and the union of the distinct non-zero singular values of two independent anti-GUE matrices, one of order n and the other of order $(n+1)$. We do not know of any particularly compact descriptions for the joint pdf for the distribution of the singular values in either setting, but instead express each as a sum. In terms of the $n \times n$ GUE, this sum involves 2^n terms, corresponding to the number of ways that n singular values can be assigned signs. In contrast, there are $\binom{n}{\lfloor n/2 \rfloor}$ ways the singular values can be partitioned between an $n \times n$ anti-GUE and a $(n+1) \times (n+1)$ anti-GUE. Neither sum is particularly compact, but the second involves asymptotically fewer terms by a factor of $\sqrt{2\pi n}$. The result is thus to be interpreted primarily as structural: it is this extra structural information, rather than the expressions themselves, that can be used to computational advantage.

Our main tool is to express relevant probability densities in terms of determinants, and then to recognize evaluations that induce a structured sparsity and permit writing the resulting determinants as products. In addition to appearing in Forrester's work on gap probabilities, where [13, Eq. (2.6)] is equivalent to our main result, the same pattern occurs in existing proofs of the applications discussed in Section 5 ([28, Ch. 20] and [29]), as well as related problems about enumerative properties of orientable maps ([20, 23, 19]). The emphasis of the present work is to show that all of these results are consequences of the same underlying structural decomposition.

In practice, when evaluating integrals of symmetric functions, it is often convenient to write the integral as a sum of terms that are equal by symmetry, and then to consider only a single term. This is the case, for example, in [28, Ch. 15]), where Mehta considers integrals related to complex Ginibre ensembles. We give presentations of (1) and (4) that are suitable for desymmetrization.

Lemma 2. *The joint probability density for the eigenvalues of the* GUE *can be represented as*

$$p_n^{\mathrm{H}}(x_1, x_2, \ldots, x_n) = c_n^{\mathrm{H}} \sum_{\pi \in \mathfrak{S}_n} \det\left(\left(x_{\pi_i}^{i+j-2} \right)_{1 \le i,j \le n} \right) \exp\left(-\frac{1}{2} \sum_{i=1}^n x_i^2 \right) \prod_{i=1}^n \mathrm{d}x_i, \tag{8}$$

where the sum is taken over all permutations of the indices $\{1, 2, \ldots, n\}$.

Proof. We recognize the product $\prod_{1\le i<j\le n}|x_i-x_j|^2$ in (1) as the square of the Vandermonde determinant, and use multiplicativity and invariance under transposition to obtain

$$\prod_{1\le i<j\le n}|x_i-x_j|^2=\det\Big(\big(x_i^{j-1}\big)_{1\le i,j\le n}\big(x_j^{i-1}\big)_{1\le i,j\le n}\Big).$$

Multiplying the matrices inside the determinant gives the expression.

$$\prod_{1\le i<j\le n}|x_i-x_j|^2=\det\begin{pmatrix} n & p_1(\vec{x}) & p_2(\vec{x}) & \cdots & p_{n-1}(\vec{x})\\ p_1(\vec{x}) & p_2(\vec{x}) & p_3(\vec{x}) & \cdots & p_n(\vec{x})\\ p_2(\vec{x}) & p_3(\vec{x}) & p_4(\vec{x}) & \cdots & p_{n+1}(\vec{x})\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ p_{n-1}(\vec{x}) & p_n(\vec{x}) & p_{n+1}(\vec{x}) & \cdots & p_{2n-2}(\vec{x}). \end{pmatrix}$$

By column-linearity, we can write this as a sum of determinants where each column depends on a single variable. The determinant vanishes if any variable is duplicated, so the sum can be restricted to terms in which each variable occurs in a single column, and we obtain

$$\prod_{1\le i<j\le n}|x_i-x_j|^2=\sum_{\pi\in\mathfrak{S}_n}\det\Big(\big(x_{\pi_i}^{i+j-2}\big)_{1\le i,j\le n}\Big).$$

The factor, $\exp\big(-\frac{1}{2}\sum_{i=1}^n x_i^2\big)$ is symmetric and thus common to all terms, and the result follows. $\square$

The proof relied only on the presence of the Vandermonde factor, so a similar derivation gives a corresponding presentation for the joint probability density functions of the anti-GUE given in (4).

Lemma 3. *For $r\in\{0,1\}$, the joint pdf of the $(2n+r)\times(2n+r)$ anti-*GUE *can be presented as*

$$p_{2n+r}^{\mathrm{aG}}(x_1,x_2,\dots,x_n):=c_{2n+r}^{\mathrm{aG}}\sum_{\pi\in\mathfrak{S}_n}\det\Big(\big(\theta_{\pi_i}^{2i+2j-4+2r}\big)_{1\le i,j\le n}\Big)\exp\Big(-\frac{1}{2}\sum_{j=1}^n\theta_j^2\Big)\prod_{j=1}^n \mathrm{d}\theta_i. \tag{9}$$

The identity terms of the summations in (8) and (9) determine signed measures, μ_n^{H} and μ_n^{aG}, on ordered n-tuples of real numbers given by

$$\mu_n^{\mathrm{H}}(x_1,x_2,\dots,x_n):=n!\,c_n^{\mathrm{H}}\det\Big(\big(x_i^{i+j-2}\big)_{1\le i,j\le n}\Big)\exp\Big(-\frac{1}{2}\sum_{i=1}^n x_i^2\Big)\prod_{i=1}^n \mathrm{d}x_i,$$

$$\mu_{2n+r}^{\mathrm{aG}}(\theta_1,\theta_2,\dots,\theta_n):=n!\,c_{2n+r}^{\mathrm{aG}}\det\Big(\big(\theta_i^{2i+2j-4+2r}\big)_{1\le i,j\le n}\Big)\exp\Big(-\frac{1}{2}\sum_{j=1}^n\theta_j^2\Big)\prod_{j=1}^n \mathrm{d}\theta_i,$$

that can be used to evaluate expectations of symmetric functions of eigenvalues. Unlike p_n^{H} and p_{2n+r}^{aG} which can be sampled by diagonalizing randomly generated matrices (for example the tri-diagonal matrices of [7] or [36]), it is not clear how to use μ_n^{H} and μ_{2n+r}^{aG} in Monte Carlo experiments.

We are now in a position to give the main result of the paper, an equivalent form of which was previously described by Forrester in [13, Eq. (2.6)].

Theorem 1. *The singular values of the $n\times n$* GUE *have the same distribution as the union of the distinct singular values of two anti-*GUE *matrices, one of order n, the other of order $n+1$. In terms of probability densities, for $x_1,x_2,\dots,x_n\ge 0$,*

$$\sum_{\epsilon\in\{\pm1\}^n}p_n^{\mathrm{H}}(\epsilon_1x_1,\epsilon_2x_2,\dots,\epsilon_nx_n)=\frac{1}{\binom{n}{\lfloor n/2\rfloor}}\sum_{S,T}p_n^{\mathrm{aG}}(x_{s_1},x_{s_2},\dots,x_{s_{\lfloor n/2\rfloor}})p_{n+1}^{\mathrm{aG}}(x_{t_1},x_{t_2},\dots,x_{t_{\lceil n/2\rceil}}) \tag{10}$$

$$\begin{pmatrix} 1 & x_2 & x_3^2 & x_4^3 \\ x_1 & x_2^2 & x_3^3 & x_4^4 \\ x_1^2 & x_2^3 & x_3^4 & x_4^5 \\ x_1^3 & x_2^4 & x_3^5 & x_4^6 \end{pmatrix} \sim \begin{pmatrix} 1 & x_3^2 & x_2 & x_4^3 \\ x_1^2 & x_3^4 & x_2^3 & x_4^5 \\ x_1 & x_3^3 & x_2^2 & x_4^4 \\ x_1^3 & x_3^5 & x_2^4 & x_4^6 \end{pmatrix} \qquad \begin{pmatrix} 1 & x_2 & x_3^2 & x_4^3 & x_5^4 \\ x_1 & x_2^2 & x_3^3 & x_4^4 & x_5^5 \\ x_1^2 & x_2^3 & x_3^4 & x_4^5 & x_5^6 \\ x_1^3 & x_2^4 & x_3^5 & x_4^6 & x_5^7 \\ x_1^4 & x_2^5 & x_3^6 & x_4^7 & x_5^8 \end{pmatrix} \sim \begin{pmatrix} 1 & x_3^2 & x_5^4 & x_2 & x_4^3 \\ x_1^2 & x_3^4 & x_5^6 & x_2^3 & x_4^5 \\ x_1^4 & x_3^6 & x_5^8 & x_2^5 & x_4^7 \\ x_1 & x_3^3 & x_5^5 & x_2^2 & x_4^4 \\ x_1^3 & x_3^5 & x_5^7 & x_2^4 & x_4^6 \end{pmatrix}$$

FIGURE 7. The rows and columns of $\left(x_i^{i+j-2}\right)_{1\le i,j\le n}$ are permuted to group the even monomials into two blocks. One block involves even-indexed variables, the other involves odd-indexed variables. Odd monomials are annihilated by the ϵ-summation and do not contribute to the density of singular values.

where the left sum runs over the 2^n *ways* $\{x_1, x_2, \dots, x_n\}$ *can be assigned signs to describe eigenvalues of a* GUE*, and the right sum runs over the* $\binom{n}{\lfloor n/2\rfloor}$ *ways to partition* $\{1, 2, \dots, n\}$ *into two sets* $S = \{s_1, s_2, \dots, s_{\lfloor n/2\rfloor}\}$ *and* $T = \{t_1, t_2, \dots, t_{\lceil n/2\rceil}\}$ *corresponding to the anti-*GUE *factors.*

Proof. Using Lemma 2, the density of the singular values of the GUE is given by

$$c_n^{\mathrm{H}} \sum_{\epsilon\in\{\pm1\}^n} \sum_{\pi\in\mathfrak{S}_n} \det\Big(\big((\epsilon_i x_{\pi_i})^{i+j-2}\big)_{1\le i,j\le n}\Big) \exp\Big(-\frac{1}{2}\sum_{i=1}^n x_i^2\Big) \prod_{i=1}^n \mathrm{d}x_i.$$

Both sums are finite, so the ϵ-summation can be carried out first, and since for fixed π each ϵ_i occurs in a single column, it can be applied column-by-column. Except for the determinant, the expression is invariant under sign-change, so summing over ϵ annihilates all matrix entries involving monomials of odd degree and creates a checkerboard pattern of sparsity. The rows and columns of the resulting matrix can be simultaneously permuted so that the odd columns and rows occur before the even columns and rows, and this collects the zero and non-zero entries into blocks (see Figure 7). We need only consider the direct sum of the non-zero blocks, and obtain the decomposition:

$$\sum_{\epsilon\in\{\pm1\}^n} \det\Big(\big((\epsilon_i x_{\pi_i})^{i+j-2}\big)_{1\le i,j\le n}\Big) = 2^n \det\Big(\big(x_{\pi_{2i-1}}^{2i+2j-4}\big)_{1\le i,j\le\lceil\frac{n}{2}\rceil} \oplus \big(x_{\pi_{2i}}^{2i+2j-2}\big)_{1\le i,j\le\lfloor\frac{n}{2}\rfloor}\Big).$$

The determinant on the right-hand side factors as a product of two determinants of the type occurring in (9) for anti-GUE of orders $2\left\lceil\frac{n}{2}\right\rceil$ and $2\left\lfloor\frac{n}{2}\right\rfloor + 1$, which correspond to n and $n+1$ for all choices of n (although the particular pairing depends on parity). The exponential factor and volume elements are separable, and thus the theorem is true up to a scalar factor. This scalar is necessarily unity since both sides of (10) are probability densities on the positive orthant. $\square$

Example 1. For the GUE of order 2, we have $p_2^{\mathrm{H}}(x,y) = \frac{1}{4\pi}(x^2 - 2xy + y^2)\mathrm{e}^{-x^2/2-y^2/2}$ and $\mu_2^{\mathrm{H}}(x,y) = \frac{1}{2\pi}(y^2 - xy)\mathrm{e}^{-x^2/2-y^2/2}$. Since $p_2^{\mathrm{H}}(a,b) = \frac{1}{2}\mu_2^{\mathrm{H}}(a,b) + \frac{1}{2}\mu_2^{\mathrm{H}}(b,a)$ for all a and b, they induce the same density on 2–sets. While $p_2^{\mathrm{H}}(x,y)$ is symmetric about $y = x$ and $y = -x$, the signed measure $\mu_2^{\mathrm{H}}(x,y)$ is symmetric about $y = \frac{1}{2}x$ and $y = -2x$ (see Figure 8).

Example 2. The singular values of the 2×2 GUE have density $\frac{1}{2}f(x,y) + \frac{1}{2}f(y,x)$ where

$$f(x,y) := \mu_2^{\mathrm{H}}(x,y) + \mu_2^{\mathrm{H}}(-x,y) + \mu_2^{\mathrm{H}}(x,-y) + \mu_2^{\mathrm{H}}(-x,-y) = \frac{2}{\pi}y^2\,\mathrm{e}^{-x^2/2-y^2/2}$$

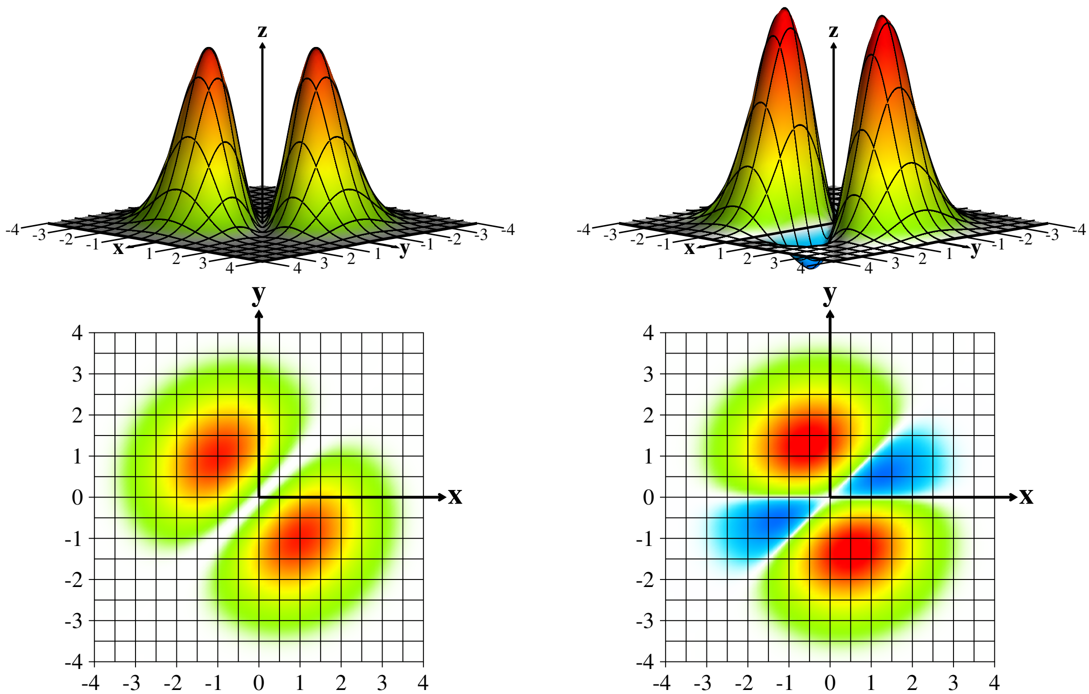

FIGURE 8. The density $p_2^{\mathrm{H}}(x, y)$ (left) and signed measure $\mu_2^{\mathrm{H}}(x, y)$ (right).

is itself a probability density. It is the product of densities of independent $X \sim \chi_1$ and $Y \sim \chi_3$ random variables. Figure 9 plots $g(x, y) = \frac{f(x,y)}{f(0,\sqrt{2})}$ as the product $g(x, y) = g(x, \sqrt{2}) \times g(0, y)$.

Remark. The proof did not use the independent Gaussian nature of the matrix entries in any essential way. As in [13, Eq. 2.6] and [11, Sec 8.4.1], a similar decomposition applies when working with any density of the form $c \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{i=1}^{n} w(|x_i|)$. The authors are unaware of any immediate applications involving these more general distributions.

Remark. In related work, previous authors have often been forced to invoke a case analysis, depending on the parity of n. This is the case, for example, in the motivating work of Jackson and Visentin [20, 23], in which their factorization applies directly only when n is even, with odd n recovered by polynomiality. Avoiding this complication is the principal reason that we prefer to describe the component distributions in terms of anti-GUE instead of $\mathrm{LUE}^{(\pm 1/2)}$ distributions.

4.1. **Bi-diagonal Representations.** A consequence of Theorem 1 is that the singular values of the GUE inherit bi-diagonal representations from anti-GUE and LUE matrices. In fact, for a given n, we have two distinct bi-diagonal models for the singular values of the order n GUE, one using the natural representation of the odd anti-GUE component that emphasizes the one-parameter nature of the GUEs, and a second obtained by considering this component in terms of Laguerre–$(+1/2)$ matrices. Figure 10 shows the relationship between these representations. Using the LUE representation of

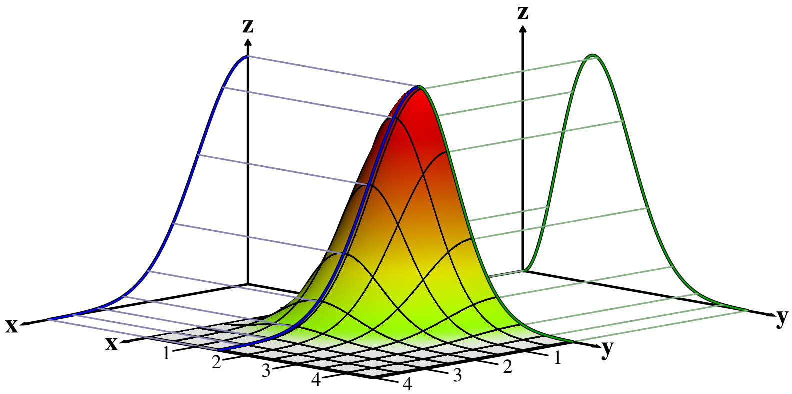

FIGURE 9. The function $g(x,y) = \frac{e}{2} y^2 e^{-x^2/2 - y^2/2}$ is separable with a maximum value of $g(0,\sqrt{2}) = 1$ and satisfies the relation $g(x,y) = g(x,\sqrt{2}) \times g(0,y)$. The factors correspond to the fact that the singular values of the 2×2 GUE have the same distribution as independently distributed χ_1 and χ_3 random variables.

the factors, we have the direct sum decomposition

$$\mathrm{GUE}_n^{(s)} \sim \begin{pmatrix} \chi_{n_1} & & & & \\ \chi_{n_1-1} & \chi_{n_1-2} & & & \\ & \ddots & \ddots & & \\ & & \chi_4 & \chi_3 & \\ & & & \chi_2 & \chi_1 \end{pmatrix} \oplus \begin{pmatrix} \chi_{n_2} & & & & \\ \chi_{n_2-3} & \chi_{n_2-2} & & & \\ & \ddots & \ddots & & \\ & & \chi_4 & \chi_5 & \\ & & & \chi_2 & \chi_3 \end{pmatrix} \tag{11}$$

where $n_1 = 2\left\lceil \frac{n}{2} \right\rceil - 1$ and $n_2 = 2\left\lfloor \frac{n}{2} \right\rfloor + 1$. This gives even powers of the determinant of GUE matrices as simple a structure as determinants of Wishart matrices, and is the basis of Section 5.1.

4.2. **Relationship with Combinatorics of Orientable Maps.** Our interest in Theorem 1 stem from the second author's attempts to develop a more natural understanding of what turned out to be an equivalent result from enumerative combinatorics. In fact both Theorem 1 and the relationship between Hermite and Laguerre level densities from Section 3 can be stated as results about map enumeration, although the equivalence is non-trivial.

Combinatorially, the moments of the GUE_n count rooted embeddings of graphs in orientable surfaces in which each face is painted with one of n colors. Similarly, when $n+a$ is a positive integer, the moments of the $\mathrm{LUE}_n^{(a)}$ count rooted embeddings of face-bipartite graphs in orientable surfaces such that each face of one class is painted with one of n colors, and each face of the other class is painted with one of $m = n+a$ colors. Scalings of these moments for large n are dominated by embeddings with a maximum number of faces. For planar maps, these are necessarily the duals of trees, which are enumerated by Catalan numbers. In this context, the relationship between the semicircle law and the quarter-circle law can be seen as a statement that all trees are bipartite, so it follows that one-part moments of the two ensembles have the same leading order behavior.

In [20], Jackson and Visentin used permutation representations of orientable maps to derive expressions for generating series of several classes of such maps in terms of irreducible characters of the

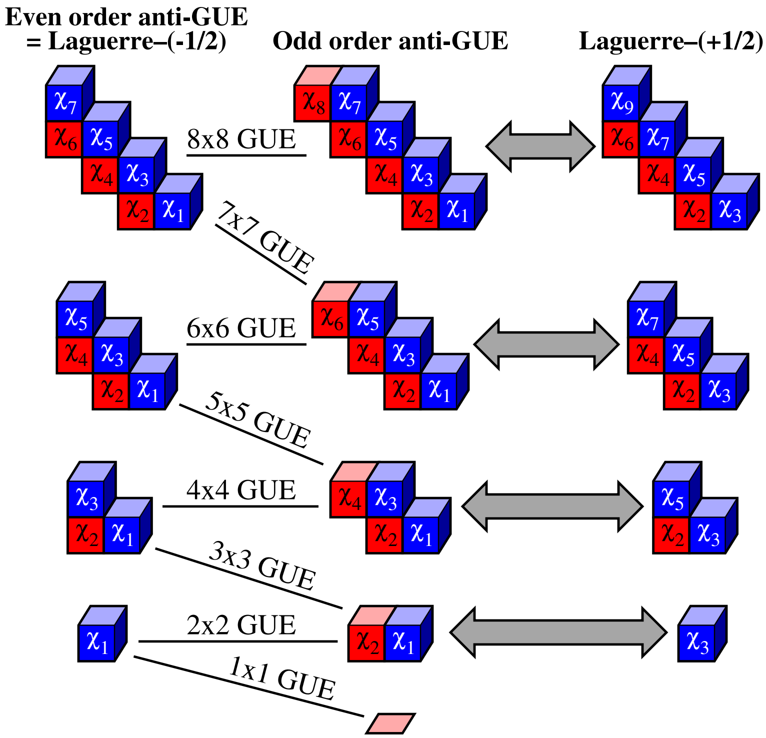

FIGURE 10. Bi-diagonal models for the singular values of GUEs. Each GUE takes its singular values from two independent blocks corresponding to an even and an odd order anti-GUE. Each odd-order anti-GUEs (second column) has an alternate representation when considered as a Laguerre–$(+1/2)$ matrix (third column).

symmetric group, and manipulated these expressions *via* their relationships with Schur functions. In this framework, they exhibited a sparsity pattern and factorization for determinantal representations of products of the form $V(\vec{x})^2 s_\theta(\vec{x})$, with s_θ a Schur function, and used this factorization to obtain a functional relationship between generating series. Their construction applies directly only when n is an even integer, but the functional identity is extended to odd n with the observation that moments depend polynomially on n. In contrast, by working with $V(\vec{x})^2$ directly in the proof of Theorem 1, we have been able to treat even and odd values of n simultaneously, and have avoided technical manipulations of irreducible characters of the symmetric group.

The enumerative result was later interpreted in terms of matrix models arising in the study of 2–dimensional quantum gravity in [19], and extended in [23] to a form equivalent to Theorem 1, although the authors were unaware of the random matrix interpretation of this extension. The generating series they considered are effectively the cumulant generating functions for the GUE_n and $\mathrm{LUE}_n^{(a)}$ densities, (1) and (2), taken to have functional dependence on n and a, although they did not interpret them in this way. In fact, the combinatorial interpretation for the cumulant generating function for $\mathrm{LUE}_n^{(a)}$ can only be established directly when a is a non-negative integer, since $m = n + a$ is to be interpreted as the cardinality of a set of colors. The functional identity, LISTING 1. MATLAB code for partitioning the singular values of GUE_n

```
% We probabilistically un−mix singular values sampled from the nxn GUE
% to produce samples distributed as the union of two Laguerre ensembles.

% t = number of samples, n = order of GUE
t = 100000; n = 7; outlist = zeros(t,n);

% We need a list of all partitions of {1,2,..., n} into two balanced parts.
% We generate all subsets of size floor (n/2) and their complements.

cbinom = nchoosek(n,floor(n/2));        % − a central binomial coefficient
parta = nchoosek([1:n], floor (n/2));   % − subsets of {1,2,..., n}
partb = zeros(cbinom,ceil(n/2));        % − their complements
for prep = 1:cbinom
    partb(prep,:) = setdiff ([1: n], parta(prep ,:));
end
partitions = [parta partb];
P = zeros(1,cbinom);

for rep = 1:t;
    % Sample singular values from the GUE of appropriate size
    G = randn(n)+i*randn(n); A = (G+G')/2; eiglist = sort(abs(eig(A)));

    % We'll need the differences of the squares of the eigenvalues
    singdiffs = ( eiglist .^2*ones(1,n) − ( eiglist .^2*ones(1,n ))');

    % The n eigenvalues can be partitioned in binom(n,floor(n/2)) ways.
    % Compute the relative densities with common factors ommited.
    for prep = 1:cbinom
        P(prep) = (abs(prod(eiglist (parta(prep ,:))))/ prod(prod(singdiffs (parta(prep ,:), partb(prep ,:)))))^2;
    end;

    % Separate the singular values with each partition occuring proportionally to its density
    outlist (rep ,:) = eiglist ( partitions ( find (cumsum(P)>rand*sum(P),1),:))';
end

figure (1); hist (reshape( outlist (:,1:3),1,[]),100);
figure (2); hist (reshape( outlist (:,4:7),1,[]),100);
```

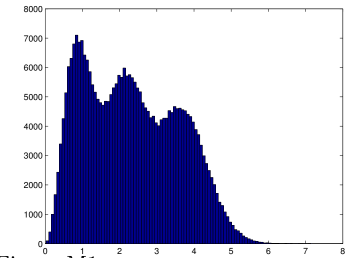

Figure M1

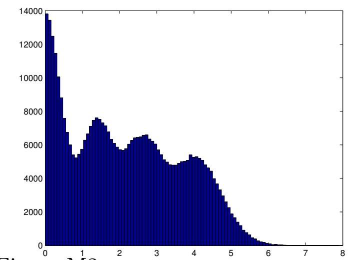

Figure M2

however, applies only when $a = \pm\frac{1}{2}$. This disconnect is resolved by applying the observation that both the generating series for maps, and the moments of the bi-diagonal model of the Laguerre ensemble, (2), depend polynomially on a.

4.3. **Observing the Decomposition.** Theorem 1 shows that it is impossible to distinguish between the singular values of an $n \times n$ GUE and a mixture of the singular values of two anti-GUE of appropriate sizes. A natural question is how this can be observed numerically, and whether this kind of decomposition has a signature that can be used to identify (or discount) related decompositions. As we saw in Examples 1 and 2, the product structure is not visible directly for the pdf of the GUE_2, but only on a desymmetrized transform.

It appears to be an interesting algorithmic question in general to determine when a collection of data can be partitioned into two (or more) independent sets. In the particular case of the GUE, we know the exact distributions of the factors, and can thus use conditional probabilities to generate two independent sets from their union. Listing 1 provides a sample of MATLAB code that demonstrates this. After a precomputation of the matrices `parta`, `partb`, and `partitions` holding a list of appropriately sized subsets and their complements taken from $\{1, 2, \ldots, n\}$, the main loop generates t samples of the singular values of GUE_n. For each sample, relative probabilities are computed that this data arose from each of the $\binom{n}{\lfloor n/2 \rfloor}$ partitions of the values between anti-GUE_n and anti-GUE_{n+1} factors. Choosing a random partition with weight proportional to these probabilities allows us to partition the values into the columns of `outlist`, such that the first $\lfloor n/2 \rfloor$ columns are distributed as the positive singular values of an anti-GUE_n and the final $\lceil n/2 \rceil$ are independently distributed as the positive singular values of an anti-GUE_{n+1}. The embedded figures, M1 and M2 show histograms approximating the level densities for the parts when $n = 7$, and should be compared with the theoretical level densities of $\text{LUE}_3^{(+1/2)}$ and $\text{LUE}_4^{(-1/2)}$ from Figure 6.

5. Applications

The structure exhibited by Theorem 1 provides new explanations for existing observations about the distribution of the determinant and extreme singular values of the GUE. We do not believe they have previously been recognized as consequences of the same underlying structure.

5.1. **The Determinant of the GUE.** While the determinant of the GUE depends on the signs of eigenvalues, its absolute value does not. The bi-diagonal model for the singular values of the GUE, given in (11), and illustrated in Figure 11 permits us to write it as a product of independent χ-distributed random variables. We obtain directly the expected values of even powers of the determinants of GUE matrices, and by invoking duality between k and n when computing the expected value of $\det M^k$ for $M \sim \text{GUE}_n$ we obtain expected values of odd powers as well. A second consequence is a direct explanation for the asymptotic log-normality of the absolute value of large GUE matrices, which had been previously concluded *via* technical computations in [5] and [35].

The distribution of the determinant of the GUE was previously studied by Mehta and Normand [29], who examined the Mellin transform of its even and odd parts, using sparsity to write each as a product of determinants. The distribution of the absolute value of the determinant corresponds to the transform of the even part, and using our decomposition we can quickly re-derive this part of their conclusion: although Mehta and Normand did not interpret the factors probabilistically, they are the Mellin transforms of χ-distributed random variables, corresponding to the diagonal entries of our bi-diagonal model for the singular values of the GUE. We do not presently have a corresponding

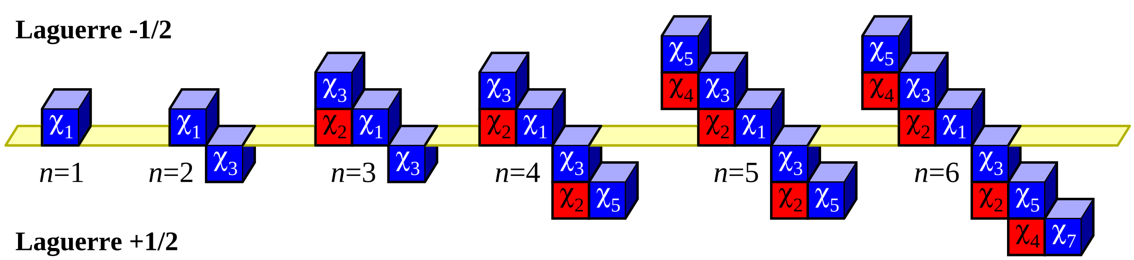

Figure 11. Bi-diagonal models for the singular values of the GUE_n. Only diagonal entries (white on blue) contribute to the absolute value of the determinant.

decomposition to describe the Mellin transform of the odd part of the pdf, but odd moments of the determinant of GUE_n are either zero for reasons of symmetry when n is odd, or are recoverable using a duality that reverses the roles of the order of a matrix and the power of its determinant.

To describe the distribution of the absolute value (or even powers) of the determinant of the GUE, it is sufficient to describe the distribution of the products of the singular values. Using the bi-diagonal representation, (11), the product of the singular values is identified with the product of the independently χ-distributed diagonal entries of these matrices (see Figure 11). The following theorem is an immediate consequence.

Theorem 2. *For $M \sim \mathrm{GUE}_n$, the absolute value of the determinant of M is distributed as the product of independent random variables $\prod_{i=1}^n X_i$ where $X_i \sim \chi_{2\lfloor \frac{i}{2} \rfloor + 1}$.*

From Theorem 2 we obtain immediate expressions for expected values of even moments of the determinant in terms of the moments of χ-distributed random variables, which are not altered by considering products of singular values instead of eigenvalues. Symmetry implies that odd moments of determinants of odd order GUE vanish, and it remains only to determine the expected values of odd moments of GUE of even orders. For this we invoke the following Lemma, which is a special case of a duality principal described by Dumitriu in [6, Theorem 8.5.3], where a more general result was derived using the machinery of symmetric function theory (see [26, 33]), based on the observation that powers of the determinant can be expressed in terms of evaluations of Jack symmetric functions.

Lemma 4 (Dumitriu [6, part of Theorem 8.5.3])**.** *If n and k are positive integers, at least one of which is even, then*

$$\mathrm{E}\left[\det(M_n^k)\right] = (-1)^{nk/2}\,\mathrm{E}\left[\det(M_k^n)\right],$$

where $M_n \sim \mathrm{GUE}_n$ and $M_k \sim \mathrm{GUE}_k$.

We thus obtain division-free expressions for the moments of the determinant of the GUE. These moments were previously given by Mehta and Normand (for a different choice of normalization) in [29] as products of ratios of Γ-functions, and with a more direct derivation by Andrews *et al.* in [1, Theorem 1] as products of ratios of factorials. In both cases, the moments were listed in four cases, depending on the parities of both n and k.

Corollary 1. *The expected values of powers of the determinant of the $M \sim \mathrm{GUE}_n$ are given by:*

$$\mathrm{E}\left[\det M^k\right] = \begin{cases} \displaystyle\prod_{i=1}^{n}\prod_{j=1}^{k/2}\left(2\left\lfloor\frac{i}{2}\right\rfloor + 2j - 1\right) & \text{if } k \text{ is even} \\ \displaystyle(-1)^{nk/2}\prod_{i=1}^{k}\prod_{j=1}^{n/2}\left(2\left\lfloor\frac{i}{2}\right\rfloor + 2j - 1\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ and } k \text{ are both odd.} \end{cases} \tag{12}$$

Remark. When n and k are both even, Corollary 1 provides two valid formulae. For $M \sim \mathrm{GUE}_6$ the first formula gives $\mathrm{E}\left[\det M^4\right] = (1\cdot3\cdot3\cdot5\cdot5\cdot7)(3\cdot5\cdot5\cdot7\cdot7\cdot9)$, which interpolates between $n = 5$ and $n = 7$ entries of the $k = 4$ row of Table 1, while the second formula presents the factorization in the form $(1\cdot3\cdot3\cdot5)(3\cdot5\cdot5\cdot7)(5\cdot7\cdot7\cdot9)$, which interpolates between $k = 3$ and $k = 5$ entries of the $n = 6$ column of the same table.

TABLE 1. For $M \sim \mathrm{GUE}_n$, we write $\mathrm{E}\left[\det(M^k)\right]$ as a product of odd integers.

	n=1	n=2	n=3	n=4	n=5	n=6	n=7
k=1	0	−1	0	1·3	0	−1·3·5	0
k=2	1	1·3	1·3·3	1·3·3·5	1·3·3·5·5	1·3·3·5·5·7	1·3·3·5·5·7·7
k=3	0	−1·3·3	0	1·3·3·3·5·5	0	−1·3·3·3·5·5·5·7·7	0
k=4	1·3	1·3·3·5	1·3·3·3·5·5	1·3·3·5·3·5·5·7	1·3·3·5·5·3·5·5·7·7	1·3·3·5·5·7·3·5·5·7·7·9	1·3·3·5·5·7·7·3·5·5·7·7·9·9
k=5	0	−1·3·3·5·5	0	1·3·3·5·5·3·5·5·7·7	0	−1·3·3·5·5·3·5·5·7·7·5·7·7·9·9	0

Remark. A consequence of Corollary 1 is that the expected value of the $2k$–th power of the determinant of a matrix sampled from the $n \times n$ GUE is a product of odd integers, none of which exceeds $2\left\lfloor \frac{n}{2} \right\rfloor + 2k - 1$. These factorizations are given explicitly in Table 1. This can be considered a signature of the fact that the singular values of the GUE have a bi-diagonal model with independent χ-distributed diagonal entries. By contrast, moments of the determinants of GOE and GSE matrices of even order can involve larger prime factors. For example, $\mathrm{E}_{\mathrm{GOE}_2}\left[\det(M^6)\right] = (3^2)(5^2)(167)$ and $\mathrm{E}_{\mathrm{GSE}_4}\left[\det(M^6)\right] = (3^6)(5^2)(7^2)(11)(347)$, with appropriate scaling.

In [5], Delannay and Le Caër used the Mellin transform of the distribution of the determinant of the GUE from [29] to establish the asymptotic log-normality of its absolute value, and presented analogous computations for the GOE. Tao and Vu gave new parallel derivations of the distributions of $\log|\det(A)|$ when A is distributed as GUE_n or GOE_n, and proved asymptotic normality in both cases ([35, Theorem 4]). They speculated that normality cannot be explained as a consequence of the existence of underlying independent random variables. In fact the logarithms of the diagonal entries of our bi-diagonal model provide such variables. This is exactly analogous to the case of Wishart matrices, analyzed by Goodman in [18], but here the χ-distributed factors occur only with odd degrees of freedom. Properties of logarithms of χ-distributed random variables give the GUE half of [35, Theorem 4] as a corollary.

Corollary 2 (Tao and Vu [35, part of Theorem 4])**.** *With $M_n \sim \mathrm{GUE}_n$ we have the central limit theorem,*

$$\frac{\log|\det M_n| - \frac{1}{2}\log n! + \frac{1}{4}\log n}{\sqrt{\frac{1}{2}\log n}} \xrightarrow{\mathrm{d}} N(0,1)$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution.

Proof sketch. From Theorem 2, if $M_n \sim \mathrm{GUE}_n$, then $\log|\det(M_n)| \sim \sum_{i=1}^{n} \log|X_i|$, is a sum of n independent random variables with $X_i \sim \chi_{2\lfloor \frac{i}{2} \rfloor + 1}$. Now the expected value of the logarithm of a χ_k-distributed random variable, X, is given by

$$\mu_k := \mathrm{E}\left[\log(X)\right] = \frac{2^{1-k/2}}{\Gamma(k/2)} \int_0^\infty \log(x) x^{k-1} \mathrm{e}^{-x^2/2}\, \mathrm{d}x = \frac{1}{2}\psi\left(\frac{k}{2}\right) + \frac{1}{2}\log 2$$

where $\psi(x) := \frac{\mathrm{d}}{\mathrm{d}x} \ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function, which satisfies the summation formula

$$\sum_{l=0}^{n-1} \psi\Big(l + \frac{1}{2}\Big) = \Big(n - \frac{1}{2}\Big)\psi\Big(n + \frac{1}{2}\Big) - \frac{\gamma}{2} - \log 2 - n.$$

Applying the summation to the even and odd terms of our expression for $\log|\det(M_n)|$ we conclude

$$\mathrm{E}\left[\log|\det M_n|\right] = \frac{n}{2}\psi\Big(\Big\lceil\frac{n}{2}\Big\rceil + \frac{1}{2}\Big) + \frac{n}{2}\log 2 - \Big\lceil\frac{n}{2}\Big\rceil = \frac{1}{2}\log\frac{n!}{\sqrt{2\pi n}} + O(n^{-1}),$$

and hence that

$$\lim_{n\to\infty} \mathrm{E}\left[\log|\det M_n|\right] - \frac{1}{2}\log n! + \frac{1}{4}\log n = -\frac{1}{4}\log 2 - \frac{1}{4}\log\pi = -0.459469\ldots.$$

Similarly, the variance of the logarithm of a χ_k-distributed random variable, X, is given by

$$\sigma_k^2 := \mathrm{E}\left[(\log X - \mu_k)^2\right] = \frac{1}{4}\psi_1\Big(\frac{k}{2}\Big),$$

where $\psi_1(x) := \frac{\mathrm{d}}{\mathrm{d}x}\psi(x)$ is the trigamma function, which satisfies the recurrence

$$\sum_{l=0}^{n-1} \psi_1\Big(l + \frac{1}{2}\Big) = \Big(n - \frac{1}{2}\Big)\psi_1\Big(n + \frac{1}{2}\Big) + \psi\Big(n + \frac{1}{2}\Big) + \gamma + \frac{\pi^2}{4} + 2\log 2.$$

Again summing over even and odd terms, we get

$$\begin{aligned}\mathrm{Var}\big[\log|\det M_n|\big] &= \frac{n}{4}\psi_1\Big(\Big\lceil\frac{n}{2}\Big\rceil + \frac{1}{2}\Big) + \frac{1}{2}\psi\Big(\Big\lceil\frac{n}{2}\Big\rceil + \frac{1}{2}\Big) + \frac{\gamma}{2} + \log 2\\ &= \frac{1}{2}\log n + \frac{1}{2}(\gamma + 1 + \log 2) + O(n^{-2}),\end{aligned}$$

and obtain the limit

$$\lim_{n\to\infty} \mathrm{Var}\big[\log|\det M_n|\big] - \frac{1}{2}\log n = \frac{1}{2}(\gamma + 1 + \log 2) = 1.1351814\ldots.$$

Asymptotic normality, and the stated central limit theorom, follow by checking the Lyapunov condition for fourth moments. Letting β_k denote the fourth central moment of a χ_k distributed random variable, X, we have

$$\beta_k = \mathrm{E}\left[(\log X - \mu_k)^4\right] = \frac{3}{16}\psi_1\Big(\frac{k}{2}\Big)^2 + \frac{1}{16}\psi_3\Big(\frac{k}{2}\Big).$$

Since the polygamma function $\psi_3(x) = \psi'''(x)$ satisfies the bound $\psi_3(x) \le \psi_1(x) \le 1$ when $x \ge 2$, we conclude that with $s_n^2 = \mathrm{Var}\big[\log|\det M_n|\big]$

$$\sum_{l=1}^{n} \beta_{2\lfloor\frac{l}{2}\rfloor+1} = O(s_n^2) \qquad \text{so} \qquad \lim_{n\to\infty}\frac{1}{s_n^4}\sum_{l=1}^{n}\beta_{2\lfloor\frac{l}{2}+1\rfloor} = 0. \qquad \square$$

Andrews *et al.* also provide relatively compact expressions for the determinant of the GOE for even order matrices as [1, Eq (23)] and odd order matrices as [1, Eq (24)], with less compact forms derived by Delannay and Le Caër in [5] and summarized by Mehta as [28, Eq. (26.5.11), Eq. (26.6.15), and Eq. (26.6.16)]. After observing numerically that the moments for odd order matrices have only small prime factors, we identified the following theorem.

Theorem 3. *For $M \sim \mathrm{GOE}_{2n+1}$, the determinant of M has the same moments as the product of independent random variables $\sqrt{2}\,X\prod_{i=1}^{n} Y_i$ where $X \sim N(0,1)$ and $Y_i \sim \chi^2_{2i+1}$.*

Proof. The distributions of the determinant and our purported product are both symmetric about zero, so it is sufficient to show that even moments agree. Beginning with [1, Eq. (24)],

$$\mathrm{E}_{\mathrm{GOE}_{2n+1}}\left[\det M^{2u}\right] = 2^u(2u-1)!!\prod_{j=1}^{2u}\frac{(2n+2j-1)!!}{(2j-1)!!},$$

we recognize $(2u-1)!!$ as the expected value of X^{2u} for $X \sim N(0,1)$. By rewriting the product as

$$\prod_{j=1}^{2u}\frac{(2n+2j-1)!!}{(2j-1)!!} = \prod_{i=1}^{n}\left(\prod_{k=0}^{2u-1}(2i+1+2k)\right)$$

we obtain a division-free expression for the moments of the determinant, and observe that the factors are the expected values of Y_i^{2u} for $Y_i \sim \chi^2_{2i+1}$. □

Remark. In [3], Bornemann and La Croix give a direct interpretation for the factors in Theorem 3. A corresponding description of the distribution of the modulus of the determinant for even order matrices then allowed them to derive the central limit theorem for the GOE analog to Corollary 2.

5.2. **Extreme Singular Values and the Condition Number of the** GUE**.** We use the condition number of a matrix to motivate the study of the distributions of the largest and smallest singular values of the GUE. From Theorem 1, these are related to singular values of Laguerre ensembles with parameters $\pm 1/2$. This perspective unifies some asymptotic results and also suggests the need for additional special functions for describing the singular value of Laguerre ensembles with non-integer parameters. Of particular note, the smallest singular value of the GUE, which is associated with the bulk scaling limit, is described in terms of the smallest singular values of Laguerre ensembles, which are associated with hard-edge limits, giving the GUE a sort of virtual hard edge.

The *condition number* of a matrix predicts stability in numerical linear algebra, and is given by the ratio of its largest and smallest singular values. In practice, fluctuations of the largest singular value are small, and the distribution of the condition number can be approximated by considering only the smallest singular value (see the first author's analysis of the corresponding problem for Laguerre ensembles in [10]). For the purpose of analyzing the condition number, the signs of eigenvalues introduce noise which we can ignore by partitioning the singular values of the GUE according to Theorem 1. We can then analyze both the smallest and largest singular values as functions of independent quantities. This makes the product structures of the eigenvalue counting functions in the limits into extensions of corresponding finite factorizations.

The earliest results we have identified along these lines involve gap probabilities in the bulk scaling limit of the GUE. The probability that the smallest singular value is at least s is also the probability that there are no eigenvalues between $-s$ and s. In the bulk-scaling limit, this is known as the *gap probability*, and, by translation invariance, becomes independent of the particular interval chosen. A more general problem is to describe the eigenvalue counting function, $E_2(k; s)$, giving the probability that a random interval of length $2s$ contains precisely k eigenvalues. The corresponding problem for the GOE was given a Fredholm determinantal representation by Gaudin in [16]. This was adapted to the CUE by Dyson in [9], and extended by Mehta and des Cloizeaux in [27] to encompass $k \neq 0$. Bornemann provides an excellent summary of the computational implications of this and related results in [2] where the following appears as his equation (5.7)

$$E_2(k; s) = \sum_{j=0}^{k} E_+(j; s)E_-(k-j; s). \tag{13}$$

Note that E_+ and E_- are not themselves eigenvalue counting functions, but defined instead in terms of the decomposition of the sine kernel by its orthogonal actions on even and odd functions. In fact, the right side of (13) can also be interpreted as a limit of counting functions for singular values. When we showed this to Bornemann, he observed that the finite version can be obtained using kernel methods parallel to the derivation of the limiting case, and that this is essentially the content of Forrester's observations about the GUE and related ensembles in [13].

Most of the required observations are already present in [28, Ch. 6], where Mehta notes the characteristic checkerboard sparsity pattern as the reason that the determinantal representation of $E_2(0; s)$ factors. This factorization is not a consequence of the relationship between the bulk-scaling limits of the GUE and CUE (such a factorization is somewhat less surprising for the CUE, because of properties of compact Lie groups, as explored by Rains [31, 32]). In fact a similar factorization occurs for GUE of finite order, and we recognize the factors as corresponding to the complementary cdfs of the smallest singular values of the two component anti-GUEs implicit in Theorem 1.

An initial analysis of extreme singular values is harmed by the inclination to partition eigenvalues according to sign. The largest singular value is then a function of the largest and smallest eigenvalues, while the smallest singular values is determined by the least positive and greatest negative eigenvalues. For asymptotically large GUE matrices, the largest and smallest eigenvalues are at soft-edges, and their distributions are described by the Tracy-Widom law. In the large n limit, these two soft-edges become independent, and the cumulative distribution function for the largest singular value factors as the product of cumulative distribution functions for the two edges. Our intuition suggests that for moderately-sized matrices, the largest and smallest eigenvalues are essentially independent, so we expect a near factorization for large but finite n. To interpret (13) as a limit we need to change perspectives. Instead of two identical distributions (those of the largest and smallest eigenvalues) becoming independent, the finite version involves two independent distributions becoming identical (the soft edges of $\mathrm{LUE}^{(+1/2)}$ and $\mathrm{LUE}^{(-1/2)}$).

By contrast, the smallest singular value is either the least positive eigenvalue, or the greatest negative eigenvalue, and these two quantities do not become independent for large matrices. This makes the factorization of the complementary cdf of the smallest singular value much more surprising.

The symmetry between smallest and largest singular values is most easily phrased in terms of singular value counting functions. We denote these by S, and define them as analogs to the E discussed previously. For a subset $J \subseteq \mathbb{R}_+$, we let $S_2^{(n)}(k, J)$ denote the probability that exactly k singular values of GUE_n lie in the set J. Similarly, we let $S^{(n)}_{\mathrm{LUE}_\pm}(k; J)$ denote the corresponding probabilities that exactly k singular values of the bi-diagonal matrix modeling the $\mathrm{LUE}_n^{(\pm 1/2)}$, recall (3), lie in J. Note that S_{LUE_-} and S_{LUE_+} can also be considered as the even and odd elements of a one-parameter family, since we could have defined them equivalently by relating $S^{(n)}_{\mathrm{LUE}_-}$ to the singular values of the order $2n$ anti-GUE and $S^{(n)}_{\mathrm{LUE}_+}$ to the order $2n+1$ anti-GUE. With this notation, the following corollary to Theorem 1, equivalent forms of which were considered by Forrester in [13], is immediate.

Corollary 3. *For any measurable subset $J \subseteq \mathbb{R}_+$, the singular value counting functions of* GUE_n *can be expressed in terms of* LUE *counting functions* via

$$S_2^{(n)}(k; J) = \sum_{j=0}^{k} S^{(n_1)}_{\mathrm{LUE}_+}(j; J) S^{(n_2)}_{\mathrm{LUE}_-}(k-j; J), \tag{14}$$

where $n_1 = \lfloor \frac{n}{2} \rfloor$ *and* $n_2 = \lceil \frac{n}{2} \rceil$.

Remark. The similarity between (13) and (14) is intentional, but also slightly overstated. On passing to limits, the rôles of the subscripts '+' and '−' are reversed. The Laguerre $+1/2$ ensembles are associated with the action of the sine kernel on odd functions, which correspond to the E_- factors in (13). Similarly, on passing to the limit, the S_- factors in (14) correspond to E_+ factors in (13).

Forrester summarizes these results into a generating series identity encompassing all k simultaneously. We take the opposite emphasis, and believe that via their relationship to condition numbers, the two particular cases $k = 0$ and $k = n$ are of special interest. In these cases, the sum consists of a single non-zero term, so the result takes the form of a factorization.

Example 3. Suppose that X, Y, and Z are the minimum singular value of matrices sampled from GUE_7, and nominal 3×3.5 and 4×3.5 matrices (corresponding to $\mathrm{LUE}_3^{(+1/2)}$ and $\mathrm{LUE}_4^{-1/2}$ respectively). The cumulative distribution function for X, Y, and Z are plotted in Figure 12 (top). By specializing (14) to $n = 7$ and $k = 1$, we find that

$$1 - \mathrm{P}(X \le x) = \big(1 - \mathrm{P}(Y \le x)\big)\big(1 - \mathrm{P}(Z \le x)\big).$$

To observe this graphically, we take logarithms, and note that $\log\big(1 - \mathrm{P}(X \le x)\big)$ is the average of $2\log\big(1 - \mathrm{P}(Y \le x)\big)$ and $2\log\big(1 - \mathrm{P}(Z \le x)\big)$, a relationship plotted in Figure 12 (bottom).

One of the initial aims of our investigation was to describe the manner in which the distribution of the smallest singular value of the GUE_n depends on n. We had hoped that by specializing (14) to $k = 0$ we could bootstrap from corresponding descriptions for Laguerre ensembles. For integral values of a, the distribution of the smallest singular value of $\mathrm{LUE}_n^{(a)}$ is described by a confluent hypergeometric function of matrix argument [6, 24]. The series expansions of these functions are, however, non-convergent when a is not an integer, and we are unsure what the appropriate analog is in this situation. Similarly when a is integral, Forrester and Hughes showed in [12] that relevant probabilities could be computed as $a \times a$ determinants of matrices with generalized Laguerre polynomials as entries. This allows a natural generalization to non-integer n, but only when a is an integer. Similarly the recurrence they identified from the double Wronskian structure of the determinants in not closed in this case.

The distributions can still be described for all a as determinants involving special functions. In particular, a direct computation gives the probability that there are no singular values less than s in terms of determinants of Hankel matrices of upper incomplete Γ-functions. Letting

$$F(a, n, s) := \det\big(\Gamma(i + j - 1 + a, s)\big)_{1 \le i,j \le n}$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} \mathrm{e}^{-t}\, \mathrm{d}t$ is the upper incomplete gamma function, we can integrate the Vandermonde term by term to see that the complementary cdf for the smallest singular value of the GUE is

$$\mathrm{P}_{\mathrm{GUE}_n}(\sigma_{\min} \ge s) = \frac{F\big(-\frac{1}{2}, \lceil \frac{n}{2} \rceil, \frac{s^2}{2}\big)}{F\big(-\frac{1}{2}, \lceil \frac{n}{2} \rceil, 0\big)} \times \frac{F\big(+\frac{1}{2}, \lfloor \frac{n}{2} \rfloor, \frac{s^2}{2}\big)}{F\big(+\frac{1}{2}, \lfloor \frac{n}{2} \rfloor, 0\big)}.$$

For small values n these can be evaluated directly, but the matrices become ill-conditioned as n grows. These expressions can be generalized to any $a > 1$, but only when n is an integer.

It should be noted that the computation of $S_2^{(n)}\big(k; (0, s)\big)$ is well-suited to the numerical Fredholm techniques described by Bornemann in [2]. In this setting, it is unclear under what conditions, if any, it is preferable to work with the Laguerre factors instead of the GUE expression directly.

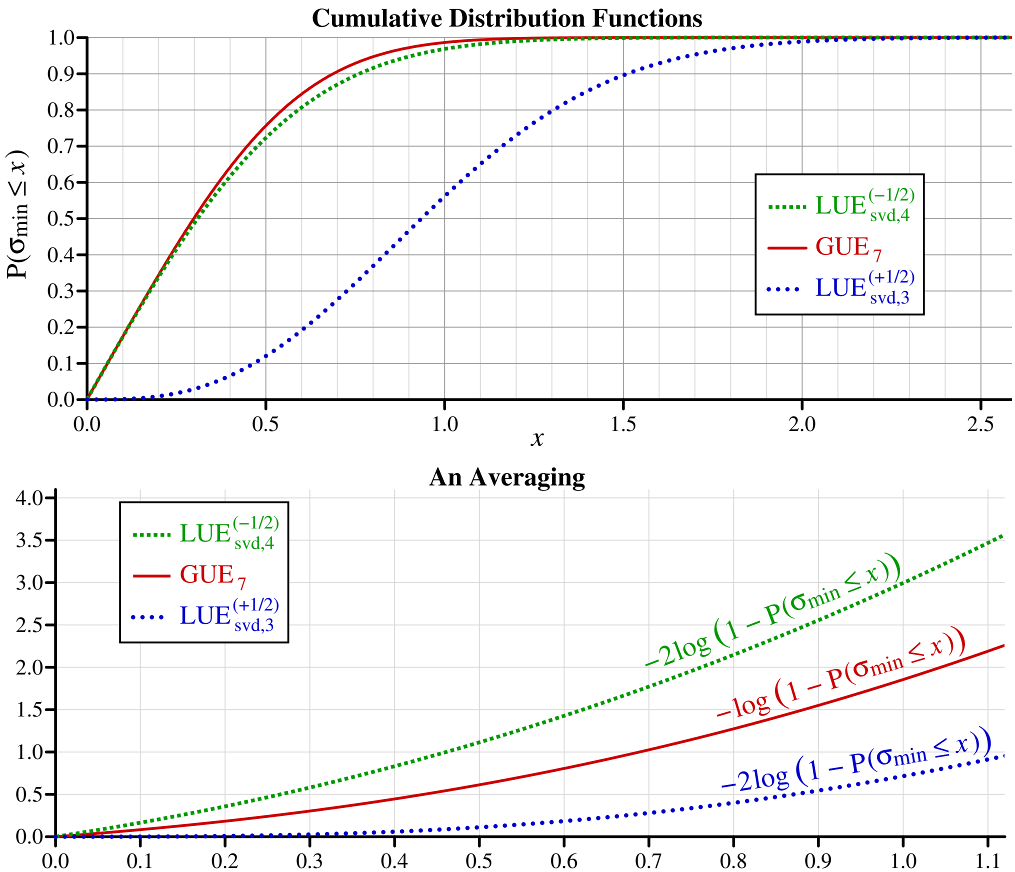

FIGURE 12. The cdfs for $\sigma_{\min}$ of GUE_7, $\mathrm{LUE}^{(-1/2)}_{\mathrm{svd},4}$, and $\mathrm{LUE}^{(+1/2)}_{\mathrm{svd},3}$ (top) are related via the logarithms of their complements (bottom).

6. Relationship to Complex Ginibre Ensembles

We close by noting that Theorem 1 also has parallels involving complex Ginibre ensembles (see [28, Ch. 15] for a discussion of these ensembles). A slight modification of the proof of Theorem 1 shows that the magnitudes of the eigenvalues of Ginibre matrices are independent χ-distributed random variables, each having a different even number of degrees of freedom. Furthermore, the phases of the eigenvalues of sufficiently high powers of such matrices are also independent. We speculate that the Ginibre ensembles could be a bridge to connect our observations to results of Rains involving powers of compact Lie groups, and to the unitary groups in particular (see [31, 32]), the connection with which was pointed out to us in discussion with Paul Bourgade. We outline some of the relevant properties of Ginibre ensembles, and sketch some of the reasons that we think they may lie at the center of the theory.

To give a more concrete motivation for considering Ginibre matrices, we note that instead decomposing the singular values of the GUE, we could equally well have decomposed the eigenvalues of the square of the GUE. In fact, it is this formulation that most closely matches the combinatorial identities of Jackson and Visentin [19, 20, 23]. The maps involved are enumerated by even moments of the GUE, but the relevant maps can also be embedded injectively into the class of bipartite maps, and when enumerated as such are naturally related to moments of the complex Ginibre ensemble. In fact this may be the more natural setting for analyzing map combinatorics, since the combinatorially related triangulation conjecture from [22, 21] can formulated in terms of Ginibre matrices, but does not appear to have a formulation in terms of the GUE.

The main result of this section, Theorem 4, was previously described by Kostlan in [25]. We rederive it here in a manner intended to emphasize the similarity with Theorem 1.

If no symmetry is imposed on a square matrix with independent complex Gaussian entries of unit variance (i.e. real and imaginary parts independently each have variance $\frac{1}{2}$), then the resulting eigenvalues are generically complex valued. The joint eigenvalue density for such an ensemble was derived by Ginibre, after whom such matrices are named, in [17]. It is supported on $\mathbb{C}^n$, and is given by

$$p_n^{\mathrm{G}}(z_1, z_2, \ldots, z_n) = c_n^{\mathrm{G}} \prod_{1\le i<j\le n} |z_i - z_j|^2 \, \exp\Big(-\frac{1}{2}\sum_{i=1}^n |z_i|^2\Big) \prod_{i=1}^n \mathrm{d}z_i, \tag{15}$$

which differs from the joint density for the GUE only in its support. As with Theorem 1, we can show that the magnitudes of the eigenvalues of the Ginibre ensemble are independent, or with a slight modification that the eigenvalues of large powers of Ginibre matrices are independent.

As with the GUE it is possible to evaluate expectations of symmetric functions of the magnitude of eigenvalues by de-symmetrizing relevant integrals. In this case, proceeding as with the proof of Theorem 1, and using the fact that $|z|^2 = z\bar{z}$, integrals can be taken against

$$n!\, c_n^{\mathrm{G}} \det \begin{pmatrix} 1 & z_2^* & z_3^{*2} & \cdots & z_n^{*\,n-1} \\ z_1 & z_2 z_2^* & z_3 z_3^{*2} & \cdots & z_n z_n^{*\,n-1} \\ z_1^2 & z_2^2 z_2^* & z_3^2 z_3^{*2} & \cdots & z_n^2 z_n^{*\,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} z_2^* & z_3^{n-1} z_3^{*2} & \cdots & z_n^{n-1} z_n^{*\,n-1} \end{pmatrix} \exp\Big(-\frac{1}{2}\sum_{i=1}^n z_i z_i^*\Big) \prod_{i=1}^n \mathrm{d}z_i.$$

Switching to polar co-ordinates with the substitutions $z_j = r_j \mathrm{e}^{\mathrm{i}\theta_j}$ and $\mathrm{d}z_j = r\,\mathrm{d}r_j \mathrm{d}\theta_j$, each phase variable occurs only in a single column of the matrix. Integrating over the phases annihilates all non-diagonal matrix entries and shows that we could have integrated against the density

$$\prod_{i=1}^n r_i^{2i-1} \mathrm{e}^{-\frac{1}{2}r_i^2} \, \mathrm{d}r_i.$$

By recognizing the factors of this density, we recover a result previously described by Kostlan.

Theorem 4 (Kostlan [25])**.** *The magnitudes of the eigenvalues of the $n \times n$ complex Ginibre ensemble have the same distribution as a set of independent χ random variables with $2, 4, \ldots, 2n$ degrees of freedom.*

From this information, and the fact that the ensemble is invariant under multiplication by a complex phase, we immediately obtain the known distribution of the determinant of the Ginibre ensemble, and the distribution of a randomly chosen eigenvalue. In particular, the pdf for the

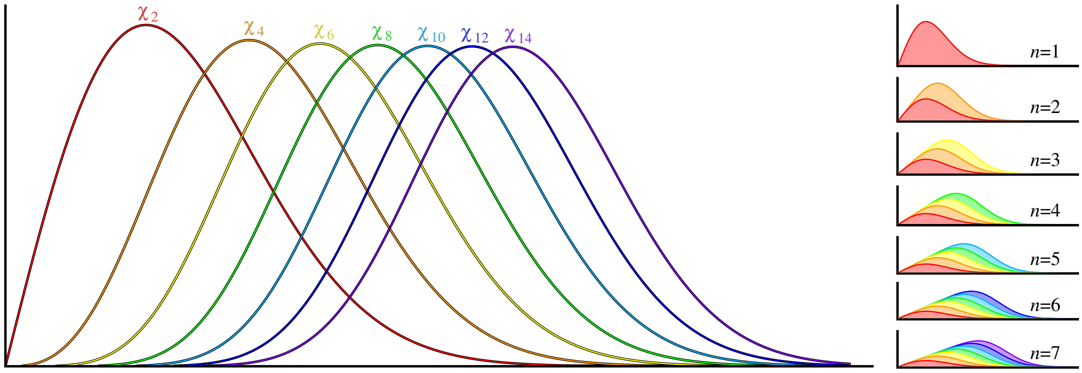

FIGURE 13. The pdfs of the magnitude of a randomly chosen eigenvalue from the $n \times n$ complex Ginibre ensemble (right) is the average of the pdfs of χ-distributed random variables with consecutive even numbers of degrees of freedom (left).

magnitude of a randomly chosen eigenvalue is given by the average of the densities of χ-distributed random variables (see Figure 13), and takes the form

$$\frac{x}{n}\frac{\Gamma(n, \frac{x^2}{2})}{\Gamma(n)}.$$

Paralleling Rains' results about the eigenvalues of compact Lie groups ([31, 32]), a slight extension shows that for an $n \times n$ Ginibre matrix, the phases of the k–th power of the eigenvalues are independent and uniformly distributed for every $k \geq n$. Together these give a weak form of the circular law.

7. Related Questions

- Is there a practical way to identify when a set-valued random variable can be generated as a union of independent sets? In particular, given a collection of samples of a set-valued random variable can we determine if the same sample distribution can be generated as a union of two smaller collections.
- Are there hidden independencies for the GOE or GβE for any $\beta \neq 2$?
- Quantitatively, the absolute value of the determinant of the $n \times n$ GUE can be sampled naïvely as a function of n^2 independent Gaussian random variables. Similarly, a χ_k random variable can be sampled as a function of k independent Gaussian random variables, so using Theorem 2 we can sample the absolute value of the determinant as a function of $n^2 - \left\lfloor \frac{(n-1)^2}{2} \right\rfloor$ independent Gaussian random variables. This sequence appears in the Online Encyclopedia of Integer Sequences as A074148, where a comment suggests that it also arises in the context of Cartan decompositions. To what extent can our main theorem be viewed as a relationship between the Unitary Group of order n and the Orthogonal Groups of order n and $n+1$? To make this question concrete, we have exhibited two ensembles of random matrices defined in terms of n^2 Gaussian random variables. One, the GUE$_n$, is invariant under conjugation by $U(n)$, while the other, anti-GUE$_n$$\otimes$anti-GUE$_{n+1}$ is invariant under conjugation by $O(n) \times O(n+1)$.

- The decomposition of Theorem 1 raises the question, to what extent can one ensemble be transformed into the other. It would appear that this cannot be accomplished deterministically. There is naturally a $2^n : \binom{n}{\lfloor \frac{n}{2} \rfloor}$ ambiguity. Any element-wise action would also carry with it an implicit mapping between $O(n) \otimes O(n+1)$ and $U(n)$.

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