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A NOTE ON THE GENERICITY OF SIMULTANEOUS STABILIZABILITY AND POLE ASSIGNABILITY

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1. INTRODUCTION

In this note, we examine the question of the genericity of simultaneous stabilizability, strong simultaneous stabilizability, and simultaneous pole assignability. The principal contribution of this note is to present simple proofs of some previously known results. In addition, we prove one new result and present some lemmas on generic greatest common divisors that may be of independent interest.

As is customary, let $\mathbf{R}(s)$ denote the field of rational functions with real coefficients; let $\mathbf{R}[s]$ denote the ring of polynomials with real coefficients; and let $\mathbf{S}$ denote the ring of proper stable rational functions with real coefficients. It is known that $\mathbf{R}(s)$ is the field of fractions associated with both $\mathbf{R}[s]$ and $\mathbf{S}$. Let $\mathbf{M}(\mathbf{R}(s))$ denote the set of matrices (of whatever order) with elements in $\mathbf{R}(s)$; $\mathbf{M}(\mathbf{R}[s])$ and $\mathbf{M}(\mathbf{S})$ are similarly defined.

Suppose we are given plants $P_1, \cdots, P_r \in \mathbf{M}(\mathbf{R}(s))$, all having the same dimension. We say that these plants are *simultaneously stabilizable* if there exists a controller $C \in \mathbf{M}(\mathbf{R}(s))$ that stabilizes each plant p_i. (The notion of stabilization used here is that from [1,2].) The plants are *strongly simultaneously stabilizable* if there exists a $C \in \mathbf{M}(\mathbf{S})$ that stabilizes each p_i. The notion of these properties being generic was first broached in [3,4]. In [3] it was shown that a single plant P of dimension $l \times m$ is generically strongly stabilizable if $\max\{l,m\}>1$. In [4] it was shown that two plants P_1, P_2, each having dimension $l \times m$, are generically simultaneously stabilizable if $\max\{l,m\}>1$. This result was extended in [5], where it was shown that a collection of plants $P_1, \cdots, P_r$, each having dimenison $l \times m$, is generically simultaneously stabilizable if $\max\{l,m\}\geq r$. In the present note, a simple proof is given of this last result, and it is also shown that generic *strong* simultaneous stabilizability holds if $\max\{l,m\}>r$; this is a new result.

The definition of simultaneous pole assignability is a bit messy since each of the plants may have a diffetrent dynamical order, but the term is essentially self-explanatory. A precise definition is given in Section 5. The only results concerning this property are in [5], where it is shown that generic pole assignability holds if $\max\{l,m\}\geq r$, and in addition, an estimate is given of the dynamic order of a controller that achieves it. In the present note, we give a simple proof of this result as well.

2. PRELIMINARIES

In this section, we define precisely the concept of genericity used here, and state without proof two results concerning the genericity of coprimeness and of Smith forms.

Suppose $\mathbf{X}$ is a topological space. Recall that R is a *binary relation* on $\mathbf{X}$ if it is a subset of $\mathbf{X}\times\mathbf{X}$; more generally, R is an *$n-$ ary* relation on $\mathbf{X}$ if it is a subset of $\mathbf{X}^n$.

Definition 1 An n-ary relation R on $\mathbf{X}$ is *generic* if it is an open dense subset of $\mathbf{X}^n$ where the latter is endowed with the product topology derived from that on $\mathbf{X}$

In other words, R is generic if it has two properties: (i) If an n-tuple $\mathbf{x}=(x_1,\cdots,x_n)$ satisfies the relation R, then there exists a neighborhood of $\mathbf{x}$ within which every element satisfies the relation. (ii) If $\mathbf{x}$ does *not* satisfy the relation, then every neighborhood of $\mathbf{x}$ contains an element that does.

Now we state two "well-known" result without proof; they can be proved as in [6, Section 7.6].

Lemma 1 Suppose $\mathbf{R}$ is a topological ring with two properties: (i) the singleton set $\{0\}$ is closed, and (ii) the set

$$R = \{(a,b): \text{there exist } x,\ y \text{ s.t. } ax+by = 1\} \tag{1}$$

is an open dense subset of R^2. If R is also a principal ideal domain, then for any integers m,n with $m<n$, the set of matrices in $R^{m\times n}$ that have a right inverse in $R^{n\times m}$ is an open dense subset of $R^{m\times n}$.

An equivalent way of stating the above lemma is as follows: Let R be as above, and define

$$R = \{A\in R^{m\times n}: A \sim [I_m \quad 0]\}, \tag{2}$$

where $\sim$ denotes equivalence. Then R is an open dense subset of $R^{m\times n}$.

To apply Lemma 1 to our specific problems,it is necessary first to topologize the various sets in question. If $f\in\mathbf{S}$, then let

$$\|f\|_{\mathbf{S}} = \sup_{\mathrm{Re}\, s\geq 0} |f(s)| = \sup_{\omega} |f(j\omega)|. \tag{3}$$

If $f\in\mathbf{R}[s]$ and $f(s) = \sum_{i=0}^{\delta} f_i s^i$, then define

$$\|f\|_{\mathbf{R}[s]} = \sum_{i=0}^{\delta} |f_i|. \tag{4}$$

If $F\in\mathbf{S}^{m\times n}$, define

$$\|F\|_{\mathbf{S}} = \sum_{i=1}^{m}\sum_{j=1}^{n} \|f_{ij}\|_{SS}. \tag{5}$$

If $F\in\mathbf{R}[s]^{m\times n}$, its norm is defined analogously. In this way, both $\mathbf{M}(\mathbf{S})$ and $\mathbf{M}(\mathbf{R}[s])$ become metric spaces. The topology on the set $\mathbf{R}(s)^{m\times n}$ is the so-called *graph topology* defined in [7]. To sketch the basic idea, suppose $P \in \mathbf{R}(s)^{m \times n}$, and let (N,D) be any right-coprime factorization (r.c.f.) of P over the ring $\mathbf{S}$; thus $N \in \mathbf{S}^{m \times n}$, $D \in \mathbf{S}^{n \times n}$. Then a *basic neighborhood* of P consists of all plants $P_1 = N_1 D_1^{-1}$, where $\|N_1 - N\|_\mathbf{S} + \|D_1 - D\|_\mathbf{S}$ is less than a given positive number ϵ. The graph topology is the topology induced by the above base. It can be shown [7] that the graph topology is metrizable, and is induced by the so-called graph metric. Basically, in the graph metric, P_1 is close to P if P_1 has an r.c.f. that is close to an r.c.f. of P.

It is easy to see that both the rings $\mathbf{S}$ and $\mathbf{R}[s]$, topologized as above, satisfy the conditions of Lemma 1. Thus we have the following result.

Lemma 2 On both $\mathbf{S}$ and $\mathbf{R}[s]$, the mn-ary relation defined in Lemma 1 is generic.

3. SOME RESULTS ON GENERIC GREATEST COMMON DIVISORS

In this section, we state and prove some results on generic greatest common divisors in a principal ideal domain, which may also be of some independent interest. For this reason, the results below are stated in greater generality than is needed for the present application.

Throughout this section, $\mathbf{R}$ denotes a principal ideal domain which is at the same time an algebra over an infinite field $\mathbf{K}$. Clearly both $\mathbf{S}$ and $\mathbf{R}[s]$ satisfy this condition, with $\mathbf{R}$ playing the role of $\mathbf{K}$.

Lemma 3 Suppose $a,b,c,d \in \mathbf{R}$, and that g.c.d.$\{a,b,c,d\} = 1$. If $ad - bc \neq 0$, then

$$\text{g.c.d.}\{a+bk, c+dk\} = 1 \tag{6}$$

for all but a finite number of values of $k \in \mathbf{K}$.

Proof Let $r = ad - bc$, and suppose that, for some $k \in \mathbf{K}$, (6) does not hold. If $p \in \mathbf{R}$ is a common divisor of $a+bk, c+dk$, then p also divides $(a+bk)d - (c+dk)b = r$. Thus, whatever be $k \in \mathbf{K}$, the only possible common divisors of $a+bk, c+dk$ are the divisors of r. Let $p_1, \cdots, p_l$ denote the distinct prime divisors of r. We claim that each p_i can be a common divisor of $a+bk$ and $c+dk$ for at most one value of k. This is enough to show that, for all except at most l values of k, none of the p_i divides both $a+bk$ and $c+dk$. This in turn establishes (6).

To prove the claim, suppose $k_1 \neq k_2$, and that

$$p_i \mid (a+bk_1),\ p_i \mid (a+bk_2),\ p_i \mid (c+dk_1),\ p_i \mid (c+dk_2). \tag{7}$$

Then subtracting the first two relations implies that $p_i \mid (k_1 - k_2)b$, i.e. $p_i \mid b$ since $k_1 - k_2$ is a field element. This in turn shows that $p_i \mid a$. Similarly the last two relations imply successively that $p_i \mid d$ and $p_i \mid c$. Since p_i divides each of a, b, c, d, it also divides their g.c.d., which is 1. But this is absurd, since p_i is a prime element. Hence each p_i can divide both $a+bk$ and $c+dk$ for at most one value of k.

Lemma 4 Suppose $a, b, c, d \in \mathbf{R}$, and g.c.d.$\{a, b, c, d\} = q$. Then

$$\text{g.c.d.}\{a+bk, c+dk\} = q \tag{8}$$

for all but a finite number of values of $k \in \mathbf{K}$.

Proof Apply Lemma 3 to the collection $\{a/q, b/q, c/q, d/q\}$, whose g.c.d. is 1.

Lemma 5 Suppose $a_i, b_i \in \mathbf{R}$ for $i=1, \cdots, l$, and define

$$q = \underset{i}{\text{g.c.d.}}\{a_i, b_i\}. \tag{9}$$

Suppose the matrix

$$\begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_l & b_l \end{bmatrix} \tag{10}$$

has rank 2. Then

$$\underset{i}{\text{g.c.d.}}\{a_i + b_i k\} = q \tag{11}$$

for all but a finite number of values of $k \in \mathbf{K}$.

Proof Note that it can be assumed without loss of generality that $b_i \neq 0$ for all i. To see this, suppose by renumbering if necessary that $b_i = 0$ for $i=1, \cdots, m$ and that $b_i \neq 0$ for $m < i \leq l$. Then $a_i + b_i k = a_i$ for all $k \in \mathbf{K}$ whenever $1 \leq i \leq m$. As a result,

$$\underset{1 \leq i \leq l}{\text{g.c.d.}}\{a_i, b_i\} = \text{g.c.d.}\{a_1, \cdots, a_m, \underset{m < i \leq l}{\text{g.c.d.}}\{a_i, b_i\}\}, \tag{12}$$

and

$$\underset{1 \leq i \leq l}{\text{g.c.d.}}\{a_i + b_i k\} = \text{g.c.d.}\{a_1, \cdots, a_m, \underset{m < i \leq l}{\text{g.c.d.}}\ a_i + b_i k\}\}. \tag{13}$$

Hence, to prove the lemma, it is enough to show that

$$\underset{m < i \leq l}{\text{g.c.d.}}\{a_i + b_i k\} = \underset{m < i \leq l}{\text{g.c.d.}}\{a_i, b_i\} \tag{14}$$

for all but a finite number of values of $k \in \mathbf{K}$.

The proof is by induction on the integer l. The result is true for $l=2$, by Lemma 4. Suppose it is true up to $l-1$, and suppose by renumbering if necessary that the matrix

$$\begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_{m-1} & b_{m-1} \end{bmatrix} \tag{15}$$

has rank 2. Define

$$\alpha = \underset{1 \le i \le l-1}{\text{g.c.d.}} \{a_i, b_i\}. \tag{16}$$

Then, by the inductive hypothesis,

$$\underset{1 \le i \le l-1}{\text{g.c.d.}} \{a_i + b_i k\} = a_l \tag{17}$$

for all except a finite number of $k \in \mathbf{K}$ Also, since $\alpha, b_l \neq 0$, it follows from Lemma 4 that, for all but a finite number of values of k,

$$\begin{aligned} \text{g.c.d.}\{\alpha, a_l + b_l k\} &= \text{g.c.d.}\{\alpha + 0 \cdot k, a_l + b_l k\} \\ &= \text{g.c.d.}\{\alpha, a_l, b_l\} \\ &= \underset{1 \le i \le l}{\text{g.c.d.}}\{a_i, b_i\}, \end{aligned} \tag{18}$$

where the last equality follows from (16). Now, by combining (17) and (18) proves the inductive hypothesis for l.

Lemma 6 Suppose $a_i, b_{ij} \in \mathbf{R}$ for $1 \le i \le l, 1 \le j \le m$, and define

$$q = \underset{i,j}{\text{g.c.d.}}\{a_i, b_{ij}\}. \tag{19}$$

Suppose that, for some j, the matrix

$$\begin{bmatrix} a_1 & b_{1j} \\ \vdots & \vdots \\ a_l & b_{lj} \end{bmatrix} \tag{20}$$

has rank 2. Then

$$\underset{i}{\text{g.c.d.}}\{a_i + \sum_{j=1}^{m} b_{ij} k_j\} = q \tag{21}$$

for all $\mathbf{k} = (k_1, \cdots, k_m) \in \mathbf{K}^m - V$, where V is a finite union of linear varieties.

Proof It can be assumed that, for each j, we have $b_{ij} \neq 0$ for some i. Otherwise, the corresponding k_j does not appear anywhere and can be ignored.

The proof is by induction on the integer m. The result is true when $m=1$, by Lemma 5. Suppose it is true for $m-1$ "constants" k_j. To prove the result for m constants, assume by renumbering if necessary that the matrix

$$\begin{bmatrix} a_1 & b_{1m} \\ \vdots & \vdots \\ a_l & b_{lm} \end{bmatrix} \tag{22}$$

has rank 2, and select integers i_1, i_2 such that $a_{i_1} b_{i_2 m} - a_{i_2} b_{i_1 m} \neq 0$. Then

$$\begin{vmatrix} a_{i_1} + \sum_{j=1}^{m-1} b_{i_1 j} k_j & b_{i_1 m} \\ a_{i_2} + \sum_{j=1}^{m-1} b_{i_2 j} k_j & b_{i_2 m} \end{vmatrix} \tag{23}$$

equals $a_{i_1} b_{i_2 m} - a_{i_2} b_{i_1 m}$ plus a linear combination of $k_1, \cdots, k_{m-1}$. Let V_{m-1} denote the set of $(k_1, \cdots, k_{m-1})$ where this quantity equals zero. Then V_{m-1} is a linear variety in $\mathbf{K}^{m-1}$, and it is not all of $\mathbf{K}^{m-1}$ since the origin in $\mathbf{K}^{m-1}$ does not lie in V_{m-1}. Hence V_{m-1} is a linear variety in $\mathbf{K}^{m-1}$ of dimension at most $m-2$. Next, by Lemma 5, we have, whenever $(k_1, \cdots, k_{m-1}) \notin V_{m-1}$, that

$$\begin{aligned} &\underset{1 \le i \le l}{\text{g.c.d.}} \{[a_i + \sum_{j=1}^{m-1} b_{ij} k_j] + b_{im} k_m\} \\ &\qquad = \underset{1 \le i \le l}{\text{g.c.d.}} \{a_i + \sum_{j=1}^{m-1} b_{ij} k_j, b_{im}\} \end{aligned} \tag{24}$$

for all except a finite number of values of k_m. Of course, the number and values of these exceptional k_m can depend on $(k_1, \cdots, k_{m-1})$. Now, what is the value of the g.c.d. on the right side of (24)? By the inductive hypothesis, for almost all

$(\bar{k}_1, \cdots, k_m)$, this g.c.d. equals

$$\underset{1\le i\le l}{\text{g.c.d.}}\{a_1, b_{i1}, \cdots, b_{im}\} \tag{25}$$

provided that, for some j, the matrix

$$\begin{bmatrix} a_1 & b_{1j} \\ \vdots & \vdots \\ a_l & b_{lj} \\ b_{1m} & 0 \\ \vdots & \vdots \\ b_{lm} & 0 \end{bmatrix} \tag{26}$$

has rank 2. But this rank condition is easily verified. Select an arbitrary j, say $j=1$. Then $b_{i1}\neq 0$ for some integer i, and $b_{nm}\neq 0$ for some integer n (recall the first paragraph of the proof). Then

$$\begin{vmatrix} a_i & b_{i1} \\ b_{nm} & 0 \end{vmatrix} = -\, b_{i1}b_{nm}\neq 0. \tag{27}$$

Thus it has been shown that, for almost all $\mathbf{k} = (k_1, \cdots, k_m)$, (21) holds.

It only remains to show the exceptional set V is a finite union of linear varieties. This is most easily done as follows. Suppose that, for a particular choice $\mathbf{k}_0\in\mathbf{K}^m$, the condition (21) fails to hold. Then (21) also fails for all $\mathbf{k}\in\mathbf{K}^m$ such that

$$a_i+\sum_{j=1}^{m} b_{ij}k_j = a_i+\sum_{j=1}^{m} b_{ij}k_{j0}, \text{ for all } i, \tag{28}$$

which defines a linear variety in $\mathbf{K}^m$. Thus V is certainly a union of linear varieties. It is a finite union because, from (24), the projection of V along each coordinate axis consists of only a finite number of points.

Since both $\mathbf{S}$ and $\mathbf{R}[s]$ are algebras over the infinite field $\mathbf{R}$, Lemmas 3-6 apply to these rings. In addition, since there is also a topology on these rings, Lemma 6 can be strengthened.

Lemma 7 Suppose R is either $\mathbf{S}$ or $\mathbf{R}[s]$. Suppose $a_i, b_{ij} \in R$ for $1 \leq i \leq l, 1 \leq j \leq m$, and define q as in (19). Finally, suppose that for some j the matrix in (20) has rank 2. Under these conditions, the set S of elements $\mathbf{v} = (v_1, \cdots, v_m) \in R^m$ with the property that

$$\underset{i}{\text{g.c.d.}}\{a_i + \sum_{j=1}^{m} b_{ij} v_j\} = q \tag{29}$$

is an open dense subset of R^m.

Proof With q as in (19), we have that for almost all $\mathbf{v} \in R^m$,

$$\underset{i}{\text{g.c.d.}}\{a_i b_{i1} v_1, \cdots, b_{im} v_m\} = q. \tag{30}$$

Similarly, for almost all $\mathbf{v}$, the matrix

$$\begin{bmatrix} a_1 & b_{1j} v_1 \\ \vdots & \vdots \\ a_l & b_{lj} v_j \end{bmatrix} \tag{31}$$

has rank 2 if the matrix of (20) has rank 2.

In proving the lemma, the openness of the set S defined by (29) is obvious, and only the denseness requires some effort. Suppose (29) does not hold for some $\mathbf{v} \in R^m$. We will construct a sequence in S converging to $\mathbf{v}$. First, select a sequence $\{\mathbf{v}^{(n)}\}$ converging to $\mathbf{v}$ such that

$$\underset{i}{\text{g.c.d.}}\{a_i, b_{i1} v_1^{(n)}, \cdots, b_{im} v_m^{(n)}\} = q \text{ for all } n, \tag{32}$$

$$\text{rank} \begin{bmatrix} a_1 & b_{1j} v_1^{(n)} \\ \vdots & \vdots \\ a_l & b_{lj} v_j^{(n)} \end{bmatrix} = 2 \text{ for all } n. \tag{33}$$

Then, by Lemma 6, for each n there is a sequence of real m-tuples $\mathbf{k}^{(n,p)}$

converging to $(1, \cdots, 1)$ such that

$$\text{g.c.d.}\{a_i + \sum_{j=1}^{m} b_{ij} v_j^{(n)} k_j^{(n,p)}\} = q \text{ for all } p. \tag{34}$$

Finally, the sequence $\{k_1^{(n,n)} v_1^{(n)}, \cdots, k_m^{(n,n)} v_m^{(n)}\}$ lies in S for all n and converges to $\mathbf{v}$. Hence S is dense.

4. GENERICITY OF SIMULTANEOUS STABILIZABILITY

In this section, we show that, given a collection of plants $P_1, \cdots, P_r \in \mathbf{R}(s)^{l \times m}$, simultaneous stabilizability is generic if $r \leq \max\{l, m\}$, and strong simultaneous stabilizability is generic if $r < \max\{l, m\}$. In both cases, the set $\mathbf{R}(s)^{l \times m}$ is topologized via the graph topology of [7]. The first result was proved in [5], but the present proof is simpler; the second result is new.

Theorem 1 Suppose l, m are given positive integers, and equip the set $\mathbf{R}(s)^{l \times m}$ with the graph topology. Let r be a positive integer, and define the r-ary relation R on $\mathbf{R}(s)^{l \times m}$ by

$$R = \{(P_1, \cdots, P_r) \colon P_1, \cdots, P_r \text{ are simultaneously stabilizable}\}. \tag{35}$$

Then the relation R is generic if $r \leq \max\{l, m\}$.

The proof of the theorem is divided into two parts. First, it is shown that the result is true in the case where $\min\{l, m\} = 1$, i.e. the case where all plants are either single-input or single-output. Then it is shown that the general case can be reduced to this special case.

Lemma 8 Suppose $\min\{l, m\} = 1$. Then the relation R defined in (35) is generic if $r \leq \max\{l, m\}$.

Proof The proof is given for the case where $l = 1$ and $r \leq m$; the case $m = 1$ and $r \leq l$ can be treated by taking transposes, i.e. by noting that C stabilizes P if and only if C' stabilizes P'.

Given plants $P_1, \cdots, P_r$, each of dimension $1 \times m$, find left-coprime factorizations (l.c.f.'s) $(\tilde{d}_i, \tilde{N}_i)$ of P_i for $i=1, \cdots, r$. Define the matrix $Q \in \mathbf{S}^{r \times (m+1)}$ by

$$Q = \begin{bmatrix} \tilde{d}_1 & \tilde{N}_1 \\ \vdots & \vdots \\ \tilde{d}_r & \tilde{N}_r \end{bmatrix}. \tag{36}$$

Let $C \in \mathbf{R}(s)^{m \times 1}$ be a controller, and let $([b_1 \ldots b_m]', a)$ be an r.c.f. of C. Then by [1,2] the controller C stabilizes each of the plants P_i if and only if each of the return differences u_i defined by

$$\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = Q \begin{bmatrix} a \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{37}$$

is a unit of the ring $\mathbf{S}$. Turning this argument around, the set of plants $\{P_1, \cdots, P_r\}$ is simultaneously stabilizable if and only if there exist units u_i, $i=1, \cdots, r$ such that (37) has a solution in $\mathbf{M}(\mathbf{S})$ for the unknowns $a, b_1, \cdots, b_m$, where the first element (a) is nonzero. Now, if $r \leq m$, the matrix Q has more columns than rows. Hence, by Lemma 2, Q is generically right-invertible. That is, if Q is not right-invertible, it can be made so by an arbitrarily small perturbation in each of its elements, which is precisely the same as slightly perturbing each of the plants P_i in the graph topology. Thus by slightly perturbing each of the plants if necessary, we can ensure that (37) has a solution in $\mathbf{M}(\mathbf{S})$, whatever be the left side. In particular, one can choose any arbitrary set of units $\{u_1, \cdots, u_r\}$, and (37) has a solution in $\mathbf{M}(\mathbf{S})$. The only additional point to be worried about is that the first element of this solution (corresponding to a) should be nonzero so that C is well-defined. But this too can be guaranteed by perturbing the units slightly if necessary. The details of this argument are routine and are left to the reader.

The next step is to reduce the multivariable case to the case of single-output plants.

Lemma 9 Suppose l,m are given positive integers, and let $P \in \mathbf{R}(s)^{l \times m}$. Let (N,D) be an r.c.f. of P over $\mathbf{S}$. Under these conditions, the set of $\mathbf{v} \in \mathbf{R}^l$ such that $\mathbf{v}N,D$ are right-coprime is generically an open dense subset of $\mathbf{R}^l$.

Proof As always, the openness is obvious and only the denseness needs a proof. Let $n^{(1)}, \cdots, n^{(l)}$ denote the rows of the matrix N. By Lemma 2, by slightly perturbing the matrices D and N if necessary, we can ensure that the Smith form of the matrix

$$M_j = \begin{bmatrix} D \\ n^{(j)} \end{bmatrix} \tag{38}$$

is $[I_m \;\; 0]'$ for all j. In other words, this means the g.c.d. of $|D|$ and of all minors of the $m \times m$ minors of the matrix

$$M = \begin{bmatrix} D \\ N \end{bmatrix} \tag{39}$$

involving exactly one row of N is equal to 1. Now, if $\mathbf{v}$ is a $1 \times l$ vector, then D and $\mathbf{v}N$ are right-coprime if and only if the g.c.d. of all $m \times m$ minors of the matrix

$$Q = \begin{bmatrix} D \\ \mathbf{v}N \end{bmatrix} \tag{40}$$

is equal to 1. We show that this is generically the case for almost all $\mathbf{v} \in \mathbf{R}^l$. Now the matrix Q has dimensions $(m+1) \times m$, as do the matrices M_j defines in (39). Moreover, since the determinant function is multilinear, a minor of Q involving the last row can be expressed as a linear combination of the corresponding minors the various M_j. To be precise, let q_i denote the minor of Q obtained by omitting the i-th row of the matrix D, and define m_{ij} to be the minor of M_j obtained by omitting its i-th row. Then

$$q_i = \sum_{j=1}^{l} m_{ij} v_j . \tag{41}$$

Since the Smith form of M_i is $[I_m \quad 0]'$, it is true that

$$\underset{i,j}{\text{g.c.d.}} \{ |D|, m_{ij} \} = 1. \tag{42}$$

Hence, if the rank condition of Lemma 6 is satisfied, it follows that, for almost all vectors $\mathbf{v} \in \mathbf{R}^l$, we have

$$\underset{j}{\text{g.c.d.}} \{ |D|, q_j \} = 1. \tag{43}$$

To complete the proof it only remains to verify that the requisite rank condition is satisfied generically. Since $P = ND^{-1}$, it follows from Cramer's rule that the minor m_{ij} equals $\pm p_{ij} |D|$. Now the rank condition of Lemma 6 requires that, for some j, the matrix

$$\begin{bmatrix} |D| & 0 \\ \vdots & \vdots \\ |D| & 0 \\ 0 & p_{1j}|D| \\ \vdots & \vdots \\ 0 & p_{mj}|D| \end{bmatrix} \tag{44}$$

has rank 2. But this is true provided $p_{ij} \neq 0$ for some i,j, i.e. if $P \neq 0$.

The significance of Lemma 9 is in showing that, generically, a multivariable plant can be stabilized using a rank one controller. Suppose P is a given multivariable plant with an r.c.f. (N,D), and suppose one can find a vector $\mathbf{v}$ such that $\mathbf{v}N,D$ are right-coprime. Then one can find a pair of elements $(a,B) \in \mathbf{M}(\mathbf{S})$ such that $a\mathbf{v}N+BD = I$. Note that a is a column vector. The Bezout identity above implies that the controller $B^{-1}a\mathbf{v}$ stabilizes the plant P. Hence one can stabilize the plant P using a rank one controller if one can find a vector $\mathbf{v}$ such $\mathbf{v}N,D$ are right-coprime. Lemma 9 states that generically such a vector always exists, and that generically almost any vector $\mathbf{v}$ will do.

Note that the only property of the ring $\mathbf{S}$ used in the above lemma is that generically a rectangular matrix has a one-sided inverse. Thus Lemma 9 is valid over any ring satisfying the conditions of Lemma 1.

With the aid of Lemma 9, the proof of Theorem 1 can be completed.

Proof of Theorem 1 If $\min\{l,m\}=1$, then the truth of the theorem is established by Lemma 8. If not, suppose without loss of generality that $l \leq m, r \leq m$; the case $l \geq m, l \geq r$ can be handled by taking transposes. Let (N_i,D_i) be an r.c.f. of P_i, for $i = 1, \cdots, r$. Then, for each fixed i, the set of $\mathbf{v}$ such that $\mathbf{v}N_i,D_i$ are right-coprime is an open dense subset of $\mathbf{R}^l$. Moreover, since the intersection of a finite number of such sets is again open and dense, it follows that the set of $\mathbf{v}$ such that $\mathbf{v}N_i, D_i$ are right-coprime for *all* i is also an open dense subset of $\mathbf{R}^l$. Let Q_i denote the $1 \times m$ plant $\mathbf{v}P_i$. By Lemma 1, generically there is a common stabilizing controller C_1 for the collection $Q_1, \cdots, Q_r$. Now let $C = C_1\mathbf{v}$; then C stabilizes each of $P_1, \cdots, P_r$.

The advantages of the present proof over that in [5] are: (i) it is simpler, and (ii) it suggests a constructive procedure for finding a common stabiizing controller.

It is shown in [4] that the problem of simultaneously stabilizing r plants is equivalent to that of simultaneously stabilizing $r-1$ plants using a stable controller. Thus, in view of Theorem 1, it is natural to conjecture that r plants of dimension $l \times m$ are generically *strongly* simultaneously stabilizable if $r < \min\{l, m\}$. This is in fact true.

Theorem 2 Suppose l, m are given positive integers, and equip the set $\mathbf{R}(s)^{l \times m}$ with the graph topology. Define an r-ary relation on $\mathbf{R}(s)^{l \times m}$ by

$$R = \{(P_1, \cdots, P_r) \text{ that are strongly simultaneously stabilizable}\}. \tag{45}$$

Then R is generic if $r < \max\{l, m\}$.

Using Lemma 9, it is possible to restrict attention to the case where $\min\{l, m\} = 1$. Suppose $l = 1, m > r$; the other case can be handled by taking transposes.

As shown in [4], the plants $P_1, \cdots, P_r$ are stronlgly simultaneously stabilizable if and only if the $r+1$ plants $P_0 = 0, P_1, \cdots, P_r$ are simultaneously stabilizable. As before, let $(\tilde{d}_i, \tilde{N}_i)$ be any l.c.f. of P_i over **S**, and define

$$S = \begin{bmatrix} 1 & 0 \\ \tilde{d}_1 & \tilde{N}_1 \\ \vdots & \vdots \\ \tilde{d}_r & \tilde{N}_r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ & Q \end{bmatrix} \in \mathbf{S}^{(r+1)\times(m+1)}. \tag{46}$$

Now by Lemma 2, the matrix Q is generically right-invertible if $r<m$. It is a simple matter to verify that S is right-invertible if Q is. Finally, if S is right-invertible, then $0, P_1, \cdots, P_r$ are simultaneously stabilizable, whence $P_1, \cdots, P_r$ are strongly simultaneously stabilizable.

5. GENERIC POLE ASSIGNABILITY

In this section, we give a simple proof of a result from [5] concerning generic simultaneous pole assignability. In order to prove the main result, it is necessasry first to define the concept of characteristic polynomials. Suppose $P \in \mathbf{M}(\mathbf{R}(s))$; then the *characteristic polynomial* of P is the monic least common multiple of the denominators of the various minors of P. Alternatively, factor P as $ND^{-1} = \tilde{D}^{-1}\tilde{N}$, where N, D are right-coprime matrices in $\mathbf{M}(\mathbf{R}[s])$, and $\tilde{N}, \tilde{D}$ are left-coprime matrices in $\mathbf{M}(\mathbf{R}[s])$; then (within a nonzero constant) $|D|$ and $|\tilde{D}|$ are both characteristic polynomials of P. If P is proper, it is possible to give yet another equivalent definition. Let (A, B, C, E) be a minimal realization of P; then $|sI - A|$ is the characteristic polynomial of P.

Note that hereafter all factorizations are over the ring of polynomials $\mathbf{R}[s]$; this is in contrast with earlier sections where all factorizations are over the ring **S** of stable rational functions.

Consider now a feedback interconnection of a plant P and a controller C. If $|I+PC| \neq 0$, then the interconnection is well-posed, and the characteristic polynomial associated with the closed-loop transfer matrix is denoted by $\psi(P,C)$. Alternatively, let $(\tilde{D}_p, \tilde{N}_p)$ be a left-coprime factorization of P, and let (N_c, D_c) denote a right-coprime factorization of C. Then, within a nonzero constant, $\psi(P,C) = |\tilde{D}_p D_c + \tilde{N}_p N_c|$. Further, suppose P is strictly proper and that C is proper, and suppose without loss of generality that $|\tilde{D}_p|$, $|D_c|$ are both monic. Then $\psi(P,C)$ equals $|\tilde{D}_p D_c + \tilde{N}_p N_c|$.

Now we can define simultaneous pole assignability. Given a collection of strictly proper plants $P_1, \cdots, P_r$, let n_i denote the McMillan degree of P_i. Then the collection of plants is ***simultaneously pole assignable*** if there exists an integer q such that, given any set of monic polynomials $\phi_1, \cdots, \phi_r$ with degrees $\deg(\phi_i) = n_i + q$, there exists a controller C such that $\psi(P_i, C) = \phi_i$ for all i.

Theorem 3 Suppose the plants P_i has dimension $l \times m$ and McMillan degree n_i, for $i=1, \cdots, r$. Then simultaneous pole assignability is generic if $\max\{l,m\} \geq r$. Moreover, generically the integer q in the above definition can be any integer that satisfies

$$q[\max\{l,m\} - r+1] \geq \sum_{i=1}^{r} n_i - \max\{l,m\}. \tag{47}$$

Proof The proof hinges on two generic properties of polynomial matrices, apart from that in Lemma 2. First, generically a square matrix is column proper. Second, generically the highest column (or row) degrees of a matrix are all nearly equal. That is, if $A \in \mathbf{R}[s]^{k}$ and $\deg|A| = \delta$, then generically all the column degrees of A

will equal δ/k if k divides δ exactly. If k does not divide δ exactly, let β denote the integer part of δ/k, and let α denote $\delta \bmod k$; then generically α columns of A will have degree $\beta+1$ while the rest will have degree β.

As in Section 4, we first deal with the case where $\min\{l,m\}=1$. If this is not true, then Lemma 9 can be applied to convert the problem to this case. (Observe that the validity of Lemma 9 is not affected if N,D are polynomial matrices.) Accordingly, suppose $l=1, m \geq r$, and let (a_i,B_i) be an l.c.f. of P_i, with a_i monic. Note that $a_i \in \mathbf{R}[s]$, $B_i \in \mathbf{R}[s]^{1\times m}$. Form the matrix

$$Q=\begin{bmatrix} a_1 & B_1 \\ \vdots & \vdots \\ a_r & B_r \end{bmatrix} \in \mathbf{R}[s]^{r\times(m+1)}. \tag{48}$$

If $r \leq m$, then generically Q has a right inverse. Thus, given any set of monic polynomials $\phi_1, \cdots, \phi_r$, generically there exist $x \in \mathbf{R}[s]$, $Y \in \mathbf{R}[s]^{m\times 1}$ such that

$$Q\begin{bmatrix} x \\ y_1 \\ \vdots \\ y_m \end{bmatrix}=\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_r \end{bmatrix}. \tag{49}$$

So if we define $C=Yx^{-1}$, then we would have achieved the desired simultaneous pole assignment, provided C is proper.

The proof is completed by showing that if (47) holds, then generically the above C is proper. For this purpose, we make the following claim: If $\deg(\phi_i)=n_i+q$, $\deg x=q$, $\deg(y_i) \leq q$ for $i=1, \cdots, m-r$, and x is monic, then generically C is proper. To prove this claim, let

$$P(s)=\begin{bmatrix}P_1(s)\\ \vdots\\ P_r(s)\end{bmatrix}\in\mathbf{R}(s)^{r\times m}=[P^{(1)}\ \ P^{(2)}], \tag{50}$$

where $P^{(2)}\in\mathbf{R}(s)^{r\times r}$. Since P is strictly proper, its Laurent series is of the form

$$P(s)=P_0s^{-1}+\cdots \tag{51}$$

Partition P_0 as $[P_{01}\ \ P_{02}]$ where $P_{02}\in\mathbf{R}^{r\times r}$. Then generically $|P_{02}|\neq 0$. Now multiply both sides of (49) by the diagonal matrix $\mathrm{Diag}\{(a_1x)^{-1},\cdots,(a_rx)^{-1}\}$. This leads to

$$\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}+P(s)C(s)=\begin{bmatrix}\phi_1/a_1x\\ \vdots\\ \phi_r/a_rx\end{bmatrix}. \tag{52}$$

Now note that $\deg\phi_i=\deg a_i+\deg x$, and that all polynomials are monic. Hence ϕ_i/a_ix is proper and has the value 1 when $s=\infty$. Using this fact in (52) shows that PC is strictly proper, i.e. that $PC(\infty)=0$. Partition C as $[C_1\ \ C_2]$, where $C_2\in\mathbf{R}(s)^{r\times r}$. Then C_1 is proper, by the hypothesis that $\deg y_i\leq q$ for $i=1,\cdots,m-r$, whence $P^{(1)}C_1$ is strictly proper. Since $PC=P^{(1)}C_1+P^{(2)}C_2$, we see that $P^{(2)}C_2$ is also strictly proper. Now, suppose by way of contradiction that C_2 is improper. Then the Laurent series of C_2 contains a term of the form $C_{20}s^i$, where $C_{20}\neq 0$ and i is a positive integer. Since $|P_{20}|\neq 0$, it follows that $P_{02}C_{02}\neq 0$, so that $P^{(2)}C_2$ is not strictly proper. This contradiction shows that C_2 is proper.

In order to complete the proof, it is shown that if (47) holds, then generically one can always find x,Y satisfying (49) such that x is monic, $\deg x\leq q$, $\deg y_i\leq q$ for $i=1,\cdots,m-r$. Let $\bar{x},\bar{Y}$ be any particular solution of (49) corresponding to a given set of polynomials $\phi_1,\cdots,\phi_r$. Then the general solution of (49) can be written as follows: Let ρ denote $m-r+1$, and select $F\in\mathbf{R}[s]^{\rho\times\rho}$, $G\in\mathbf{R}[s]^{r\times\rho}$ such that

$$Q\begin{bmatrix}F\\G\end{bmatrix}=0_{r\times\rho},\quad \begin{bmatrix}F\\G\end{bmatrix}\sim\begin{bmatrix}I_\rho\\0\end{bmatrix}, \tag{53}$$

where ~ denotes equivalence. Then the general solution of (49) is

$$\begin{bmatrix}x\\Y\end{bmatrix}=\begin{bmatrix}\bar{x}\\\bar{Y}\end{bmatrix}+\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}\alpha_1\\\vdots\\\alpha_\rho\end{bmatrix}, \tag{54}$$

where $\alpha_1,\cdots,\alpha_\rho\in\mathbf{R}[s]$ are arbitrary. The proof consists of showing that the α_i's can be chosen such that deg $x=q$ and x is monic, and deg $y_i\leq q$ for $i=1,\cdots,m-r$.

Partition Q as $[Q_1\ \ Q_2]$ where $Q_2\in\mathbf{R}[s]^{r\times r}$. Then generically $|Q_2|\neq0$ and Q_1,Q_2 are left-coprime, in which case (G,F) is a right-coprime factorization of $Q_2^{-1}Q_1$. In particular, deg $|F|=$deg $|Q_2|$. Since each plant P_i is strictly proper, each element of the i-th row of Q_2 has degree no larger than n_i-1. Hence generically deg $|F|=$deg $|Q_2|=\sum_{i=1}^{r}(n_i-1)=N-r$, where N denotes $\sum_{i=1}^{r}n_i$. Now we consider two cases: (a) ρ divides $N-r+1$, and (b) ρ does not divide $N-r+1$. In the former case, generically the matrix F is column proper, and the column degree of each column is $(N-r+1)/\rho=:k$, except for the first column whose degree is $k-1$. By multiplying F on the right by a unimodular matrix if necessary (which can be absorbed into the free parameters $\alpha_1,\cdots,\alpha_\rho$), we can assume that F is already in Hermite form. Thus f_{11} has degree $k-1$ and f_{ii} has degree k for all $i\geq2$. Now, we proceed sequentially as follows: First, by Euclidean division, choose α_ρ such that deg

$(\bar{y}_{m-r}+f_{\rho\rho}\alpha_\rho)$ $<$ deg $f_{\rho\rho}=k$. Next, choose $\alpha_{\rho-1}$ such that deg $(\bar{y}_{m-r-1}+f_{\rho-1,\rho}\alpha_\rho+f_{\rho-1,\rho-1}\alpha_{\rho-1}$ $<$ deg $(f_{\rho-1,\rho-1}=k$. In this way, choose $\alpha_2, \cdots, \alpha_\rho$ such that deg $(y_1, \cdots, y_{m-r}) \leq k-1$. Then, by the claim above, generically $C=Yx^{-1}$ is proper with McMillan degree $q=k-1$. Finally, choose α_1 such that deg $x=k-1$ and such that x is monic; this can be done since f_{11} is $k-1$. In case (b), suppose ρ does not divide $N-r+1$, and let k denote the integer part of the fraction $(N-r+1)/\rho$. Then the first several column degrees of F can be assumed to be k, while the rest are $k+1$. By an argument similar to the above, $C=Yx^{-1}$ is generically proper with McMillan degree $q=k$. In case (a), we have

$$q \geq \frac{N-r+1}{\rho} - 1 = \frac{N-r+1-\rho}{\rho} = \frac{N-m}{\rho}, \tag{55}$$

$$\rho q \geq N-m. \tag{56}$$

In case (b), we have that $\rho(q+1) \geq N-r+1$, since $q=k$ is the integer part of $(N-r+1)/\rho$. This again leads to the same inequality (56), which is the same as (47) when $\min\{l,m\}=1$.

We have thus proved the theorem for the case when $l=1$, $m \geq r$. If $l > 1$, we can invoke Lemma 9 to find a constant row vector v such that (vN_i,D_i) are right-coprime for $i=1, \cdots, r$, where (N_i,D_i) is a right-coprime factorization of P_i. Then we apply the foregoing result.

6. CONCLUSIONS

In this paper, we have derived some results concerning the genericity of simultaneous stabilizability, simultaneous strong stabilizability, and simultaneous pole assignability. The results in the first and third category are already known [5], but the present proofs are simpler. The result concerning simultaneous strong stabilizability is new, and as far as we are able to determine, cannot be derived using the methods of [5]. In addition, we have presented some lemmas concerning generic greatest common divisors which may be of some independent interest.

In contrast with [5], the proofs here are formulated in input-output setting, without recourse to state-space realizations. As a consequence, the proofs given here suggest simple procedures for the computation of a common controller that achieves the desired property. These procedures are actually quite numerically robust, and have been applied with success to the design of reliable controllers for a jet engine. These results will be reported elsewhere.

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