Regularity aspects of the theory of infinite dimensional representations of Lie groups

by

Niels Kristian Skovhus Poulsen

Cand. Scient., University of Aarhus, Denmark
(January, 1968)

Submitted in Partial Fulfillment

of the Requirements for the

Degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

May, 1970

Signature redacted

Signature of Author ....
Department of Mathematics,
May 14, 1970

Signature redacted

Certified by ....
Thesis Supervisor

Signature redacted

Accepted by ....
Chairman, Departmental Committee
on Graduate Students

THESIS
MATH.
1970
Ph.D.

Regularity aspects of the theory of infinite dimensional representations of Lie groups

by

Niels Kristian Skovhus Poulsen

Submitted to the Department of Mathematics on May 14, 1970 in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

Abstract:

First we study intertwining sesquilinear forms on the spaces of C^∞-vectors of two representations of a Lie group in a Banach space and a Hilbert space respectively. Using regularity methods such forms are identified with certain closed densely defined intertwining operators for the representations. This gives new criteria for irreducibility and equivalence of unitary representations. The results are applied to study families of representations having a common space of C^∞-vectors, and the theory is illustrated by some examples.

Secondly we introduce C^∞-systems of imprimitivity for a unitary representation of a Lie group. The usual projection valued measure is replaced by a measure whose values are (possibly unbounded) positive operators. A C^∞-system of imprimitivity gives rise to an induced representation, and we give a complete classification of such systems. The result is a generalization of Mackey's imprimitivity theorem, and the proof is based on the regularity of a certain sesquilinear form on the space of C^∞-vectors for the representation.

Thesis supervisor: Irving E. Segal

Title: Professor of Mathematics

Acknowledgements:

The author wishes to express his gratitude to Professor Irving E. Segal, who has been a source of encouragement and inspiration during the solution of the problems of this thesis. He is also grateful to Professor Roe Goodman for many enlightening conversations on the theory of group representations.

Table of Contents

Introduction.

Let $\mathcal{S}$ denote the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}$. Let $(U(t)f)(x) = f(x+t)$ and $(V(t)f)(x) = \exp(itx)\cdot f(x)$ for $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. It is well known that the system $\{U(s),V(t) \mid s,t \in \mathbb{R}\}$ is irreducible in $L^2(\mathbb{R})$ and that the two unitary groups leave $\mathcal{S}$ invariant. Using some distribution theory one can prove the following result:

If $\beta(\cdot,\cdot)$ is a continuous sesquilinear form on $\mathcal{S}$ which is invariant under the groups:

$$\beta(\varphi,\psi) = \beta(U(t)\varphi,U(t)\psi) = \beta(V(t)\varphi,V(t)\psi)$$

for all $\varphi, \psi \in \mathcal{S}$, $t \in \mathbb{R}$, then β has to be a multiple of scalar product

$$\beta(\varphi,\psi) = \text{const} \cdot \int \varphi(x)\overline{\psi(x)}dx \quad \text{for} \quad \varphi, \psi \in \mathcal{S}.$$

On the other hand, assuming that the scalar product is essentially the only continuous invariant sesquilinear form on $\mathcal{S}$, it is easy to show that the system $\{U(s),V(t) \mid s, t \in \mathbb{R}\}$ is irreducible in $L^2(\mathbb{R})$.

Now, the unitary groups $U(\cdot)$ and $V(\cdot)$ form restrictions of a certain continuous unitary representation of the Heisenberg group [23, Ch. 2, §1] and $\mathcal{S}$ is exactly the space of C^∞ vectors for this representation. The theorem then has an interpretation as a result about the irreducibility of the group representation.

In the following we shall adopt this viewpoint. It turns out that the corresponding result holds for any strongly continuous unitary representation of a Lie group (§3, Theorem 3.1). In fact, we show that all the usual irreducibility criteria for the representation in the Hilbert space remain valid if the representation is restricted to the (Fréchet) space of C^∞ vectors.

In [29, §3] Segal proved an analogous result for the action of a quantum process on the space of smooth vectors for the energy operator. The result of Theorem 3.1 was conjectured by Segal [29] on the basis of the many similarities between quantum field theory and the theory of group representations.

The result of Theorem 3.1 is derived from much more general results on sesquilinear forms established in §2. Here we study pairs (V,U) of representations of a Lie group. V is a representation in a Banach space and U is a unitary representation in a Hilbert space. We give a complete characterization of continuous sesquilinear intertwining forms on the spaces of C^∞-vectors in terms of certain closed densely defined intertwining operators of the representations. (A more detailed description is given in the introduction of §2). This is done by means of a regularity method which is familiar from the Hilbert space theory of partial differential operators.

In §4 we study families of Banach space representations having a common space of C^∞ vectors. In case the family contains an irreducible unitary representation we get some additional information about various types of irreducibility and equivalence of all representations in the family.

§5 contains some examples. First we consider various Banach space representations of the Heisenberg group to illustrate the theory developed in §4. We characterize the space of C^∞-vectors for the regular representations of a Lie group G in $L^p(G)$. In this connection we prove a "Sobolev inequality" (§5, Lemma 5.1) which is of some independent interest.

Using elliptic operators Blattner [2] proved that a C^∞-vector for an induced representation U^L (of a Lie group) is a continuous function in case the representation L of the subgroup is finite dimensional.

In §5 we use our "Sobolev inequality" to give a complete characterization of the space of C^∞-vectors of an arbitrary induced representation. In particular we prove that a C^∞-vector is actually an infinitely differentiable function, and an analytic vector is an analytic function. Also we establish the fact that point evaluation always defines a continuous linear mapping on the (Fréchet) space of C^∞-vectors. As a consequence we get a general version of Blattner's intertwining number theorem - applicable to the case where L is not necessarily finite dimensional.

In §6 we introduce C^∞-systems of imprimitivity for a unitary representation U of a Lie group. Roughly speaking, the conventional projection valued measure is replaced by a "measure" whose values are (possibly unbounded) positive operators. Using C^∞-theory we show that such a system gives rise to an induced representation U^L. Moreover a C^∞-system of imprimitivity P for U canonically determines a "renormalized" system P_o such that the pair (U,P_o) is equivalent to the restriction of the induced pair (U^L,P^L) to a $U^L(G)$-invariant subspace. Our methods are inspired by but different from those of Bruhat [6].

A system of imprimitivity in the sense of Mackey [18] naturally gives rise to a C^∞-system of imprimitivity, and as a special case of the main theorem we get a new proof of Mackey's imprimitivity theorem (for the case of Lie groups).

§1. General results on C^∞ vectors.

In order to establish the notation it is convenient to recall some results on representation theory [6], [13], [22], [26].

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $g \to V(g)$ be a strongly continuous representation of G in a Banach space $\underline{B}$. A vector $x \in \underline{B}$ is called a C^∞ vector for V if the mapping $g \to V(g)x$ is C^∞ from G to $\underline{B}$ or equivalently if the function $g \to \langle V(g)x,f \rangle$ is C^∞ on G for each continuous linear functional $f \in \underline{B}^*$. The set of C^∞ vectors is clearly a linear subspace of $\underline{B}$ which we will denote by $\underline{D}_\infty$ or $\underline{D}_\infty(V)$.

On $\underline{D}_\infty$ we have a representation v of $\mathfrak{g}$ defined by

$$v(X)x = \frac{d}{dt} V(\exp(tX))x|_{t=0} \quad \text{for} \quad X \in \mathfrak{g} \, , \; x \in \underline{D}_\infty$$

The mapping $X \to v(X)$ is a representation of $\mathfrak{g}$ as a Lie algebra of operators having $\underline{D}_\infty$ as a common invariant dense domain, and v has a unique extension to a representation, also denoted by v, of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Let $\{X_1, X_2, \ldots, X_d\}$ be a basis for $\mathfrak{g}$, and let $v_1(X_k)$ denote the infinitesimal generator of the one-parameter group $t \to V(\exp(tX_k))$. Then $\underline{D}_\infty$ can be characterized in the following way [13, Th. 1.1]

$$\underline{D}_\infty = \bigcap_{k=1}^{d} \bigcap_{n=1}^{\infty} \underline{D}_{v_1(X_k)^n} ,$$

where $\underline{D}_{v_1(X_k)^n}$ denotes the domain of $v_1(X_k)^n$. In particular $\underline{D}_\infty$ coincides with "the maximal domain for V" employed by Segal [27], [28].

Following Goodman [13] we topologize $\underline{D}_\infty$ by the following family of semi-norms ρ_n:

$$\rho_n(x) = \sum_{1 \leq i_k \leq d} \|v(X_{i_1} \dots X_{i_n})x\|$$

for $n = 0,1,2,\dots$ (with the interpretation $\rho_o(x) = \|x\|$). Then $\underline{D}_\infty$ is a Fréchet space [13], and for $g \in G$ the restriction $V_\infty(g)$ of $V(g)$ to $\underline{D}_\infty$ is a continuous linear operator on $\underline{D}_\infty$. Using the relation $g \cdot \exp(tX) \cdot g^{-1} = \exp(Ad(g) \cdot tX)$ this can be seen directly, but it is also an immediate consequence of the closed graph theorem.

In §2 we shall find it convenient to use a different description of the topology on $\underline{D}_\infty$. The following result is due to Goodman (unpublished).

<u>Lemma 1.1</u>: Let $L \in \mathfrak{U}(\mathfrak{g})$ be an elliptic element and

let $A = \overline{v(L)}$ (cf. Corollary 1.1). Then

$$\underline{D}_{\infty} = \bigcap_{n=0}^{\infty} \underline{D}_{A^n}$$

Furthermore, the topology on $\underline{D}_{\infty}$ defined by the semi-norms $x \to \|A^n x\|$ $n = 0,1,2,\ldots$ is identical with the topology defined by the family $\{\rho_n \mid n = 0,1,\ldots\}$.

Proof: The non-trivial inclusion follows from the regularity theorem for elliptic differential operators on the group. The proof is similar to the proof of Theorem 1.1 in [13]. To prove the last part we note that each semi-norm $\|\cdot\|_n$ is continuous in the topology defined by the family $\{\rho_n\}$, and since $\underline{D}_{\infty}$ is a Fréchet space in both topologies they must coincide (the closed graph theorem). Q.E.D.

Langlands [17] proved that $A = \overline{v(L)}$ is the infinitesimal generator of a strongly continuous semi group in $\underline{B}$ (if L is suitably normalized). We shall need this result only for the elliptic element $\Delta = \sum_{k=1}^{d} X_k^2$, and we remark that for uniformly bounded representations there is a much simpler proof of this case [22].

Proposition 1.1: For each $x \in \underline{D}_{\infty}$, the mapping $g \to V_{\infty}(g)x$ is C^{∞} from G to $\underline{D}_{\infty}$.

Proof: Let $x \in \underline{D}_\infty$ and let $L = X_1^{\alpha_1} \dots X_d^{\alpha_d} \in \mathcal{U}(\mathfrak{g})$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is a set of non-negative integers. It suffices to show that the mapping $g \to v(L)V(g)x$ from G to $\underline{B}$ is C^∞ in a neighborhood of the identity e in G. Let $g(t) = \exp(t_1 X_1) \dots \exp(t_d X_d)$ for $t = (t_1, t_2, \dots, t_d) \in \mathbb{R}^d$. Then the mapping $g(t) \to t$ is an analytic coordinate system in a neighborhood of e in G [14, Ch. II]. Therefore the mapping $(s,t) \to V(g(s) \cdot g(t))x$ from $\mathbb{R}^{2d}$ to $\underline{B}$ is C^∞ in a neighborhood of $(0,0)$, but since

$$v(L)V(g(t))x = \left(\frac{\partial}{\partial s_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial s_d}\right)^{\alpha_d} V(g(s) \cdot g(t))x \big|_{s=0}$$

this completes the proof.

The following result is a generalization of Lemma 2.5 in [29].

Proposition 1.2: Let $\underline{D}$ be any dense linear subspace of $\underline{B}$ which is contained in $\underline{D}_\infty$ and invariant under the $V(g)$, $g \in G$. Then $\underline{D}$ is dense in $\underline{D}_\infty$ (i.e. in the $\underline{D}_\infty$-topology).

Proof: Since each $V_\infty(g)$ is continuous on $\underline{D}_\infty$ we can assume that $\underline{D}$ is a closed subspace of $\underline{D}_\infty$, and hence that $\underline{D}$ is complete in the $\underline{D}_\infty$-topology. Then for

$x \in \underline{D}$ and $\varphi \in C_0^\infty(G)$ we have $\int \varphi(g)V(g)x\, dg \in \underline{D}$ [5, Ch. III, §3], and we want to show that each vector in $\underline{D}_\infty$ is a limit of a sequence of vectors of this form.

For $x \in \underline{B}$ and $\varphi \in C_0^\infty(G)$ we use the notation $V(\varphi)x = \int \varphi(g)V(g)x\, dg$. (dg denotes some left-invariant Haar measure on G, and $C_0^\infty(G)$ denotes the space of all infinitely differentiable functions of compact support on G).

If $x \in \underline{B}$ there exists a sequence $\{x_n\}$ in $\underline{D}$ such that $x_n \to x$ in $\underline{B}$. Then for $L \in \mathfrak{U}(\mathfrak{g})$ we have

$$v(L)V(\varphi)x_n = V(L\varphi)x_n \to V(L\varphi)x = v(L)V(\varphi)x$$

in $\underline{B}$, but this means exactly that $V(\varphi)x_n \to V(\varphi)x$ in the $\underline{D}_\infty$-topology. Since $\underline{D}$ is closed we get $V(\varphi)x \in \underline{D}$ for all $x \in \underline{B}$ and $\varphi \in C_0^\infty(G)$.

Now let $x \in \underline{D}_\infty$. Each $v(L)$ is a continuous linear operator, on $\underline{D}_\infty$, so

$$v(L)V(\varphi)x = \int \varphi(g)v(L)V(g)x\, dg$$

Therefore, if $\{\varphi_n\} \subseteq C_0^\infty(G)$ with $\varphi_n \geq 0$, $\int \varphi_n(g)dg = 1$ and supp $\varphi_n \downarrow \{e\}$ [26], we have

$$v(L)V(\varphi_n)x = \int \varphi_n(g)v(L)V(g)x\ dg \rightarrow v(L)V(e)x$$

in $\underline{B}$ for all $L \in \mathcal{U}(\mathfrak{g})$. Since we always assume $V(e) = 1$ this shows that $V(\varphi_n)x \rightarrow x$ in $\underline{D}_\infty$.

Corollary 1.1: Let $\underline{D}$ be a dense subspace of $\underline{B}$ which is contained in $\underline{D}_\infty$ and invariant under the $V(g)$, $g \in G$. Then for each $L \in \mathcal{U}(\mathfrak{g})$, $v(L)$ has a closure $\overline{v(L)}$ in $\underline{B}$ and $\overline{v(L)} = \overline{v(L)|_{\underline{D}}}$.

Proof: Let $\hat{V}$ denote the contragredient representation of V is the sense of Bruhat [6, p. 113], the corresponding infinitisimal representation is denoted by $\hat{v}$. For $x \in \underline{D}_\infty(V)$ and $f \in \underline{D}_\infty(\hat{V})$ we have

$$\langle v(L)x,f\rangle = \langle x,\hat{v}(L^*)f\rangle \quad \text{for all} \quad L \in \mathcal{U}(\mathfrak{g}),$$

where $L \rightarrow L^*$ is the usual $*$-operation in $\mathcal{U}(\mathfrak{g})$. Hence $v(L)^* \supseteq \hat{v}(L^*)$, and since $\underline{D}_\infty(\hat{v})$ is w^*-dense in $\underline{B}^*$ [6] it follows that $v(L)$ has a closure. The last part of the statement is immediate from Proposition 1.2 since $v(L)$ is continuous on $\underline{D}_\infty$. Q.E.D.
This result is a generalization of Theorem 1 in [28].

Remark: Proposition 1.2 has a natural analogue in case

$t \to V(t)$ is a strongly continuous semi group in a Banach space. A simple modification of the proof gives the following useful result (which is well known in the case of a one-parameter unitary group in a Hilbert space.)

Corollary 1.2: Let $t \to V(t)$ be a strongly continuous semi group in a Banach space $\underline{B}$ and let A be the infinitesimal generator. Let $\underline{D}$ be a dense subspace of $\underline{B}$ which is contained in $\underline{D}_{A^n}$ (the domain of A^n) for some $n \in \mathbb{N}$ and such that $V(t)\underline{D} \subset \underline{D}$ for $t \in (0,\infty)$. Then $\underline{D}$ is a core for A^n, i.e.

$$A^n = \overline{A^n \big|_{\underline{D}}}$$

Proof: $\underline{D}_{A^n}$ is a Banach space in the graph norm and $\underline{D}$ is dense in this space. Q.E.D.

Remark: Here (and several times in the following) we implicitely made use of the fact that A^n is again a closed densely defined operator. This is known to be true for every closed densely defined linear operator (in a Banach space) with a non-void resolvent set. (See e.g. p. 602 and p. 648 of [11]).

Now again let V be a continuous representation of

G in a Banach space $\underline{B}$.

A vector $x \in \underline{B}$ is called an analytic vector for V if the mapping $g \to V(g)x$ is analytic on G or equivalently if the function $g \to \langle V(g)x,f\rangle$ is analytic on G for each $f \in \underline{B}^*$. The subspace of analytic vectors is clearly contained in $\underline{D}_\infty$, and Nelson [21, Lemma 7.1] showed that a C^∞ vector x is analytic for V iff

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \rho_n(x) < +\infty \quad \text{for some} \quad s > 0.$$

In §3 we shall need the following result [13, Prop. 2.2].

Lemma 1.2: Let x be an analytic vector for V. Then for some $s > 0$

$$V(\exp(tX_k))x = \sum_{n=0}^{\infty} \frac{t^n}{n!} v(X_k)x \quad \text{for} \quad |t| < s, \ k = 1,2,\ldots,d,$$

and the series converges absolutely in $\underline{D}_\infty$.

Finally we shall need the fact that the set of analytic vectors for V is dense in $\underline{B}$ [21].

§2. Sesquilinear Forms.

In this section we consider a continuous unitary representation U of a Lie group G in a Hilbert space $\underline{H}$. The corresponding infinitesimal representation of $\mathcal{U}(\mathfrak{g})$ on $\underline{D}_\infty(U)$ is denoted by $u(\cdot)$. Now let V be a continuous representation of G in a Banach space $\underline{B}$ and let $\beta(\cdot,\cdot)$ be a (separately) continuous sesquilinear form on $\underline{D}_{-\infty}(V) \times \underline{D}_{-\infty}(U)$, which is group invariant:

$$\beta(V(g)x, U(g)y) = \beta(x,y) \quad \text{for all}$$

$$(x,y) \in \underline{D}_{-\infty}(V) \times \underline{D}_{-\infty}(U),\ g \in G$$

The importance of the study of such intertwining forms was established by Bruhat [6], and in the present section we shall study the structure of β from a different point of view. As observed by Bruhat there exists a unique continuous linear mapping $T : \underline{D}_{-\infty}(V) \to \underline{D}_{-\infty}(U)^*$ (the anti-dual) such that $\beta(x,y) = \langle Tx, y\rangle$, and we have $TV_\infty(g) = U_\infty(-g)^*T$ for all $g \in G$. (This is true for any pair of representations in much more general spaces).

Now, because $\underline{H}$ is a Hilbert space we have natural injections

$$\underline{D}_{\infty}(U) \subseteq \underline{H} \subseteq \underline{D}_{\infty}(U)^*,$$

and it is of interest to know when T actually maps $\underline{D}_{\infty}(V)$ into $\underline{H}$.

We prove that this is always the case and in fact T maps $\underline{D}_{\infty}(V)$ into $\underline{D}_{\infty}(U)$ continuously. As the main result we establish a 1-1 correspondance between group invariant sesquilinear forms β and certain closed densely defined intertwining operators from $\underline{B}$ to $\underline{H}$. The method is based on regularity properties of the resolvent of the "elliptic" operator $u(\Delta)$ - a technique which is well known in the theory of partial differential operators.

First we give a method of producing sesquilinear forms of this type.

Proposition 2.1: Let V and V' be continuous representations of a Lie group G in Banach spaces $\underline{B}$ and $\underline{B}'$ respectively. Let T be a closable linear operator from $\underline{B}$ to $\underline{B}'$ such that

1) $\underline{D}_T \supseteq \underline{D}_{\infty}(V)$

2) $TV(g) \supseteq V'(g)T$ for all $g \in G$.

Then T maps $\underline{D}_{\infty}(V)$ into $\underline{D}_{\infty}(V')$ continuously.

Proof: Since the closure of T also satisfies 1) and 2) we may assume T is closed. If we know T maps $\underline{D}_\infty(V)$ into $\underline{D}_\infty(V')$ the closed graph theorem shows that T is automatically continuous in the Fréchet topologies of these spaces. In any event we can consider the restriction $T_\infty : \underline{D}_\infty(V) \to \underline{B}'$ which by the same argument is a continuous operator from the Fréchet space $\underline{D}_\infty(V)$ to the Banach space $\underline{B}'$. Then for $x \in \underline{D}_\infty(V)$ the mapping $g \to T_\infty V_\infty(g)x$ is C^∞ from G to $\underline{B}'$ by Proposition 1.1. On the other hand $V'(g)T_\infty x = T_\infty V_\infty(g)x$, so $T_\infty x$ is a C^∞-vector for V'. Q.E.D.

In case V' is a unitary representation in a Hilbert space $\underline{B}'$ we see that

$$\beta(x,y) = \langle Tx,y\rangle$$

defines a continuous sesquilinear intertwining form on $\underline{D}_\infty(V) \times \underline{D}_\infty(V')$.

Proposition 2.2: Let V and V' be as in Proposition 2.1, and suppose S is a continuous linear mapping of $\underline{D}_\infty(V)$ into $\underline{D}_\infty(V')$ such that $SV_\infty(g) = V'_\infty(g)S$ for all $g \in G$. Then S is closable.

Proof: In order to see that the closure of the graph of S in $\underline{B} \times \underline{B}'$ is the graph of a linear operator we assume $\{x_n\} \subseteq \underline{D}_\infty(V)$, $x_n \to 0$ in $\underline{B}$ and $Sx_n \to y$ in $\underline{B}'$. Then we have to show $y = 0$. For each $\varphi \in C_0^\infty(G)$ we have (cf. proof of Prop. 1.2) $V(\varphi)x_n \to 0$ in $\underline{D}_\infty(V)$, so $SV(\varphi)x_n = V'(\varphi)Sx_n \to 0$ in $\underline{D}_\infty(V')$. Hence $V'(\varphi)y = 0$ for all $\varphi \in C_0^\infty(G)$, so $y = 0$. Q.E.D.

This observation could be used to give a more self-contained proof of the following theorem. On the other hand, the proof presented seems to give more detailed information about the structure of T.

Now let U be a continuous unitary representation of G in a Hilbert space $\underline{H}$ and V a continuous representation of G in a Banach space $\underline{B}$.

Theorem 2.1: Let $\beta(\cdot,\cdot)$ be a continuous sesquilinear form on $\underline{D}_\infty(V) x \underline{D}_\infty(U)$ which is G-invariant:

$$\beta(V(g)x,U(g)y) = \beta(x,y) \quad \text{for all}$$

$$(x,y) \in \underline{D}_\infty(V) \times \underline{D}_\infty(U),\ g \in G .$$

Then there exists a closed linear operator T from $\underline{B}$ to $\underline{H}$ with $\underline{D}_T \supseteq \underline{D}_\infty(V)$ and such that

1) T maps $\underline{D}_\infty(V)$ into $\underline{D}_\infty(U)$ continuously

2) $\beta(x,y) = \langle Tx,y\rangle$ for all $(x,y) \in \underline{D}_\infty(V) \times \underline{D}_\infty(U)$

3) $TV(g) = U(g)T$ for all $g \in G$.

If T is required to be the closure of its restriction to $\underline{D}_\infty(V)$, T is unique.

Proof: Let $\{X_1,\ldots,X_d\}$ be a basis for the Lie algebra $\mathfrak{g}$ and let $\Delta = \sum_{k=1}^{d} X_k^2$. Then the operators $\overline{u(\Delta)}$ and $\overline{v(\Delta)}$ are infinitesimal generators of strongly continuous semigroups in $\underline{H}$ and $\underline{B}$ respectively. Actually we know that $\overline{u(\Delta)}$ is a self adjoint operator in $\underline{H}$ [22], and $\overline{u(\Delta)} \leq 0$. Since the spectrum of an infinitesimal generator is contained in a left half-plane we can choose a real number $\lambda > 0$ such that the operator

$$C = \overline{v(\lambda I-\Delta)} = \lambda I - \overline{v(\Delta)}$$

has a bounded inverse in $\underline{B}$.

This has the advantage that $x \to \|C^n x\|$ is a norm on $\underline{D}_\infty(V)$ for $n = 0,1,2,\ldots,$ and we have

$\|x\| \leq \|C^{-1}\|\cdot\|Cx\| \leq \|C^{-1}\|^2 \|C^2x\| \leq \ldots$ for all $x \in \underline{D}_\infty(V)$.

By Lemma 1.1 this family defines the topology on $\underline{D}_\infty(V)$. We note that $\underline{D}_{C^n}$ is a Banach space in the norm $\|x\|_n = \|C^n x\|$ and by Corollary 1.2. $\underline{D}_\infty(V)$ is dense in this space. The operator

$$A = \overline{u(\lambda I-\Delta)} = \lambda I-\overline{u(\Delta)}$$

is self adjoint and $A \geq \lambda I$. We note that $\underline{D}_{A^n}$ is a

Hilbert space in the scalar product

$$\langle x,y\rangle_n = \langle A^n x, A^n y\rangle, \quad x, y \in \underline{D}_{A^n},$$

and $\underline{D}_\infty(U)$ is dense in this space. (This is also clear from spectral theory).

A has a bounded inverse (since $\lambda > 0$) and the norms $y \to \|A^n y\|$ $n = 0,1,2,\ldots$ define the topology on $\underline{D}_\infty(U)$.

Since $\beta(\cdot,\cdot)$ is separately continuous it is automatically continuous because we work with Fréchet spaces (see e.g. [12, p. 17] or [25, p. 88]). Hence for some integers m and n

$$|\beta(x,y)| \leq \text{const.}\ \|C^m x\|\cdot\|A^n y\|$$

for all $(x,y) \in \underline{D}_\infty(V) \times \underline{D}_\infty(U)$. By the properties of the norms we can take $m = n$.

It follows that there exists a unique continuous linear operator S from the Banach space $(\underline{D}_{C^n}, \|\cdot\|_n)$ into the Hilbert space $(\underline{D}_{A^n}, \|\cdot\|_n)$ such that

$$\beta(x,y) = \langle Sx,y\rangle_n$$

for all $(x,y) \in \underline{D}_\infty(V) \times \underline{D}_\infty(U)$. This means

$$\beta(x,y) = \langle A^n Sx, A^n y\rangle,$$

and we use the same notation for S as an operator from $\underline{B}$ to $\underline{H}$ (S is then a possibly unbounded operator with $\underline{D}_S = \underline{D}_{A^n}$)

Now we show that S has a "nice" restriction to

$\underline{D}_\infty(V)$. For $(x,y) \in \underline{D}_\infty(V) \times \underline{D}_\infty(U)$ and $X \in \mathfrak{g}$ the mapping

$$t \to \beta(V(\exp(tX))x,y) = \beta(x,U(\exp(-tX))y)$$

is a C^∞ function on $\mathbb{R}$ (by Proposition 1.1). By differentiation and use of the invariance of $\beta(\cdot,\cdot)$ we get

$$\beta(C^m x,y) = \beta(x,A^m y) \quad \text{for} \quad m = 1,2,3,\ldots$$

For $m = n$ this gives

$$\langle SC^n x, A^{2n} y\rangle = \langle A^n Sx, A^{2n} y\rangle,$$

where we have used the fact that A is symmetric. (This is precisely the step where the unitarity of U comes in).

Since A has a bounded inverse it is obvious that A maps $\underline{D}_\infty(U)$ onto itself, and because this space is dense in $\underline{H}$ we get

$$SC^n x = A^n Sx \quad \text{for} \quad x \in \underline{D}_\infty(V).$$

Since $C^n x \in \underline{D}_\infty(V)$ the left hand side of this equation is an element in $\underline{D}_{A^n}$, hence $Sx \in \underline{D}_{A^{2n}}$ whenever $x \in \underline{D}_\infty(V)$.

Repeating the argument we find that S maps $\underline{D}_\infty(V)$ into $\underline{D}_\infty(U) = \bigcap_{n=1}^{\infty} \underline{D}_{A^n}$.

By the continuity of S in the $\|\cdot\|_n$-norms we have

$$\|SC^n x\| = \|A^n Sx\| \leq \text{const.} \|C^n x\| \quad \text{for} \quad x \in \underline{D}_\infty(V)$$

Because C has a bounded inverse, C maps $\underline{D}_\infty(V)$ onto itself, hence it follows that S has a unique extension to a continuous linear operator, also denoted by S, of $\underline{B}$ into $\underline{H}$.

Let $T_o = A^{2n}S \Big|_{D_\infty(V)}$. Since S is continuous the operator $A^{2n}S$ is closed, so T_o has a closure T. It is easily checked that T has the desired properties. Q.E.D.

<u>Corollary 2.1</u>: The operator T of Theorem 2.1 can be chosen to have the form $T = A^{2n}S$, where S is a bounded linear mapping of $\underline{B}$ into $\underline{H}$ which maps $\underline{D}_\infty(V)$ into $\underline{D}_\infty(U)$ continuously.

If $U = V$ we can get more information about T. We state the result for the case β is Hermitian.

<u>Corollary 2.2</u>: Let U be a continuous unitary representation of G in a Hilbert space H and let β be a continuous Hermitian sesquilinear form on $\underline{D}_\infty$ which is invariant under the $U(g)$, $g \in G$. Then there exists a unique self adjoint operator T in $\underline{H}$ such that

$$\beta(x,y) = \langle Tx,y \rangle \quad \text{for all} \quad x, y \in \underline{D}_\infty.$$

This operator has the following properties:

i) T leaves $\underline{D}_\infty$ invariant, and the restriction $T\big|_{\underline{D}_\infty}$ is a continuous linear operator on $\underline{D}_\infty$

ii) T is essentially self adjoint on $\underline{D}_\infty$

iii) $TU(g) = U(g)T$ for all $g \in G$.

<u>Proof</u>: Even if β is not Hermitian we can get a limit

on the possibilities of T. Since S is bounded (cf. proof of theorem 2.1) and $SA^{2n} \subseteq A^{2n}S$ we get $SA \subseteq AS$ (because A is a function of A^{2n}). Using the fact that A^{2n} is essentially self adjoint on $\underline{D}_\infty$ (clear from spectral theory) a standard calculation shows

$$\overline{(SA^{2n})} = T \subseteq A^{2n}S$$

In case β is Hermitian it follows from the construction of S that S is self adjoint, and because S commutes with A we get that SA^{2n} is essentially self adjoint. Therefore we must have

$$\overline{(SA^{2n})} = T = A^{2n}S$$

so T is essentially self adjoint on $\underline{D}_\infty$.

If T_1 is any self adjoint operator in $\underline{H}$ such that $\beta(x,y) = \langle T_1x,y\rangle$ for all x, y $\in \underline{D}_\infty$ we have $T_\infty \subseteq T_1$. It follows that $T \subseteq T_1$ and hence that $T = T_1$. Q.E.D.

§3. Irreducibility and equivalence.

Recall that two representations V and U are called weakly equivalent (in the sense of Naimark) [20] in case they have a closed densely defined injective "intertwining operator" T with a dense range. This notion is unsatisfactory in general, and one could try to restrict it by requiring T to have $\underline{D}_{-\infty}(V)$ in its domain. By the results of §2 (Prop. 2.1, 2.2) this makes the relation transitive, but apparently it loses symmetry. (Of course we could furthermore require T to have $\underline{D}_{-\infty}(U)$ in its range. Then V and U would be equivalent if and only if V_∞ and U_∞ are equivalent).

As we have seen in §2, a representation V in a Banach space is "equivalent to" a unitary representation U in a Hilbert space if and only if there exists a non-degenerate continuous sesquilinear intertwining form on $\underline{D}_{-\infty}(V) \times \underline{D}_{-\infty}(U)$. This notion seems very reasonable for certain types of representations. On the other hand it is not strong enough to identify certain representations of the Heisenberg group which ought to be identified. Therefore we shall not discuss the problem any further.

Now we apply Theorem 2.1 to prove some results on irreducibility of a unitary representation.

Let $g \to U(g)$ be a strongly continuous unitary representation of a Lie group G in a Hilbert space $\underline{H}$. We recall that U is called irreducible if it satisfies any one of the following equivalent conditions (suppose $\underline{H} \neq \{0\}$)

i) There is no closed subspace of $\underline{H}$ which is invariant under the $U(g)$, $g \in G$, other than $\{0\}$ and $\underline{H}$.

ii) If T is a continuous linear operator on $\underline{H}$ such that $TU(g) = U(g)T$ for $g \in G$, then $T = \lambda \cdot 1$ for some $\lambda \in C$.

iii) Each non-zero vector x in $\underline{H}$ is cyclic for U i.e. span $\{U(g)x \mid g \in G\}$ is dense in $\underline{H}$.

We will not be concerned with the notion of algebraic irreducibility (there is no invariant subspace of $\underline{H}$ other than $\{0\}$ and $\underline{H}$) for the following reason. If U is algebraically irreducible we have $\underline{D}_\infty(U) = \underline{H}$ and in particular, the one-parameter groups $t \to U(\exp(tX_k))$ are uniformly continuous. Then it follows that $g \to U(g)$ is uniformly continuous, and if G is a connected Lie group this implies, that $\underline{H}$ is finite-dimensional [30].

The following results show that all the criteria for irreducibility of U remain valid for the restriction U_∞ of U to $\underline{D}_\infty$.

Theorem 3.1: Let G be a Lie group and let $g \to U(g)$ be a continuous unitary representation of G in a Hilbert space $\underline{H}$. Then the following statements are equivalent.

1) U is irreducible in $\underline{H}$

2) There is no closed invariant subspace of $\underline{D}_\infty$ other than $\{0\}$ and $\underline{D}_\infty$.

3) Each non-zero vector x in $\underline{D}_\infty$ is cyclic for U_∞.

4) If T is a continuous linear operator on $\underline{D}_\infty$ such that $TU_\infty(g) = U_\infty(g)T$ for all $g \in G$, then $T = \lambda 1$ for some $\lambda \in C$.

5) If $\beta(\cdot,\cdot)$ is a continuous invariant sesquilinear form on $\underline{D}_\infty$, then $\beta(x,y) = \lambda \cdot \langle x,y \rangle$ for all $x, y \in \underline{D}_\infty$, and some $\lambda \in C$.

Remark: It follows from the proof that conditions 1), 2), and 3) are equivalent if U is any strongly continuous representation in a Banach space. This fact has been proved earlier by Bruhat [6, Proposition 2.6] in the case where G is generated by a compact neighborhood of e (and even for more general representations.) In this case our topology on $\underline{D}_\infty$ coincides with the topology used by Bruhat.

Proof: 1) => 2): Let $\underline{D}$ be a closed invariant subspace of D_∞ and suppose $\underline{D}$ contains a non-zero vector x. Since x is cyclic for U, $\underline{D}$ is dense in $\underline{H}$, and by

Proposition 1.2 $\underline{D} = \underline{D}_{\infty}$.

2) => 3): If $x \in \underline{D}_{\infty}$ is any non-zero vector, $\underline{D} = \overline{\text{span}}\ \{U_{\infty}(g)x \mid g \in G\}$ is a closed non-zero invariant subspace of $\underline{D}_{\infty}$. Hence $\underline{D} = \underline{D}_{\infty}$.

3) => 1): Let $\underline{K} \subseteq \underline{H}$ be a closed invariant subspace containing a non-zero vector x. Choose $\varphi \in C_0^{\infty}(G)$ such that $y = U(\varphi)x \neq 0$. Then the subspace span $\{U(g)y \mid g \in G\}$ is contained in $\underline{K}$ and dense in $\underline{D}_{\infty}$. Since $\underline{D}_{\infty}$ is dense in $\underline{H}$ we get $\underline{K} = \underline{H}$.

1) => 5): Let $\beta(\cdot,\cdot)$ be a continuous invariant sesquilinear form on $\underline{D}_{\infty}$. By Theorem 2.1 there exists a closed linear operator T in $\underline{H}$ such that $\beta(x,y) = \langle Tx,y\rangle$ and $U(g)T = TU(g)$ for $g \in G$. Let $T = V|T|$ be the polar decomposition of T. By uniqueness of this decomposition V and the spectral projections of $|T|$ likewise commute with all the $U(g)$, $g \in G$. Hence $T = \lambda 1$ for some $\lambda \in C$, and $\beta(x,y) = \lambda\langle x,y\rangle$ for all $x, y \in \underline{D}_{\infty}$.

5) => 1): Let T be a continuous linear operator in $\underline{H}$ and suppose $TU(g) = U(g)T$ for $g \in G$. Then $\beta(x,y) = \langle Tx,y\rangle$ is an invariant continuous sesquilinear form on $\underline{D}_{\infty}$. It follows that $T = \lambda 1$ for some $\lambda \in C$. The same argument shows the implication: 5) => 4).

4) => 5): Let $\beta(\cdot,\cdot)$ be a continuous invariant

sesquilinear form on $\underline{D}_\infty$. By Theorem 2.1 $\beta(x,y) = \langle Tx,y\rangle$ for some continuous linear operator T on $\underline{D}_\infty$, and T commutes with the $U_\infty(g)$, $g \in G$. Therefore $\beta(x,y) = \lambda\langle x,y\rangle$ for some $\lambda \in C$.

Corollary 3.1: If G is a connected Lie group the following statements are equivalent:

a) U is irreducible

b) If T is a continuous linear operator on $\underline{D}_\infty$ such that $Tu(X_k) = u(X_k)T$ for $k = 1,2,\dots,d$, then $T = \lambda 1$ for some $\lambda \in C$.

Proof: This follows from Theorem 3.1 and the following result

Proposition 3.1: Let G be a connected Lie group and let V be a continuous representation of G in a Banach space $\underline{B}$. Let T be a continuous linear operator on $\underline{D}_\infty(V)$ such that $Tv(X_k) = v(X_k)T$ for $k = 1,2,\dots,d$. Then $V(g)T = TV(g)$ for all $g \in G$.

Proof: Let x be an analytic vector for V and let $V(t) = V(\exp(tX_k))$ for $t \in \mathbb{R}$. By Lemma 1.2

$$V(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} v(X_k)^n x \quad \text{for} \quad |t| < t_o \quad (\text{for some} \quad t_o > 0)$$

and the series converges in D_∞. Since T is continuous we have

$$TV(t)x = \lim_{N\to\infty} \sum_{0}^{N} \frac{t^n}{n!} v(X_k)^n TX \quad \text{for} \quad |t| < t_o$$

In particular the series $\sum_n \frac{t^n}{n!} v(X_k)^n Tx$ converges in $\underline{B}$ and from the Taylor expansion of $V(t)$ it is clear that the limit must be $V(t)Tx$. Hence $V(t)Tx = TV(t)x$ for $|t| < t_o$.

Let $I = \{t \in \mathbb{R} \mid V(t)Tx = TV(t)x\}$ and let $\rho_o \in I$. Then $V(\rho_o)x$ is again an analytic vector, so (for some $\epsilon > 0$) $V(t)TV(\rho_o)x = TV(t)V(\rho_o)x$ for $|t| < \epsilon$. This shows that I is open and since I is also closed we have $I = \mathbb{R}$.

By Proposition 1.2 the set of analytic vectors is dense in $\underline{D}_\infty$, so $TV(t)x = V(t)Tx$ for all $t \in \mathbb{R}$ and $x \in \underline{D}_\infty$. Using the coordinate system $g(t) \to t$ for the proof of Proposition 1.1 we get $TV(g)x = V(g)Tx$ for all $x \in \underline{D}_\infty$ and all g in a neighborhood of e in G. Since G is generated by any such neighborhood this relation holds for all g in G. Q.E.D.

In particular we get the following (well known) result

<u>Corollary 3.2:</u> Let G be a connected Lie group and let $g \to V(g)$ be a continuous representation in a Banach space $\underline{B}$. For each element Z in the center of $\mathcal{U}(\mathfrak{g})$ we have

$$v(Z)V(g) = V(g)v(Z) \quad \text{for all} \quad g \in G.$$

§4. Some special Banach space representations.

In this section we study families of Banach space representations having a common space of C^∞ vectors. If the family contains an irreducible unitary representation, the results in §2 give some additional information about irreducibility of all representations in the family.

Theorem 4.1: Let $g \to V(g)$ be a continuous representation of a Lie group G in a Banach space $\underline{B}$. Let $\|\cdot\|'$ be a continuous norm on $\underline{D}_\infty(V)$ (i.e. in the $\underline{D}_\infty$-topology) and suppose there exists a non-negative real valued function $c(\cdot)$ on G which is bounded on some neighborhood of $e \in G$ and such that

$$\|V_\infty(g)x\|' \leq c(g)\|x\|' \quad \text{for all } g \in G,\ x \in \underline{D}_\infty .$$

Let $\underline{B}'$ denote the completion of $\underline{D}_\infty$ in the norm $\|\cdot\|'$, and let $V'(g)$ be the extension to $\underline{B}'$ of $V_\infty(g)$ for $g \in G$. Then $g \to V'(g)$ is a continuous representation of G in $\underline{B}'$, and $\underline{D}_\infty(V') \supseteq \underline{D}_\infty(V)$.

Proof: For each $g \in G$, $V_\infty(g)$ has a unique extension to a continuous linear operator $V'(g)$ on the Banach space $\underline{B}'$, and $\|V'(g)\|' \leq c(g)$ for $g \in G$. In particular, $\|V'(g)\|' \leq$ const. for all g in a neighborhood of e in G. It is easily seen that V' has the group property $V'(g_1g_2) = V'(g_1)V'(g_2)$, so in order to prove the continuity it suffices to show that the mapping $g \to V'(g)x$ is continuous at $e \in G$ for each $x \in \underline{D}_\infty$.

By the continuity of $\|\cdot\|'$ there exists a $k > 0$ such that (for some $n > 0$)

$$\|x\|' \leq k \sum_{m=0}^{n} \rho_m(x) \quad \text{for all} \quad x \in \underline{D}_\infty .$$

Here ρ_m denotes the semi-norm determined by V as defined in §1. For each $x \in \underline{D}_\infty$ $V'(g)x = V_\infty(g)x$, so the continuity is clear from Proposition 1.1. In fact the mapping $g \to V'(g)x$ is C^∞ from G to $\underline{B}'$. Therefore $\underline{D}_\infty(V') \supseteq \underline{D}_\infty$, and if v' denotes the infinitesimal representation on $\underline{D}_\infty(V')$ it is easily seen that

$$v'(L)x = v(L)x \quad \text{for} \quad x \in \underline{D}_\infty(V) \quad \text{and} \quad L \in \mathfrak{U}(\mathfrak{g}).$$

<u>Corollary 4.1</u>: Let V be an irreducible representation, and let $\|\cdot\|'$ be a norm on $\underline{D}_\infty$ which satisfies the hypothesis of Theorem 4.1. Suppose that the topology on $\underline{D}_\infty$ defined by the semi-norms $x \to \|v(L)x\|'$, $L \in \mathfrak{U}(\mathfrak{g})$ is equivalent to the original one. Then V' is an irreducible representation in $\underline{B}'$ and $\underline{D}_\infty(V') = \underline{D}_\infty(V)$.

<u>Proof</u>: The original topology on $\underline{D}_\infty$ is defined by the semi-norms $x \to \|v(L)x\|$, $L \in \mathfrak{U}(\mathfrak{g})$. In the proof of Theorem 4.1 we noticed that $v(L)x = v'(L)x$ for $x \in \underline{D}_\infty$, hence it follows that $\underline{D}_\infty$ is a closed subspace of

$\underline{D}_\infty(V')$. On the other hand Proposition 1.2 shows that $\underline{D}_\infty$ is dense in $\underline{D}_\infty(V')$, so they must coincide. Since V is irreducible it follows from Theorem 3.1 (see the remark after the theorem) that V' is irreducible. Q.E.D.

Now we recall some definitions. Let V_1 and V_2 be strongly continuous representations of G in Banach spaces $\underline{B}_1$ and $\underline{B}_2$ respectively. A (separately) continuous sesquilinear form β on $\underline{B}_\infty \times \underline{B}_2$ is called an intertwining form for V_1 and V_2 if

$$\beta(V_1(g)x, V_2(g)y) = \beta(x,y) \quad \text{for all} \quad (x,y)\in \underline{B}_1 \times \underline{B}_2,\ g \in G.$$

A continuous linear mapping T of $\underline{B}_1$ into $\underline{B}_2$ is called an intertwining operator for V_1 and V_2 if

$$TV_1(g) = V_2(g)T \quad \text{for all} \quad g \in G.$$

Let $g \to U(g)$ be a continuous irreducible unitary representation of G in a Hilbert space $\underline{H}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two (not necessarily distinct) norms on $\underline{D}_\infty(U)$ each of which satisfies the conditions in Corollary 4.1. Let V_1 and V_2 denote the corresponding irreducible representations in $\underline{B}_1$ and $\underline{B}_2$ respectively. With this notation we have the following result:

Corollary 4.2:

a) The vector space of intertwining sesquilinear forms for V_1 and V_2 is at most one dimensional.

b) If V_1 and V_2 has a non-zero intertwining operator

T, then $\underline{B}_1 \subseteq \underline{B}_2$ and there exists a $\lambda \neq 0$ such that $Tx = \lambda x$ for all $x \in \underline{B}_1$.

Proof: Let β be an intertwining sesquilinear form for V_1 and V_2. Then the restriction of β to $\underline{D}_\infty$ is continuous and it is invariant under the $U_\infty(g)$, $g \in G$. By Theorem 3.1

$$\beta(x,y) = \lambda\langle x,y\rangle \text{ for all } x, y \in \underline{D}_\infty.$$

Since β is uniquely determined by its values on $\underline{D}_\infty$ this proves a).

Suppose $T : \underline{B}_1 \to \underline{B}_2$ is an intertwining operator for V_1 and V_2. $\|Tx\|_2 \leq c\|x\|_1$ for all $x \in \underline{B}_1$. Then by Proposition 2.1 T leaves $\underline{D}_\infty$ invariant, and the restriction of T to $\underline{D}_\infty$ is a continuous linear operator on this space. By Theorem 3.1 there exists a $\lambda \in C$ such that $Tx = \lambda x$ for $x \in \underline{D}_\infty$. Hence $|\lambda|\ \|x\|_2 \leq c\|x\|_1$ on $\underline{D}_\infty$, and if $T \neq 0$ we have $\lambda \neq 0$.
Therefore $\underline{B}_1 \subseteq \underline{B}_2$ and the inclusion map is continuous. Clearly $Tx = \lambda x$ for $x \in \underline{B}_1$.

Note that if T maps $\underline{B}_1$ onto $\underline{B}_2$ the spaces coincide and the two norms are equivalent.

§5. Examples.

It is known that many of the spaces used in the theory of partial differential operators have a natural connection with group representations. As an illustration of the theory in §4 we choose some of these spaces as examples.

Example 1: Let G be the Heisenberg group, i.e. the group of all real 3×3-matrices of the form

$$g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

The Lie algebra is generated by elements X, Y and Z satisfying the commutation relations

$$[X,Y] = Z, \quad [X,Z] = [Y,Z] = 0.$$

The standard representation (or Schrödinger representation) of G is realized in $L^2(\mathbb{R})$ in the following way:

$$(U(g)f)(x) = e^{ic}e^{ibx}f(x+a), \quad f \in L^2(\mathbb{R}).$$

Using Goodman's theorem (cf. §1) it is easily seen that the space of C^∞ vectors for the representation $g \to U(g)$ is exactly the Schwartz space $\mathcal{S}$. On $\mathcal{S}$ the infinitesimal representation u is given by

$$u(X) \equiv iP = \frac{d}{dx}, \quad u(Y) \equiv iQ = ix, \quad u(Z) = i1$$

where we have introduced the conventional operators P and Q, and the topology on $\mathcal{S}$ can be defined by the semi-norms

$$\varphi \to \|Q^n P^m \varphi\|_2, \quad n,m = 0,1,2,\ldots$$

The following discussion is easily modified to include the case $p = +\infty$, but for simplicity we will assume $1 \leq p < \infty$.

It is well known that the L^p-norm $\|\cdot\|_p$ is continuous on $\mathcal{S}$, and using a Sobolev lemma it is easily seen that the topology on $\mathcal{S}$ can be defined by the semi-norms

$$\varphi \to \|Q^n P^m \varphi\|_p, \quad n,m = 0,1,2,\ldots$$

By Theorem 4.1 and Corollary 4.1 the representation U_∞ in $\mathcal{S}$ has an extension to a strongly continuous irreducible representation V_p in $L^p(\mathbb{R})$ and $\underline{D}_\infty(V_p) = \mathcal{S}$.

If β is a non-zero intertwining sesquilinear form for V_p and V_q we get from Corollary 4.2 that

$$\beta(\varphi,\psi) = \lambda \cdot \int \varphi(x)\overline{\psi(x)}dx \quad \text{for all } \varphi, \psi \in \mathcal{S}.$$

This is possible if and only if $\frac{1}{p} + \frac{1}{q} = 1$.

If T is an intertwining operator for V_p and V_q it follows from Corollary 4.2 that $T = 0$ or $p = q$ in which case $T = \lambda 1$ for some $\lambda \in C$.

So although the representations all have the same space of C^∞ vectors they are far from being equivalent in the classical sense. The example could be generalized by introducing a tempered weight function in the definition of the L^p-norm. Instead of doing this we consider a slightly different situation.

Let k be a tempered weight function on $\mathbb{R}$ in the sense of Hörmander [15, pp. 34-37], i.e. k is a positive real function and there exist constants C and N such that

$$k(x+y) \leq (1+C|x|)^N k(y) \quad \text{for} \quad x, y \in \mathbb{R}.$$

For $1 \leq p < \infty$ we let $B_{p,k}$ denote the space of all tempered distributions f such that the Fourier transformed $\hat{f}$ is a function and

$$\|f\|_{p,k} \equiv \{\int |k(x)\hat{f}(x)|^p dx\}^{1/p} < \infty$$

Then $B_{p,k}$ is a Banach space with the norm $\|\cdot\|_{p,k}$, and $\mathcal{S}$ is dense in this space. The norm $\|\cdot\|_{p,k}$ is continuous on $\mathcal{S}$ and in fact the topology on $\mathcal{S}$ can be defined by the semi-norms:

$$\varphi \to \|Q^n P^m \varphi\|_{p,k}, \quad n,m = 0,1,2,\ldots$$

For $\varphi \in \mathcal{S}$ we have $\|U_\infty(g)\varphi\|_{p,k} \leq (1+C|b|)^N \|\varphi\|_{p,k}$

It follows from Theorem 4.1 and Corollary 4.1 that the representation in $\mathcal{S}$ has an extension to a strongly continuous irreducible representation V in $B_{p,k}$ and $\underline{D}_\infty(V) = \mathcal{S}$.

Again one can compare the different representations. Properties of the spaces $B_{p,k}$ can be found in [15].

The same results hold for all the different irreducible representations of G [23], and they remain valid for any (finite) number of degrees of freedom. For other examples of a similar nature see [10].

Example 2: Let $G = \mathbb{R}^d$ and let V_p denote the regular representation of G in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

$$(V_p(t)f)(x) = f(x+t), \quad f \in L^p.$$

Then it is easily seen that

$$\underline{D}_{\infty}(V_p) = \{f \in C^{\infty}(\mathbb{R}^d) \mid D^{\alpha}f \in L^p \text{ for all } \alpha\} \quad ;$$

here $D^{\alpha} = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_d})^{\alpha_d}$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$ is any set of non-negative integers.

For $q \geq p$ $\|\cdot\|_q$ is a continuous norm on $\underline{D}_{\infty}(V_p)$. The representation given in Theorem 4.1 is just V_q, and we have $\underline{D}_{\infty}(V_p) \subseteq \underline{D}_{\infty}(V_q)$ and the inclusion map is continuous.

Again the spaces $\underline{D}_{\infty}(V_p)$ are well known. For details we refer to Schwartz [24, p. 199].

This example can be generalized to arbitrary Lie groups. The description of $\underline{D}_{\infty}$ remains valid but the analogy is complete only if G is unimodular. (Proposition 5.1).

Let G be a Lie group and let $1 \leq p < \infty$. We form $L^p(G)$ for some right invariant Haar measure on G. The (right-) regular representation V_p in $L^p(G)$ is defined by $(V_p(g)f)(x) = f(x\cdot g)$ for $g \in G$, $f \in L^p(G)$. Then $\|V_p(g)f\|_p = \|f\|_p$, and $g \to V_p(g)$ is a strongly continuous representation of G in $L^p(G)$.

Each X in $\mathfrak{g}$ defines a left invariant differential operator $\tilde{X}$ on G in the usual way

$$(\tilde{X}f)(x) = \frac{d}{dt} f(x\cdot\exp(tX))|_{t=0}$$

if f is a differentiable function on G. Let $\{X_1,\ldots,X_d\}$ be a basis in $\mathfrak{g}$. We use the multi-index notation $\tilde{X}^\alpha = \tilde{X}_1^{\alpha_1} \ldots \tilde{X}_d^{\alpha_d}$ if $\alpha = (\alpha_1,\ldots,\alpha_d)$ is a set of non negative integers. As usual $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. We let Δ denote the modular function on G. Specifically, Δ is defined by the equation:

$$\int_G \varphi(g\cdot x)dx = \Delta(g)^{-1} \int_G \varphi(x)dx$$

for $g \in G$ and $\varphi \in C_o(G)$. (Here $C_o(G)$ denotes the space of continuous functions with compact support). Then we have the following Sobolev inequality.

<u>Lemma 5.1</u>: For $1 \leq p < \infty$ there exists a constant $C > 0$ such that

$$\|f\cdot\Delta^{\frac{1}{p}}\|_\infty \leq C\cdot \sum_{|\alpha|\leq d} \|\tilde{X}^\alpha f\|_p .$$

for all differentiable functions f for which the right hand side is finite.

Proof: Suppose we had the following inequality:

$$|f(e)| \leq C \cdot \sum_{|\alpha| \leq d} \|\tilde{X}^{\alpha} f\|$$

Then by left invariance of the operators $\tilde{X}^{\alpha}$

$$|f(g)| \leq C \sum_{|\alpha| \leq d} \{\int_G |\tilde{X}^{\alpha} f(g \cdot x)|^p dx\}^{1/p} = \Delta(g)^{-\frac{1}{p}} C \sum_{|\alpha| \leq d} \|\tilde{X}^{\alpha} f\|_p$$

for all $g \in G$.

Hence it suffices to study the situation in a neighborhood of e in G.

Let (U,φ) be a local coordinate system in a neighborhood of e in G

$$\varphi(g) = (x_1(g), \ldots, x_d(g)) \in \mathbb{R}^d.$$

Using the notation $f^* = f \circ \varphi^{-1}$ we have [14, p. 10]

$$(\tilde{X}_i f)(g) = \sum_{j=1}^{d} (\tilde{X}_i x_j)(g) \left(\frac{\partial f^*}{\partial x_j}\right)_{\varphi(g)}$$

The vector fields $\{\tilde{X}_1, \ldots, \tilde{X}_d\}$ induce a basis in the tangent space at each point of G. In particular, the matrix $\{(\tilde{X}_i x_j)(g)\}$ is non-singular for each $g \in U$. Now, define

$$Y_i = \sum_{j=1}^{d} (\tilde{X}_i x_j)^*(x) \frac{\partial}{\partial x_j} \quad \text{on} \quad \varphi(U).$$

Then Y_i is an analytic vector field, and the set $\{Y_1, \ldots, Y_d\}$ is linearly independent at each point x in $\varphi(U)$. We have $Y_i f^* = (\tilde{X}_i f)^*$, so

$(\tilde{X}^{\alpha} f)^* = Y_1^{\alpha_1} \ldots Y_d^{\alpha_d} f^*$ for all α.

Let $x = \varphi(e)$ and let $K \subseteq U$ be a compact set such that $\varphi(K) = \{y \in \mathbb{R}^d \mid |x-y| \leq r\}$ for some $r > 0$. Then

$$\int_K |\tilde{X}^\alpha f(g)|^p dg = \int_{\varphi(K)} |Y_1^{\alpha_1} \cdots Y_d^{\alpha_d} f^*(y)|^p F(y)dy,$$

where the density F is a positive non-vanishing C^∞ function [14, Ch. X].

Therefore it suffices to prove the following inequality:

$$|f^*(x)| \leq C \sum_{|\alpha| \leq d} \{ \int_{\varphi(K)} |Y_1^{\alpha_1} \cdots Y_d^{\alpha_d} f^*(y)|^p F(y)dy\}^{1/p}$$

By changing the constant C the function F can be neglected, so the following lemma completes the proof.

<u>Lemma 5.2</u>: Let $V \subseteq \mathbb{R}^d$ be an open set and let $x \in V$. Suppose $Y_1,\ldots,Y_d$ are C^∞ vector fields on V which are linearly independent at each point of V. Then for $1 \leq p < \infty$ and $r > 0$ with $K(x,r) = \{y \mid |x-y| \leq r\} \subseteq V$ there exists a constant $C > 0$ such that

$$|f(x)| \leq C \cdot \sum_{|\alpha| \leq d} \{ \int_{K(x,r)} |Y_1^{\alpha_1} \cdots Y_d^{\alpha_d} f(y)|^p dy\}^{1/p}$$

for all $f \in C^d(V)$.

<u>Proof</u>: Let $Y_i = \sum_{j=1}^{d} a_{ij}(y) \frac{\partial}{\partial y_j}$ where $\det \{a_{ij}(y)\} \neq 0$ for all $y \in V$.

Let $\{b_{ij}(y)\} = \{a_{ij}(y)\}^{-1}$. Then the b_{ij} are C^∞ functions on V and

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{d} b_{ij}(y)Y_j$$

Let $f \in C^d(V)$. Choose $g \in C_o^\infty(V)$ with $\operatorname{supp} g \subseteq K(x,r)$ and such that $g(y) = 1$ in a neighborhood of x. Then $h = f \cdot g \in C_o^d(V)$ and $\operatorname{supp} h \subseteq K(x,r)$, so

$$f(x) = h(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \frac{\partial^d}{\partial y_1 \cdots \partial y_d} h(y)dy$$

Therefore

$$|f(x)| \leq \int_{K(x,r)} \left|\frac{\partial^d}{\partial y_1 \cdots \partial y_d} h(y)\right|dy$$

$$\leq C \cdot \sum_{|\alpha|\leq d} \int_{K(x,r)} |Y_1^{\alpha_1} \cdots Y_d^{\alpha_d} f(y)|dy,$$

where C depends only on the behavior of derivatives of g and the b_{ij} $i,j = 1,2,\ldots,d$.

Then Hölder's inequality completes the proof. Q.E.D.

Now we are ready to characterize the space of C^∞ vectors for the regular representation in $L^p(G)$.

<u>Proposition 5.1</u>: Let V_p denote the regular representation of G in $L^p(G)$ for $1 \leq p < \infty$. Then

$$\underline{D}_\infty(V_p) = \{f \in C^\infty(G) \mid \tilde{X}^\alpha f \in L^p(G) \text{ for all } \alpha\}$$

and $C_o^\infty(G)$ is dense in $D_\infty(V_p)$.

If G is unimodular we have $\underline{D}_\infty(V_p) \subseteq \underline{D}_\infty(V_q)$ for $q \geq p$ and the inclusion map is continuous.

Proof: Since the proof of the first part is similar to the proof of Theorem 5.1 (below) we omit the details. The second part follows from Lemma 5.1 and the Riesz convexity theorem [11].

Example 3: The space of smooth vectors of an induced representation of a Lie group.

In [2] Blattner developed the basic theory of induced representations of not necessarily separable groups. The theory was applied to study the intertwining number of two representations of a Lie group. Blattner's methods are based on properties of the space of C^∞ vectors of the induced representation, and in this section we give a complete characterization of this space (Theorem 5.1).

Since we use some definitions and results from Blattner's paper, it is convenient to follow Blattner's notation [2].

Let G be a Lie group and let H be a closed subgroup. We choose some right invariant Haar measures on G and H and we let Δ and δ denote the respective modular functions. Let $M = G/H$ denote the right coset space and let π be the projection of G onto M.

Suppose L is a continuous unitary representation of H in a Hilbert space $\underline{h}$, and let F^* be the set of functions f from G to $\underline{\mathfrak{h}}$ satisfying the following conditions:

i) $f(\cdot)$ is measurable

ii) $f(\xi\cdot x) = \Delta(\xi)^{-\frac{1}{2}}\delta(\xi)^{\frac{1}{2}}L(\xi)f(x)$ for $\xi \in H$ and $x \in G$.

iii) $\|f(\cdot)\|^2$ is locally integrable

Each such function f defines a Radon measure μ_f on M via the equation

$$\int_G \|f(x)\|^2\varphi(x)dx = \int_M (\tau\varphi)(p)d\mu_f(p)$$

where $\varphi \in C_o(G)$ and $(\tau\varphi)(\pi(x)) = \int_H \varphi(\xi\cdot x)d\xi$. We set $\|f\| = \mu_r(M)^{1/2}$ and $F = \{f \in F^* \mid \|f\| < \infty\}$. If we identify functions in F which are equal locally almost everywhere (l.a.e.) we get a Hilbert space $\underline{H}^L$, and the representation U^L of G is defined in the following way:

$$(U^L(g)f)(x) = f(x\cdot g) \quad \text{for} \quad f \in \underline{H}^L.$$

Let $C^\infty(G,\underline{\mathfrak{h}})$ denote the space of infinitely differentiable functions from G to $\underline{\mathfrak{h}}$. Then we have the following result:

Theorem 5.1:

$$\underline{D}_\infty(U^L) = \{f \in C^\infty(G,\underline{\mathfrak{h}}) \mid \tilde{X}^\alpha f \in \underline{H}^L \text{ for all } \alpha\}$$

Proof: Let $f \in C^\infty(G,\underline{\mathfrak{h}})$ and $X \in \mathfrak{g}$ and suppose

$f \in \underline{H}^L$ and $\tilde{X}f \in \underline{H}^L$. Then for $t \neq 0$

$$\frac{1}{t}[f(y\cdot\exp(tX))-f(y)] = \frac{1}{t}\int_0^t \tilde{X}f(y\cdot\exp(sX))ds$$

Hence by Hölder's inequality (suppose $t > 0$):

$$\|\frac{1}{t}[f(y\cdot\exp(tX))-f(y)] - \tilde{X}f(y)\|^2$$
$$\leq \frac{1}{t}\int_0^t \|\tilde{X}f(y\cdot\exp(sX))-\tilde{X}f(y)\|^2 ds.$$

If $\varphi \in C_0^+(G)$ we get

$$\int_G \|\frac{1}{t}[f(y\cdot\exp(tX))-f(y)] - \tilde{X}f(y)\|^2\varphi(y)dy$$
$$\leq \int_G \frac{1}{t}\int_0^t \|\tilde{X}f(y\cdot\exp(sX))-\tilde{X}f(y)\|^2 ds\varphi(y)dy$$
$$= \frac{1}{t}\int_0^t \|\tilde{X}f(y\cdot\exp(sX))-\tilde{X}f(y)\|^2\varphi(y)dy\ ds$$

By definition of the norm in $\underline{H}^L$ this gives

$$\|\frac{1}{t}[U^L(\exp(tX))f-f] - \tilde{X}f\|^2$$
$$\leq \frac{1}{t}\int_0^t \|U^L(\exp(sX))\tilde{X}f-\tilde{X}f\|^2 ds \to 0 \quad \text{as} \quad t \to 0,$$

since the integrand is a continuous function of s.

In other words, $\tilde{X}f$ is in the domain of the infinitesimal generator $u_1(X)$ of the one parameter group $t \to U^L(\exp(tX))$ and $u_1(X)f = \tilde{X}f$. If $\tilde{X}^\alpha f \in \underline{H}^L$ for all α, it follows that f is in the domain of all powers of the operators $u_1(X)$, $X \in \mathfrak{g}$. Then by Goodman's theorem we have $f \in \underline{D}_\infty$ and $u(D)f = \tilde{D}f$ for all $D \in \mathfrak{U}(\mathfrak{g})$ (For $D \in \mathfrak{U}(\mathfrak{g})$ we let $\tilde{D}$ denote the corresponding left invariant differential operator on G).

In order to prove the other inclusion we introduce the functions $\epsilon(\varphi,v)$, $\varphi \in C_0^\infty(G)$ and $v \in \underline{V}$ [2, p. 82] defined as follows:

$$\epsilon(\varphi,v)(x) = \int_H \varphi(\xi\cdot x)\delta(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}}L(\xi^{-1})v \, d\xi,$$

and we let $\underline{D} = \text{span}\,\{\epsilon(\varphi,v) \mid \varphi \in C_0^\infty(G),\ v \in \underline{V}\}$.

Then $\underline{D} \subseteq \underline{D}_\infty$ [2, Lemma 6] and clearly $\underline{D}$ is invariant under the $U^L(g)$, $g \in G$. On the other hand $\underline{D}$ is dense in $\underline{H}^L$ [2, Lemma 2], so by Proposition 1.2 we get that $\underline{D}$ is dense in $\underline{D}_\infty$.

It is easy to see that $\epsilon(\varphi,v) \in C^\infty(G,\underline{V})$ for all $\varphi \in C_0^\infty(G)$ and $v \in \underline{V}$, so all functions in $\underline{D}$ are infinitely differentiable. To complete the proof we need the following result:

Lemma 5.3: There exists a constant $C > 0$ such that

$$\|\Delta(x)^{1/2}\cdot f(x)\| \leq C\cdot \sum_{|\alpha|\leq d} \|\tilde{X}^\alpha f\|$$

for all $x \in G$, $f \in \underline{D}$.

The proof will be given later.

Let $f \in \underline{D}_\infty(U^L)$. Then there exists a sequence $\{f_n\} \subseteq \underline{D}$ such that for all $D \in \mathcal{U}(\mathfrak{g})$:

$$\|\tilde{D}f_n - u(D)f\| \to 0 \text{ as } n \to \infty.$$

It follows from the proof of Proposition 1 in [2] that for each $D \in \mathcal{U}(\mathfrak{g})$ there exists a subsequence $\{f_{n_k}\}$

(depending on D) such that $\tilde{D}f_{n_k} \to u(D)f$ l.a.e.

Let $\mathcal{S}(G,\underline{V})$ denote $C^\infty(G,\underline{V})$ as a topological vector space (with the usual topology). By Lemma 5.3 $\{f_n\}$ is a Cauchy sequence in $\mathcal{S}(G,\underline{V})$, so there exists a unique function $f_o \in \mathcal{S}(G,\underline{V})$ such that $f_n \to f_o$; i.e. $\tilde{D}f_n \to \tilde{D}f$ uniformly on compact sets for all $D \in \mathcal{U}(\mathfrak{g})$. Then $\tilde{D}f_o = u(D)f$ l.a.e., so we may assume $f = f_o$. Hence f is infinitely differentiable and $\tilde{D}f = u(D)f$ for all $D \in \mathcal{U}(\mathfrak{g})$.

Proof of Lemma 5.3: By the proof of Lemma 5.1 there exists a compact neighborhood K of e in G and a constant $C_K > 0$ such that:

$$\|f(e)\| \leq C_K \sum_{|\alpha|\leq d} \{\int_K \|\tilde{X}^\alpha f(y)\|^2 dy\}^{1/2}$$

for all differentiable functions from G to $\underline{V}$. Let $\varphi_o \in C_o^+(G)$ such that $\varphi_o = 1$ on K. Taking $C = C_K\|\tau\varphi_o\|_\infty^{-\frac{1}{2}}$ and $\varphi = \|\tau\varphi_o\|_\infty^{-1}\varphi_o$ we have $0 \leq \tau\varphi \leq 1$ and

$$\|f(e)\| \leq C \sum_{|\alpha|\leq d} \{\int_G \|\tilde{X}^\alpha f(y)\|^2 \varphi(y)dy\}^{1/2}$$

By the left invariance of $\tilde{X}^\alpha$ we get

$$\|f(x)\| \leq C \sum_{|\alpha|\leq d} \{\int \|\tilde{X}^\alpha f(x\cdot y)\|^2 \varphi(y)dy\}^{1/2}$$

$$= \Delta(x)^{-\frac{1}{2}} C \sum_{|\alpha|\leq d} \{\int_G \|\tilde{X}^\alpha f(y)\|^2 \varphi(x^{-1}y)dy\}^{-1/2}$$

If $\tilde{X}^\alpha f \in \underline{H}^L$ for all α with $|\alpha| \leq d$ this gives

$$\|f(x)\| \Delta(x)^{1/2} \leq C \sum_{|\alpha| \leq d} \|\tilde{X}^\alpha f\| \quad \text{for all } x \in G.$$

Q.E.D.

<u>Corollary 5.1</u>: For all $f \in \underline{D}_\infty(U^L)$ we have

$$\|\Delta(x)^{1/2} f(x)\| \leq C \sum_{|\alpha| \leq d} \|\tilde{X}^\alpha f\| \quad \text{for all } x \in G.$$

In particular, for each fixed $x \in G$, $f \to f(x)$ is a continuous linear mapping from $\underline{D}_\infty$ to $\underline{\mathfrak{h}}$.

<u>Corollary 5.2</u>: If f is an analytic vector for U^L, then f is an analytic function from G to $\underline{\mathfrak{h}}$.

<u>Proof</u>: By Nelson's characterization of the analytic vectors (cf. §1) there exists a constant $C > 0$ such that

$$\|\tilde{X}_{i_1} \dots \tilde{X}_{i_n} f\| \leq C^n \cdot n! \quad \text{for all } n$$

Let $K \subseteq G$ be any compact set. By Corollary 5.1 we have (we can neglect Δ on K)

$$\sup_{x \in K} \|\tilde{X}^\alpha f(x)\| \leq \text{const.} \sum_{|\beta| \leq d} \|\tilde{X}^\beta \tilde{X}^\alpha f\|$$

$$\leq \text{const.} \sum_{|\beta| \leq d} C^{|\alpha+\beta|} |\alpha+\beta|! \quad \text{for all } \alpha$$

Therefore there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} \|X^\alpha f(x)\| \leq C_K^{|\alpha|} \, |\alpha|! \quad \text{for all } \alpha.$$

This completes the proof (see e.g. [1] or [21, Th. 2]).

Q.E.D.

Remark: The "converse" is not true. There exists an analytic function f on $\mathbb{R}$ such that $f^{(n)} \in L^2(\mathbb{R})$ for $n = 0,1,2,\ldots,$ but f is not an analytic vector for the regular representation. (For example one can show that the function F given in [16, p. 177] has these properties).

Blattner's intertwining number theorem:

As an application of our Sobolev lemma we get a version of Blattner's intertwining number theorem. We have the following situation [2, §5, Th. 3]:

For $i = 1,2$ H_i is a closed subgroup of G with modular function δ_i. L_i is a unitary representation of H_i in a Hilbert space $\underline{\mathfrak{h}}_i$, and U^{L_i} denotes the corresponding induced representation of G. Let $\mathfrak{R}(U^{L_1}, U^{L_2})$ denote the space of intertwining operators of U^{L_1} and U^{L_2} and let $I(U^{L_1},U^{L_2}) = \dim \mathfrak{R}(U^{L_1},U^{L_2})$.

For each $T \in \mathfrak{R}(U^{L_1},U^{L_2})$ we define a linear mapping F_T from $C_o^\infty(G)$ into $\mathcal{L}(\underline{\mathfrak{h}}_1,\underline{\mathfrak{h}}_2)$ as follows:

$F_T(\varphi)v = (T \quad \varepsilon(\varphi,v))(e)$ for $v \in \underline{\mathfrak{h}}_1$ and $\varphi \in C_o^\infty(G)$

If f is any function on G and $(\xi_1,\xi_2) \in H_1 \times H_2$ we let

$$(\rho_{\xi_1,\xi_2} f)(x) = f(\xi_1^{-1} x \xi_2) \quad \text{for } x \in G.$$

For each relatively compact open set $O \subseteq G$ we give $C_o^\infty(O)$ the topology induced by the norm $\varphi \to \sum_{|\alpha| \leq d} \|\tilde{X}^\alpha \varphi\|_\infty$, and we give $C_o^\infty(G)$ the corresponding inductive limit topology.

Theorem 5.2: Let $\mathfrak{M}$ denote the subspace of maps $F \in \mathcal{L}(C_o^\infty(G), \mathcal{L}(\underline{\mathfrak{v}}_1, \underline{\mathfrak{v}}_2))$ such that

$$F(\rho_{\xi_1,\xi_2}\varphi) = \delta_1(\xi_1)^{\frac{1}{2}}\delta_2(\xi_2)^{\frac{1}{2}}\Delta(\xi_1\xi_2^{-1})^{\frac{1}{2}}L_2(\xi_2)F(\varphi)L_1(\xi_1^{-1})$$

for all $(\xi_1,\xi_2) \in H_1 \times H_2$ and all $\varphi \in C_o^\infty(G)$. Then the map $T \to F_T$ is a faithful linear map of $\mathcal{R}(U^{L_1}, U^{L_2})$ into $\mathfrak{M}$. In particular we have $I(U^{L_1}, U^{L_2}) \leq \dim \mathfrak{M}$.

Proof: Blattner's proof applies word for word, using the inequality from Corollary 5.1 instead of the elliptic inequality used by Blattner. Q.E.D.

§6. C^∞-systems of imprimitivity.

In [18] Mackey introduced "systems of imprimitivity" for unitary representations of locally compact groups, and it is well known that Mackey's imprimitivity theorem gives a rather complete analysis of certain important types of group representations. Mackey proved the theorem in the separable case using measure theoretic arguments, and the theorem was later proved in general by Loomis and Blattner [4].

In this section we introduce C^∞-systems of imprimitivity for unitary representations of Lie groups, and we show that such a system gives rise to an induced representation. Roughly speaking the conventional projection valued measure is replaced by a "measure" whose values are (possibly unbounded) positive operators. Similar systems have been introduced by Davies [7] who has also given a discussion of the possible physical applications of the theory (see also [8]).

A system of imprimitivity in the sense of Mackey naturally gives rise to a C^∞-system of imprimitivity, and as a special case of the main theorem of this section we get a new proof of Mackey's imprimitivity theorem (for the case of Lie groups). First we recall Mackey's definition of a system of imprimitivity for a given group representation.

Let G be a Lie group and let $M = G/H$ where $H \subseteq G$ is a closed subgroup. As usual $C_o(M)$ denotes the space of complex valued continuous functions with compact support on M, and $C_o^\infty(M)$ denotes the subspace of infinitely differentiable functions. Here M is given the usual analytic structure [14, Ch. II], and G acts (on the right) as a Lie transformation group of M. We let $g \to R(g)$ denote the corresponding representation of G in $C_o(M)$, i.e.

$$(R(g)\psi)(\pi(a)) = \psi(\pi(a\cdot g)) \quad \text{for} \quad g \in G,\ \psi \in C_o(M)$$

Assume $g \to U(g)$ is a continuous unitary representation of G in a Hilbert space $\underline{K}$.

Definition 6.1: A system of imprimitivity for U based on M is a $*$-homomorphism $P(\cdot)$ of the $*$-algebra $C_o(M)$ (under the pointwise operations) into the $*$-algebra $\mathcal{L}(\underline{K})$ of all continuous linear operators on $\underline{K}$ such that

i) $P(C_o(M))\underline{K}$ is dense in $\underline{K}$

ii) $U(g)P(\psi)U(g)^{-1} = P(R(g)\psi)$ for $\psi \in C_o(M)$, $g \in G$.

As an example, let L be a continuous unitary representation of H and let U^L be the corresponding induced representation of G in $\underline{H}^L$ (cf. §5, Ex. 3). For $\psi \in C_o(M)$ we define $P^L(\psi)$ on $\underline{H}^L$ by setting

$$(P^L(\psi)f)(a) = \psi(\pi(a))\cdot f(a).$$

Then $P^L(\psi)$ is well-defined, and it is easily seen that $P^L(\cdot)$ is a system of imprimitivity for U^L. We note some important properties of the pair (U^L, P^L).

If $\varphi \in C_o(G)$ and $f_1, f_2 \in \underline{H}^L$ it follows from the definition of the scalar product in $\underline{H}^L$ that

$$<P^L(\tau\varphi)f_1, f_2> = \int_G \varphi(a)<f_1(a), f_2(a)>da$$

Here τ is the mapping of $C_o(G)$ onto $C_o(M)$ defined in §5, Example 3, and we remark that τ also maps $C_o^\infty(G)$ onto $C_o^\infty(M)$ [6].

By Corollary 5.1 $\beta(f_1, f_2) = <f_1(e), f_2(e)>$ defines a continuous sesquilinear form on $\underline{D}_\infty(U^L)$, and $<f_1(a), f_2(a)> = \beta(U^L(a)f_1, U^L(a)f_2)$ for all $a \in G$, $f_1, f_2 \in \underline{D}_\infty$. Finally if $\psi \in C_o^\infty(M)$ it follows from Theorem 5.1 that $P^L(\psi)$ leaves $\underline{D}_\infty$ invariant and hence defines a continuous linear operator on this space. The following definition as well as the proof of Theorem 6.1 grew out of these observations.

Now let U be any continuous unitary representation of G in a Hilbert space $\underline{K}$. We let $\mathcal{L}(\underline{D}_\infty)$ denote the space of all continuous linear operators on $\underline{D}_\infty = \underline{D}_\infty(U)$.

<u>Definition 6.2</u>: A C^∞-system of imprimitivity for U based on M is a linear mapping $P(\cdot)$ from $C_o^\infty(M)$ into $\mathcal{L}(\underline{D}_\infty)$ such that

1) $P(C_o^\infty(M))\underline{D}_\infty$ is dense in $\underline{K}$
2) $U(g)P(\psi)U(g^{-1}) = P(R(g)\psi)$ for all $g \in G$, $\psi \in C_o^\infty(M)$
3) For each $x \in \underline{D}_\infty$ and $\psi \geq 0$ we have

$$<P(\psi)x, x> \geq 0, \quad \text{and} \quad \sup_{0 \leq \psi \leq 1} < P(\psi)x, x> < \infty.$$

P is called normalized in case

$$\sup_{0 \leq \psi \leq 1} \langle P(\psi)x,x\rangle = \|x\|^2 \quad \text{for all} \quad x \in \underline{D}_\infty.$$

First we show that this is a generalization of the usual notion.

Lemma 6.1: Let $P(\cdot)$ be a system of imprimitivity for U based on M. Then for $\psi \in C_0^\infty(M)$ the operator $P(\psi)$ leaves $\underline{D}_\infty$ invariant, and the map $\psi \to P(\psi)|_{\underline{D}_\infty}$ is a normalized C^∞-system of imprimitivity for U.

Proof: Since $P(\cdot)$ is a $*$-homomorphism it is clear that $P(\cdot)$ is positivity preserving, and it is well known that $\|P(\psi)\| \leq \|\psi\|_\infty$ for all $\psi \in C_0(M)$ [9]. From ii) we see that the kernel of $P(\cdot)$ is invariant under the $R(g)$, $g \in G$. Since $P(\cdot)$ is non-trivial, an application of the Stone-Weierstrass theorem shows that $P(\cdot)$ is injective, so we get $\|P(\psi)\| = \|\psi\|_\infty$ [9]. Let $E = \sup\{P(\psi) \mid \psi \in C_0^\infty(M),\ 0 \leq \psi \leq 1\}$. Then it is easily seen that E is a projection. If $Ex = 0$ we get $P(\psi)x = 0$ for all $\psi \in C_0^\infty(M)$. Then by i) $x = 0$, so $E = I$.

For $X \in \mathfrak{g}$ we let $\hat{X}$ denote the corresponding vector field on M. Then for $\psi \in C_0^\infty(M)$ we have

$$\frac{1}{t}[R(\exp(tX))\psi - \psi] \to \hat{X}\psi$$

uniformly on M as $t \to 0$. Now let $x \in \underline{D}_\infty$ and

$\psi \in C_0^\infty(M)$. Then

$$U(\exp(tX))P(\psi)x = P(R(\exp(tX))\psi)U(\exp(tX))x$$

and since the right hand side is differentiable at $t = 0$ it follows that $P(\psi)x$ is in the domain of $u_1(X)$, and

$$u_1(X)P(\psi)x = P(\hat{X}\psi)x + P(\psi)u(X)x$$

Then (by induction) it follows from Goodman's theorem that $P(\psi)x$ is a C^∞-vector for U. Thus $P(\psi)$ leaves $\underline{D}_\infty$ invariant, and by the closed graph theorem $P(\psi)$ is continuous on $\underline{D}_\infty$. Q.E.D.

Remark: Suppose P is a system of imprimitivity for U in the sense of Davies [7], i.e. P is a positivity preserving linear map from $C_0(M)$ into $\mathcal{L}(\underline{K})$ satisfying 2) (or rather ii) of Def. 6.1). Then P is automatically continuous from the inductive limit topology on $C_0(M)$ to the norm topology on $\mathcal{L}(\underline{K})$. Therefore, for $\psi \in C_0^\infty(M)$, the proof of Lemma 6.1 shows that $P(\psi)$ leaves $\underline{D}_\infty$ invariant and $P(\psi)|_{\underline{D}_\infty}$ is a continuous linear operator on $\underline{D}_\infty$.

In such cases we shall not distinguish between the operator $P(\psi)$ and its restriction to $\underline{D}_\infty$. Also we call P a C^∞-system of imprimitivity for U in case the restrictions give such a system.

If $P(\cdot)$ is a C^∞-system of imprimitivity for U the $P(\psi)$ can be unbounded operators in $\underline{K}$, and examples show that it is no longer true that U is equivalent to an induced representation. Therefore the natural question arises: "What is the structure of the pair (U,P)". We shall now describe a method of producing C^{∞}-systems of imprimitivity which will turn out to be the most general one.

Example 6.1: Let L be a continuous unitary representation of the subgroup H and let U^L be the corresponding induced representation of G. If U^L is reducible we obtain a new pair (U_1,P_1) by restricting (U^L,P^L) to a $U^L(G)$-invariant subspace $\underline{H}$ of $\underline{H}^L$, i.e. we let

$$U_1(g) = U^L(g)\Big|_{\underline{H}} \quad \text{for} \quad g \in G$$

$$P_1(\psi) = EP^L(\psi)\Big|_{\underline{H}} \quad \text{for} \quad \psi \in C_0^{\infty}(M)$$

where E is the projection of $\underline{H}^L$ onto $\underline{H}$. Then P_1 is a normalized C^{∞}-system of imprimitivity for the subrepresentation U_1, and if E does not commute with the $P^L(\psi)$ P_1 is not an algebra homomorphism. (Lemma 6.2). Now suppose T is a continuous linear operator on $\underline{D}_{\infty}(U_1)$ which commutes with the $U_1(g)$, $g \in G$ and let

$$P_2(\psi) = TP_1(\psi)T \quad \text{for} \quad \psi \in C_0^{\infty}(M)$$

Then if T is symmetric in $\underline{H}$ and 0 is not an eigenvalue for T^* it can be verified that P_2 is a

C^∞-system of imprimitivity for U_1, and the operators $P_2(\psi)$ are possibly unbounded.

Before we state the main theorem of this section we recall that two pairs (U_1,P_1) and (U_2,P_2) are called unitarily equivalent if there exists a unitary operator W such that

$$WP_1(\psi)W^{-1} = P_2(\psi) \quad \text{for all} \quad \psi \in C_o^\infty(M)$$

and

$$WU_1(g)W^{-1} = U_2(g) \quad \text{for all} \quad g \in G.$$

Now let U be a continuous unitary representation of G in a Hilbert space $\underline{K}$ and let $M = G/H$ where H is a closed subgroup of G.

Theorem 6.1: Let P be a C^∞-system of imprimitivity for U based on M. Then there exists a normalized C^∞-system of imprimitivity P_o for U and a continuous unitary representation L of H such that the pair (U,P_o) is unitarily equivalent to the restriction of (U^L,P^L) to a $U^L(G)$-invariant subspace $\underline{H}$ of $\underline{H}^L$. Moreover P_o can be chosen in such a way that

$$P(\psi) = TP_o(\psi)T \quad \text{for all} \quad \psi \in C_o^\infty(M),$$

where T is a continuous linear operator on $\underline{D}_\infty$ which is essentially self adjoint in $\underline{K}$, T^* is 1-1 and $TU(g) = U(g)T$ for all $g \in G$.

Proof: For each $x \in \underline{D}_\infty$ the map $\varphi \to \langle P(\tau\varphi)x,x\rangle$ is a positive linear functional on $C_0^\infty(G)$. By a well known result [5] it extends uniquely to a non-negative measure $\mu_{x,x}$ on G. Therefore (by polarization) there exists a family $\{\mu_{x,y} \mid x,y \in \underline{D}_\infty\}$ of Radon measures on G such that

$$\langle P(\tau\varphi)x,y\rangle = \int_G \varphi(a)d\mu_{x,y}(a) \quad \text{for} \quad \varphi \in C_0^\infty(G)$$

For $g \in G$ we have

$$\int_G \varphi(a\cdot g)d\mu_{x,y}(a) = \langle P(\tau\varphi)U(g^{-1})x,U(g^{-1})y\rangle$$

For $X \in \mathfrak{g}$ this gives

$$\int_G (\tilde{X}\varphi)d\mu_{x,y} = -\langle P(\tau\varphi)u(X)x,y\rangle - \langle P(\tau\varphi)x,u(X)y\rangle$$

$$= -\int_G \varphi d\mu_{u(X)x,y} - \int_G \varphi d\mu_{x,u(X)y} \,.$$

Therefore, if $\tilde{D}$ is any left-invariant differential operator on G it follows that there exists a Radon measure μ on G (depending on x, y and $\tilde{D}$) such that

$$\int \tilde{D}\varphi \; d\mu_{x,y} = \int \varphi \; d\mu \quad \text{for all} \quad \varphi \in C_0^\infty(M).$$

Thus all distribution derivatives of $\mu_{x,y}$ are measures, so by a result due to Schwartz [24, p. 191] $\mu_{x,y}$ is an infinitely differentiable function on G. In other words, there exist C^∞-functions $h_{x,y}$ on G such that

$$d\mu_{x,y}(a) = h_{x,y}(a)da \quad \text{for} \quad x,y \in \underline{D}_\infty.$$

Here da denotes some right invariant Haar measure.

Now let $\beta(x,y) = h_{x,y}(e)$ for $x,y \in D_\infty$. Then $\beta(\cdot,\cdot)$ is a sesquilinear form on $\underline{D}_\infty$ and

$$\beta(U(g)x,U(g)y) = h_{x,y}(g) \quad \text{for all} \quad g \in G.$$

This gives the following fundamental formula:

$$\langle P(\tau\varphi)x,y\rangle = \int_G \varphi(a)\beta(U(a)x,U(a)y)da$$

for all $x,y \in \underline{D}_\infty$ and $\varphi \in C_o^\infty(G)$.

Let $\{\varphi_n\} \subseteq C_o^\infty(G)$ be the usual δ-sequence. Then $\langle P(\tau\varphi_n)x,y\rangle \to \beta(x,y)$ for all $x,y \in \underline{D}_\infty$, and by the principle of uniform boundedness β is a continuous sesquilinear form on $\underline{D}_\infty$. Also note that $\beta(x,x) \geq 0$ for all $x \in \underline{D}_\infty$.

Pick a right invariant Haar measure on H and let δ be the modular function. Δ denotes the modular function on G, and we let $\rho(\xi) = \delta(\xi)\Delta(\xi^{-1})$ for all $\xi \in H$. By the definition of τ (see §5, Ex. 3) we have $\tau\varphi = \tau(\delta(\xi^{-1})\varphi(\xi^{-1}\cdot))$ for all $\xi \in H$ and $\varphi \in C_o^\infty(G)$. Thus

$$\int_G \varphi(a)\beta(U(a)x,U(a)y)da = \int_G \varphi(\xi^{-1}a)\delta(\xi^{-1})\beta(U(a)x,U(a)y)da$$

$$= \int_G \varphi(a)\delta(\xi)^{-1}\Delta(\xi)\beta(U(\xi\cdot a)x,U(\xi\cdot a)y)da$$

Using a δ-sequence $\{\varphi_n\} \subseteq C_o^\infty(G)$ we get the relation

$$(*) \quad \beta(U(\xi)x,U(\xi)y) = \rho(\xi)\beta(x,y)$$

for $\xi \in H$, $x,y \in \underline{D}_\infty$.

As already noted β is a continuous pseudo scalar product on $\underline{D}_\infty$, so $\ker\beta = \{x \in \underline{D}_\infty \mid \beta(x,x) = 0\}$ is a closed linear subspace of $\underline{D}_\infty$. Then $\underline{\mathfrak{h}}_o = \underline{D}_\infty/\ker \beta$ is a Fréchet space and for $x \in \underline{D}_\infty$ we let $[x] = x + \ker \beta$ denote the corresponding equivalence class. Also $\underline{\mathfrak{h}}_o$ is a pre-Hilbert space with the scalar product

$$<[x],[y]> = \beta(x,y)$$

and we let $\underline{\mathfrak{h}}$ denote the Hilbert space completion of $\underline{\mathfrak{h}}_o$ in the corresponding norm.

It is obvious from the relation (*) that $\ker \beta$ is invariant under the $U(\xi)$, $\xi \in H$, and we let

$$L_o(\xi)[x] = [\rho(\xi)^{-1/2}U(\xi)x] \quad \text{for} \quad \xi \in H,\ x \in \underline{D}_\infty.$$

Then $L_o(\xi)$ is a well defined isometric linear operator on $\underline{\mathfrak{h}}_o$ and $L_o(\xi_1)L_o(\xi_2) = L_o(\xi_1\xi_2)$. Take $L(\xi) = \overline{L_o(\xi)}$ for $\xi \in H$. Then $\xi \to L(\xi)$ is a unitary representation of H in $\underline{\mathfrak{h}}$, and since

$$<L(\xi)[x],[y]> = \rho(\xi)^{-1/2}\beta(U(\xi)x,y)$$

for all $x,y \in \underline{D}_\infty$, it follows that L is continuous.

For each $x \in \underline{D}_\infty$ we define a function f_x from G to $\underline{\mathfrak{h}}$ as follows: $f_x(a) = [U(a)x]$ for $a \in G$. Then $f_x(a\cdot g) = f_{U(g)x}(a)$ for $a, g \in G$, and f_x has the properties:

i) f_x is infinitely differentiable from G to $\underline{\mathfrak{h}}$

ii) $f_x(\xi\cdot a) = \rho(\xi)^{1/2}L(\xi)f_x(a)$ for $\xi \in H$, $a \in G$

iii) $\|f_x(\cdot)\|^2 = \beta(U(\cdot)x,U(\cdot)x)$ is locally integrable.

Therefore the mapping $C : x \to f_x$ is a linear mapping of $\underline{D}_\infty$ into F^* (cf. §5, Ex. 3). Actually C maps $\underline{D}_\infty$ into $\underline{H}^L$, and by definition of the norm in $\underline{H}^L$ we have

$$\|Cx\|^2 = \sup_{0\leq\tau\varphi\leq 1} \int \varphi(a)\|f_x(a)\|^2 da = \sup_{0\leq\tau\varphi\leq 1} \langle P(\tau\varphi)x,x\rangle$$

for all $x \in \underline{D}_\infty$. Furthermore

$$\langle P(\tau\varphi)x,y\rangle = \langle P^L(\tau\varphi)Cx,Cy\rangle \quad \text{for } x,y \in \underline{D}_\infty,\ \varphi \in C_o^\infty(G)$$

Let $\underline{H}$ denote the closure of $C\,\underline{D}_\infty$ in $\underline{H}^L$. It follows that C has a closure as a mapping from $\underline{K}$ into $\underline{H}$. Therefore C has a closure $\overline{C}$ as a mapping from $\underline{K}$ into $\underline{H}^L$. If $\overline{C}y = 0$ we have $y \perp P(C_o^\infty(M))\underline{D}_\infty$, hence $y = 0$.

Now the rest of the proof is standard. Let $\overline{C} = WT$ be the polar decomposition. Then T is a positive self adjoint operator in $\underline{K}$ such that $TU(g) = U(g)T$ for $g \in G$, and W is an isometry of $\underline{K}$ into $\underline{H}^L$ such that $W^*W = I$ and $WW^* = E$ where E is the projection of $\underline{H}^L$ onto $\underline{H}$. By the definition of C we have

$$CU(g) = U^L(g)C \quad \text{for all } g \in G,$$

and the same is true for $\overline{C}$. Therefore $\underline{H}$ is invariant under the $U^L(g)$, $g \in G$, and because W has the same intertwining property as $\overline{C}$ we have

$$WU(g)W^{-1} = U^L(g)|_{\underline{H}} \quad \text{for all } g \in G.$$

Here we consider W as a unitary mapping of $\underline{K}$ onto $\underline{H}$.

Since $\underline{D}_T \supseteq \underline{D}_\infty$ we know (cf. §2) that T leaves $\underline{D}_\infty$ invariant and $T \mid_{\underline{D}_\infty}$ is a continuous linear operator on $\underline{D}_\infty$. Also T is essentially self adjoint on $\underline{D}_\infty$. Now let

$$P_1(\psi) = E\, P^L(\psi) \mid_{\underline{H}} \quad \text{for} \quad \psi \in C_0^\infty(M)$$

If we let $P_0(\psi) = W^{-1}P_1(\psi)W$ it is clear that P_0 is a normalized C^∞-system of imprimitivity for U and since $P(\psi) = C^*P^L(\psi)C$ it is easily verified that $P(\psi) = TP_0(\psi)T$ on $\underline{D}_\infty$. Q.E.D.

For the sake of completeness we include a discussion of the uniqueness of such representations, leaving out some of the computational details.

First we note that P_0 is canonical in the following sense. If P_0' is any other normalized C^∞-system for U such that $P(\psi) = T'P_0'(\psi)T'$ for all $\psi \in C_0^\infty(M)$, where T' is an operator having the same properties as T (in Theorem 6.1), then the pair (U,P_0') is unitarily equivalent to the pair (U,P_0). In fact, the operator $T'T^{-1}$ extends uniquely to a unitary operator.

Now suppose the pair (U,P_0) (of Theorem 6.1) is unitarily equivalent to two restrictions (U_i,P_i) of (U^{L_i},P^{L_i}) to closed group invariant subspaces $\underline{H}_i$ of $\underline{H}^{L_i}$, where $\{f_i(e) \mid f_i \in \underline{H}_i \cap \underline{D}(U^{L_i})\}$ is dense in the representation space $\underline{V}_i$ of L_i for $i = 1,2$.

Then L_1 and L_2 are unitarily equivalent.

In order to see this let V be a unitary mapping of $\underline{H}_1$ onto $\underline{H}_2$ which maps (U_1,P_1) into (U_2,P_2). Since U_i is a subrepresentation of U^{L_i} we have $\underline{D}_\infty(U_i) = \underline{H}_i \cap \underline{D}_\infty(U^{L_i})$ $i = 1,2$, and V maps $\underline{D}_\infty(U_1)$ onto $\underline{D}_\infty(U_2)$. Let $f_i \in \underline{D}_\infty(U_i)$ $i = 1,2$. Then for $\varphi \in C_0^\infty(G)$ we have

$$<P^{L_2}(\tau\varphi)Vf_1,f_2> = \int \varphi(a)<(Vf_1)(a),f_2(a)>da$$
$$= <P^{L_1}(\tau\varphi)f_1,V^*f_2> = \int \varphi(a)<f_1(a),(V^*f_2)(a)>da$$

Using an approximate identity we get

$$<(Vf_1)(e),f_2(e)> = <f_1(e),(V^*f_2)(e)>$$

Therefore $V_0 : f_1(e) \to (Vf_1)(e)$ is a well defined closable and invertible linear mapping from a dense subspace of $\underline{\mathfrak{h}}_1$ onto a dense subspace of $\underline{\mathfrak{h}}_2$. Using the relations $(i = 1,2)$

$$(U^{L_i}(\xi)f_i)(e) = f_i(\xi) = \rho(\xi)^{\frac{1}{2}}L_i(\xi)f_i(e) \quad \text{for} \quad \xi \in H$$

it is easy to see that

$$V_0L_1(\xi)f_1(e) = L_2(\xi)V_0f_1(e) \quad \text{for all} \quad \xi \in H.$$

Using the polar decomposition it follows that L_1 and L_2 are unitarily equivalent. This completes the discussion. Before we turn to the usual case we prove the following simple result.

<u>Lemma 6.2</u>: Let $\mathfrak{A}$ be a *-algebra of bounded linear operators in a Hilbert space $\underline{B}$ and let E be a

projection in $\underline{B}$ such that the mapping $A \to EAE$ is a $*$-homomorphism of $\mathfrak{A}$ into $\mathcal{L}(E\underline{B})$. Then $E \in \mathfrak{A}'$.

<u>Proof</u>: If we represent operators in $\underline{B}$ as 2×2-matrices relative to the decomposition $\underline{B} = E\underline{B} \oplus (I-E)\underline{B}$ it is easy to see that $\mathfrak{A}$ must be represented by diagonal matrices. Hence $EA(I-E) = (I-E)AE$ for all $A \in \mathfrak{A}$. Q.E.D.

In case P is an ordinary system of imprimitivity for U we get a much stronger result, known as Mackey's imprimitivity theorem [4], [18].

<u>Corollary 6.1</u>: Let U be a continuous unitary representation of G in a Hilbert space $\underline{K}$ and let P be a system of imprimitivity for U based on $M = G/H$. Then there exists a continuous unitary representation L of H (unique up to equivalence) such that the pair (U,P) is unitarily equivalent to the pair (U^L, P^L).

<u>Proof</u>: By Lemma 6.1 P defines a normalized C^∞-system of imprimitivity for U (also denoted by P), so we already know L exists. In the following we use the notation from the proof of Theorem 6.1.

The mapping $C : x \to f_x$ is now isometric and $\overline{C} = W$. Therefore we have

$$WP(\psi)W^{-1} = P^L(\psi)\big|_{\underline{H}} \quad \text{for all} \quad \psi \in C_o^\infty(M).$$

Because P is a *-homomorphism Lemma 6.2 shows $EP^L(\psi) = P^L(\psi)E$ for all $\psi \in C_o^\infty(M)$. We know already E commutes with the $U^L(g)$, $g \in G$, thus E is induced from a bounded operator in $\underline{\mathfrak{h}}$ [3]. In fact there exists a unique projection E_o in $\underline{\mathfrak{h}}$ such that

$$(Ef)(a) = E_o f(a) \quad \text{for} \quad f \in \underline{H}^L.$$

On the other hand $(Ef_x)(e) = f_x(e)$ for $x \in \underline{D}_\infty$, and we have constructed $\underline{\mathfrak{h}}$ as the completion of such vectors. Hence $E = I$ and W is the desired unitary mapping. The uniqueness of L is clear from our general discussion. Q.E.D.

Remark: It is not necessary to appeal to Blattner [3] in order to get the projection E_o. We have already seen that we can define a mapping E_o as follows

$$E_o f(e) = (Ef)(e), \quad f \in \underline{D}_\infty(U^L)$$

Using the relation

$$\|P^L(\tau\varphi)Ef\|^2 \leq \|E\|^2 \|P^L(\tau\varphi)f\| \quad \varphi \in C_o^\infty(G)$$

it is easily seen that

$$\|E_o f(e)\| \leq \|E\| \cdot \|f(e)\| \quad \text{for all} \quad f \in \underline{D}_\infty(U^L)$$

Therefore E_o extends uniquely to a bounded linear operator in $\underline{\mathfrak{h}}$, and it is clear that this extension is a projection.

REFERENCES

[1] J. Barros-Neto: Analytic composition kernels for Lie groups; Pacific J. Math. 12 (1962), pp. 661-678.

[2] R. T. Blattner: On induced representations, Amer. J. Math. 83 (1961), pp. 79-98.

[3] R. T. Blattner: On a Theorem of G. W. Mackey, Bull. Amer. Math. Soc. 68 (1962), pp. 585-587.

[4] R. J. Blattner: Positive definite measures, Proc. Amer. Math. Soc. 14 (1963), pp. 423-428.

[5] N. Bourbaki: Integration, Hermann, Paris 1952.

[6] F. Bruhat: Sur les Représentations Induites des Groupes de Lie, Bull. Soc. Math. France 84 (1956), pp. 97-205.

[7] E. B. Davies: Systems of imprimitivity; preprint.

[8] E. B. Davies and J. T. Lewis: An operational approach to the axiomatic foundations of quantum theory; to appear.

[9] J. Dixmier: Les C*-algèbres et leur représentations Gauthier-Villars, Paris 1969.

[10] J. Dixmier: Sur les Représentations Unitaires des Groupes de Lie Nilpotents III, Can. J. Math. 10 (1958), pp. 321-348.

[11] Dunford and Schwartz: Linear operators, vol. I Interscience Publishers, Inc., New York.

[12] I. M. Gelfand und N. J. Wilenkin: Verallgemeinerte Funktionen (Distributionen) IV, VEB Deutscher Verlag der Wissenschaften, Berlin 1964.

[13] R. Goodman: Analytic and entire vectors for representations of Lie groups, Trans. Amer. Math. Soc. (1969).

[14] S. Helgason: Differential Geometry and Symmetric Spaces, Academic Press, New York 1962.

[15] L. Hörmander: Linear Partial Differential Operators, Springer, Berlin 1963.

[16] Y. Katznelson: An introduction to harmonic analysis Wiley, New York 1968.

[17] R. P. Langlands: On Lie semi-groups, Canadian J. Math. 12 (1960), pp. 686-693.

[18] G. W. Mackey: Imprimitivity for representations of locally compact groups I; Proc. Nat. Acad. Sci. 35 (1949), pp. 537-545.

[19] G. W. Mackey: Induced representations I; Ann. Math. 55 (1952) pp. 101-139.

[20] G. W. Mackey: Infinite-dimensional group representations; Bull. Amer. Math. Soc. 69 (1963), pp. 628-686.

[21] E. Nelson: Analytic vectors, Ann. Math. 70 (1959), pp. 572-615.

[22] E. Nelson and W. F. Stinespring: Representation of elliptic operators in an enveloping algebra, Amer. J. Math. 81 (1959), pp. 547-560.

[23] L. Pukanszky: Leçons sur les Représentations des Groupes. Dunod, Paris 1967.

[24] L. Schwartz: Théorie des distributions, Hermann, Paris 1966.

[25] H. H. Schaefer: Topological Vector Spaces, Macmillan, New York 1966.

[26] I. E. Segal: A class of operator algebras which are determined by groups, Duke Math. Journ. 18 (1951), pp. 221-265.

[27] I. E. Segal: A theorem on the measurability of group-invariant operators, Duke Math. Journ. 26 (1959), pp. 549-552.

[28] I. E. Segal: An Extension of a Theorem of L. O'Raifeartaigh, J. Funct. Anal. 1 (1967), pp. 1-21.

[29] I. E. Segal: Nonlinear Functions of Weak Processes II, J. Funct. Anal., to appear.

[30] I. M. Singer: Uniformly continuous representations of Lie groups, Ann. Math. 56 (1952), pp. 242-247.

Biographical Note:

Niels Kristian Skovhus Poulsen was born on March 17, 1941 in Ikast, Denmark. He attended public schools in Ikast and Herning and graduated from Herning Gymnasium in 1960 (studentereksamen).

The author was a student and a "student-instructor" at Aarhus Universitet, where he received the Cand. Scient. degree in mathematics in January 1968.

From September 1968 he was enrolled as a graduate student at M.I.T., where he was supported in part by a Teaching Assistantship and by a Research Assistantship.

The author is on leave from Matematisk Institut, Aarhus Universitet, where he holds a position as universitetsadjunkt. He expects to do research in mathematics at M.I.T. for one more year, before returning to Denmark.