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LINEAR SMOOTHING FOR DESCRIPTOR SYSTEMS[†]

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Abstract

The linear smoother for an n-th order two-point boundary value descriptor system (TPBVDS) [1] is formulated by using the general linear estimator solution developed by the method of complementary models in [2]. The smoother is shown to take the form of a $2n^{th}$ order TPBVDS. By employing the solutions to generalized Riccati equations [3] it is shown that the smoother dynamics can be decomposed into two n^{th} order descriptor form equations. The implementation of the smoother solution is also addressed.

1. Introduction

The class of descriptor systems was introduced by Luenberger [1] to describe the dynamics of certain linear systems for which the standard state-space representation is not applicable. The properties of processes of this type have been studied by a number of investigators (e.g. [4], [5] and [6]) and the optimal control problem has been examined in [7]. In this paper we discuss the fixed-interval smoothing problem for an n^{th} order TPBVDS. The solution is formulated as a $2n^{th}$ order TPBVDS by an application of the general linear estimator solution developed in [2] via the method of complementary models. The implementation of the smoother solution is also addressed. A general solution for a "regular"* TPBVDS is established along with a condition for well-posedness. This general solution is developed as a linear combination of (i) the solution to stable forward and backward recursions corresponding to processes with standard state space dynamics and (ii) the two-point boundary value. Following the methodology in [8] for the decomposition of the smoother for two-point boundary value processes satisfying state-space form dynamics, the dynamics of the $2n^{th}$ order TPBVDS are also decomposed into two n^{th} order descriptor form representations. This decomposition can be viewed as the generalization to descriptor systems of the Mayne-Fraser smoothing formula [13] for standard linear systems. An application of the general solution to these two lower order descriptor form systems yields the smoothed estimate.

Some aspects of the smoother and its implementation remain unresolved. First, the forward/backward general solution is only applicable for a regular TPBVDS. General conditions for regularity of the smoother solution have not yet been established. Second, the decomposition of the $2n^{th}$ order TPBVDS smoother into two n^{th} order descriptor forms requires the existence of solution into two n^{th} order descriptor forms requires the existence of solutions to "generalized" Riccati equations [3], and general conditions for the existence of such solutions have not been established.

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*Conditions for regularity are given in Section 2.

2. Two Point Boundary Value Descriptor System (TPBVDS)

2.1 The Model

In this section the model for the discrete linear stochastic TPBVDS is introduced, and a forward/backward form of its general solution is developed. Let u_k be an mx1 white sequence on [0,K-1] with constant covariance matrix Q. Let E, A and B be nxn, nxn and nxm matrices respectively with the pair {E,A} comprising a regular pencil [9]. Thus, E is not required to be invertible. Let V be an nx2n matrix written in nxn partitions as $[V^0 : V^K]$, and let v be an nx1 random vector uncorrelated with u and with covariance matrix Π_v. Then the discrete TPBVDS satisfies the difference equation:

$$Ex_{k+1} = Ax_k + Bu_k \tag{1a}$$

with two-point boundary condition

$$v = V^0x_0 + V^Kx_K . \tag{1b}$$

The realization of the smoother developed in [2] by the method of complementary models requires an operator representation of the process dynamics and boundary condition. The appropriate operator representation for (1) is developed as follows. Let the set of points between 0 and K-1 be represented by $\Omega = [0,K-1]$, and let the boundary of this region be defined as $\partial\Omega = \{0,K\}$. With D representing the unit delay, the dynamics of the nx1 vector process in (1a) are given by the first order difference operator:

$$L: l_2^n(\Omega \cup \partial\Omega) \rightarrow l_2^n(\Omega) \tag{2a}$$

defined notationally as

$$L = (D^{-1}E - A) \tag{2b}$$

and operationally as

$$(Lx)_k = Ex_{k+1} - Ax_k . \tag{2c}$$

Viewing the matrix V as the operator

$$V: R^{2n} \rightarrow R^n \tag{2d}$$

the dynamics and boundary condition in (1a) and (1b) can be expressed as

$$Lx = Bu \tag{3a}$$

$$Vx_b = v \tag{3b}$$

where

$$x_b = \begin{bmatrix} x_0 \\ x_K \end{bmatrix} . \tag{3c}$$

2.2 A General Solution for the TPBVDS

A general solution for the TPBVDS in (1) is formulated here as a linear combination of two stable recursions, one forward and one backward. This form of the solution both lends insight into the properties of descriptor systems defined by (1) (e.g. well-posedness) and will provide a means for implementing the smoother that is developed later in Section 3.

Given that $\{E,A\}$ comprises a regular pencil there exist nonsingular nxn matrices T and F such that

$$FET^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} \tag{4a}$$

and

$$FAT^{-1} = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} \tag{4b}$$

where A_f is $n_1 x n_1$, A_b is $n_2 \times n_2$, $n = n_1 + n_2$, and all eigenvalues of A_f and A_b lie within the unit circle. These properties of A_f and A_b can be obtained as a modification of the decomposition of a regular matrix pencil as described in [9]. The decomposition in [9] splits the pencil into forward dynamics corresponding to a pencil of the type $zI-\tilde{A}_f$ and into backward dynamics corresponding to $z^{-1}I-\tilde{A}_b$ where $\tilde{A}_b$ is nilpotent. The modification (4) simply shifts the unstable forward modes of $\tilde{A}_f$ into the backward dynamics A_b. Employing T in (4) to define the equivalence transformation

$$\begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix} \equiv Tx_k \tag{5}$$

and multiplying the dynamics in (1a) on the left by F, (1) becomes decoupled as

$$x_{f,k+1} = A_f x_{f,k} + B_f u_k \tag{6a}$$

and

$$x_{b,k} = A_b x_{b,k+1} - B_b u_k \tag{6b}$$

where

$$\begin{bmatrix} B_f \\ B_b \end{bmatrix} \equiv FB \quad . \tag{6c}$$

Note that A_f is forward stable and A_b is backward stable. Given the transformation in (5), the boundary condition in (1b) takes the form

$$v = [V_f^0 \vdots V_b^0] \begin{bmatrix} x_{f,0} \\ x_{b,0} \end{bmatrix} + [V_f^K \vdots V_b^K] \begin{bmatrix} x_{f,K} \\ x_{b,K} \end{bmatrix} \tag{7a}$$

where

$$[V_f^0 \vdots V_b^0] \equiv V^0 T^{-1} \text{ and } [V_f^K \vdots V_b^K] \equiv V^K T^{-1} \; . \tag{7b}$$

Employing the forward/backward representation of the dynamics of the TPBVDS in (6), a general solution to (1) is derived as follows. Define $x_{f,k}^0$ as the solution to (6a) with zero initial condition and $x_{b,k}^0$ as the solution to (6b) with a zero final condition. Note that both can be computed by numerically stable recursions. With Φ_f the transition matrix for A_f and Φ_b the transition matrix for A_b, one can write

$$x_{f,k} = \Phi_f(k,0)x_{f,0} + x_{f,k}^0 \tag{8a}$$

and

$$x_{b,k} = \Phi_b(k,K)x_{b,K} + x_{b,k}^0 \; . \tag{8b}$$

Substituting for $x_{f,K}$ and $x_{b,0}$ from (8a) and (8b) into (7a) and solving for $x_{f,0}$ and $x_{b,K}$ gives

$$\begin{bmatrix} x_{f,0} \\ x_{b,K} \end{bmatrix} = F_{fb}^{-1} \left\{ v - V_f^K x_{f,K}^0 - V_b^0 x_{b,0}^0 \right\} \tag{9a}$$

where

$$F_{fb} = [V_f^0 + V_f^K \Phi_f(K,0) \vdots V_b^0 \Phi_b(0,K) + V_b^K] \; . \tag{9b}$$

Finally substituting for $x_{f,0}$ and $x_{b,K}$ from (9a) into (8a) and (8b), it can be shown that the solution to (6) is given by

$$\begin{bmatrix} x_{f,b} \\ x_{b,k} \end{bmatrix} = \Phi_{fb}(k)F_{fb}^{-1} \left\{ v - V_f^K x_{f,K}^0 - V_b^0 x_{b,0}^0 \right\} + \begin{bmatrix} x_{f,k}^0 \\ x_{b,k}^0 \end{bmatrix} . \tag{10a}$$

where

$$\Phi_{fb}(k) = \left[\begin{array}{c:c} \Phi_f(k,0) & 0 \\ \hdashline 0 & \Phi_b(k,K) \end{array}\right] \tag{10b}$$

Applying the inverse of the transformation in (5), the original process x_k is recovered by

$$x_k = T^{-1} \begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix} \; . \tag{11}$$

In this way, we have have constructed a stable, forward/backward two-filter recursive implementation of the general solution for the TPBVDS defined by (1a) and (1b) with the only constraint being that $\{E,A\}$ form a regular pencil.

The well-posedness condition for (1a) and (1b) is that F_{fb} in (9b) be invertible. That is, for x to be uniquely defined on [0,K] by the input u_k and boundary value v, F_{fb} must be invertible. This invertibility criterion can be used to show that if E is singular, (1b) must be a two-point (or multi-point) boundary condition (i.e. (1b) cannot be an initial condition) as follows.

First note the E singular implies that A_b is singular and consequently that Φ_b is singular. For an initial value problem, $V^0 = I$ and $V^K = 0$, and therefore, $V_f^K = 0$ and $V_b^K = 0$. Thus, for an initial value problem F_{fb} becomes

$$\begin{aligned} F_{fb} &= [V_f^0 \vdots V_b^0 \; \Phi_b(0,K)] \\ &= [V_f^0 \vdots V_b^0] \begin{bmatrix} I & 0 \\ 0 & \Phi_b(0,K) \end{bmatrix} \end{aligned}$$

Clearly, a necessary condition for F_{fb} to be nonsingular is that $\Phi_b(0,K)$ be nonsingular. However, this does not hold when E is singular.

2.3 Green's Identity

In addition to the operator representation of the TPBVDS in (3) the smoother solution we develop later also requires the formal adjoint difference operator L^+ which is obtained through Green's identity for L on Ω [2]. Green's identity for discrete processes is obtained from summation by parts of the inner product

$$\begin{aligned} \langle \gamma, Lx \rangle_{l_2^n[0,K-1]} &= \sum_{K=0}^{K-1} \gamma_k'(Lx)_k \\ &= \sum_{K=0}^{K-1} \gamma'_k(Ex_{k+1} - Ax_k) \; . \end{aligned} \tag{12}$$

Summation by parts can be interpreted as the counterpart of integration by parts as follows:

Intergration by Parts	Summation by Parts
$\int_0^T u\,dv = uv\Big\vert_0^T - \int_0^T v\,du$	$\sum_{k=0}^{K-1} u_{k+1}(v_{k+1}-v_k) =$
	$(u_K v_K - u_0 v_0) - \sum_{k=0}^{K-1} v_k(u_{k+1}-u_k)$

(13)

The term on the right hand side of the identity for summation by parts in (13) has been obtained simply by shifting the index of summation on the left hand side and adding $(u_k v_k - u_0 v_0)$ to account for the shift. To put (12) into the form of (13), we perform the same type of index shifting to write

$$\sum_{k=0}^{k-1} (Ex_{k+1}-Ax_k)'\gamma_k = \sum_{k=0}^{K-1} x'_k(E'\gamma_{k-1}-A\gamma_k)$$

$$- x_0'E\gamma_{-1} + x_K'E'\gamma_{K-1}. \tag{14}$$

Defining

$$L^{\dagger} = DE' - A', \tag{15a}$$

$$x_b = \begin{bmatrix} x_0 \\ x_K \end{bmatrix}, \quad \gamma_b = \begin{bmatrix} \gamma_{-1} \\ \gamma_{K-1} \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} -E' & 0 \\ 0 & E' \end{bmatrix}. \tag{15b,c,d}$$

Green's identity can be written directly from (14) as

$$\langle Lx,\gamma\rangle_{l_2^n[0,K-1]} = \langle L^{\perp}\gamma,x\rangle_{l_2^n[0,K-1]} + \langle x_b,\Gamma\gamma_b\rangle_{R^{2n}} \tag{16}$$

We will see later that the estimator can be expressed in a simpler form if written in terms of a shifted version of γ, which we denote by λ:

$$\lambda_{k+1} \equiv \gamma_k \text{ (i.e. } \lambda = D\gamma) . \tag{17a}$$

In terms of the shifted process λ, γ_b is given by

$$\lambda_b \equiv \gamma_b = \begin{bmatrix} \lambda_0 \\ \lambda_K \end{bmatrix} . \tag{17b}$$

Given Green's identity as expressed in (15) and (16), in the next section we establish an internal difference realization of the smoother for the discrete TPVBDS.

3. The Smoother and Two-Filter Implementation

3.1 The Smoother as a TPBVDS

The observations for the fixed-interval smoothing problem are comprised of an observation of the process x at each point in the interval $[0,K-1]$ as well as a boundary observation as described below. Let C be a pxn matrix, and let W be a full rank qx2n matrix with $q\leq n$, with the rows of W linearly independent of those of V in (2.3) and with qxn partitions:

$$W = [W^0 \vdots W^K] . \tag{18}$$

Let r_k be a pxl white noise process over $[0,K-1]$ whose covariance matrix R is nonsingular and constant on $[0,K-1]$. Let r_b be a qxl random vector with nonsingular covariance matrix Π_b. In addition, u, v, r and r_b are assumed to be mutually orthogonal. The observations are defined by a process y_k

$$y_k = Cx_k + r_k \text{ on } [0,K-1] \tag{19a}$$

and a qxl boundary observation

$$y_b = Wx_b + r_b . \tag{19b}$$

The minimum variance estimator of x given the observations y and y_b can be written by substituting from the operator description of the TPBVDS in (3) and Green's identity in (15) and (16) into the operator form for the estimator in equation (3.17) of [2]. In this case, the adjoints of B, C, W and V are all simply given by matrix transpositions. The formal adjoint difference operator $L^{\perp}$, the matrix* Γ, and x_b and γ_b (temporarily γ will be used in place of λ) have all been determined in the derivation of Green's identity. The resulting smoother dynamics are given by

$$\left[\begin{array}{c|c} E & 0 \\ \hline 0 & E' \end{array}\right]\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\gamma}_{k-1} \end{bmatrix} = \left[\begin{array}{c|c} A & BQB' \\ \hline -C'R^{-1}C & A' \end{array}\right]\begin{bmatrix} \hat{x}_k \\ \hat{\gamma}_k \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline C'R^{-1} \end{array}\right] y_k \tag{20a}$$

with boundary condition

$$W'\Pi_b^{-1}y_b = (W'\Pi_b^{-1}W + V'\Pi_v^{-1}V)\begin{bmatrix} \hat{x}_0 \\ \hat{x}_K \end{bmatrix} + \Gamma\begin{bmatrix} \hat{\gamma}_{-1} \\ \hat{\gamma}_{K-1} \end{bmatrix} \tag{20b}$$

As mentioned earlier, it will be convenient to write the smoother in terms of the shifted process λ defined in (17b). In this way the apparent four-point boundary condition in (20b) becomes a two-point boundary condition. Furthermore, if we were to specialize to the case of causal processes ($E=I$, $V^0=I$, $V^K=0$), the smoother takes the traditional form of the discrete fixed-interval smoother (see e.g. [10]). Thus, in terms of λ and λ_b (20a) and (20b) become

$$\left[\begin{array}{c|c} E & 0 \\ \hline 0 & E' \end{array}\right]\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_k \end{bmatrix} = \left[\begin{array}{c|c} A & BQB' \\ \hline -C'R^{-1}C & A' \end{array}\right]\begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_{k+1} \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline C'R^{-1} \end{array}\right] y_k \tag{21a}$$

$$\begin{bmatrix} W^{0'} \\ W^{K'} \end{bmatrix}\Pi_b^{-1}y_b = \left[\begin{array}{c|c} V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 & -E' \\ \hline V^{K'}\Pi_v^{-1}V^0 + W^{K'}\Pi_b^{-1}W^0 & 0 \end{array}\right]\begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} V^{0'}\Pi_v^{-1}V^K + W^{0'}\Pi_b^{-1}W^K & 0 \\ \hline V^{K'}\Pi_v^{-1}V^K + W^{K'}\Pi_b^{-1}W^K & E \end{array}\right]\begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix} \tag{21b}$$

Defining

$$\mathcal{E} = \left[\begin{array}{c|c} E & -BQB' \\ \hline 0 & -A' \end{array}\right] \tag{22a}$$

$$\mathcal{A} = \left[\begin{array}{c|c} A & 0 \\ \hline C'R^{-1}C & -E' \end{array}\right] \tag{22b}$$

and

$$\mathcal{B} = \left[\begin{array}{c} 0 \\ \hline C'R^{-1} \end{array}\right], \tag{22c}$$

the smoother dyanmics in (21a) can be written in descriptor form as

$$\mathcal{E}\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_{k+1} \end{bmatrix} = \mathcal{A}\begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_k \end{bmatrix} + \mathcal{B}y_k . \tag{23}$$

*In [2] Γ is denoted by E. A change in notation is required since E is used here in defining the dynamics of the TPBVDS in (1a).

3.2 Impelmentation of the Smoother

Assuming* that the pair $\{E,A\}$ is regular, one could immediately employ available numerical techniques [11] to find matrices F and T analogous to those in (4a) and (4b) to obtain a solution for $\{\hat{x},\hat{\lambda}\}$ as described in (8) through (11). Rather than taking this numerical route directly, we will take an intermediate step by which (23) is further simplified revealing some additional underlying structure of the smoother dynamics. As will become apparent, this intermediate step has been motivated by the two-filter fixed-interval smoother solutions for non-descriptor form systems (i.e. systems for which $E = I$) [8]. In particular, we will find 2nx2n matrix sequences F_k and T_k which decompose E and A as

$$F_k E T_{k+1}^{-1} = \begin{bmatrix} E_{f,k} & 0 \\ 0 & A_{b,k} \end{bmatrix} \tag{24a}$$

and

$$F_k A T_{k+1}^{-1} = \begin{bmatrix} A_{f,k} & 0 \\ 0 & E_{b,k} \end{bmatrix} \tag{24b}$$

Given this decomposition and defining

$$\begin{bmatrix} \hat{x}_{f,k} \\ \hat{x}_{b,k} \end{bmatrix} = T_k \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_k \end{bmatrix} \tag{25}$$

the $2n^{th}$ order smoother dynamics in (23) become decoupled into two n^{th} order descriptor forms

$$E_{f,k}\, \hat{x}_{f,k+1} = A_{f,k}\, \hat{x}_{f,k} + B_{f,k}\, y_k \tag{26a}$$

and

$$E_{b,k}\, \hat{x}_{b,k+1} = A_{b,k}\, \hat{x}_{b,k} + B_{b,k}\, y_k \tag{26b}$$

where

$$\begin{bmatrix} B_{f,k} \\ B_{b,k} \end{bmatrix} = F_k\ \mathcal{B}. \tag{26c}$$

It can be shown that in order to achieve the decomposition in (26a) and (26b), the partitions of F_k and T_k denoted by

$$F_k = \begin{bmatrix} F_{11}^{\ k} & F_{12}^{\ k} \\ F_{21}^{\ k} & F_{22}^{\ k} \end{bmatrix} \quad \text{and} \quad T_k = \begin{bmatrix} T_{11}^{\ k} & T_{12}^{\ k} \\ T_{21}^{\ k} & T_{22}^{\ k} \end{bmatrix} \tag{27}$$

must satisfy

$$E_{f,k}\, T_{11}^{\ k+1} = F_{11}^{\ k}\, E \tag{28a}$$

$$E_{b,k}\, T_{22}^{\ k} = F_{22}^{\ k}\, E' \tag{28b}$$

$$E_{f,k}\, T_{12}^{\ k+1} = -\, F_{11}^{\ k}\, BQB' + F_{12}^{\ k}\, A' \tag{28c}$$

$$E_{b,k}\, T_{21}^{\ k} = F_{21}^{\ k} A + F_{22}^{\ k}\, C'R^{-1}C \tag{28d}$$

$$A_{b,k}\, T_{21}^{\ k+1} = F_{21}^{\ k}\, E \tag{28e}$$

$$A_{f,k}\, T_{12}^{\ k} = F_{12}^{\ k} E' \tag{28f}$$

$$A_{b,k}\, T_{22}^{\ k+1} = -\, F_{21}^{\ k}\, BQB' + F_{22}^{\ k}\, A' \tag{28g}$$

$$A_{f,k}\, T_{11}^{\ k} = F_{11}^{\ k} A + F_{12}^{\ k}\, C'R^{-1}C \tag{28h}$$

Motivated by the decomposition of the smoother dynamics for state-space form two-point boundary value systems in [8] and [12], choose

$$T_k = \begin{bmatrix} \theta_{f,k} & -E' \\ \theta_{b,k} E & I \end{bmatrix}. \tag{29}$$

Employimg (29) along with the eight relations in (28) yields

$$E_{f,k} = E\, \theta_{f,k}^{-1} \tag{30a}$$

$$A_{f,k} = A[I - (C'R^{-1}C + \theta_{f,k})^{-1} C'R^{-1}C]\, \theta_{f,k}^{-1} \tag{30b}$$

$$B_{f,k} = -A(C'R^{-1}C + \theta_{f,k})^{-1}\, C'R^{-1} \tag{30c}$$

$$E_{b,k} = E' \tag{30d}$$

$$B_{b,k} = C'R^{-1} \tag{30e}$$

$$A_{b,k} = A'[I + (BQB' + \theta_{b,k}^{-1})\ BQB^{-1}] \tag{30f}$$

$$F_k = \left[\begin{array}{c|c} I & -A(C'R^{-1}C + \theta_{f,k})^{-1} \\ \hline A'(BQB' + \theta_{b,k}^{-1})^{-1} & I \end{array}\right] \tag{30g}$$

where θ_f and θ_b satisfy the following generalized Riccati equations:

$$E\theta_{f,k+1}^{-1}\, E' = A(C'R^{-1}C + \theta_{f,k})^{-1}\, A' + BQB' \tag{31a}$$

$$E'\theta_{b,k}\, E = A'(BQB' + \theta_{b,k+1}^{-1})^{-1}\, A + C'R^{-1}C\ . \tag{31b}$$

Equations (30a) through (30f) along with equations (26a) and (26b) completely define the decoupled dynamics for the transformed processes $\hat{x}_{f,k}$ and $\hat{x}_{b,k}$.

An expression for the boundary condition for the tranformed processes can be developed as follows. First rewrite the boundary condition (21b) for the untransformed smoother processes in a more compact form:

$$W'\Pi_b^{-1} y_b = \left[\begin{array}{c|c} V_0 & -E' \\ \hline V_{0,K} & 0 \end{array}\right] \begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix} + \left[\begin{array}{c|c} V_{0,K}' & 0 \\ \hline V_K & E' \end{array}\right] \begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix} \tag{32}$$

*An outstanding problem is the determination of the conditions for which $\{E,A\}$ is regular.

With the inverse of equivalence transformation T_k given by

$$T_k^{-1} = \left[\begin{array}{c|c} I & \theta_{f,k}^{-1} E'\theta_{f,k} \\ \hline -\theta_{b,k} E' & \theta_{f,k} \end{array}\right] \left[\begin{array}{c|c} Z_k^{-1} & 0 \\ \hline 0 & \tilde{Z}_k^{-1} \end{array}\right] \tag{33a}$$

where

$$Z_k = \theta_{f,k} + E'\theta_{b,k} E \tag{33b}$$

$$\tilde{Z}_k = (\theta_{b,k} E\theta_{f,k}^{-1} E' + I)\theta_{f,k} \ , \tag{33c}$$

the boundary condition (32) can be written in terms of the transformed processes $\hat{x}_f$ and $\hat{x}_b$ as

$$W'\Pi_b^{-1} y_b$$

$$= \left[\begin{array}{c|c} (V_0 + E'\theta_{b,0}E)\, Z_0^{-1} & (V_0\theta_{f,0}^{-1} - I)E'\theta_{f,0}\tilde{Z}_0^{-1} \\ \hline V_{0,K}Z_0^{-1} & V_{0,K}\theta_{f,0}^{-1} \ E'\theta_{f,0}\tilde{Z}_0^{-1} \end{array}\right] \begin{bmatrix} \hat{x}_{f,0} \\ \hat{x}_{b,0} \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} V'_{0,K}\, Z_K^{-1} & V'_{0,K}\theta_{f,K}^{-1} E'\theta_{f,K}\tilde{Z}_K^{-1} \\ \hline (V_K - E'\theta_{b,k}E)Z_K^{-1} & (V_K\theta_{f,K}^{-1} + I)E'\theta_{f,K}\tilde{Z}_K^{-1} \end{array}\right] \begin{bmatrix} \hat{x}_{f,K} \\ \hat{x}_{b,K} \end{bmatrix} \tag{34}$$

Given the expression for the inverse transformation in (33a), the estimate $\hat{x}$ can be recovered from the forward and backward processes by inverting (25) to give

$$\hat{x}_k = Z_k^{-1} \hat{x}_{f,k} + \theta_{f,k}^{-1} E'\theta_{f,k}\tilde{Z}_k^{-1}\hat{x}_{b,k} \ . \tag{35}$$

That is, the estimate is linear combination of the transformed forward and backward processes.

Of course to recover $\hat{x}_k$ via (35) we must first solve (26) with boundary condition (34) to get $\hat{x}_f$ and $\hat{x}_b$. However, the general solution presented in Section 2 is only applicable for the case that E_f, E_b, A_f, A_b, B_f and B_b are constant. To satisfy this condition, we can evaluate (30a) through (30f) with the steady state solutions to the generalized Riccati equations for θ_f and θ_b in (31a) and (31b).

The objective has been to transform the smoother dynamics into two lower order descriptor forms. The potential benefits are twofold. The first is that the computation required to compute the two n^{th} order solution is less than that required for a single $2n^{th}$ order system. The second and most important is that the descriptor form dynamics of the lower order systems may be more easily analyzed in terms of whether or not they represent a regular pencil. For instance, if we assume that a steady state solution θ_f exists for (31a) then the forward dynamics (26a) become

$$E\theta_f \hat{x}_{f,k+1} = A[I - (C'R^{-1}C + \theta_f)^{-1}C'R^{-1}C]\theta_f \hat{x}_{f,k}$$

$$- A(C'R^{-1}C + \theta_f)^{-1}C'R^{-1}Cy_k$$

A question which is unanswered thus far is: given that $\{E,A\}$ forms a regular pencil, under what conditions do $\{E, A[I-(C'R^{-1}C + \theta_f)^{-1}C'R^{-1}C]\}$ form a regular pencil?

4. The Smoothing Error

In addition to providing a general representation of the estimator dynamics and boundary condition, the method of complementary models as employed in [2] also yields a representation for the estimation error dynamics and boundary condition. Employing equations (3.20) and (3.23) in [2], the smoothing error for the TPBVDS can be shown to satisfy descriptor form dynamics:

$$\mathcal{E} \begin{bmatrix} \tilde{x}_{k+1} \\ -\hat{\lambda}_{k+1} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \tilde{x}_k \\ -\hat{\lambda}_k \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C'R^{-1} \end{bmatrix} \begin{bmatrix} u_k \\ r_k \end{bmatrix} \tag{36a}$$

with boundary condition

$$V'\Pi_v^{-1} v - W'\Pi_b^{-1} r_b = \left[\begin{array}{c|c} V_0 & -E' \\ \hline V_{0,K} & 0 \end{array}\right] \begin{bmatrix} \tilde{x}_0 \\ -\hat{\lambda}_0 \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} V'_{0,K} & 0 \\ \hline V_K & E' \end{array}\right] \begin{bmatrix} \tilde{x}_K \\ -\hat{\lambda}_K \end{bmatrix} \tag{36b}$$

where the error is defined as $\tilde{x} = x - \hat{x}$, $\mathcal{E}$ and $\mathcal{A}$ are defined in (22), and the boundary condition (36b) is written with the same notation as that used to express the smoother boundary condition in (32).

The covariance of the smoothing error can be computed by an application of the difference equations formulated in the Appendix. In order to apply those equations, the error dynamics (36a) must first be transformed to the decoupled forward/backward form in (6a) and (6b) with the boundary condition transformed accordingly (see (7a) and (7b)).

There is an alternative to directly transforming (36a) into forward/backward form. That is, given the similarity between the error dynamics (36a) and the smoother dynamics (23), the transformations T_k and F_k in (29) and (30) can be used to first decouple the error dynamics into two nth order descriptor forms

$$\begin{bmatrix} e_{f,k} \\ e_{b,k} \end{bmatrix} = T_k \begin{bmatrix} \tilde{x}_k \\ -\hat{\lambda}_k \end{bmatrix} \tag{37}$$

and then solve for the error covariance in terms of these dynamically decoupled processes.

5. CONCLUSION

The smoother and smoothing error equations for a TPBVDS have been derived. A stable method for obtaining a numerical solution for a regular TPBVDS as a linear combination of forward and backward recursions and the two-point boundary value has been developed along with a well-posedness condition.

The smoother for an n^{th} order system is shown to be a $2n^{th}$ order TPBVDS. With respect to the smoother, the following remain as areas for further investigation: (1) well-posedness or smoothability conditions [14] need to be established, (2) conditions under which the $2n^{th}$ order TPBVDS smoother is regular as defined in Section 2, (3) conditoins for existence of solutions to the generalized Riccati equations (31a) and (31b), and (4) how might the smoother decomposition simplify the determination of smoother regularity.

Since the 1-D representation of the discretized dynamics of 2-D systems can be written in the form of a multi-point boundary value descriptor system (MPBVDS) [12], an extension to multi-point problems of the smoother solution developed here would be quite useful. A general solution for MPBVDS similar to the forward/backward form for TPBVDS in Section 2 has been developed in [12]. However, the extension of the smoother solution to problems of this type is still under investigation.

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Appendix: The Covariance of a TPBVDS

A set of difference equations from which the covariance of a TPBVDS can be computed are presented in this appendix. Since the smoother error is given as a TPBVDS in (36), these equations are especially useful in evaluating the performance of the TPBVDS smoother derived in this paper.

The starting point for the development here is the general solution in (10a) for the TPBVDS defined by (1a) and (1b). Recall that the boundary value v is assumed to be orthogonal to the input u_k throughout $\Omega = [0, K-1]$. Thus, v is orthogonal to $x^0_{f,k}$ and $x^0_{b,k}$ in (10a) for all k in Ω so that the covariance of x_k in (11) can be written as a linear combination of Π_v, the covariance of v, and the following three covariances:

(1) $$P^0_f(n,k) \equiv E[x^0_{f,n} x^{0\prime}_{f,k}] \quad , \tag{A.1a}$$

(2) $$P^0_b(n,k) \equiv E[x^0_{b,n} x^{0\prime}_{b,k}] \tag{A.1b}$$

and

(3) $$P^0_{fb}(n,k) \equiv E[x^0_{f,n} x^{0\prime}_{b,k}] \quad . \tag{A.1c}$$

Difference equations for each of these three covariances can be derived by substituting expressions for x^0_f and x^0_b represented as weighted sums of u_k on Ω into each of the expectations in (A.1). In this manner, it can be shown that

(1) $$P^0_f(n,k) = \Phi_f(n,k) P^0_f(k,k) \tag{A.2a}$$

where

$$P^0_f(k+1,k+1) = \Phi_f(k+1,k) P^0_f(k,k) \Phi'_f(k+1,k) + B_{f,k} Q_k B'_{f,k} ; \quad P^0_f(0,0) = 0 \quad . \tag{A.2b}$$

(2) $$P^0_b(n,k) = P^0_b(k,k) \Phi'_b(k,n) \tag{A.3a}$$

where

$$P^0_b(k-1,k-1) = \Phi_b(k-1,k) P^0_b(k,k) \Phi'_b(k-1,k) + B_{b,k} Q_k B'_{b,k} ; \quad P^0_b(K,K) = 0 \tag{A.3b}$$

and (3) for $n > k$

$$P^0_{fb}(n,k) = \Pi^0_{fb,n} \Phi'_b(k,n) - \Phi_f(n,k) \Pi^0_{fb,k} \tag{A.4a}$$

where

$$\Pi^0_{fb,k+1} = \Phi_f(k+1,k) \Pi^0_{fb,k} \Phi'_b(k+1,k) + B_{f,k} Q_k B'_{b,k} ; \quad \Pi^0_{fb,0} = 0 \quad . \tag{A.4b}$$

and for $n \leq k$,

$$P^0_{fb}(n,k) = 0 \quad . \tag{A.4c}$$

Given these three covariances, the covariance of x_k:

$$P_k = E[x_k x'_k] = T_k^{-1} \left\{ E \begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix} [x'_{f,k} : x'_{b,k}] \right\} T_k^{-1\prime} \tag{A.5a}$$

can be written as

$$P_k = T_k^{-1} \left\{ \Phi_{fb}(k) F_{fb}^{-1} \Pi_v F_{fb}^{-1} \Phi'_{fb}(k) + \Xi(k) + \Xi(k)' + \Theta(k) + \left[\begin{array}{c:c} P^0_f(k,k) & 0 \\ \hdashline 0 & P^0_b(k,k) \end{array} \right] \right\} T_k^{-1\prime} \tag{A.5b}$$

where

$$\Xi(k) = -\Phi_{fb}(k) F_{fb}^{-1} \{ V_f^K [P^0_f(K,k) + P^0_{fb}(K,k)] + V_b^0 [P^{0\prime}_{fb}(k,0) + P^0_b(0,k)] \} \tag{A.5c}$$

and

$$\Theta(k) = \Phi_{fb}(k) F_{fb}^{-1} [V_f^K : V_b^0] \left[\begin{array}{c:c} P^0_f(K,K) & P^0_{fb}(K,0) \\ \hdashline P^{0\prime}_{fb}(K,0) & P^0_b(0,0) \end{array} \right] \left[\begin{array}{c} V_f^{K\prime} \\ \hdashline V_b^{0\prime} \end{array} \right] . \tag{A.5d}$$

Thus, the covariance of the process x_k can be computed for any point k in Ω given the solution of the three matrix difference equations (A.2), (A.3) and (A.4).