**1.138J/2.062J/18.376J, WAVE PROPAGATION**

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C. C. Mei

Homework no. 2

Due November 9, 2006.

**In all exercises, please describe the physical meaning of your mathematical results. Use graphics if it can help the explanation. If you do any numerical computations, feel free to use Matlab.**

(II) **Bragg scattering by a periodic wall**. A long and straight channel of averaged width $2b$ have wavy walls in $-L < x < L$ described by $y = b + a \sin Kx$ and $y = -b - a \sin Kx$. The corrugations have a small amplitude, : $Ka = \epsilon \ll 1$ and the corrugate dpart is long $KL \gg 1$.

Consider a plane sound wave incident from $x \sim -\infty$.

$$\phi_{inc} = A_o e^{ikx - i\omega t}, \quad x \sim -\infty. \tag{H.4.1}$$

Use the symmetry and consider only one half of the channel $0 < y < b + a \sin Kx$.

1. Derive the approximate boundary condition on the wall that there is no normal flux

$$\nabla\phi \cdot \vec{n} = 0 \quad \text{on} \quad y = b + \sin x \tag{H.4.2}$$

Use the fact that the unit normal to the surface $F(x, y) = 0 = y - b - a \sin Kx$ is

$$\vec{n} = \frac{\nabla F}{|\nabla F|} \tag{H.4.3}$$

For small dorrugations anything on the wall can be approximated by Taylor expansion

$$f(x, b + a \sin x) = f(x, b) + a \sin Kx \left(\frac{\partial f}{\partial y}\right)_{y=b} + ... \tag{H.4.4}$$

Derive the envelope equations for right-going and left-going (reflected) plane waves when Bragg resonance is nearly satisfied.

2. Assume that both incident and reflected waves can be comparably significant due to Bragg reflection

$$\phi = Ae^{ikx-i\omega t} + * + Be^{-ikx-\omega t} + * \tag{H.4.5}$$

Derive the equations for the slow variations of the envelopes $A(X,T), B(X,T)$, where $X = \epsilon x, T = \epsilon t$. Show that the results are similar to those in the lectures, but give the new coupling coefficient.

3. To understand better the solution to the diffraction problem check (do not send in) (3.25) and (3.29) chapter 5, and plot $R(X=0)$ for a wide range of $\Omega L$.

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(I). **Disperiosn curve and band gaps.** For a periodically laminated elastic slab, the wave equation for sinusoidal wave is

$$\frac{d}{dx}\left[\mathcal{E}(x)\frac{dU}{\partial x}\right] + \frac{\omega^2}{C^2}U(x) = 0 \tag{H.4.6}$$

Consider weak but periodic elasticity so that

$$\mathcal{E}(x) = 1 + \sum_m b_m e^{imKx} \tag{H.4.7}$$

where $b_m = O(\epsilon) \ll 1$ and $m = \pm\infty, ... \pm 3, \pm 2, \pm 1$ but $b_0 = 0$. In other words $m$ is an integer but non- zero. Note that for real $\mathcal{E}$ we must have

$$b_{-n} = b_n^* \tag{H.4.8}$$

1. Use a naive perturbation expansion to show that at the second order there will be forcing terms representing reflected wave of the type

$$e^{-ikx-i\omega t} \tag{H.4.9}$$

if

$$\boxed{k + nK = -k.} \tag{H.4.10}$$

This is the general Bragg condition, of, if $k = \pm K/2, \pm K, \pm 3K/2, ...$

2. Use Bloch's theorm and assume the solution to be

$$U(x) = e^{ikx} \sum_n V_n e^{inKx} \qquad \text{(H.4.11)}$$

where $n$ covers all integers including 0. Show that $V_n$ is governed by the infinite set of algeraic equations:

$$\left( \frac{\omega^2}{C^2} - (k + nK)^2 \right) V_n - \sum_m b_m (k + (n-m)K)(k + nK) V_{n-m} = 0 \qquad \text{(H.4.12)}$$

3. Rewrite (H.4.12) for $n = -N$, and then again for $n = 0$.

4. Show that for all $n \neq N$, $V_n$ is much smaller than $V_0$ since $m = O(\epsilon)$.

5. Show from the two last equations that if $k = NK/2$ then $V_N$ can be of the same order of magnitude as $V_0$.

6. Let $k = NK/2 + k', \omega = NK/2 + \omega'$ where $(k', \omega') = O(\epsilon)$. Show that $\omega'$ and $k'$ are related by equations for hyperbolas, hence confirm Figure 2 for the band gaps.

7. What is the band gap width for each $N$?