AN ELECTRONIC DIFFERENTIAL ANALYZER

by

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(1943)

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(1943)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
(1948)

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SECTION I

ABSTRACT

An electronic differential analyzer system has been developed of complete mathematical generality, and a functioning model, capable of solving ordinary differential equations of orders through the fourth, both linear and non-linear, has been constructed and operated. The electronic differential analyzer has a high speed of operation and is extremely flexible with regard to equation parameters and initial-conditions. This flexibility permits rapid investigation of wide ranges of equation solutions and the quick qualitative determination of the nature of these solutions with regard to periodicity, instability, and discontinuities; it further permits rapid cut and try adjustment of unknown initial conditions to fit prescribed final conditions.

The development of this differential analyzer has required the invention of two new computing elements, an electronic function generator and an electronic multiplier. These components together with the balance of the differential analyzer have been used in the solution of a number of representative differential equations of the linear and non-linear types.

Comparison of observed and calculated solutions reveals an accuracy of from 1 to 5 percent, depending upon the equation solved. This is completely adequate for a great many engineering problems. The observed precision of the solutions ranges from 0.002 to 0.1 percent. An analysis of the errors introduced into differential equation solutions by the frequency limitations of the computing elements, such as the integrators and adders, has been made and the results of this analysis verified experimentally.

The cost of construction of an electronic differential analyzer can be expected to lie between $4000 and $20,000, depending upon the range of problems to be treated. It is the feeling of the author that this electronic differential analyzer should find considerable application in mathematics, physics, and engineering.

Acknowledgement

The author wishes to express his profound appreciation to Professor Henry Wallman for his guidance and encouragement throughout the course of this work. Professor Wallman proposed the development of an electronic differential analyzer to the author as a thesis topic, and he has given most generously of his time for discussion and constructive criticism.

The author acknowledges the interest of the Directors of the Research Laboratory of Electronics, Professor J. A. Stratton and Professor A.G. Hill, under whose auspices this work has been carried out.

The author also acknowledges many helpful discussions with Mr. Ronald E. Scott and numerous other colleagues in the Research Laboratory of Electronics on the various problems encountered in this research.

Finally the author's appreciation and thanks go to his wife, Lois, who typed this thesis through the middle of a Boston summer.

CONTENTS

SECTION II

INTRODUCTION

History

This thesis is concerned with the development of an electronic differential analyzer. In order that the position of this work in the field of mathematical machines may be clearly understood it is worthwhile to review very briefly past work in this field.

Early in the history of mathematics as a science the development of mathematical machines was begun. These machines were developed as labor saving devices to perform routine operations, which mathematicians could do without machines, with a considerable saving in time and energy. The ancient abacus, the slide rule, and the modern desk calculator are all machines of this type. The differential analyzer, a machine to solve ordinary differential equations, is one of the more recent developments.

2.1 Previous Work on Differential Analyzers

The first conception of a differential analyzer appears to be due to Lord Kelvin toward the end of the 19th century.[1] This idea was independently rediscovered by Vannevar Bush at the Massachusetts Institute of Technology in 1925.[2,3] The first comprehensive differential analyzer was built in 1930.[4] This first machine and those

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1 Sir William Thomson (Lord Kelvin)- a series of papers, and one by his brother James Thomson, published in Proceedings of the Royal Society, v.24, 262-275, Feb. 1876.

2 Bush, V., Gage, F.D., and Stewart, H.R., "A Continuous Integraph," Jour. Frank. Inst., v.208, 63-84, 1927.

3 Bush, V., and Hazen, H.L., "Integraph Solution of Differential Equations," Jour. Frank. Inst., v.208, 575-615, 1927.

4 Bush, V., "The Differential Analyzer. A New Machine for Solving Differential Equations," Jour. Frank. Inst., v.212, 447-488, 1931.

developing immediately from it were characterized by complete generality, the ability to solve any ordinary differential equation, subject only to the limitations of the number of machine components available.[5,6,7,8] These machines utilize mechanical shaft rotations for the various dependent and independent variables, and are rather slow in their operation, requiring from 20 to 40 minutes to run through a typical problem solution and from 2 to 24 hours of preparation or set-up time before the solution of each new problem. The accuracies obtainable with these machines may in certain cases be as good as 0.1%, although in other cases the accuracy is not so good as this. Machines with such performance are rather costly to construct and operate.

It was soon recognized by a number of English workers that there would be considerable utility for a differential analyzer of more moderate accuracy, perhaps 1 to 5%, which could be constructed and operated at a considerably reduced cost. A number of small differential analyzers were built along this general line.[9,10,11]

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5 Travis, Irven, "Differential Analyzer Eliminates Brain Fag," Machine Design, 15-18, July 1935.

6 Hartree, D.R., " The Differential Analyzer," Nature, v.135, 940, June 1935.

7 Rosseland, Svein, "Mechanische Integration von Differentialgleichungen," Die Naturwissenschaften, 27 Jahrg., Heft 44, 729-735, 1939.

8 Kuehni, H.P., and Peterson, H.A., "A New Differential Analyzer," Trans. A.I.E.E., v.63, 221-228, May 1944.

9 Hartree, D.R., and Porter, A., "The Construction of a Model Differential Analyzer," Mem. and Proc. Manchester Lit. and Phil. Soc., v.79, 51-72, July 1935.

10 Massey, H.S.W., Wylie, J., and Buckingham, R.A., "A Small Scale Differential Analyser: Its Construction and Operation," Proc. Royal Irish Acad., v.45, 1-21, 1938.

2.2 Present Trends in Differential Analyzers

Two of the more recently developed differential analyzers have been aimed at strengthening some of the weaknesses of the early mechanical machines.[8,12] These new machines are considerably larger than those previously developed and can thus handle problems of much greater complexity. A considerable increase in flexibility has been achieved by interconnecting the various mechanical computing elements by means of electrical servomechanisms. The set-up of these analyzers is thus accomplished by suitable electrical rather than mechanical interconnections. These machines afford for many cases even greater accuracy than was previously possible, of about 0.01% in favorable cases; they have a running time per solution of about 20 minutes and an initial set-up time of from 20 minutes to 2 hours. They are, however, extremely expensive to build and operate; for example the use of the M.I.T. Differential Analyzer for the period of one hour on a problem requiring the entire machine costs about $50.

The solving of a completely new problem on a differential analyzer introduces a third very important time period in addition to the initial set-up time and final solution times indicated above. That is the time necessary to determine what range of initial conditions and equation parameters are of importance. In order to determine this information it may frequently be necessary to run a large number of exploratory solutions. This situation becomes particularly difficult

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11 Lennard-Jones, J.E., Wilkes, M.V., and Bratt, J.B., "The Design of a Small Differential Analyser," Proc. Camb. Phil. Soc., v.35 (III), 485-493, July 1939.

12 Bush, V., and Caldwell, S.H., "A New Type of Differential Analyzer," Jour. Frank. Inst., v.240, 255-325, Oct. 1945.

in those problems for which not all the initial conditions, but instead some of the final values of the desired solution, are known. For these situations it is necessary to guess at the unknown initial conditions, run a solution, see how the final values thus obtained differ from the desired final values and then attempt to readjust the initial conditions to correct the observed deviations. This process may well require 30, 100, or more trial solutions before the desired final solution is obtained. The time and money thus consumed on even the most modern of the differential analyzers is so great as to render the solution of problems of this type impractical, yet there are a multitude of problems of this nature which are continually confronting the physicist, mathematician, and engineer.

As a result of the increased interest during the recent war in the automatic control of all manner of mechanical devices such as aircraft, marine craft, guided missiles, and the like, there has recently been considerable work done in the field of specialized differential analyzers, frequently called simulators.[13,14,15,16,17,18]

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13 Instruction Booklet prepared by the Bell Telephone Laboratories for the Western Electric M-IX antiaircraft gun director.

14 Ragazzini, J.R., Randall, R.H., and Russell, F.A., "Analysis of Problems in Dynamics by Electronic Circuits," Proc. I.R.E., v.35, 444-452, 1947.

15 Scientific Research and Experiment Department, Admiralty Computing Service Report, "Solution of Differential Equations by an Electronic Differential Analyzer."

16 Korn, G.A., "Elements of D-C Analogue Computers," Electronics, Apr. 1948, p. 122.

17 Philbrick, G.A., "Designing Industrial Controllers by Analog," Electronics, June 1948, p. 108.

These differential analyzers are designed to solve the differential equations associated with the motion or operation of a particular device or perhaps class of devices, and thus fall far short of complete mathematical generality. They utilize electrical voltages as the dependent variables and time as the independent variable. Because these machines are concerned with simulating the operation of various physical devices, they normally operate on a real time scale. This results in solution times of a few seconds to a few minutes depending on the characteristics of the device being simulated. Short set-up time is not of great importance in simulators because most of the operation is concerned with the solution of a single differential equation.

All of the differential analyzers mentioned above belong to the general class of measurement, or continuous-variable, machines. One of the simplest mathematical machines of this type is the common slide rule. The other broad class of mathematical machines are those of the counting, discrete-variable, or digital, type; a standard desk calculating machine is of this type. Currently there is a large amount of development being done on high speed electronic discrete-variable or digital machines.[19, 20] One of the new fields which will be opened by the ultimate development of these machines is that of partial differential equations which are not separable into ordinary

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18 Reeves Instrument Corp., Booklet on an Electronic Analogue Computer; <u>Electronics</u>, p. 231, Apr. 1948.

19 Rockett, F., "Selective Sequence Digital Computer for Science," <u>Electronics</u>, p. 138, Apr. 1948.

20 Burks, A.W., "Electronic Computing Circuits of the ENIAC," <u>Proc</u>. <u>I.R.E.</u>, v.35, 756-767, Aug. 1947.

differential equations. This type of work is completely beyond the scope of the existing differential analyzers. At the present time it would appear that the digital machines will be even more costly than any existing calculators and will therefore, for some time to come, find their principal application to those problems of complexity beyond the scope of existing machines.

The present trends in the field of differential analyzers can be briefly summarized as follows. The continuous variable differential analyzers are being expanded in the direction of (1) greater accuracy, (2) greater size - to handle more complicated problems - and (3) consequently greater cost. Some work is being done on simulators which have (1) reduced accuracy, (2) reduced cost, but are (3) specialized in the types of differential equations they can handle. The field of digital differential analyzers is just beginning to develop; these machines will be characterized by (1) the ability to solve hitherto insoluble problems, (2) extremely high speeds of operation, (3) highest accuracies, (4) large size, and (5) high cost.

2.3 Qualitative Description of Electronic Differential Analyzer

Early in the fall of 1945 it was felt that there was a considerable need for a differential analyzer of somewhat different characteristics from any then in existence or under development.[21] There appeared to be the need for a machine having the following characteristics: (1) moderate accuracy, of perhaps 1 to 10%, (2) large reduction in cost over the existing differential analyzers, (3) high speed of

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21 This need was first recognized and pointed out to the author by Prof. H. Wallman, Department of Mathematics, Massachusetts of Technology, who suggested development of such a machine as a thesis program.

operation, (4) complete mathematical generality, and (5) above all, extreme flexibility to permit the rapid investigation of wide ranges of equation parameters and initial conditions.

A differential analyzer of this type bears the same relation to the larger differential analyzers that a slide rule bears to a desk calculating machine. Its uses are numerous. (a) It can be used as a slide rule is used to give rapid solutions of moderate accuracy to the differential equations encountered by the engineer, physicist, and mathematician. In this role it is useful not only in solving non-linear equations or equations with variable coefficients, but also in solving higher order ordinary differential equations with constant coefficients, which are very tedious to handle analytically. (b) Such a differential analyzer can also be used as an adjunct to one of the larger differential analyzers. It can be used to carry out the time-consuming exploratory solutions necessary to determine those ranges of equation parameters and initial conditions of interest. This preliminary work could be done at a great saving in time and money, and then, if warranted, the larger and more accurate machine could be used to obtain the final desired solution. (c) Such a differential analyzer, by nature of its moderate cost and great flexibility, is very useful as a teaching tool in the fields of mathematics, engineering and physics.

It appears clear that such a machine should be electronic in nature. The differential analyzer described in this thesis has been developed to fulfil this need. From time to time during the process of this development it has been apparent that a number of other investigators are planning a similar machine.[22,23,24,25,26,27] These papers, however indicate that the work described has not been developed as far as the electronic differential analyzer of this thesis.

Philosophy of a Continuous Variable Differential Analyzer

2.4 The Feedback Concept

In order that the problems facing the development of a differential analyzer may be understood a brief discussion of the general idea behind continuous variable differential analyzers as conceived by Vannevar Bush will be given here.[4,12]

This idea can best be presented by considering a particular example of a differential equation and indicating the general procedure

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22 Mynall, D.J., "Electrical Analogue Computing," Electronic Engineer, four parts, June--Sept., 1947.

23 Bruk, I.S., "A Device for the Solution of Ordinary Differential Equations," Comptes Rendues de l'Académie des Sciences de l'URSS., v.LIII, 6, 523-526, 1946.

24 MacKay, D.M., Nature, v.159, Jan. 22, 1947.

25 Koehler, J.S., "An Electronic Differential Analyzer," Jour. of Applied Physics, v.19, no.2, 148-155, Feb. 1948.

26 Korolkov, N.V., "The results of the Development and Testing of an Experimental Apparatus for the Solution of Systems of Differential Equations," Bull. Acad. Sci. U.R.S.S., Classe Sci., p. 585-596, 1947.

27 Gradstein, I.S., "The Solution of Systems of Linear Equations by L.I. Guttenmaher's Electrical Models," Bull. Acad. Sci. U.R.S.S., Classe Sci., Tech., p. 529-584, 1947

for obtaining its solution on a differential analyzer. Consider the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0; \qquad (1)$$

it will be assumed that physical quantities have been chosen to be used as the dependent and independent variables and that devices are available to perform the mathematical operations such as addition, multiplication, integration and the like. For the mechanical differential analyzers the physical quantity used as a variable is shaft rotation and the various units of the differential analyzer must therefore be capable of adding, subtracting, integrating, etc. shaft rotations. In an electronic differential analyzer the variables are voltages, and units are required to perform the necessary mathematical operations on voltages. The general procedure for solving a differential equation is to isolate the highest derivative in the equation. Thus for Eq. (1) one writes

$$\frac{d^2y}{dx^2} = -(\frac{dy}{dx} + y). \qquad (2)$$

From this point on the step by step procedure of setting up this differential equation is illustrated with block diagrams in Fig. 1. In Fig. 1(a) a particular shaft or terminal post is assumed to be the highest derivative, in this case $\frac{d^2y}{dx^2}$. In Fig. 1(b) this point is connected to a cascade of two units which perform the mathematical operation of integration with respect to x. The outputs of these two units are then $\frac{dy}{dx}$ and y respectively. In Fig. 1(c) these outputs are then supplied to a third unit which performs addition; its output is $\frac{dy}{dx} + y$. This output is put through a fourth unit which changes

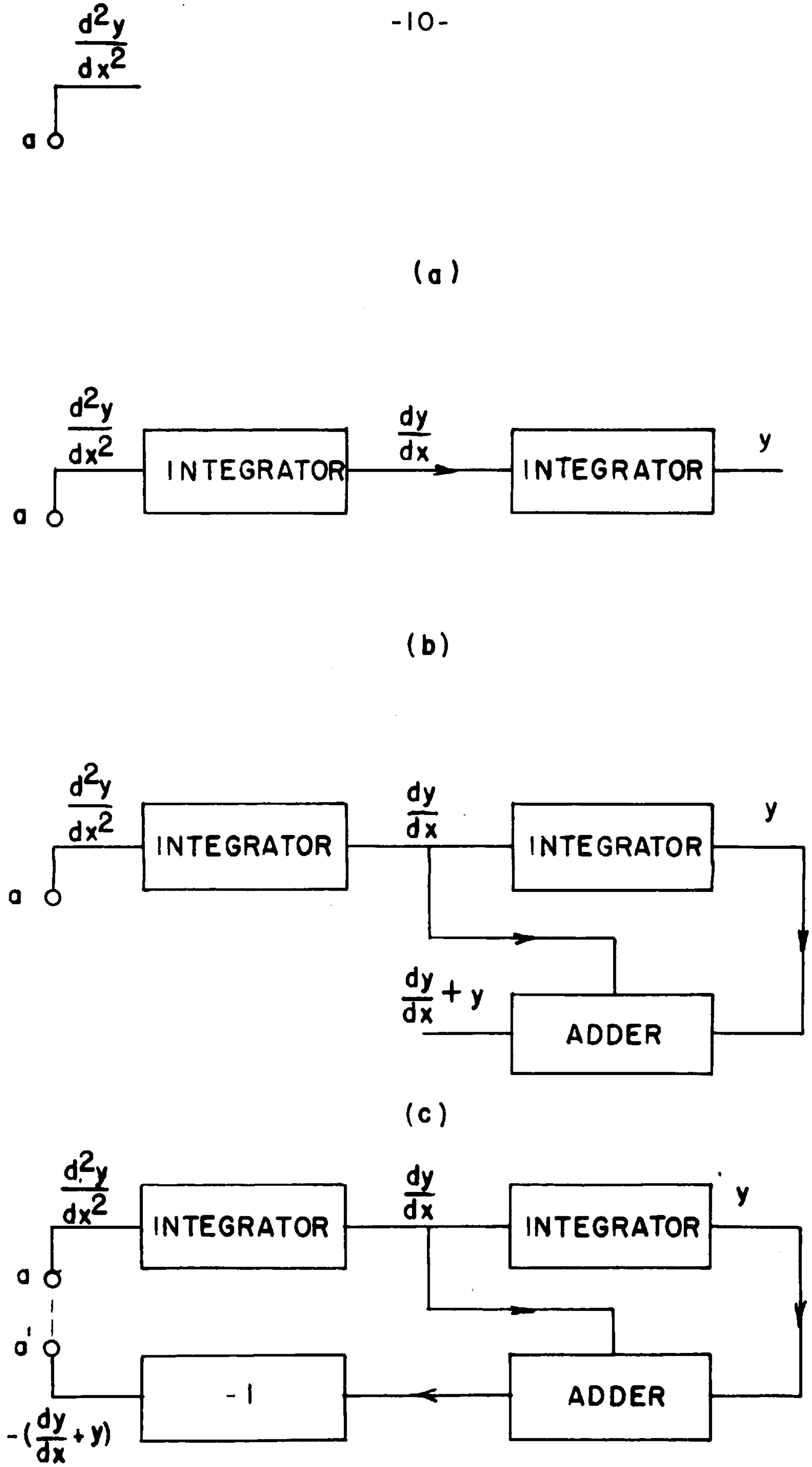

FIG. 1 Set-up of simple differential equation

the algebraic sign, forming the quantity $-(\frac{dy}{dx} + y)$ as shown in Fig. 1(d). This output appears at the terminal or shaft a' but according to Eq. (2) is equal to $\frac{d^2y}{dx^2}$. Therefore to solve the differential equation one connects the terminals or shafts a-a' thus closing the feedback loop. If now the independent variable is permitted to vary, the various shafts or voltages in this interconnection of units are constrained to vary in such manner as to solve the desired differential equation. In order to obtain a useful solution for a particular case it is necessary to have some process for giving the various dependent variables in this set-up their proper initial values before the solution process is begun, and some means has to be available to present the results of each solution in a form easily comprehensible by the operator.

The fundamental principle of the process outlined above for the solution of a differential equation is that of feedback. The process consists of (1) assuming one of the unknown dependent variables, (2) performing such operations on this quantity as are necessary to generate all the other unknowns in the differential equation, (3) generating such functions of the independent variable as may be required for the equation, and (4) interconnecting these generated quantities in the manner specified by the differential equation being solved.

2.5 The Possibility of Different Systems

It should be recognized, in particular, that the use of integration in the example considered was not of fundamental importance. One might as well, on the face of it, rewrite Eq. (1) in the form

$$y = -(\frac{dy}{dx} + \frac{d^2y}{dx^2}). \qquad (3)$$

When written in this way it is apparent that one could as well solve this differential equation by using units which perform differentiation of the dependent variables. Such a set-up is indicated in block form in Fig. 2.

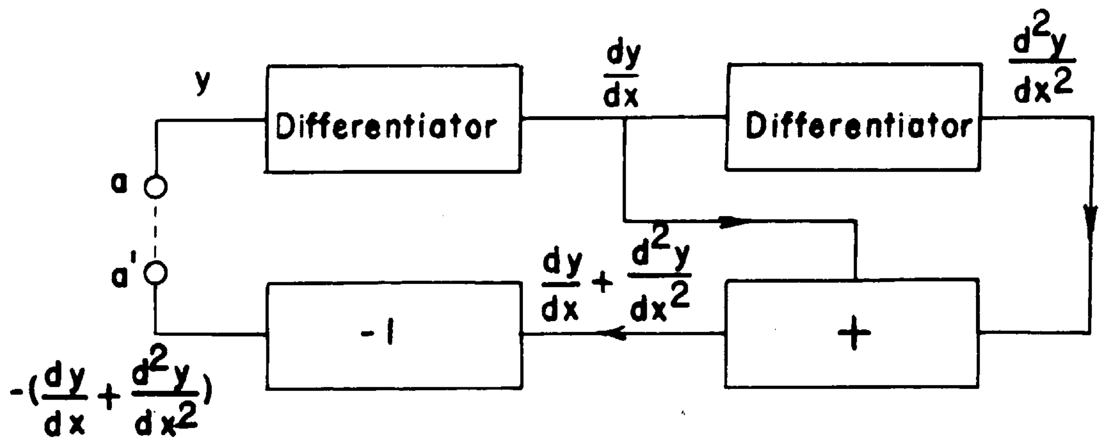

Fig. 2 Alternate set-up of simple differential equation

Carrying this procedure one step further one can see that by using a combination of integration and differentiation there is a large number of possible ways in which any differential equation can be solved on a differential analyzer. For any particular case the system used will depend largely upon the difficulty of realizing practical units to perform the mathematical operations required.

Statement of the Thesis Problem

The basic problems in the development of an electronic differential analyzer can now be stated. They are:

(1) The choice of a system to be used in the differential analyzer. This choice must result in a system whose component units can be

realized economically by electronic means. It must be a completely general system permitting the solution of any ordinary differential equation.

(2) The components for this system must be developed into a form giving satisfactory performance, and if some components are not available new ones must be invented.

(3) A model differential analyzer must be built and tested as a functioning unit.

(4) This differential analyzer should then be used to solve a variety of typical problems. The test problems must be so chosen as to verify the full mathematical generality of the machine.

(5) Finally an investigation of the accuracies obtainable with the differential analyzer, the principal sources of errors and steps that can be taken to minimize them.

SECTION III

SYSTEMS

Before entering into a detailed discussion of the various possible differential analyzer systems, we restate the principal features the electronic differential analyzer is desired to have:

(1) moderate accuracy, of perhaps 1-10%,

(2) low cost of construction and operation,

(3) complete mathematical generality, so that any ordinary differential equation can be handled,

(4) high speed of operation, and

(5) above all, extreme flexibility to permit rapid investigation of wide ranges of equation-parameters and initial-conditions.

The relative merits of various possible differential analyzer systems will now be considered in the light of these basic requirements.

Possible Systems

3.1 The Choice of Variables

Electrical voltage is the most natural choice as the physical quantity to represent the dependent variables in view of the requirements of high speed operation and flexibility. Electrical voltage can be varied as rapidly, with existing techniques, as any physical quantity known. If the various components of a differential analyzer are designed to perform mathematical operations on voltages, the interconnection of these various units can be easily accomplished by means of patch cords or cables.

Having chosen to use voltages for the dependent variables one still has the possibility of using another physical quantity for the independent variable. It is apparent from the general discussion of the feedback method of solving differential equations given in the introduction that some component in any differential analyzer must perform the operation of integration or differentiation of the dependent variables with respect to the independent variable. The set-up indicated in Fig. 1 requires only integration, while the set-up of Fig. 2 utilizes only differentiation, but it is manifest that either one or both of these operations must be performed in any differential analyzer no matter what its nature.

Having voltages as dependent variables, a possible choice for the independent variable is also a voltage. If this is to be the case, it will be necessary to develop electronic units to perform either integration of one voltage with respect to another voltage or differentiation of one voltage with respect to another. Let us consider the case of integration. A unit which performs integration of one voltage with respect to another, that is the operation

$$W = \int P dQ, \tag{4}$$

where P and Q are voltages, will henceforth be designated as a general integrator. The differential dQ can also be written as

$$dQ = \frac{dQ}{dt} \cdot dt , \tag{5}$$

where dt is the time differential. Thus the integral of Eq. (4) can be rewritten

$$W = \int (P\frac{dQ}{dt})dt . \tag{6}$$

It is apparent from this that general integration of one voltage with respect to another can be accomplished by integration with respect to time of the product of a voltage and the derivative of a voltage with respect to time. The wheel and disc integrators used in the mechanical differential analyzers perform integration with respect to time of the product of a shaft rotation and a shaft angular velocity.

All general integrators known to the author involve in their construction the operation of multiplication. If one chooses to perform general integration in an electronic differential analyzer with voltages as variables, it would appear to be necessary to have a circuit or device which gives the product of two voltages as a component part of every integrator. It is unfortunately true that multiplication is the most difficult mathematical operation that one is called on to perform electrically, for although it is easy to obtain an electronic device which will have a product term in its output, it is very difficult to isolate this term; more will be said about this in Section IV.

Although there is an element of advantage in building an electronic differential analyzer using general integrators, because then the analogy between it and the large mechanical machines would be exact, the difficulties of realizing a practical electronic general integrator are great enough to warrant investigation of other possible schemes. These are based on the fact that although the operation of multiplication is a very difficult one electronically, integration and differentiation, with respect to time, are very easily performed.

Because differentiation and integration are easily performed on voltages with respect to time, it was decided to use time as the independent variable in this electronic differential analyzer.

3.2 The Limitation on the Use of Time Differentiators

With time as the independent variable there still remains the choice of whether differentiation, as in Fig. 2, or integration as in Fig. 1, or some combination of the two should be employed.

Fortunately a clear-cut answer can be given to this question by considering the limitations which are encountered in any practical time differentiator due to the finite bandwidths of practical amplifiers.

Suppose one has an ideal differentiator. The input and output of such a unit is related by Eq. (7)

$$e_2 = K\frac{de_1}{dt} . \tag{7}$$

The Laplace transform of the transfer characteristic for this unit is then[28]

$$\mathcal{L}\left(\frac{e_2}{e_1}\right) = \frac{e_2}{e_1}(s) = sK . \tag{8}$$

This shows that for real frequencies, $s = j\omega$, the transfer characteristic of an ideal differentiator is

$$\frac{e_2}{e_1}(\omega) = j\omega K , \tag{9}$$

that is, it has a phase characteristic which is constant at +90 degrees and a magnitude characteristic which increases directly with frequency. Such a magnitude characteristic is plotted in decibels versus the logarithm of the frequency in Fig. 3 and is labelled ideal.*

It is evident that this ideal characteristic can never be achieved in practice since it requires infinite gain at infinite frequency. Rather it is known that due to unavoidable stray capacities any physical

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28 Gardner, M.F., and Barnes J.L., Transients in Linear Systems, Vol. I, John Wiley & Sons, New York, 1942, 126-130.

* It is interesting to note that any high-pass filter can be thought of as a time differentiator over a portion of the frequency range; similarly any low-pass filter will have the characteristic of a time integrator for some frequencies. A time integrator or differentiator can also be thought of as an amplifier whose amplitude-frequency characteristic has been rotated to a slope of minus or plus 6db/octave respectively.

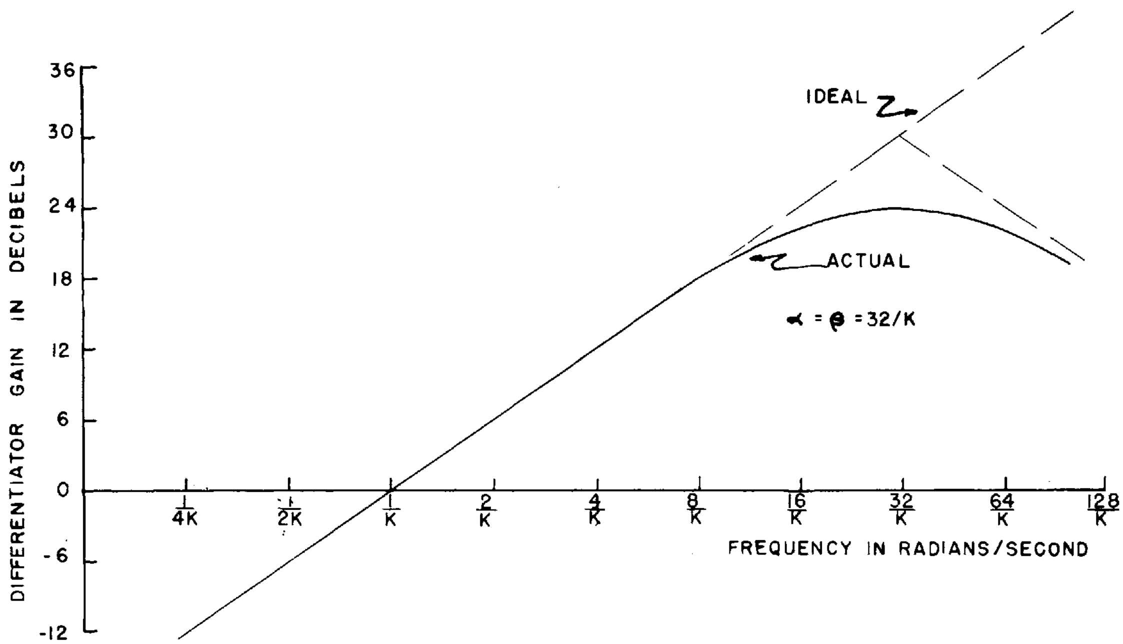

Fig. 3 Log-db characteristics of time-differentiators

characteristic at high frequencies has a slope of at least -6 db per octave rather than the ideal slope of +6 db per octave.[29] A realizable characteristic of this type is also plotted in Fig. 3. The simplest time-differentiator characteristic is therefore of the form

$$\frac{e_2}{e_1}(j\omega) = \frac{j\omega K}{(1 + j\omega\alpha)(1 + j\omega\beta)} , \qquad (10)$$

where normally both

$$\alpha << K , \qquad (11)$$

and

$$\beta << K . \qquad (12)$$

For the characteristic sketched in Fig. 3, $\alpha = \beta$, but in general the relative values of these two high frequency time constants will

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[29] Bode, Hendrik W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand Co., New York, 1945, Chapt. XVII.

depend on the practical circuit used. The important point is that at least two such time constants must always be present in order that the requirement of physical realizability be satisfied.

The importance of this limitation will be realized by considering a simple differential equation for which the solution is easily calculated. Such an equation is

$$(K)\,\frac{dy}{dt} - y = 0\,. \tag{13}$$

The characteristic equation of this differential equation is

$$(sK - 1) = 0\,, \tag{14}$$

and therefore this equation has the solution

$$y = Ce^{t/K}\,, \tag{15}$$

where C is a constant determined by the initial condition of the solution. The differential analyzer set-up necessary to solve this equation, using a time differentiator, is indicated in Fig. 4.

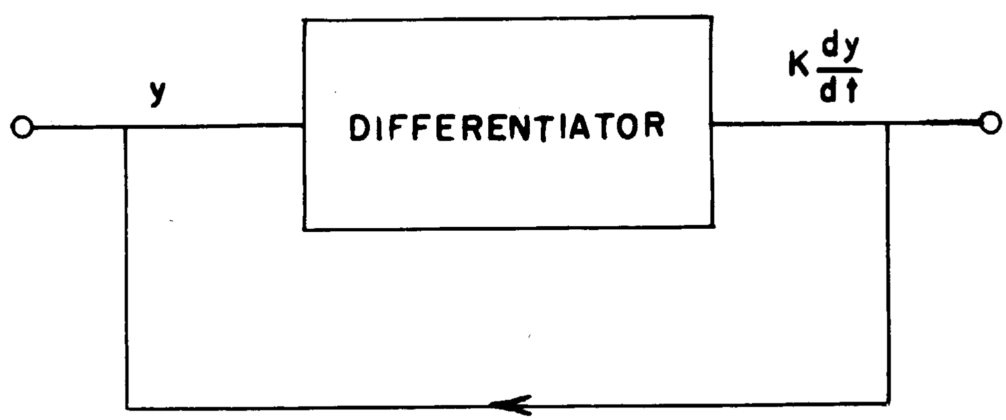

Fig. 4 Block diagram of set-up for equation $K\,\frac{dy}{dt} - y = 0$.

If the time-differentiator were ideal, this set-up would give the solution of Eq.(15). If one assumes that the differentiator has the simplest realizable characteristic, given by Eq. (10), one observes by making $e_1 = e_2$ and letting $j\omega = s$ that the characteristic equation of the differential equation actually solved is

$$(1 + s\alpha)(1 + s\beta) = sK , \tag{16}$$

which has roots

$$s_1 = \frac{K-\alpha-\beta}{2\alpha\beta} + \sqrt{\left(\frac{K-\alpha-\beta}{2\alpha\beta}\right)^2 - \frac{1}{\alpha\beta}} , \tag{17}$$

$$s_2 = \frac{K-\alpha-\beta}{2\alpha\beta} - \sqrt{\left(\frac{K-\alpha-\beta}{2\alpha\beta}\right)^2 - \frac{1}{\alpha\beta}} . \tag{18}$$

If now, as is usually the case, Eqs. (11) and (12) are satisfied, these roots are

$$s_1 \approx \frac{1}{K} , \tag{19}$$

$$s_2 \approx \frac{K}{\alpha\beta} . \tag{20}$$

The solution obtained by the differential analyzer set-up of Fig. 4 is therefore

$$y = C_1 e^{t/K} + C_2 e^{tK/\alpha\beta} . \tag{21}$$

Comparing this solution with the desired one of Eq. (15), one sees that there is an error term $C_2 e^{tK/\alpha\beta}$. In theory at least it is possible to adjust the initial conditions so as to make C_2 equal to zero. In practice, however, this is completely impossible since the time-constant of the error term is much smaller than the time-constant of the desired term according to the assumptions of Eqs. (11) and (12). It is therefore not possible to obtain a usable solution to eq. (13) with a differential analyzer employing time-differentiators. This is an extremely serious limitation on the generality of such a differential analyzer, which is encountered whenever one attempts to solve a differential equation whose characteristic equation has roots with positive real parts, that is in the right half of the complex s-plane.

The conclusion therefore is that because of the limitations imposed on physically realizable time-differentiators by the bandwidth limitations of amplifiers, a general electronic differential analyzer cannot be designed using differentiators as a basic unit.

One is forced therefore to use time-integrators as the basic unit of an electronic differential analyzer.

Details of Electronic Differential Analyzer System

For the reasons indicated above, the system chosen for the electronic differential analyzer of this thesis utilizes voltages as the dependent variables and time as the independent variable. The basic functional operation for this analyzer is time integration.

3.3 Solution Display

The solutions obtained from such an analyzer are in the form of a voltage as a function of time or a voltage as a function of another voltage, if one dependent variable is plotted as a function of another. An electronic device well suited to the plotting of one voltage as a function of another is a cathode-ray tube. Such a tube is used to display the solutions obtained by the differential analyzer. In displaying a solution as a function of the independent variable time, a periodic voltage varying linearly with time (a saw tooth wave) is applied to the horizontal deflecting plates of the tube, and a voltage corresponding to the desired solution is applied to the vertical deflecting system.

3.4 Repetition Rate

In order that the solution displayed on the cathode-ray tube face appear as a stationary curve to the observer's eye, it is necessary that the solutions traced on the screen be repeated at a reasonably high rate. If the cathode-ray tube screen employs the common P-1 phosphor, this repetition rate should be at least 30 c.p.s. If one of the more persistant screens such as the P-7 phosphor, which was developed for radar PPI displays, is used, a repetition rate as low a 1 c.p.s. might be employed.* In the differential analyzer described here the repetition rate is 60 c.p.s. This is convient as it permits use of the a-c power mains as a standard source of synchronizing signals.

3.5 Sequence of Operation

At this repetition frequency a new solution is run-off every 1/60 of a second, and these solutions are displayed superimposed one upon the other on the face of the output cathode-ray tube. In order that this display appear as a single line on the output screen it is necessary that each successive solution be identical with its predecessor as long as the same solution is being displayed. This imposes the requirement that the initial conditions at the beginning of each solution be identical with those of the preceding solution. In general the voltages in the differential analyzer at the end of a solution

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* It will be shown in Section V that component limitations at high and low frequencies introduce errors into solutions obtained on an electronic differential analyzer. The choice of a repetition rate is a compromise; a high repetition rate requires excellent high-frequency response of all analyzer components while a low repetition rate places the emphasis on the low-frequency response.

will not be the same as those required by the initial conditions; it is therefore necessary to allow some time between the end of each solution and the beginning of the next solution to permit the reestablishment of the proper initial conditions. In this differential analyzer this is accomplished by allotting approximately 1/120 of a second for the solution of the differential equation and 1/120 of a second for the reestablishing of the initial values prior to the next solution. Fig. 5 is a sketch of the solution of a typical equation as it would appear in this analyzer if two complete solution periods are displayed. Figs. 6(a) and 6(b) are photographs of observed solutions of the differential equation

$$\frac{dy}{dt} = ky \; . \qquad (22)$$

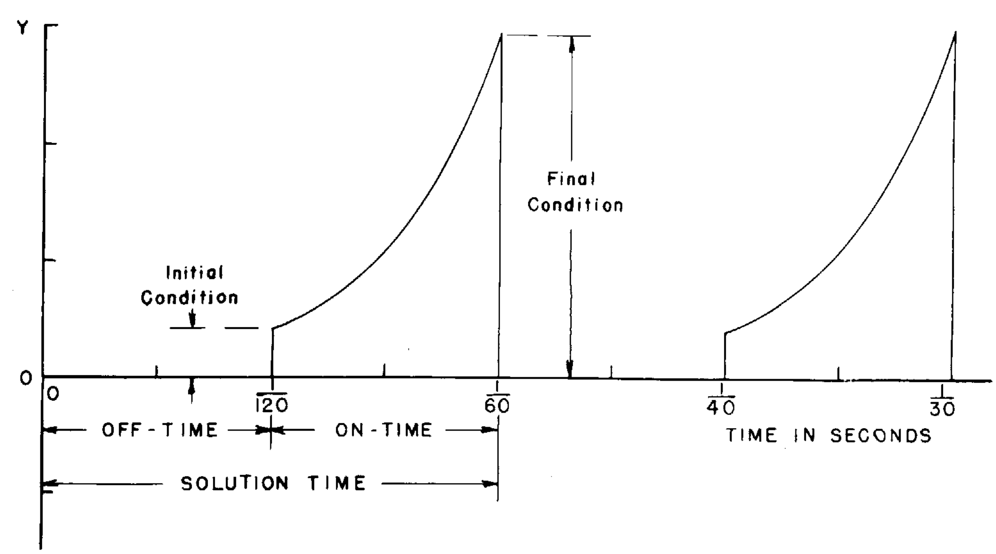

Fig. 5 Appearance of a typical differential analyzer solution as a function of time.

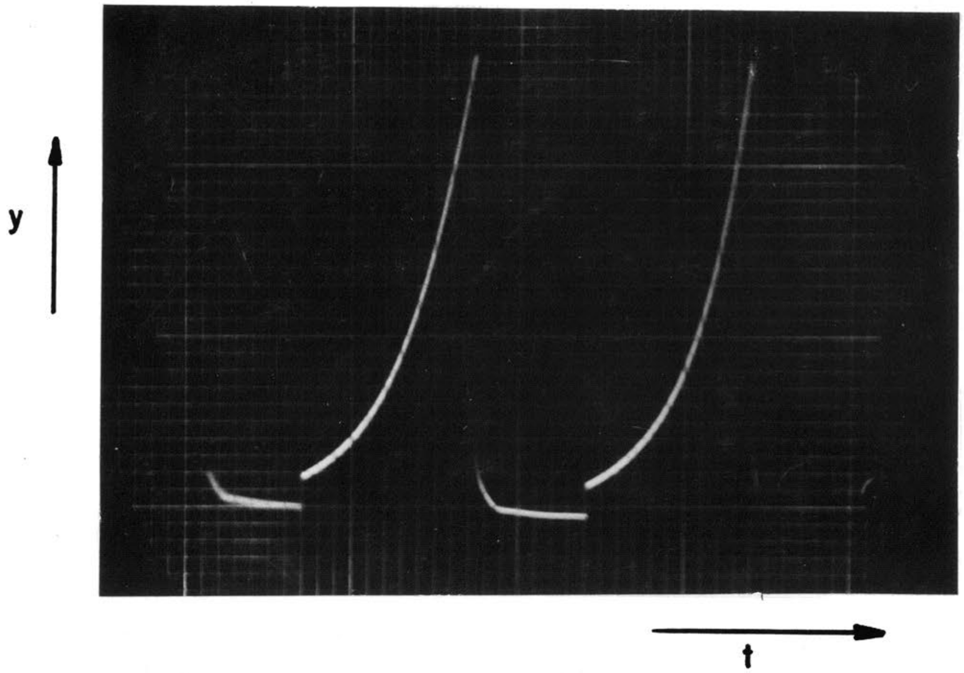

Fig. 6(a) Two complete periods of the solution of the equation $\frac{dy}{dt} = ky$.

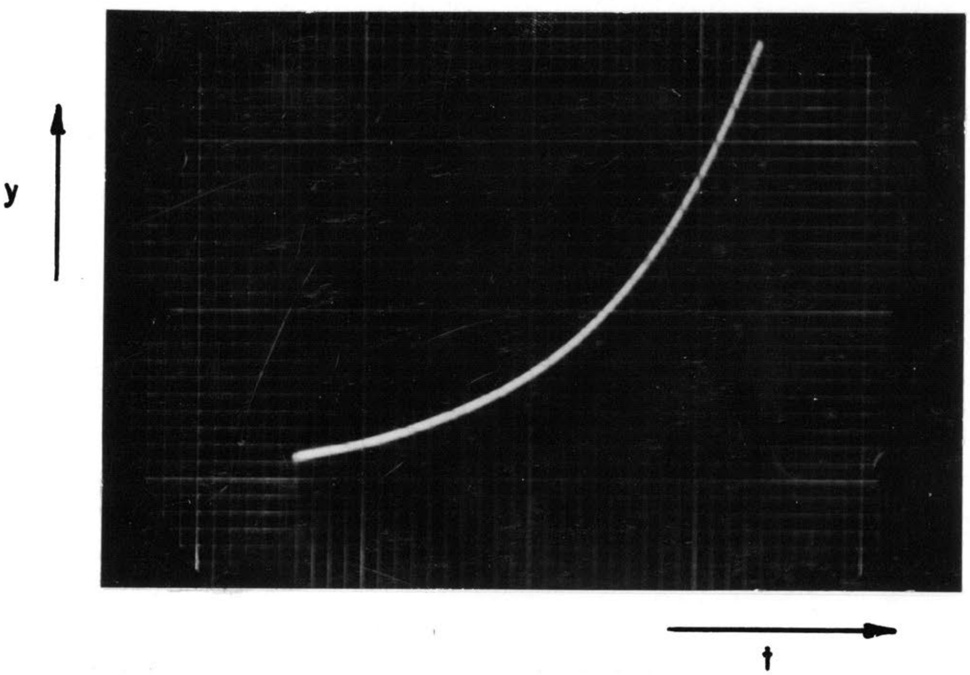

Fig. 6(b) Solution of $\frac{dy}{dt} = ky$ as normally displayed, with off period blanked out by an intensity gate.

Fig. 6(a) displays two complete solution periods, showing both the on- and off-times. Fig. 6(b) gives the solution as normally displayed in the differential analyzer; here a blanking voltage has been applied to the intensity grid of the cathode-ray tube during the off-time of the solution period so that only the solution is displayed.

3.6 Initial Conditions

To satisfy the initial conditions of the differential equations requires establishment of specified initial values of all voltages throughout the differential analyzer at the beginning of each solution period. This is accomplished by forcing all voltages in the analyzer to a constant reference level, usually close to zero volts, during the off-time. At the beginning of the on-time for the differential analyzer voltage steps are introduced at the proper points throughout the machine to set the desired initial values. These initial value voltage steps remain constant throughout the solution time; they are adjustable by the operator through ordinary potentiometers. The operator varying these initial condition potentiometers sees instantaneously the effect of his adjustment, since a new solution is traced out every 1/60 of a second. Fig. 7 is a triple exposure photograph of the solution of Eq. (22) for three different initial values.

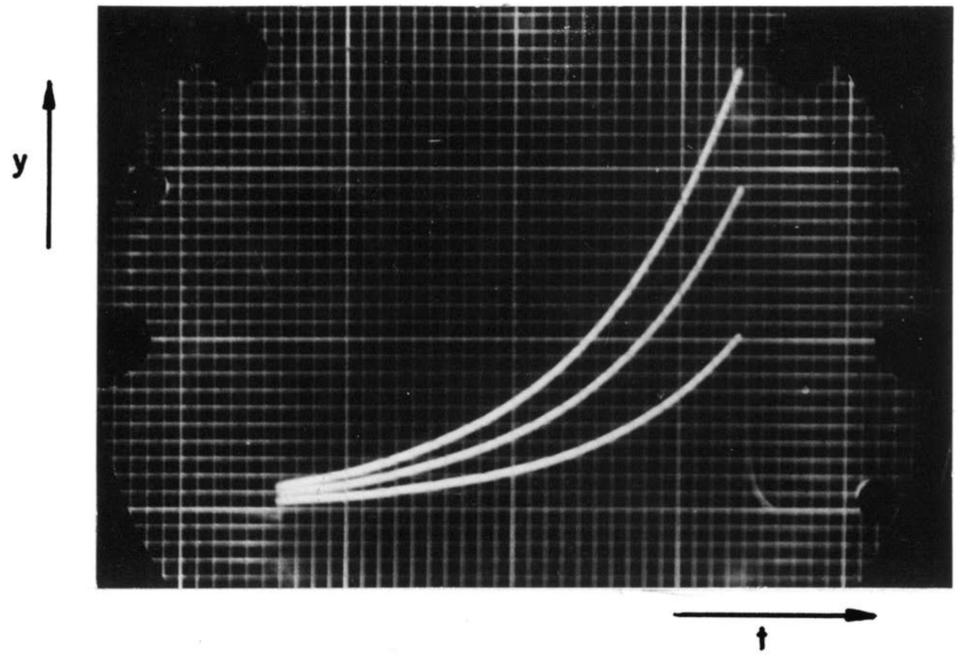

Fig. 7 Triple exposure photograph of the solutions of $\frac{dy}{dt} = ky$ for three different values of y_o.

Units Required For Electronic Differential Analyzer

3.7 Ordinary Differential Equations with Constant Coefficients

Having settled on the broad general features of the electronic differential analyzer one can now consider in detail what units had to be developed for its realization. In this regard it is probably easiest to start out with the special case of ordinary linear differential equations with constant coefficients. The additional components necessary to solve the more difficult non-linear equations and equations with variable coefficients will be considered later.

An ordinary linear differential equation with constant coefficients can always be written in the form

$$\sum_{n=0}^{m} A_n \frac{d^n y}{dt^n} = F(t), \qquad (n = 1,2,\ldots,m) \tag{23}$$

where

(1) F(t) is an arbitrary driving function of time,

(2) the coefficients A_n are constants, and

(3) m is the order of the differential equation.

For the purpose of solution on the differential analyzer this equation can be rewritten by isolating the highest derivative:

$$\frac{d^m y}{dt^m} = -\frac{1}{A_m} \sum_{n=0}^{m-1} A_n \frac{d^n y}{dt^n} + \frac{F(t)}{A_m}. \qquad (24)$$

A block diagram of a general set-up to solve a differential equation of this form with the electronic differential analyzer is given in Fig. 8.

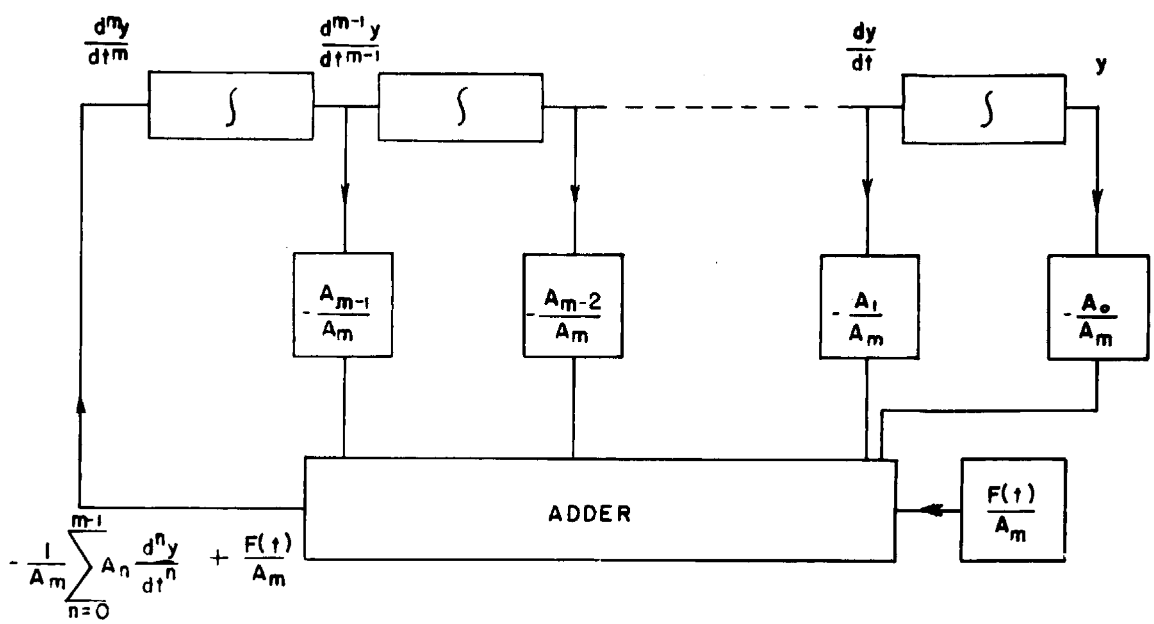

Fig. 8 Block diagram of set-up for solution of an ordinary linear differential equation of order m with constant coefficients

There are four basic units required by this set-up. First there are the blocks labeled with the integral sign in Fig. 8; these are units which perform integration with respect to time. Second there are the boxes labeled $-\frac{A_{m-1}}{A_m}$; these are amplifiers with constant gains; a negative algebraic sign is obtained by a 180 degree phase shift between the input and output of the amplifiers. Third there is the box marked adder, which is to be capable of forming the sum of a number of voltages. Finally there is the box $\frac{F(t)}{A_m}$, which is to generate a voltage that is an arbitrary function of time.

There are a number of other units not indicated in Fig. 8 which will also be necessary in an electronic differential analyzer. First there must be some means of introducing initial conditions into the set-up shown. Second, since the solutions of this equation are to be repeated periodically in time, there must be some means of turning the differential analyzer on and off at the proper times and of restoring the final voltage values to zero at the end of every solution on-time. These last two functions are closely related and can be combined into a single unit which will be called an initial condition and gate pulse generator. Other necessary units are (1) a viewing scope on which the solutions can be displayed to the operator, (2) calibrating equipment to permit quantitative measurements on the solutions obtained, and (3) power supplies to run the various units.

The block diagram of Fig. 8 does not represent the only interconnection of units which will give a solution to Eq. (23). Some of the scale-factor amplifiers might be incorporated into the integrators or the adder, and it is not necessary to do all the summing in a single adder. Partial sums might first be formed and then combined in a final adder. All of these variations, however, are of minor nature and will not introduce any change in the types of components required. A summary list of the components necessary to solve ordinary linear differential equations with constant coefficients on the differential analyzer is therefore:

(1) Time integrators,

(2) Adders,

(3) Amplifiers to provide scale factor and sign changes,

(4) Initial condition and gate pulse generator,

(5) Viewing cathode-ray tube oscilloscope,

(6) Calibrating equipment to permit quantitative measurement of

(a) Time, the independent variable, and

(b) Voltages, the dependent variables,

(7) A Function generator to generate arbitrary voltage functions of time,

(8) Power supplies.

The additional apparatus needed for differential equations with variable coefficients and non-linear differential equations will now be considered.

3.8 Ordinary Linear Differential Equations with Variable Coefficients

For linear equations with variable coefficients Eq. (23) is still applicable provided one removes the restriction that the coefficients A_n be constants and write instead that

$$A_n = A_n(t) \ . \tag{25}$$

The block diagram of Fig. 8 must now be modified as shown in Fig. 9, where every constant coefficient box has been replaced by (a) a function generator box to generate $-\frac{A_n(t)}{A_m(t)}$, and (b) a unit performing multiplication of one voltage by another.

The presence of variable coefficients requires the use of one new

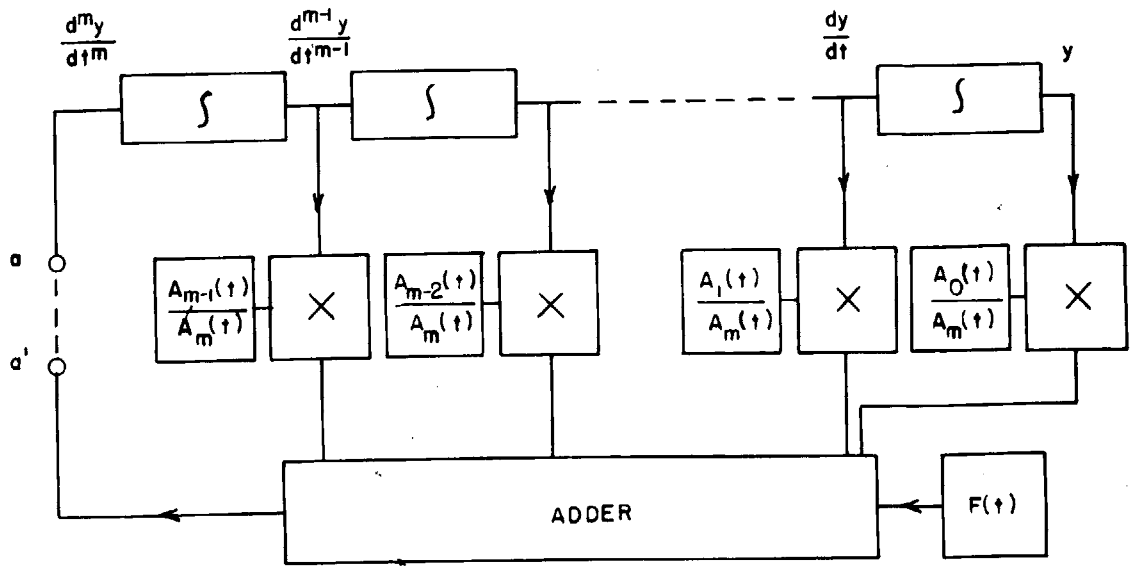

Fig. 9 Block diagram of set-up of differential analyzer for the solution of a linear differential equation with variable coefficients.

type of unit, an electronic multiplier,* plus additional function generators to generate the variable coefficients as voltage functions of time.

3.9 Non-Linear Differential Equations

The most general type of ordinary differential equation can be written in the form

$$F\left(\frac{d^m y}{dt^m}, \ldots, \frac{dy}{dt}, y, t,\right) = 0. \qquad (26)$$

* We see, therefore, that the choice of time as the independent variable, which permits the use of time-integration rather than general-integration of the voltage variables, does not completely obviate the necessity for electronic multiplication. It does, however, reduce the number of multiplications required to that minimum explicitly indicated by the differential equation. In view of the difficulty of performing multiplication this is a very worthwhile step. Thus in the present system, no multipliers are required in the solution of a linear differential equation with constant coefficients, no matter how high the order of the equation. If a voltage independent variable and general integrators were used, the solution of an ordinary linear differential equation with constant coefficients of order m would require m electronic multipliers as components in the necessary general integrators.

The only new operation required in going from a linear differential equation with variable coefficients to this most general ordinary differential equation, is that it is now necessary to generate functions of the dependent as well as the independent variables. Thus the non-linear equation

$$\frac{d^2y}{dt^2} = F(y) , \tag{27}$$

would be set up according to the block diagram of Fig. 10.

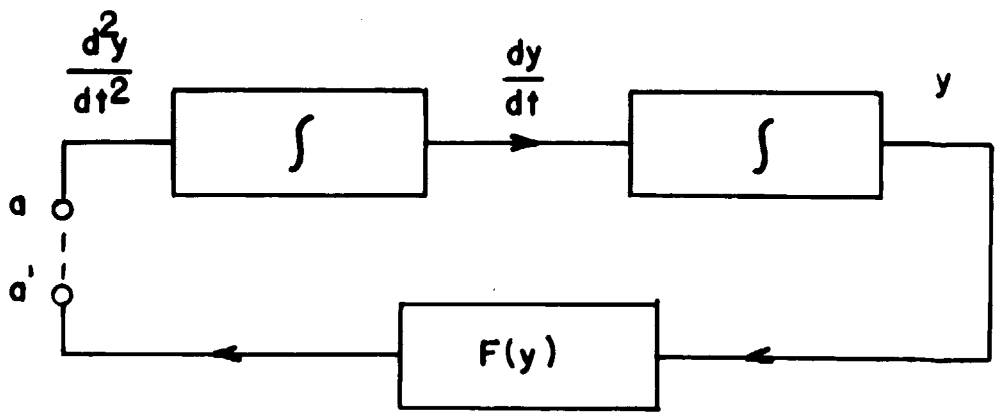

Fig. 10 Block diagram of set-up for the solution of the non-linear differential equation $\frac{d^2y}{dt^2} = F(y)$ on a differential analyzer.

The new component required is the unit labelled F(y). This is a unit which will generate a voltage that is an arbitrary function of any input voltage. It will be designated here as an arbitrary function generator; its analog in the mechanical differential analyzers is the input table. Note that an arbitrary function generator can be made to generate functions of the independent variable, time, by connecting to its input a voltage varying linearly with time over the solution period, that is, a saw tooth wave.

For a differential analyzer of full generality then, two additional items must be added to the list of components on page (29); they are

(9) Multipliers, units whose output voltage is proportional to the product of two input voltages, and

(10) Arbitrary function generators, units that generate an output voltage which is an arbitrary function of the input voltage.

Of the components listed, these last two did not exist when the work of this thesis was begun. It has been necessary to invent electronic devices to perform the operations of multiplication and function generation.

Although some basic circuits for the performance of such operations as time integration and addition have been described by other workers, it has been necessary in this development to go into considerable detail concerning the characteristics of these units in order that sources of error be identified and studied.

SECTION IV

DIFFERENTIAL ANALYZER COMPONENTS

Multiplication

4.1 Important Multiplier Characteristics

As has been indicated, multiplication is a difficult operation to perform electronically. Much previous work has been done on this subject, and it will be reviewed briefly here. First, however, it is worthwhile to investigate the desired objectives of the multiplying unit.

(a) Since voltages are the dependent variables, the multiplier must form the product of two voltages.

(b) For complete mathematical generality the multiplier must be capable of treating input voltages of both algebraic signs, that is, it must be a four-quadrant multiplier.

(c) Because the repetition frequency of the differential equation solution is to be 60 c.p.s., the multiplier must be capable of dealing with frequencies much higher than 60 c.p.s., at least as high as perhaps 60 Kc/s.

(d) The accuracy should be as high as possible.

(e) The size and cost must be kept to a minimum.

(f) The balancing adjustments in the multiplier should be as stable and easy to control as possible.

That any four-quadrant multiplier will involve some form of balancing can be easily seen by considering the block diagram of such a unit in Fig. 11.

Fig. 11 Block diagram of electronic multiplier.

This unit must have the characteristic that if the input voltages are E_1 and E_2, the output voltage is kE_1E_2, where k is a constant. In particular, if E_1 is zero, the output voltage must be zero no matter what the value of E_2. Therefore there must exist in the multiplier some means to balance the output to zero when E_1 is zero. A similar argument can be applied to the case for which E_1 is non-zero and E_2 zero. Any four-quadrant multiplier will thus have somewhere hidden in its design at least two balances. The nature of these balances is probably the most important single feature of a multiplier. This point can be illustrated by considering the case of a multiplier for which these balances are not perfect. Such a multiplier might have the characteristic

$$\text{multiplier output} = E_1E_2 + .025E_1 + .025E_2, \qquad (28)$$

assuming (1) E_1 is one input voltage,

(2) E_2 is the second input voltage, and

(3) the maximum permissible value of either E_1 or E_2 is unity.

When E_1 and E_2 both have their maximum values of unity, the output of this multiplier will be correct to within 5%, which represents usable accuracy. Consider, however, what happens when E_1 is held constant at its maximum value and E_2 is reduced to 0.1. The desired product is then 0.1, but the output of this multiplier is 0.128; the error is now 28%. This clearly shows that the reliability with which the zero balances can be made imposes most stringent limitations on the usable dynamic range of the multiplier.

4.2 Previous Multiplier Developments

A linear potentiometer can be used to perform multiplication, as indicated in Fig. 12.[30]

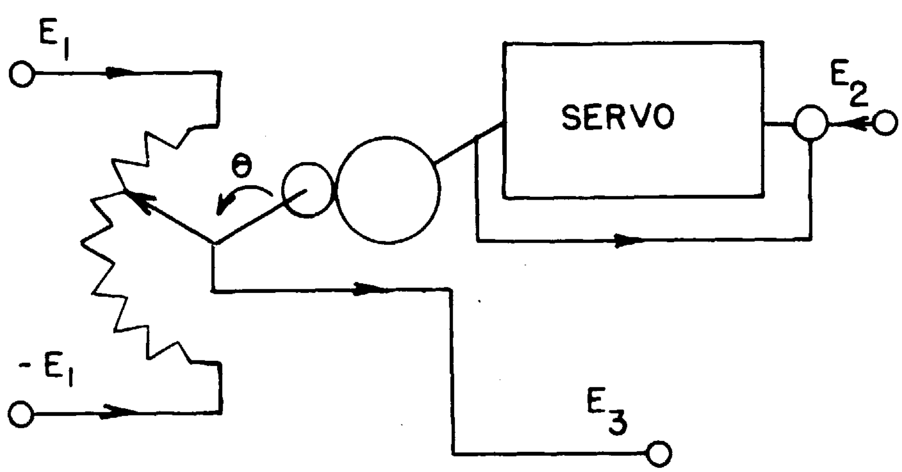

Fig. 12 Servo-driven multiplier.

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30 Murray, F.J., The Theory of Mathematical Machines, King's Crown Press, New York, 1st Edition, 1947, p. 25.

For this set-up

$$E_3 = k \cdot \theta E_1 \; . \qquad (29)$$

If a servomechanism is used to position the potentiometer shaft, θ is proportional to the voltage E_2; then

$$E_3 = KE_1E_2 \; . \qquad (30)$$

This multiplying circuit has been used in fire-control computers; it is not satisfactory for the present application because its operating speed is limited by the mechanical shaft and motor inertia.

Because of our requirement of high speed, all mechanical or electromechanical multiplying schemes can be ruled out at the outset. There exist, however, a wide variety of proposed methods for performing multiplication with all-electronic circuits. One such idea is shown in Fig. 13.

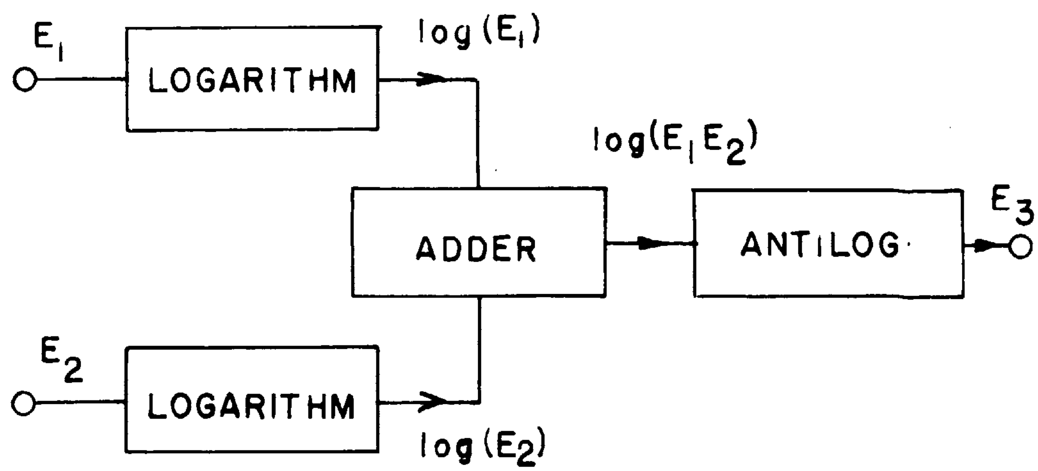

Fig. 13 Block diagram of logarithmic multiplier

The two input voltages E_1 and E_2 are first applied to units which form their logarithms. These logarithms are then added and supplied to a device which forms the antilogarithm; the output from this unit, E_3, is the desired product. Kallmann[31] and Wingate[32] have developed multipliers of this type. Kallmann used welded germanium crystal rectifiers to achieve the necessary exponential current-voltage characteristics, while Wingate employed 6AL5 diodes. A logarithmic multiplier has the major disadvantage, from the point of view of a general mathematical machine, that it is limited to one-quadrant multiplication. There is also the difficulity of finding an accurate logarithmic device. The 6AL5's used by Wingate had to be operated at currents of the order of 0.1 μa., which has the disadvantages of any low level device,--sensitivity to hum pickup, etc. The characteristics of the germanium diodes were hard to match and maintain with age.

Another general class of multipliers can be described as double-modulation multipliers. The concept employed is to impress the input voltages E_1 and E_2 through suitable modulating circuits on a common carrier signal, which is then detected in a manner to give the desired product as the output. Since amplitude, frequency and many forms of pulse modulation can be applied to this scheme, the ramifications of this method are numerous.

Siebert[33] has developed a multiplier of this general class employing double amplitude-modulation as indicated in Fig. 14. In this circuit a 400 c.p.s. carrier signal is passed through two linear amplitude

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31 Kallmann, H.E., "Log Bridge and Ratio Meter," Radiation Laboratory Internal Group Report 41, July 16, 1945.

32 Wingate, S.A., Master's Thesis in Electrical Engineering, Massachusetts Institute of Technology, 1946.

33 Siebert, W.M., Master's Thesis in Electrical Engineering, Massachusetts Institute of Technology, Feb. 1948.

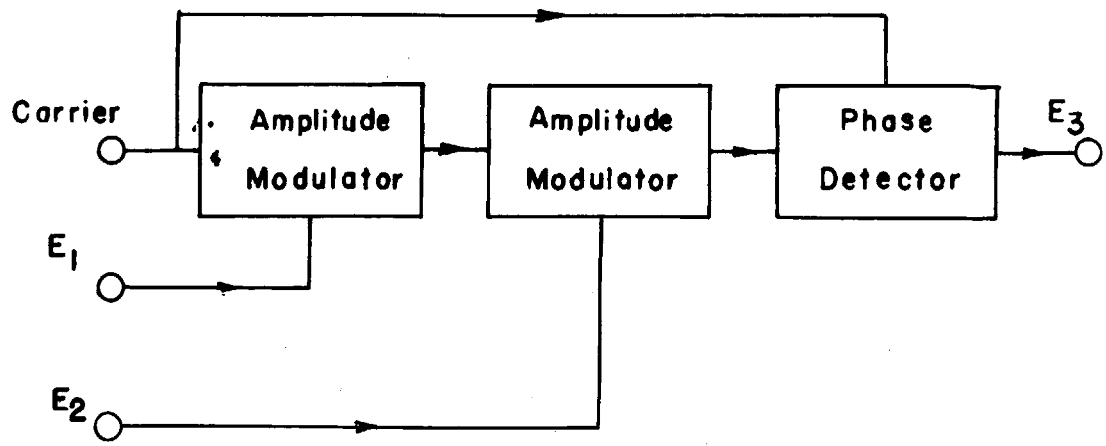

Fig. 14 Siebert multiplying circuit.

modulators and then detected in a phase detector; in this manner four-quadrant multiplication is obtained.

Double pulse-modulation has been used by Ham[34] in the development of a general integrator. A block diagram of this circuit is shown in Fig.15.

In this circuit the pulse width is made proportional to $\frac{dE_1}{dt}$ and the height to E_2. The pulse area is therefore

$$\text{pulse area} = E_2 \cdot \frac{dE_1}{dt} , \qquad (31)$$

which when integrated with respect to time gives the output

$$E_3 = \int E_2 dE_1 . \qquad (32)$$

Two such integrators can be combined, as in the mechanical differential analyzer, to perform multiplication according to the relation

$$E_1 E_2 = \int E_2 dE_1 + \int E_1 dE_2 . \qquad (33)$$

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34 Ham, J.M., Master's Thesis in Electrical Engineering, Massachusetts Institute of Technology, 1947.

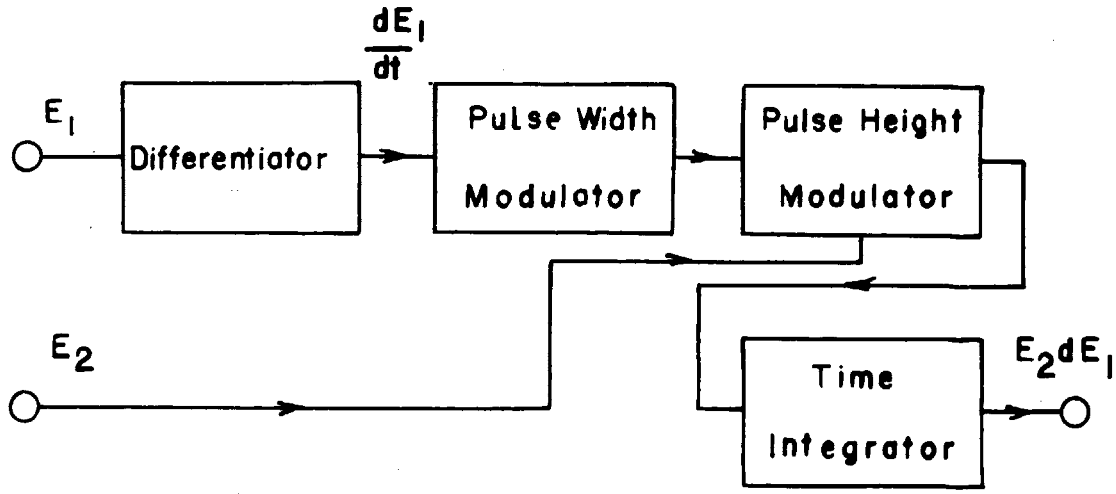

Fig. 15 Ham general integrator.

The principal limitation of any modulation multiplier from the point of view of rapid computation is the fact that the speed of the multiplier is limited by the carrier frequency employed. It is probably desirable to have the carrier frequency at least ten times the high frequency cut-off of the multiplier. If, for example, frequencies up to 60 Kc/s are to be treated, the Siebert multiplier should employ a carrier frequency of 600 Kc/s, and the Ham general integrator a pulse repetition rate of about the same value. If the problems presented by this carrier frequency can be overcome without too great circuit complexity, either of these schemes might well be employed in an electronic differential analyzer.

Another possibility for electronic multiplication is the difference-of-squares system.[35] A block diagram of such a multiplier is shown in Fig. 16.

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35 Murray, op. cit., p. 31, reference 30.

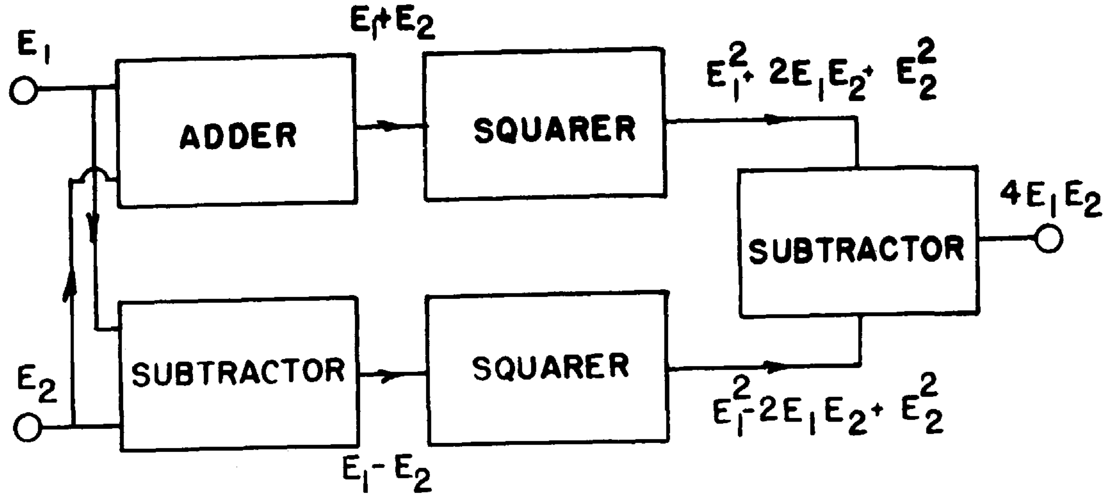

Fig. 16 Difference-of-squares multiplier.

This circuit requires the formation of the sums and differences of the two input voltages E_1 and E_2. These two voltages, $E_1 + E_2$ and $E_1 - E_2$ are then supplied to two identical squaring circuits. The outputs of these two squaring circuits, $E_1^2 + 2E_1E_2 + E_2^2$ and $E_1^2 - 2E_1E_2 + E_2^2$, are then subtracted, yielding the desired output, $4E_1E_2$. This circuit is sometimes referred to as the four-squares multiplier.

A number of multipliers of this type were built by the author using balanced modulators as the square-law elements. They can be made to have very high operating speed and a satisfactory dynamic range. Difficulties were encountered in attempting to maintain the two square-law circuits identical. The balanced modulators suffered from the usual troubles of vacuum tube parameters changing with age, filament voltage, plate voltage, and the like. If these difficulties could be overcome, or if a more satisfactory type of square-law circuit could be developed, the difference-of-squares multiplier would be entirely satisfactory for the electronic differential analyzer. For the present the new multiplying circuit described below, called the crossed-fields electron-beam multiplier, appears to afford a more satisfactory solution to the problem.

4.3 The Crossed-Fields Electron-Beam Multiplier

A beam of electrons moving in vacuum has extremely low inertia and its three degrees of freedom afford a number of possibilities for high speed multiplication. The standard cathode-ray tube provides a beam of electrons and electronic means of controlling its motion in cartesian coordinates. A sketch of such a tube is shown in Fig. 17.

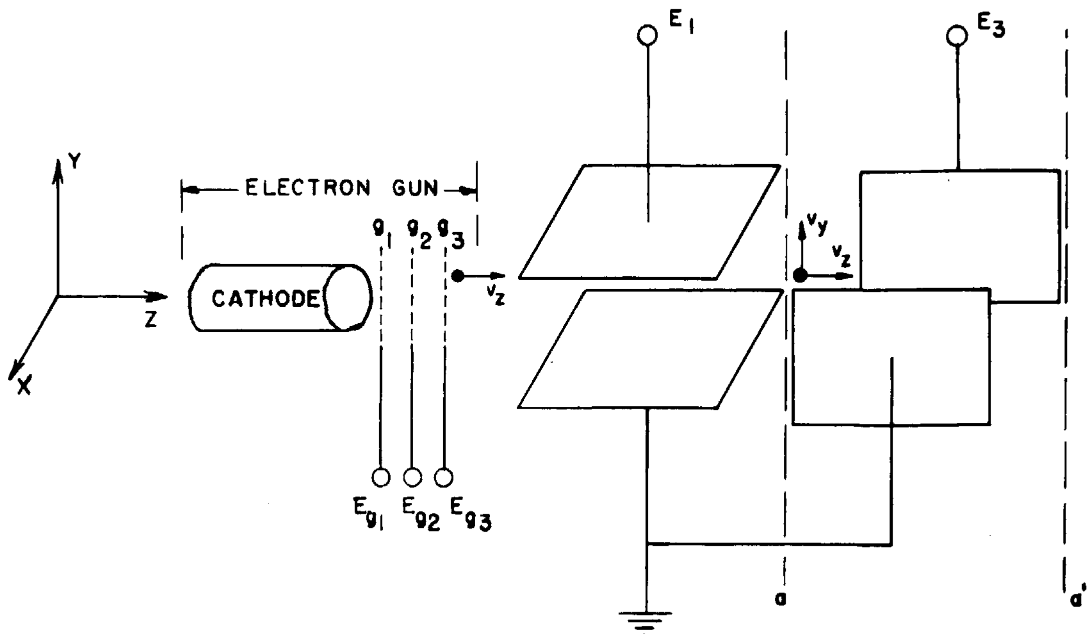

Fig. 17 Geometry of a typical electrostatic deflection cathode-ray tube.

The cathode and first three grids of this structure make up that portion of a cathode-ray tube normally referred to as the electron gun. It is the purpose of this electron gun to form a narrow beam of electrons having a velocity along the axis of the tube $\bar{k}v_z$, where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors in the x,y,z directions respectively. The number of electrons in this beam at any instant, that is the beam intensity, is controlled by the voltage on grid No. 1, E_{g1}. The magnitude of the axial speed v_z is proportional to the square root of the voltage on the third and final grid of the electron gun, E_{g3}.

4.31 Principal of Operation

If the electron beam leaving the electron gun with velocity $\bar{k}v_z$ passes between the first pair of deflecting plates across which there is a potential E_1, the electron beam is given a component of velocity in the y-direction $\bar{j}v_y$, according to the relation

$$\bar{j}v_y \propto \frac{\bar{j}E_1}{v_z} \quad . \tag{34}$$

The electron beam is then moving in the region a-a' with velocity

$$\bar{v} = \bar{k}v_z + \bar{j}v_y \ . \tag{35}$$

In this region a-a' there is a force $\bar{F}_{xe}$ acting on the electrons in the beam in the x-direction due to the electrostatic field between the x-deflecting plates;

$$\bar{F}_{xe} = e\,\bar{\mathcal{E}}_x \ , \tag{36}$$

where e = the electron charge, 1.60×10^{-19} coulombs, and

$\bar{\mathcal{E}}_x$ = the electrostatic field between the x-deflecting plates due to the impressed voltage E_3,

$$|\bar{\mathcal{E}}_x| \propto E_3 \ . \tag{37}$$

If an axial magnetic field $\bar{k}B_z$ is also present in the region a-a' (such a field might be generated for example by a coil so wound as to have its axis of symmetry coincident with the z-axis of the cathode-ray tube), there is an additional force $\bar{F}_{xm}$ exerted on the electron beam in this region:

$$\bar{F}_{xm} = e(\bar{j}v_y \times \bar{k}B_z) \ , \tag{38}$$

where the x denotes vector cross product. The vector $\bar{F}_{xm}$ points in the x-direction. The speed v_y is proportional to E_1 and the magnitude of the axial field B_z can be made proportional to some other external voltage E_2; then one can write from Eq. (38)

$$|\bar{F}_{xm}| \propto E_1 \cdot E_2 \, . \tag{39}$$

Looking at Eqs. (36) and (38) one sees that in the region a-a' there are two forces $\bar{F}_{xe}$ and $\bar{F}_{xm}$ acting on the electron beam in the x-direction because of the crossed electric and magnetic fields.

If some means is available to make these two forces equal and opposite, then

$$\bar{\mathcal{E}}_x = \bar{j}v_y \times \bar{k}B_z \tag{40}$$

and from Eqs. (37) and (39) one finds

$$E_3 = k(E_1 \cdot E_2) \; ; \tag{41}$$

the voltage E_3 is thus proportional to the product of the two input voltages E_1 and E_2.

In the arrangement described above the only forces acting to deflect the electron beam in the x-direction are those in the region a-a'. If these forces add up to zero, there is no x-deflection of the electron beam. This fact can be used to bring about the desired quality of the forces $\bar{F}_{xe}$ and $-\bar{F}_{xm}$ as indicated in Fig. 18.

At the face of the cathode-ray tube screen two photocells V_1 and V_2 are located. Between these two photocells a vertical partition is placed coincident with the y-axis of the cathode-ray tube. If no x-forces act on the electron beam, it will strike somewhere along the edge of this partition. The outputs of the two photocells are subtracted and fed to an amplifier which is connected to the electro-

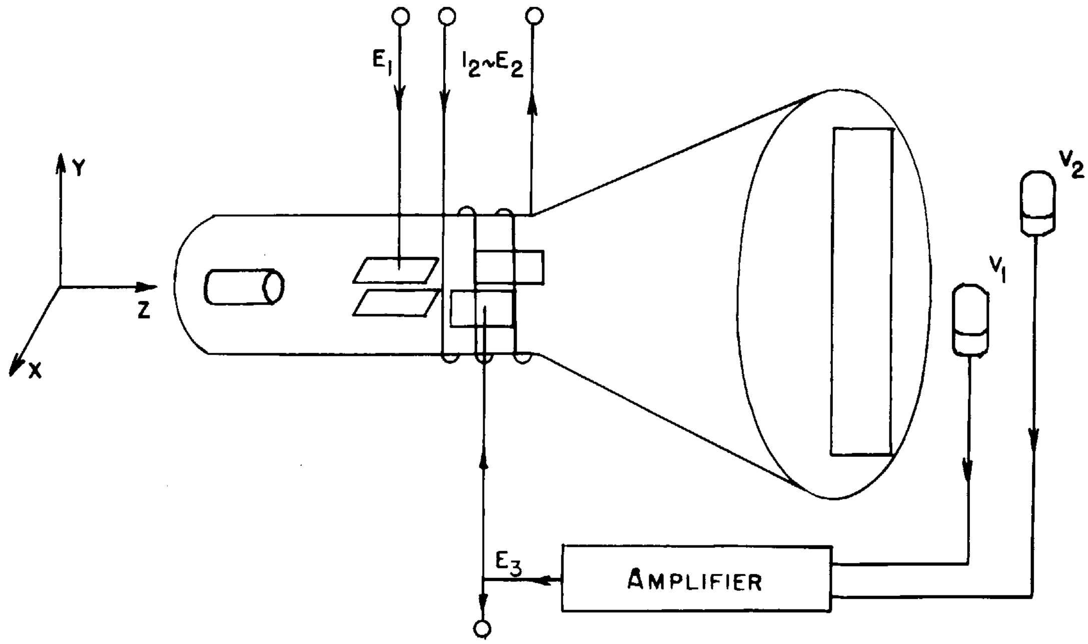

Fig. 18 Feedback method of equating the forces in the crossed-fields electron-beam multiplier.

static x-deflecting plates. For no x-deflection the outputs of the two photocells are equal and opposite so that there is no voltage E_3. This is the case when the magnetic force, $\bar{F}_{xm}$, is zero, that is whenever either B_z or v_y are zero. If both B_z and v_y are non-zero, there is a magnetic force and the electron beam is deflected to the right or left of the partition on the output screen depending upon the relative signs of the two inputs E_1 and E_2. Such a motion of the electron beam results in increased output for one photocell and decreased output from the other. The difference of these outputs is fed back to the electrostatic plates as E_3 with the proper phase to oppose the magnetic force. If the gain around the feedback loop is made sufficiently large, then the feedback voltage E_3 is proportional to the magnetic force and thus to the product $E_1 \cdot E_2$.*

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* The fundamental characteristic of this multiplier is the use of the crossed electric and magnetic fields in the region a-a', while the manner in which the forces $\bar{F}_{xe}$ and $\bar{F}_{xm}$ are made equal is of secondary importance.

The magnitude of the magnetic force $|\bar{F}_{xm}|$ is proportional to the product of E_1 and E_2, and the force changes direction when either E_1 or E_2 change sign; therefore four-quadrant multiplication is obtained.

The speed of response for this multiplier is determined by (a) the speed with which the voltage E_1 on the y-deflecting plates and the field B_z can be varied, and (b) by the speed with which the feedback loop can respond. More will be said below concerning these points in discussing the crossed-fields multiplier that has been built and tested. It is relevant now to inquire, however, as to the balances required by this multiplier. These can be separated by considering the two cases for which the inputs E_1 and E_2 are each made independently zero.

First assume that both input voltages E_1 and E_2 are zero; is the output voltage zero? In any practical circuit the answer is certainly no. E_3 is the voltage at the output of an amplifier and even though the amplifier input is zero there is always a certain amount of noise voltage present. Some sources of noise can be controlled, while other sources, such as shot noise in tubes and thermal agitation noise in resistors are ultimate limiting factors. Suppose now the voltage E_1 is made non-zero while the voltage E_2 is kept zero. Then there is a y-deflection of the electron beam. In order that this deflection cause no change in the light intensity on the two photocells, the photocell partition has to be oriented to coincide with the path of the electron beam. If this path is not a straight line, an error results unless the partition can be so shaped as to take account of this effect. Even though the partition is made to fall exactly along the path corresponding to zero fields in the region a-a', some output is obtained unless care is taken to keep stray fields in this region to a minimum. A common source of such stray fields are the 60 c.p.s. power mains; these fields can be reduced by proper magnetic shielding. Finally consider the third case, in which E_1 is zero but E_2 and correspondingly the axial magnetic field is non-zero. Here zero output requires zero velocity in the y-direction for zero voltage on the y-deflecting plates. One requirement then is that the electron gun produce a truly axial electron beam. If this is not the case, a small biasing voltage may have to be introduced on the y plates for compensation. Another requirement is that the magnetic field be exactly axial. For any practical coil this is only approximately true, and as a result there are some forces on the electron beam due to cross-products between the field and the axial velocity $\bar{k}v_z$.

Although the paragraph above contains a rather long list of possible sources of error in this multiplying scheme, considerations of this type are necessary if a true evaluation of the worth of any multiplying scheme is to be obtained. The incisive question is what happens in the three cases for which the output of an ideal multiplier should be zero, namely:

(a) both inputs zero,

(b) input No. 1 maximum and input No. 2 zero, and

(c) input No. 1 zero and input No. 2 maximum.

It is important to note from the above discussion in the crossed-fields electron-beam multiplier that the adjustments and sources of error depend upon rather stable matters, such as the geometry of the cathode-ray tube structure and the physical location of the error-sensing photocells. Inherently variable processes, such as primary emission from the cathode, or secondary emission from some other surface, enter into this multiplier only in the secondary manner in that they can cause variations in the gain around the feedback loop; provided this gain is always kept high these effects are negligible. This stability has been borne out by measurements made on the practical multiplier described below.

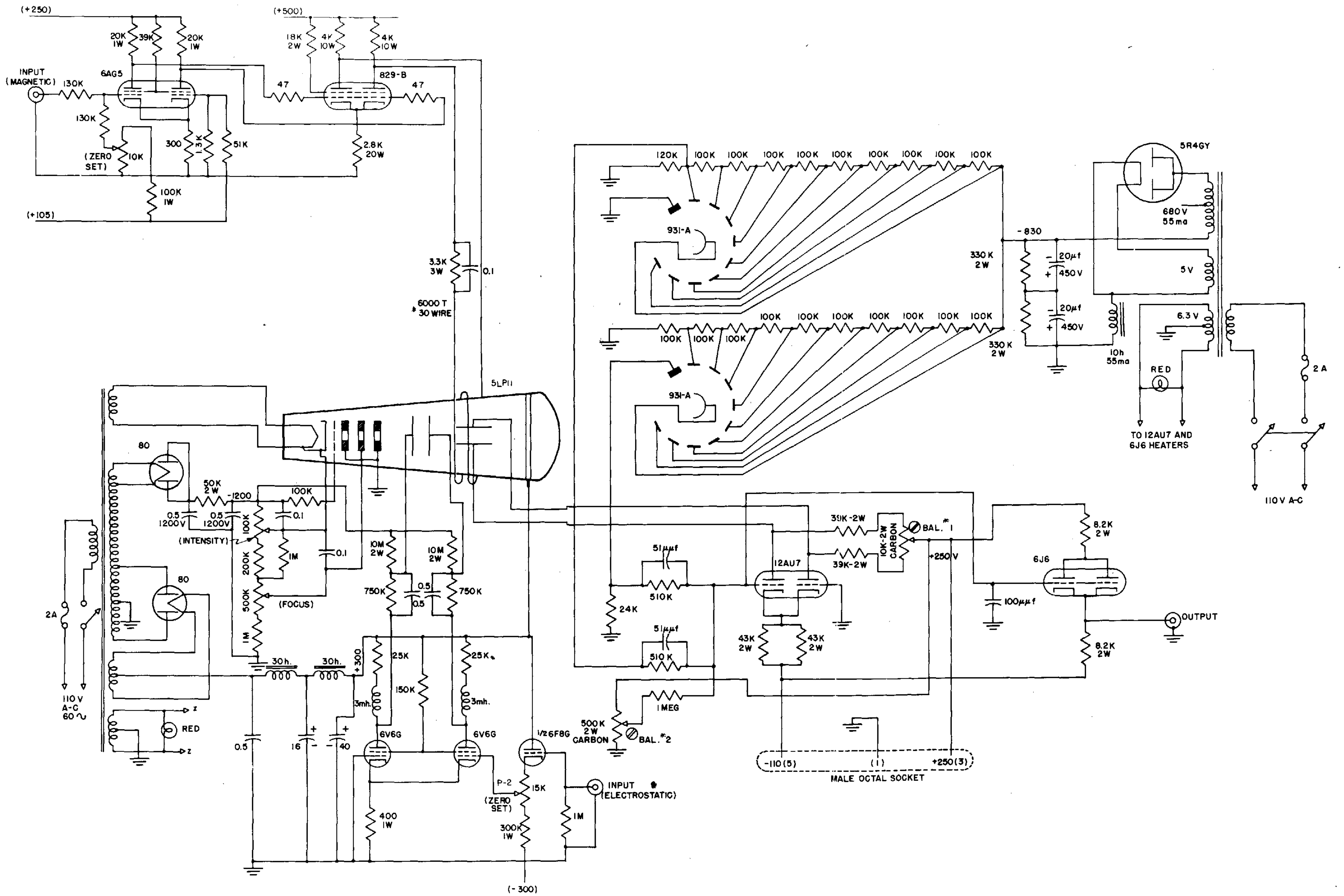
4.32 Practical Multiplying Unit

The complete circuit diagram of a crossed-fields multiplier is shown in Fig. 19.

This unit is built around a Dumont 208 oscilloscope employing a 5LP11 cathode-ray tube. The power supply and centering controls for this tube are those normally provided in the Dumont instrument. The P-11 screen was chosen because of its short persistence, and high light output in the blue and ultra-violet range where the 931-A photomultiplier tube is most sensitive. For the fastest possible response a P-5 phosphor could have been utilized in either this unit or the function generator; the speeds obtained with the P-11 phosphor were found to be adequate in this differential analyzer.

The error sensing unit consists of two RCA 931-A photomultiplier tubes. The outputs of these two tubes are subtracted and amplified by a 12AU7 cathode-coupled phase inverter. This tube drives the 5LP11 horizontal deflecting plates directly to close the feedback loop. The difference voltage from the photomultiplier tubes, which is the desired product, is also connected to a 6J6 cathode follower providing a low impedance output terminal.

The 931-A photomultiplier tube provides a convenient source of stable d-c gain. An alternative feedback loop which has been employed



FIG·19 CROSSED-FIELDS ELECTRON-BEAM MULTIPLIER is indicated in block form in Fig. 20.

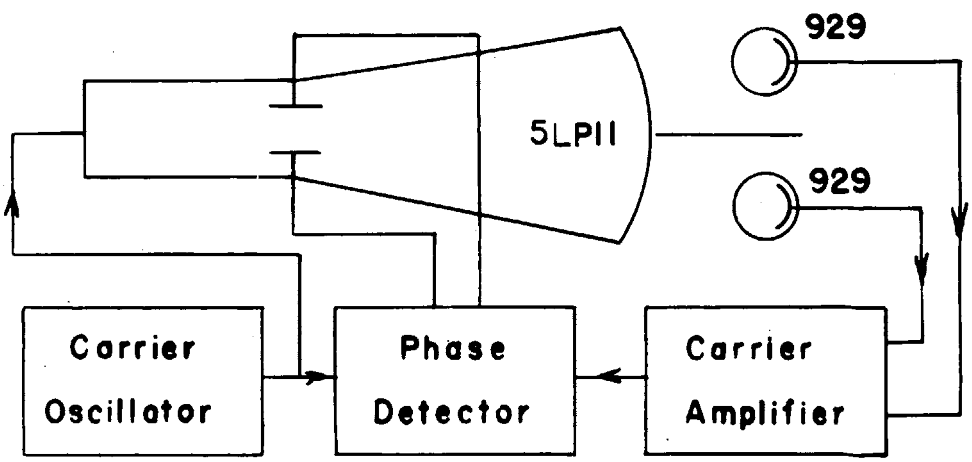


Fig. 20 Carrier feedback loop.

In this circuit the beam intensity of the cathode-ray tube and thus the light output from the spot at which the electron beam strikes the tube face is modulated at a carrier frequency, (of about 400 Kc/s). The output of the pickup photocells is modulated at the carrier frequency, and the difference of their outputs is amplified by a band-pass radio-frequency amplifier. The output of this amplifier is phase-detected and applied to the horizontal deflecting plates to close the feedback loop. Similar results have been obtained from both these feedback loops, but the unit shown in Fig. 19 is preferred because of its simplicity.

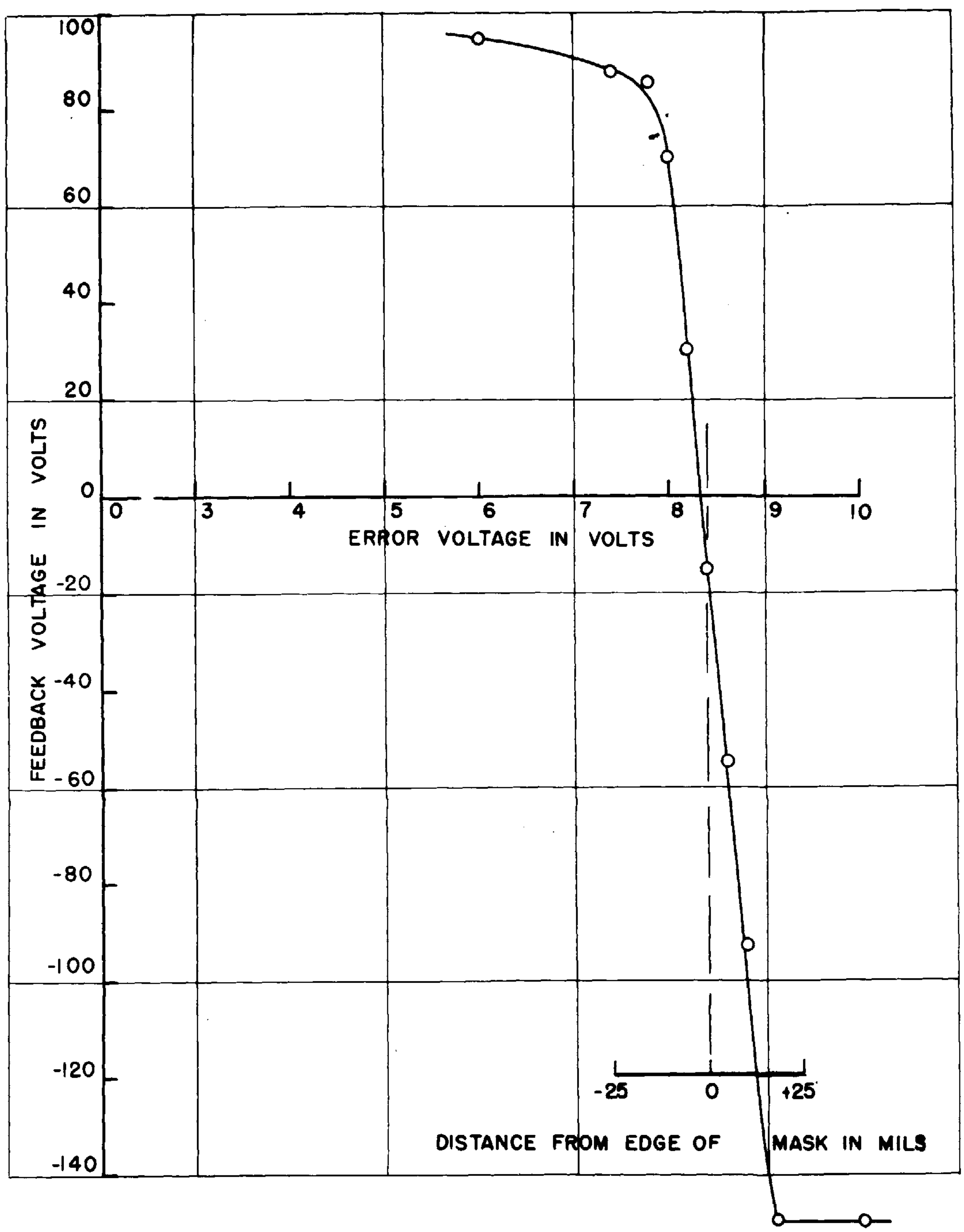
The multiplier input to the vertical deflecting plates is made through a pair of 6V6's connected to form a cathode-coupled phase inverter. These tubes form the last stage of amplification normally present in the Dumont oscilloscope; the only modification made was to disconnect the low level amplifying stages. The centering adjustment associated with this amplifier is very convenient in setting the zero balance on this input.

The axial magnetic field is generated by the coil L_1, which is wound about the horizontal deflecting plates of the 5LP11 cathode-ray tube. This coil is driven by a two stage d-c amplifier as shown. The unbalanced input voltage to this amplifier, E_m, is connected to a cathode-coupled phase inverter of two 6AG5's. This stage in turn drives the 829-B output amplifier. All of the tubes in this unit are operated Class A_1.

4.33 Measured Multiplier Characteristics

The measured characteristics of the open feedback loop are plotted in Figs. 21 and 22. To obtain Fig. 21 the feedback loop was broken at the cathode-ray tube plates and a plot made of the output voltage of the feedback loop as ordinate against the horizontal plate voltage as abscissa. The second plot, Fig. 22, gives the loop gain as ordinate against the horizontal plate voltage as abscissa. The horizontal plate voltage could also be labelled horizontal spot displacement for the cathode-ray tube beam, since horizontal plate voltage and horizontal spot displacement are linearly related by the constant deflection sensitivity of about 50 volts per inch. Fig. 22 could be obtained by differentiating the curve of Fig. 21. It was measured in this case independently by applying a small a-c signal to the horizontal deflecting plates and measuring the a-c output of the feedback loop as a function of the d-c voltage on the horizontal plates.

It is important to observe that the gain of the feedback loop is only constant over a very narrow region, corresponding to very small



FIG· 21 MULTIPLIER OPEN-LOOP CHARACTERISTIC

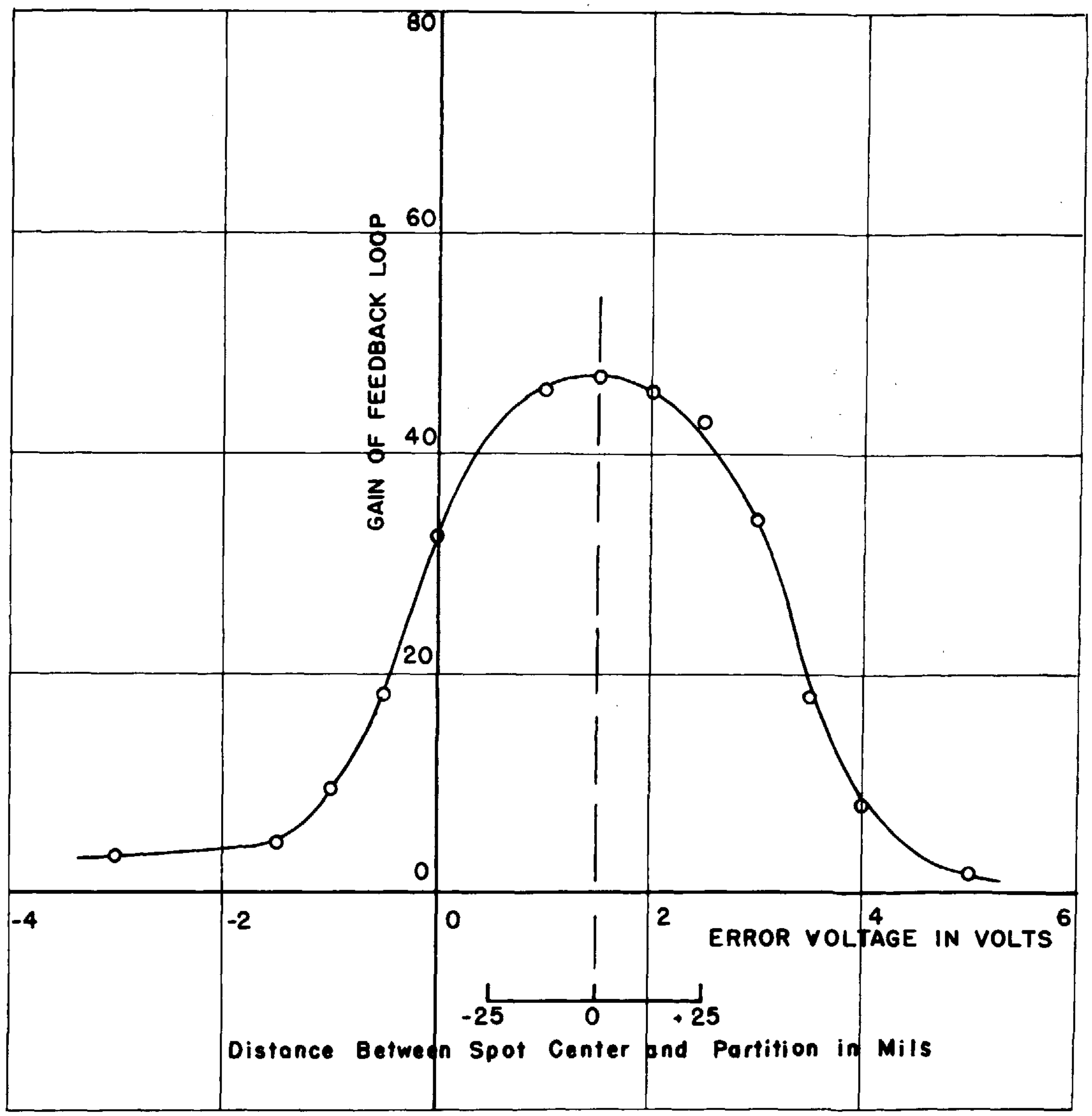


FIG. 22 LOOP GAIN OF MULTIPLIER FEEDBACK CIRCUIT

displacement of the electron beam from its zero-signal position directly below the vertical partition. This non-linearity in the feedback loop is reflected in the high frequency response of the multiplier.

The balance characteristic of the multiplier is described by saying that the output has the form

$$\text{Output} = E_o = k(E_m \cdot E_e + .009E_e + .015E_m + .0005), \qquad (42)$$

where k = 0.66.

This shows that both inputs are balanced-out to within 1.5% of the full-scale output and that the residual noise in the output of the unit is .05% of the full scale output. Fig. 23 is a triple exposure photograph of the crossed-fields multiplier output for three different conditions which should produce zero output. The signal used in making this test was a 240 c.p.s. sine wave, which was synchronized with the line frequency of 60 c.p.s. The output observed for both E_e and E_m zero is principally shot noise amplified by the 931-A photomultipliers. There is also a small amount of power supply hum present. When E_m is zero and E_e is maximum, some second harmonic of the signal E_e is observed in the output. This is caused by the fact that there is some change in gain of the feedback loop as a function of the vertical position of the cathode-ray beam. This results in some output from the feedback loop because of the fact that the gains from the two photocells are not identical. The residual output observed for the case of E_e zero and E_m maximum appears to be caused primarily by non-uniformities of the magnetic field and defocusing effects.

The speed of the multiplier is best measured by observing its output rise time for a square wave input signal. Photographs of the

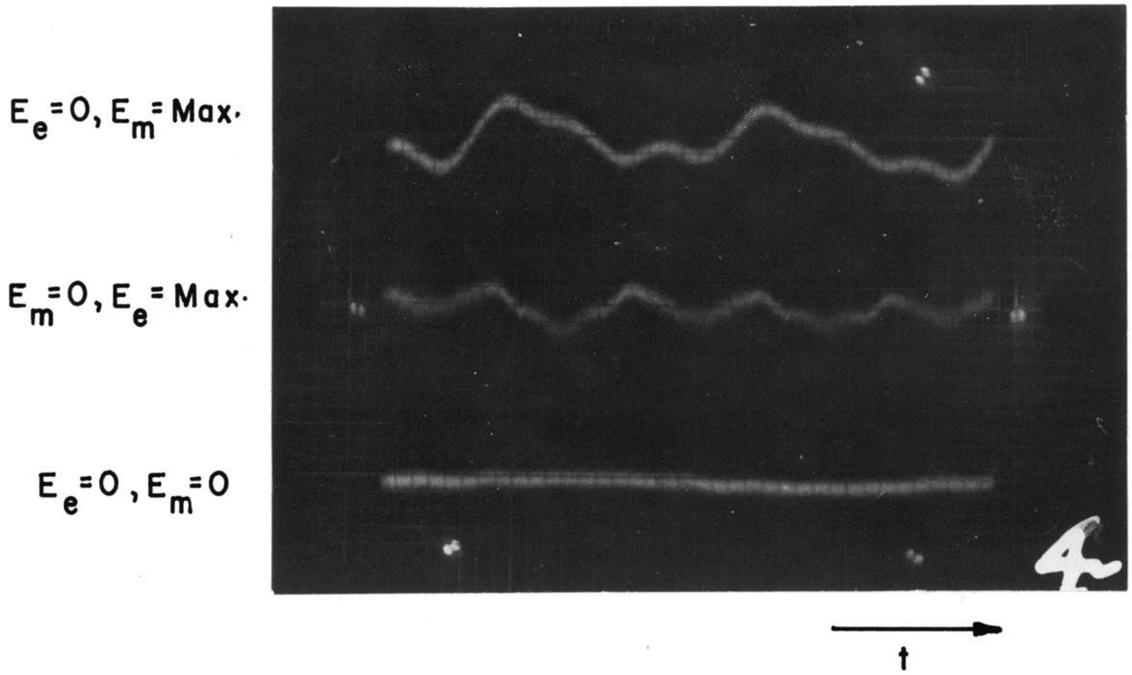
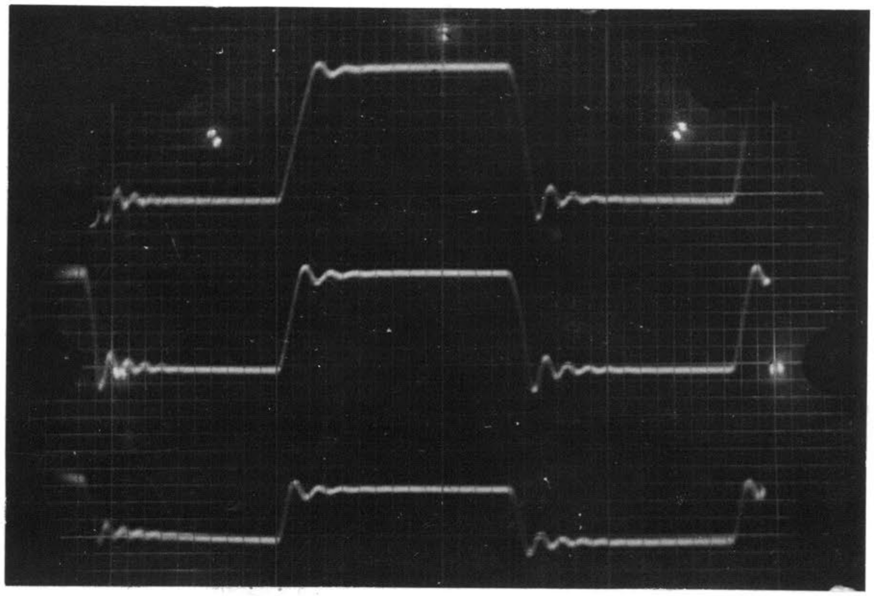


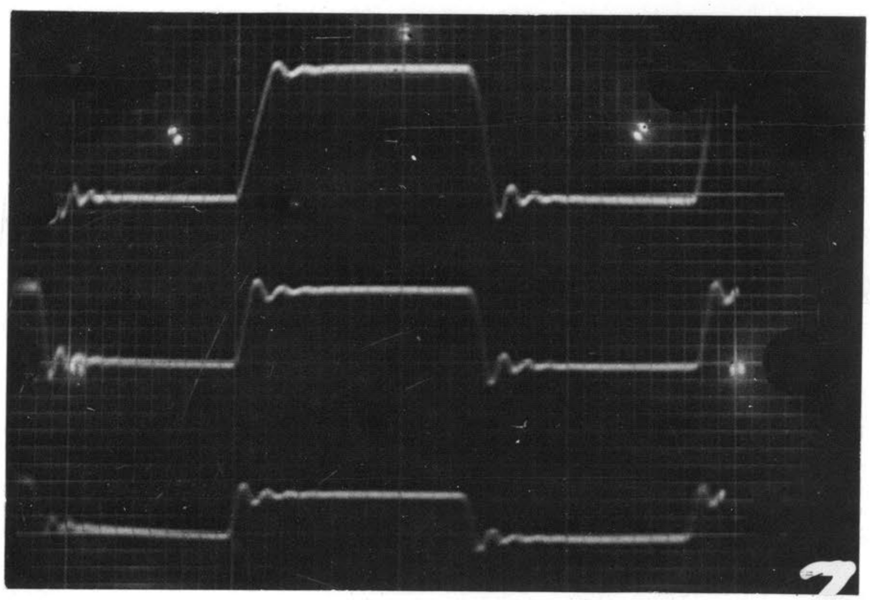
Fig. 23 "Zero output" of crossed-fields multiplier.

square wave response of the crossed-fields multiplier are shown in Figs. 24 and 25. Fig. 24(a) shows a multi-exposure photograph of the multiplier output when E_m is a constant and E_e is a 10 Kc/s square wave of fixed amplitude, for various values of E_m. Fig. 24(b) is a similar photograph for fixed E_m and variable amplitude of square wave. One notes that in both cases the rise time remains constant at about 5 µsec for small amplitude outputs, but increases linearly with output for larger amplitudes. The reason for this behavior lies in the nature of the feedback loop output as a function of displacement from the zero position, as indicated in Fig. 21. For small signals this output varies linearly with displacement of the cathode-ray beam, and a linear response is obtained. Because of the high-frequency time-constants in the feedback loop, however, there is a maximum rate of change of output voltage with time which the unit can generate.



|← 50 μsec →|

Fig. 24(a) Multiplier square wave response; E_e 10 Kc/s square wave, E_m three different values.



|← 50 μsec →|

Fig. 24(b) Multiplier square wave response; E_m constant, E_e a variable amplitude 10 Kc/s square wave.

If the input to the multiplier requires that this rate be exceeded, the feedback loop can no longer follow; and the cathode-ray tube beam is moved away from equilibrium into a region of very low feedback gain. As long as the cathode-ray tube beam is completely to one side or the other of the vertical partition the output voltage of the feedback unit increases or decreases at a constant rate, independent of the error present. The non-linear response shown in Fig. 24 is thus the result of the non-linear characteristic of Fig. 21.

This effect can be overcome by redesigning the error sensing unit of the feedback loop to give a constant loop gain for all possible errors. One possibility would be to replace the partition used here with a photographic film having uniformly varying density from complete opacity along a vertical center line to transparency along the edges of the cathode-ray tube screen.

The output of the multiplier with E_e constant and E_m a 500 c.p.s. square wave is given in Fig. 25. The rise time for this channel is considerably longer than for the electrostatic input channel being about 450 μsec. The limiting factor in this case is the amplifier driving the axial field coil L_1. For the multiplier built, this coil has an inductance of about 2 henrys and requires a maximum current of about ±25 milliamperes. The design of magnetic deflecting circuits for cathode-ray tubes is amply discussed elsewhere.[36,37] Through suitable redesign it should be possible to increase the speed of

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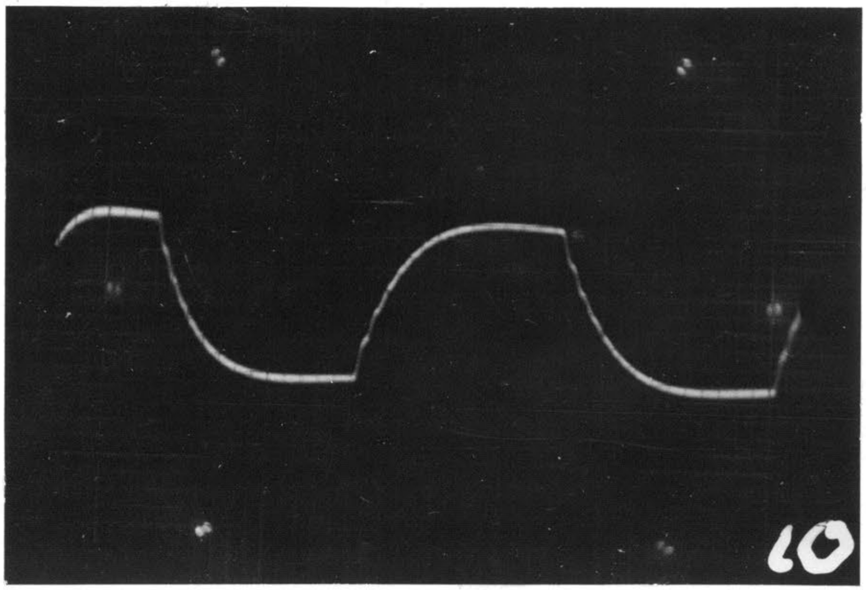
[36] M.I.T. Radar School Staff, Principles of Radar, McGraw-Hill, New York, 1946, 2nd Edition, Chap. III.

[37] Schade, O.H., "Magnetic Deflection Circuits for Cathode-Ray Tubes," RCA Review, v. 8, p. 506, September 1947.



|←1000 μsec→|

Fig. 25 Multiplier square wave response; E_e constant, E_m a 500 c.p.s. square wave.

response of the axial field to give a rise time of a few microseconds. It will be shown in Section VI that for many cases this lack of speed in a single channel of a multiplier is not a very serious drawback.

Photographs of the complete multiplier characteristics are given in Figs. 26 and 27. A qualitative measure of response linearity is given by these photographs, which are taken by photographing on a cathode-ray tube screen plots of the input versus output of the multiplier with fixed input on one channel and variable input on the other.

A more quantitative check of the multiplier linearity can be made by subtracting the input from the output and amplifying the difference. The photographs of Figs. 28 and 29 are obtained in this manner. In each case a sinusoidal input is applied to one channel and the second channel input is held constant. Fig. 28 shows plots versus time of the input, output, and difference between the input

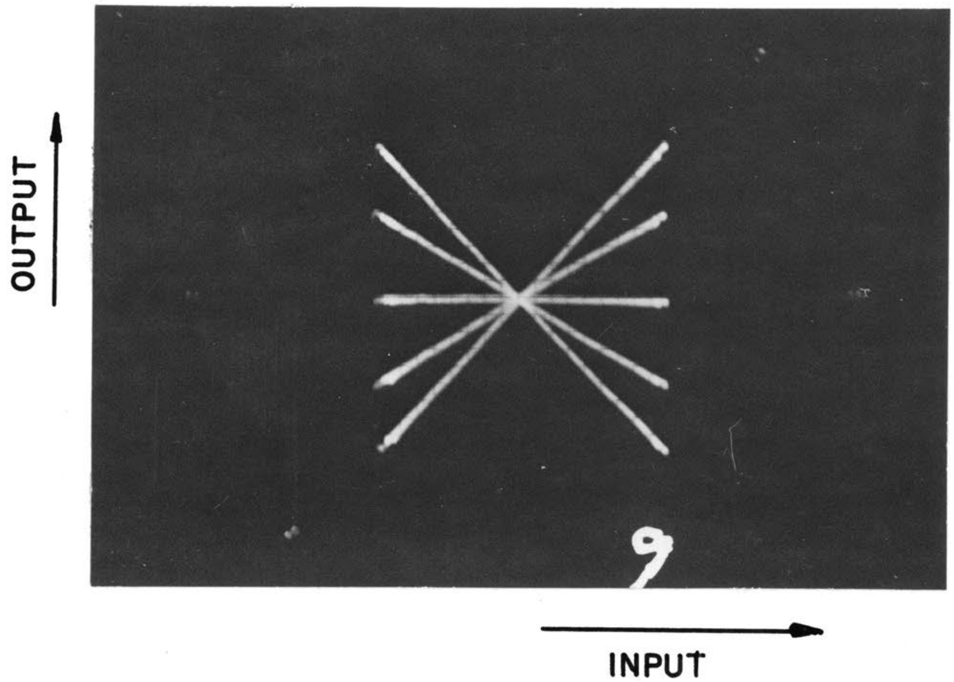


Fig. 26 Multiplier input-output characteristic; sine wave input for E_e and various constant values for E_m.

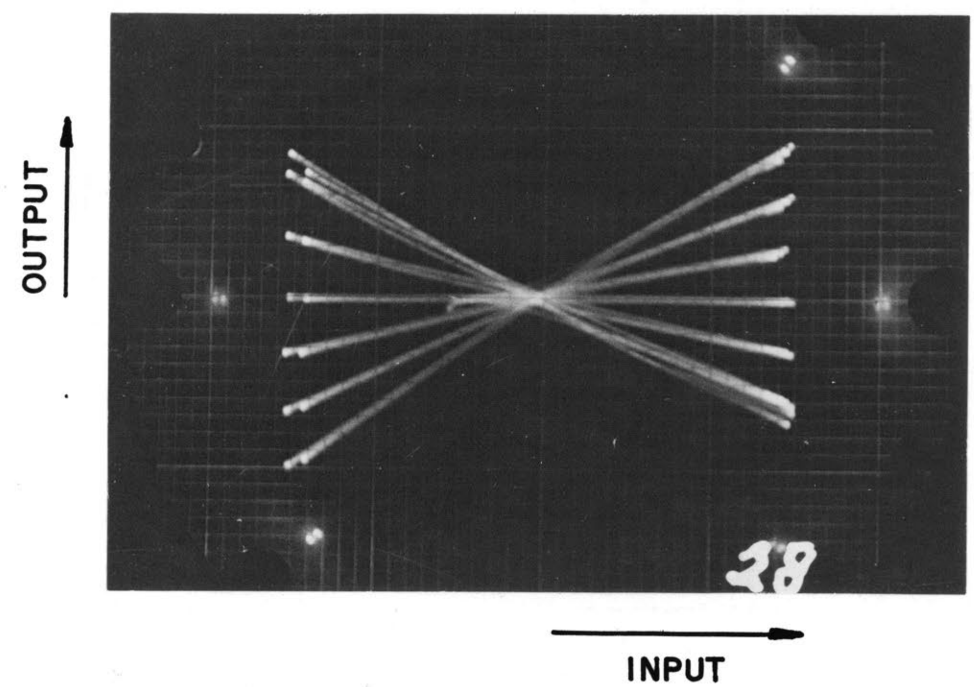


Fig. 27 Multiplier input-output characteristics for various fixed values of E_e and sine wave input for E_m.

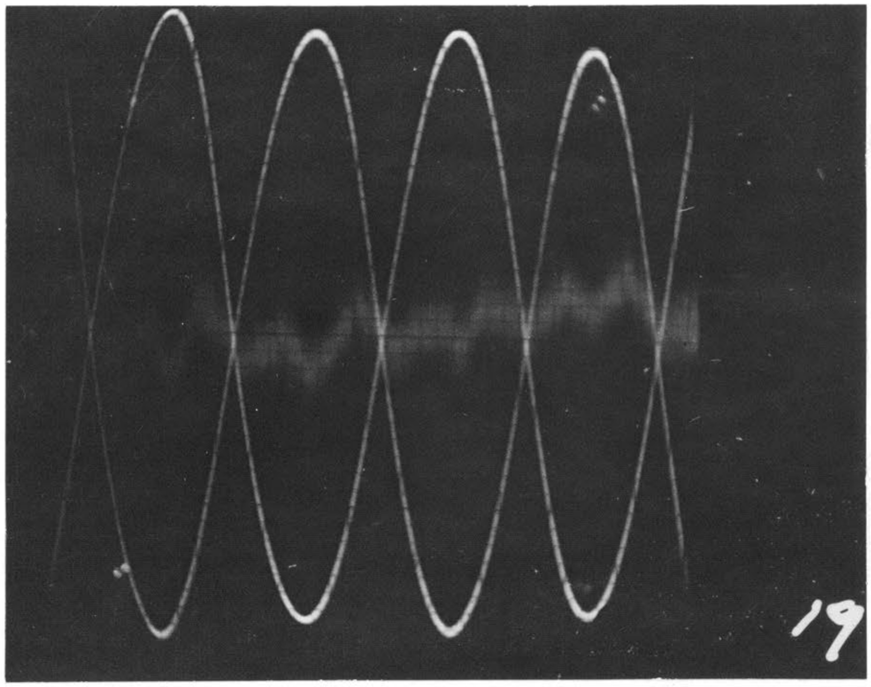


Fig. 28 Multiplier linearity check, E_m held constant.

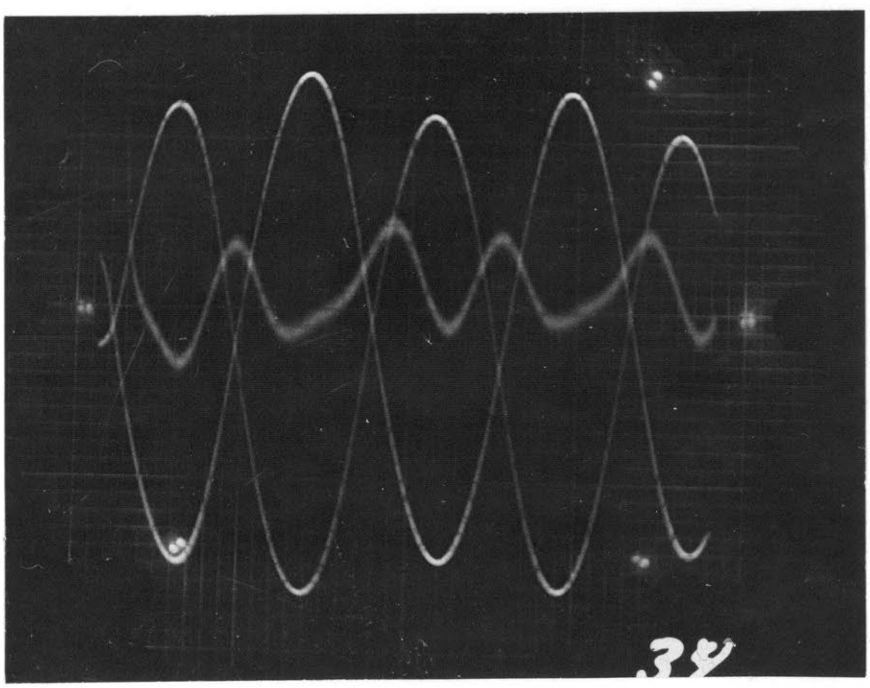


Fig. 29 Multiplier linearity check, E_e held constant.

and output for E_m held constant. The three photographs are superimposed by a triple exposure and the difference signal was amplified by an extra factor of ten so that it would be clearly visible. In addition to the shot noise from the photomultiplier tubes there is about 1.2% second harmonic distortion evident.

Fig. 29 shows the similar situation for the case E_e constant and E_m variable.

Fig. 30 shows the output of the multiplier plotted versus time for two sinusoidal input signals having a frequency ratio of 8 : 1. This pattern is a useful one for quickly adjusting the zero-balances of the multiplier. Fig. 31 shows the effect of an unbalance in the magnetic input and Fig. 32 shows the effect of unbalancing the electrostatic input.

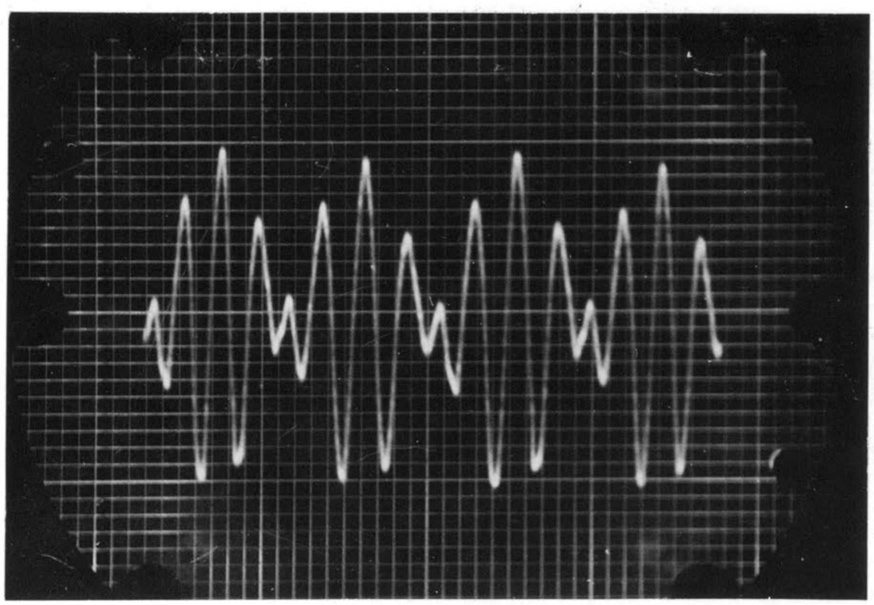


Fig. 30 Multiplier output versus time for E_e a 480 c.p.s. sine wave and E_m a 60 c.p.s. sine wave.

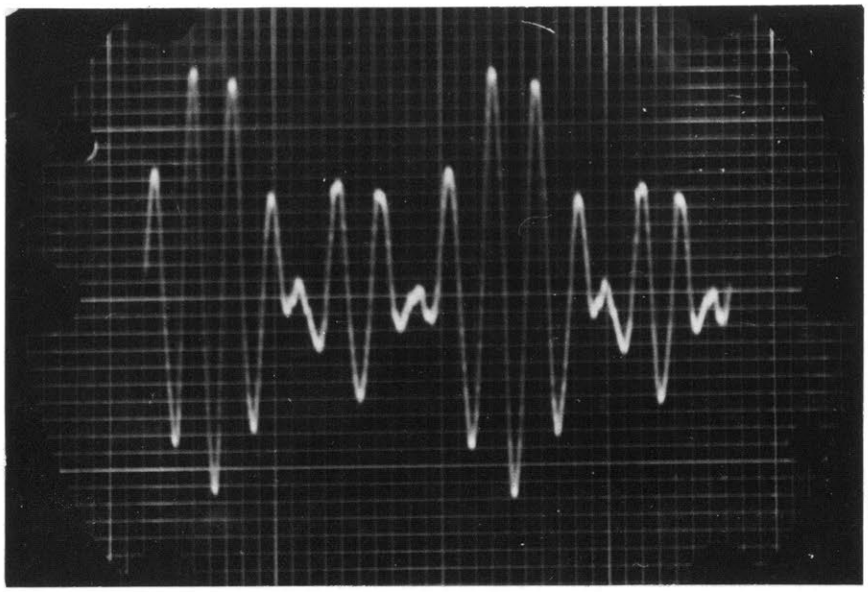


Fig. 31 Same as Fig. 30 with electrostatic input unbalanced.

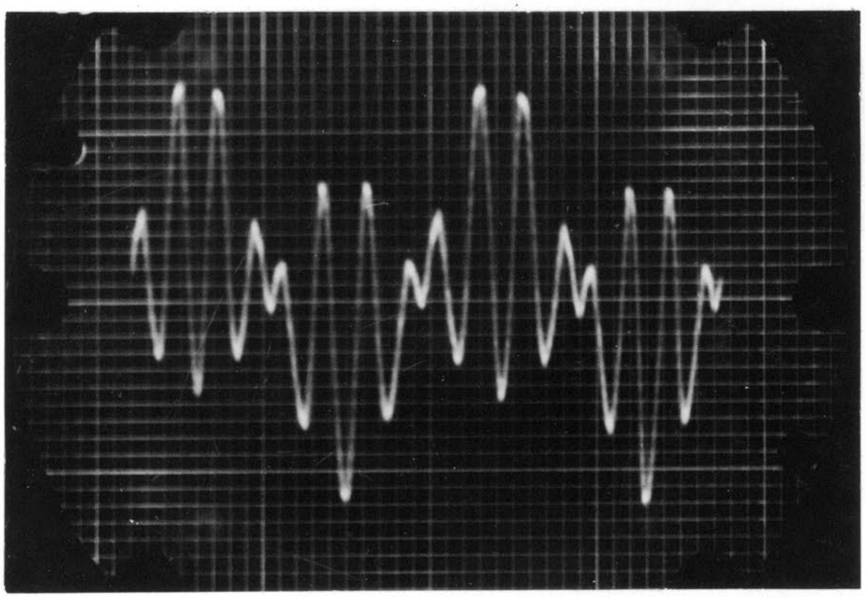


Fig. 32 Same as Fig. 30 with magnetic input unbalanced.

In summary then, a crossed-fields multiplier has been built and tested. Its output is linear in each input to within 2% and the zero-balance errors amount to about 2.5% of the full scale input. The causes of the various zero errors have been determined; it should be possible without much redesign to reduce these errors by a factor of about ten. The balance adjustments are easy to establish and maintain.

The application of this multiplier to the solution of differential equations with variable coefficients is described in Section VI.

There are a few further modifications of this multiplying method which may be worthy of further consideration. One is the possibility of building a special cathode-ray tube with the error-sensing detector inside the evacuated envelope of the tube. One such scheme would be to mount two parallel collector plates within the tube, where the tube face is now located, to replace the photocells used in the present model. Use might also be made of the fact that the velocity $\bar{j}v_y$ is inversely related to the axial velocity of the electron beam, according to Eq. (34). Since v_z is controlled by the voltage on the third grid of the electron gun E_{g3} there is a possibility of obtaining division in addition to multiplication, at least over a limited range, in the same tube.

A number of other schemes for multiplying with special cathode-ray tube structures have been considered. Unless a long-term development project were envisaged, however, the time necessary to construct satisfactory vacuum tubes is such as to render these ideas unattractive.

Function Generation

4.4 Some Methods of Function Generation

A second unit indicated as necessary in the discussion of Section III is the arbitrary function generator, or input table. Such a unit must be capable of generating an output voltage that is an arbitrary function of the input voltage. As has been pointed out, such a function generator can be made to generate functions of time by making the input signal a voltage which is linearly related to time.[38]

It should be recognized, however, that there are other possible ways of generating voltage functions of time without the use of an arbitrary function generator. One well known method is to approximate the desired time function by a power- or Fourier-series expansion. A series of voltages varying as t, t^2, t^3, etc., can be generated and then combined in adding and subtracting circuits to approximate the desired function. Alternately a harmonic synthesizer might be constructed, generating a series of sine waves of controllable phase and amplitude with frequencies harmonically related to the repetition rate of the differential analyzer. Combinations of these voltages could be made to approximate any desired function of time according to well known Fourier-series techniques. A harmonic synthesizer of this type has been developed in the Biology Department at the Massachusetts Institute of Technology.

Another possibility is to utilize the technique frequently employed on the mechanical differential analyzers to generate functions which

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[38] Welti, G., Masters Thesis in Mechanical Engineering at the Massachusetts Institute of Technology, Spring 1948.

can be specified as solutions of differential equations. In this manner any function of time for which the differential equation can be found can be generated on an auxiliary portion of the electronic differential analyzer. This technique is used to generate the functions $\sin\omega t$ and $\cos\omega t$ in the solution of the Mathieu and Hill equations described in Section VI. This method permits one to generate functions of any variable on the mechanical differential analyzer, where the independent variable is arbitrary. On this electronic differential analyzer it is only applicable to the generation of functions of time, because time is always the independent variable. This is the only place where the use of time as the independent variable appears to impose any real restriction on the electronic differential analyzer. As will be seen this restriction is only minor and can easily be avoided.

4.5 Arbitrary Function Generation

To generate an arbitrary function the mechanical machines utilize an input table. The desired function is drawn on the table and an operator manually tracks this curve with a cross-hair as the machine moves. Some work has been done on automatic means of tracking the desired function, but in so far as the author is aware no general application of such a scheme has been made.[39] Because of the high speed of operation required, a function generator for the electronic differential analyzer must be entirely automatic in its operation.

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39 Hazen, H.L., Jaeger, J.J., and Brown, G.S., "An Automatic Curve Follower," Rev. Sci. Inst., v.7, 353-357, September 1936.

The task of reproducing a completely arbitrary wave shape at a repetition frequency of 60 c.p.s. might at first glance seem extremely difficult. Television, which has been generating arbitrary functions of not one, but two variables, at such repetition frequencies suggests the answer.

One simple scheme for the generation of arbitrary functions is indicated in Fig. 33.

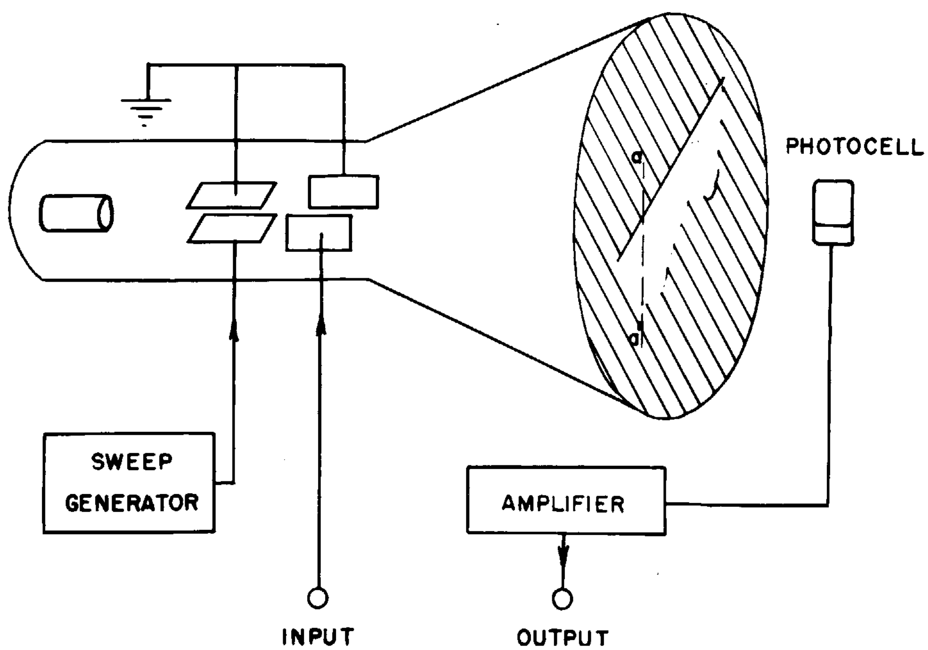

Fig. 33 A simple arbitrary function generator.

For this unit the function to be generated is cut out in the form of an aperture in some opaque material which is placed in front of a cathode-ray tube screen. A photocell is located in front of this aperture, and its output, suitably amplified, forms the output voltage of the function generator. The electron beam is forced to sweep over the vertical line a-a' continuously at a high frequency; the light radiated by the portion of the vertical line that is visible through the function aperture is proportional to the length of the visible line. The photocell has an output linearly proportional to this light, and the horizontal position of the line is proportional to input voltage applied to the x-deflecting plates of the cathode-ray tube. If the aperture is so made to have its width linearly related to the function to be generated, then the photocell amplifier output is proportional to the plotted function of the input voltage.[40, 41, 42]

This function generator is simple in its construction, but has the disadvantage of depending on the uniformity of the cathode-ray tube screen, the photocell characteristic, and the cathode-ray tube electron beam.

4.6 The Feedback Function Generator

The function generator shown in Fig. 34 was first developed by the author in July 25, 1947. It has been described independently by other investigators in this country and England.[43, 44, 45]

4.61 Principal of Operation

The difference between this system and that of Fig. 33 is the use of a feedback circuit to reduce the sensitivity of the output to the instable charactcristics of the cathode-ray tube and photocell.

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40 Koehler, op. cit., reference 25.

41 Gilson, W.E., "Medical Stimulus Circuits," Electronics, p. 99 July 1945.

42 A function generator of this type has been built by H. Logemann in the Biology Department of the Massachusetts Institute of Technology.

43 MacKay, op. cit., reference 24.

44 Mynall, op. cit., reference 22.

45 Sunstein, D.E., "The photoformer," presented at the National Convention of the Institute of Radio Engineers in March 1948.

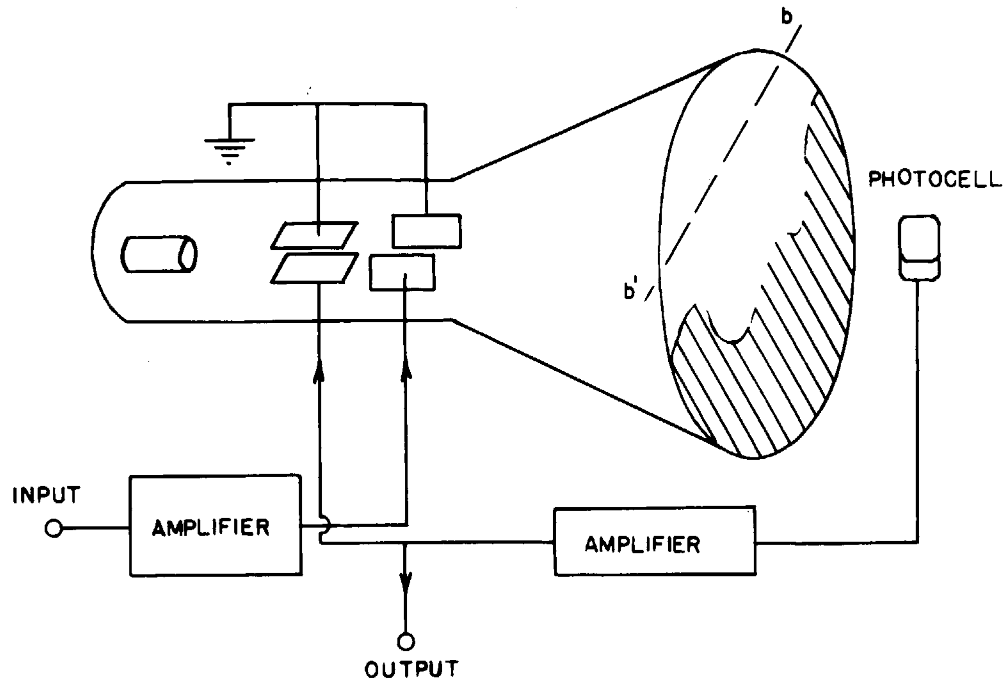


Fig. 34 Feedback function generator.

In Fig. 34 the desired function is again cut out of some opaque material; this time in the form of a mask rather than an aperture. This mask is placed across the face of a cathode-ray tube. The output of a photocell located in front of the masked cathode-ray tube screen is fed through an amplifier to the vertical deflecting system of the tube. The phase of this amplifier is chosen to give a downward deflection of the cathode-ray tube beam when the amount of light striking the photocell is increasing. Finally a biasing voltage at the output of this amplifier is so chosen that the electron beam strikes the screen along the line b-b' if no light enters the photocell. The line b-b' must everywhere be above the function mask. If the feedback loop is closed, then as long as the electron beam strikes the screen above the function mask the photocell develops an output forcing the electron beam down toward the mask. As the electron beam reaches the mask the light-spot on the screen begins to be obscured by the edge of the mask; as a result the light striking the photocell, and correspondingly the voltage forcing the electron beam downward, is reduced. The beam takes up an equilibrium position such that the light striking the photocell generates just enough voltage at the amplifier output to hold the beam stationary.

If the electron beam is now moved in a horizontal direction by a voltage applied to the horizontal deflecting system, the beam is constrained to follow the silhouetted function mask at every point. If the horizontal position of the beam and the input to the horizontal deflecting amplifier are linearly related, the output of the feedback loop is the plotted function of the input voltage.

4.62 Practical Function Generating Unit

The circuit diagram of a practical arbitrary function generator is given in Fig. 35. This unit, like the crossed-fields multiplier, was built, for convenience, around a Dumont 208 oscilloscope. The cathode-ray tube utilized is the 5LP11.

A 931-A photomultiplier tube is used as the detecting photocell in Fig. 35. Its output is amplified by the 6J6 phase inverter and applied directly to the vertical deflecting plates of the cathode-ray tube. The zero signal operating point is adjusted by the biasing potentiometer P-1 so that the unit tracks over the entire range of the function mask. The input to the function unit utilizes the last stage of the horizontal amplifier normally employed in the model 208 oscilloscope. This consists of a push-pull 6V6 cathode-coupled phase inverter. The zero point for this input can be set by the potentiometer marked P-2.

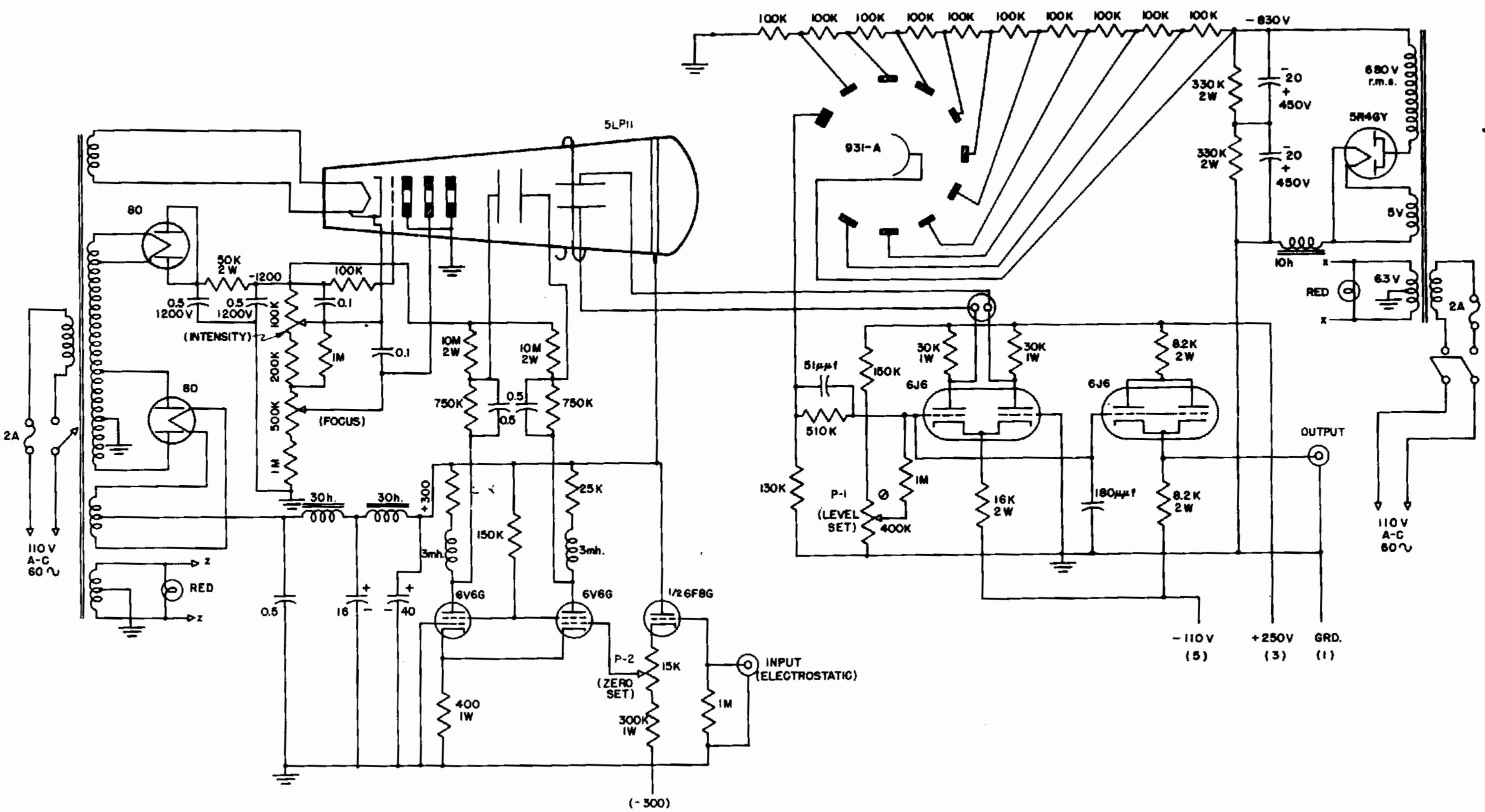


Fig.35 Practical circuit of the feedback function generator

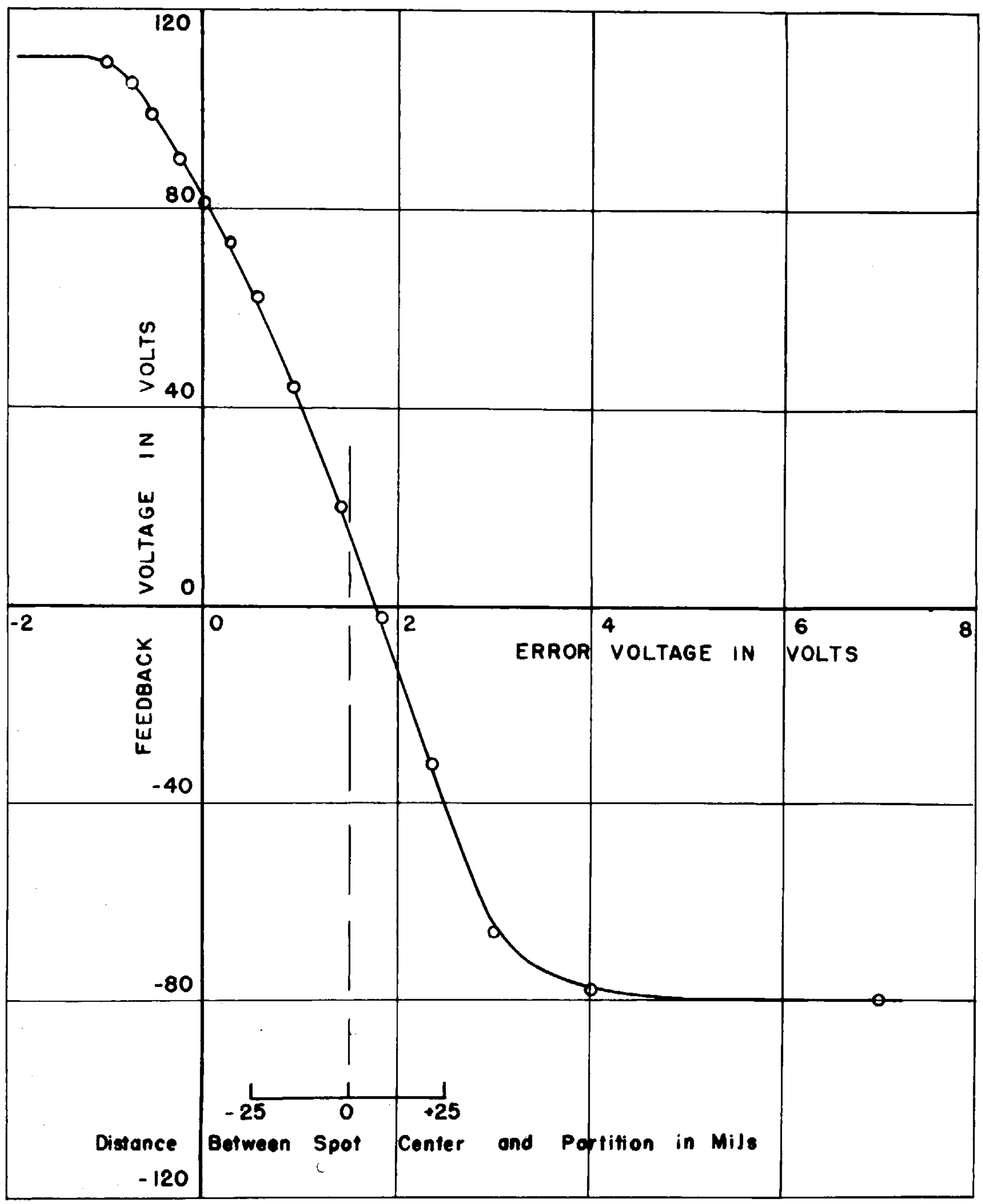


FIG. 36 FUNCTION GENERATOR OPEN-LOOP CHARACTERISTIC

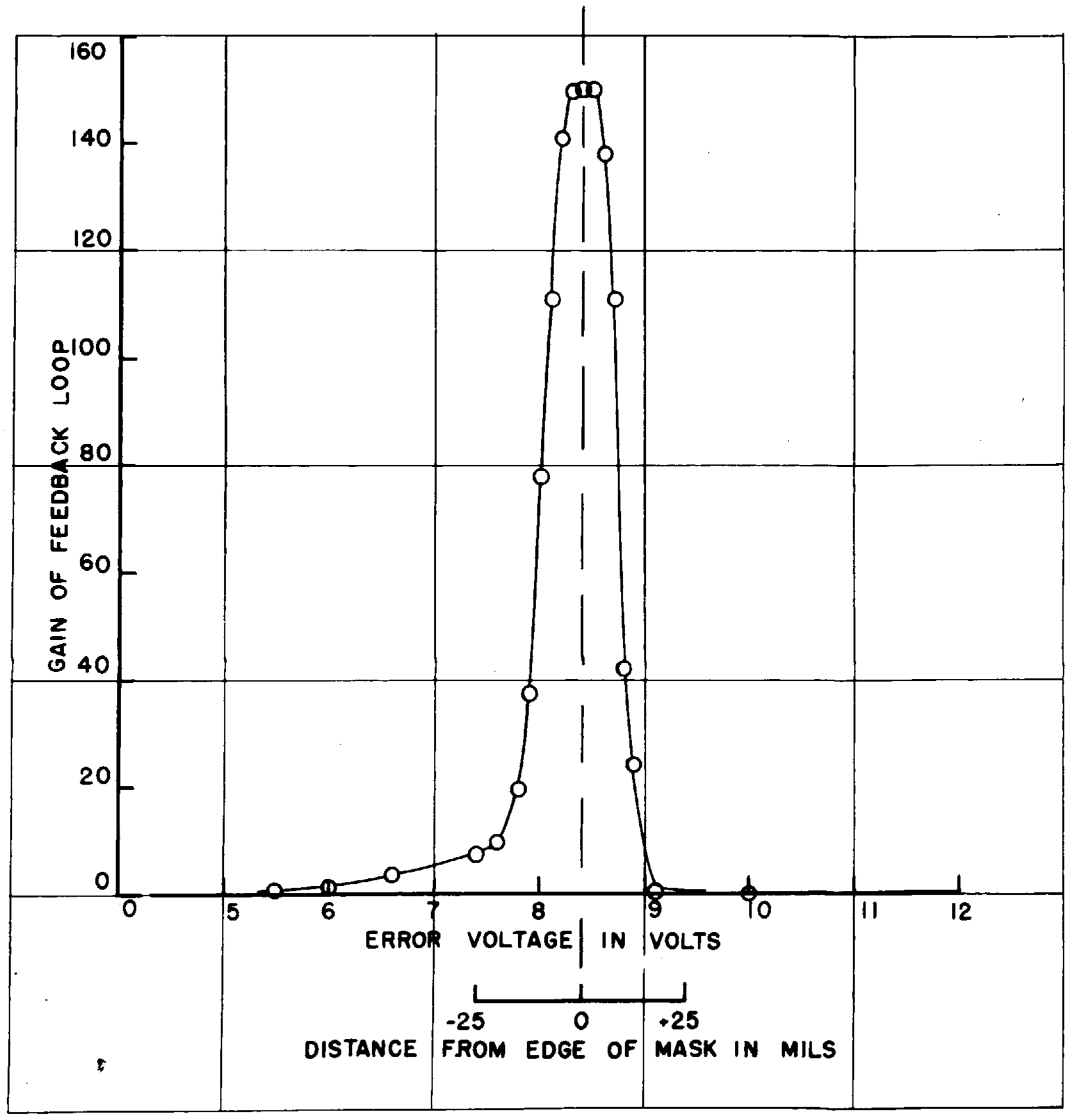


FIG. 37 FUNCTION GENERATOR LOOP GAIN

4.63 Measured Function Generator Characteristics

The open circuit characteristic of the feedback loop is plotted in Fig. 36; the 6J6 is disconnected from the cathode-ray tube plates. This figure gives the measured plate to plate voltage of the 6J6 as the electron beam is moved from below to above a portion of the function mask. Note that here, as in the crossed-fields multiplier, the error sensing is linear over only a very small region about the edge of the function mask. Fig. 37 is a plot of the feedback loop gain over the same range of spot positions.

A test of the speed of a function generator can be made by generating the step-function shown in Fig. 38. The technique used is to place a mask of this function in the function generator and then generate it as a function of time by applying a linear sweep voltage to the function generator input. The output can then be studied on the linear time base

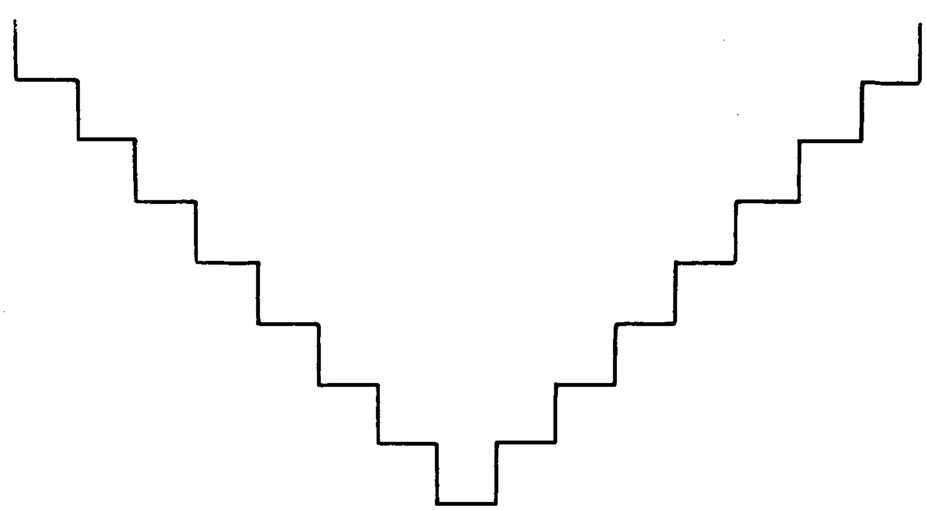
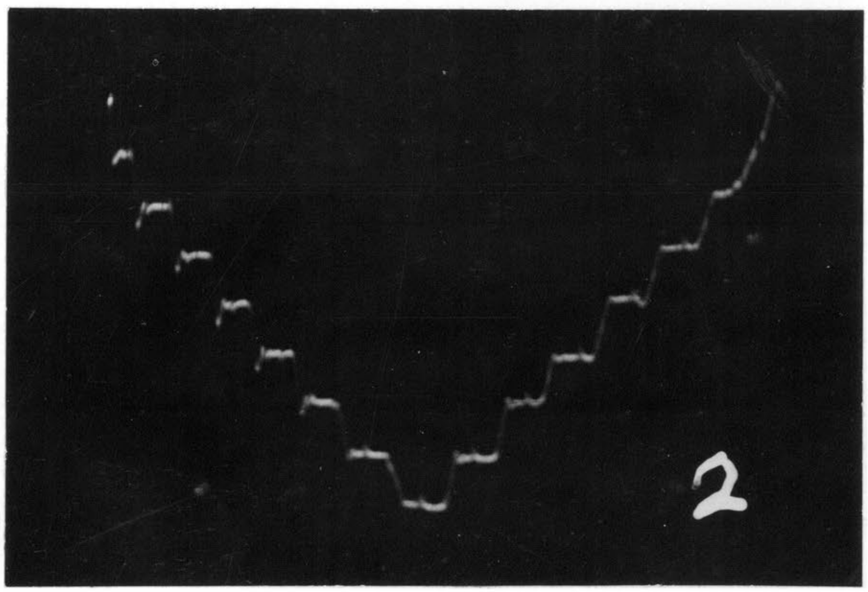


Fig. 38 Function generator test mask.

of a standard cathode-ray oscillograph. A photograph of the results of such a test for the function generator of Fig. 35 is given in Fig. 39. In this test the feedback loop gain was experimentally adjusted to give the best observable response. One notes that the non-linearity of the error sensing device in the feedback loop causes the rise time to be a function of the amplitude as was the case with the multiplier. The rise times indicated in Fig. 39 range from 8 to 16 microseconds while the fall times run from 4 to 16 microseconds.



→| |← 50 μsec.

Fig. 39 Response of feedback function generator to test mask of Fig. 38, plus 50 μsec marker pips.

That there is an optimum gain for the feedback loop of this function generator is recognized by considering the nature of the electron beam as it strikes the cathode-ray tube screen. This beam ideally has sharply defined edges; in practice the electron density of the beam cross section is as sketched in Fig. 40.

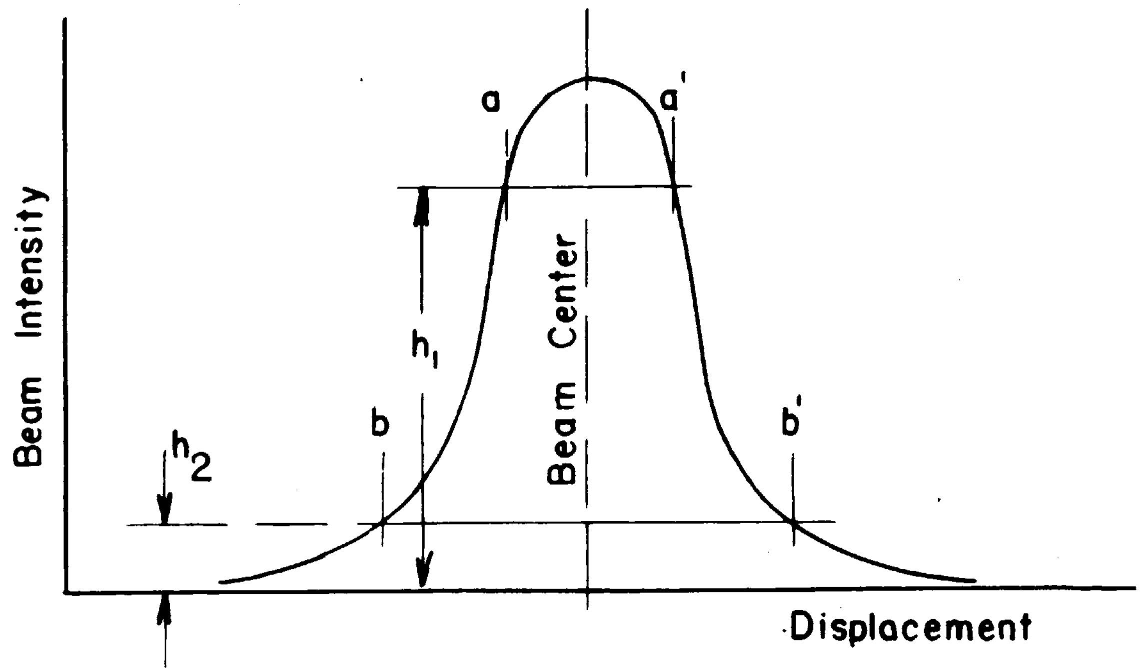


Fig. 40 Probable distribution of electron density through the cross section of a cathode-ray tube electron beam.

As a result the light intensity radiated from the cathode-ray tube screen is a similar function of position across the diameter of the spot. If the gain of the feedback loop is moderate, a light intensity corresponding to a height h_1 is necessary to hold the electron beam to the mask edge and the effective diameter of the electron beam is a-a'. If the feedback gain is reduced, sufficient output is not available to permit tracking over the entire function mask. If, on the other hand, the gain is increased too much, the intensity necessary to hold the cathode-ray tube beam to the mask may be reduced to some value such as h_2 in Fig. 40. For this condition then, because of the bell shaped intensity destribution of the electron beam, the effective diameter of the electron beam has been increased to b-b'. This effectively blunts the function generator error sensing device; the broad beam cannot follow fine detail on the function mask and resolution is lost. This effect is illustrated in Fig. 41, which is a triple exposure photograph of the function generator output, generating the test pattern as a function of time, for feedback gains that are optimum, greater than optimum, and less than optimum.

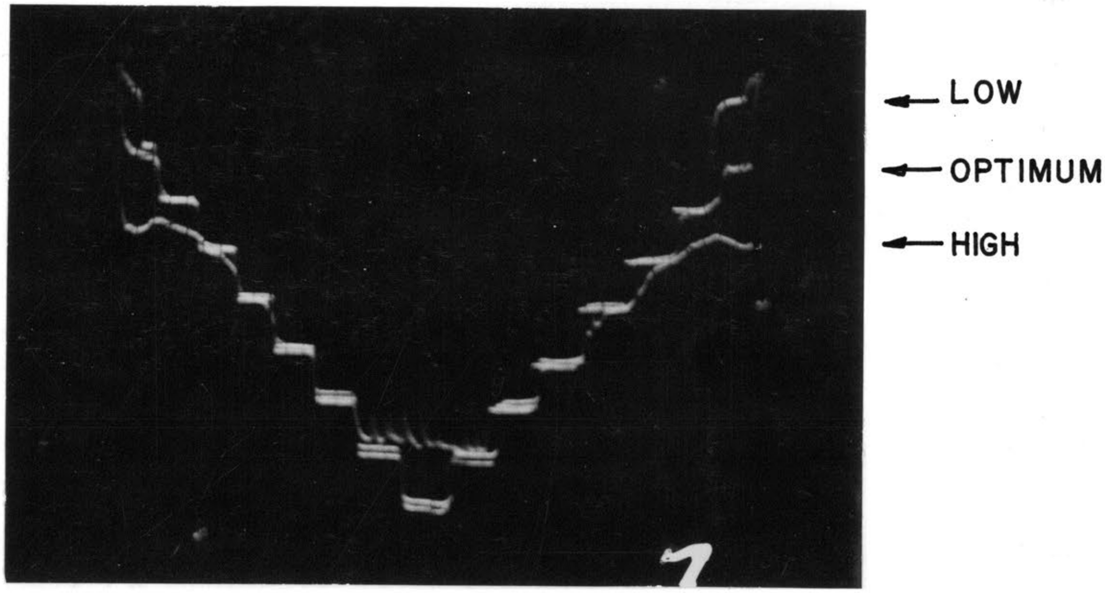


Fig. 41 Effect of feedback loop gain on the generation of a function test pattern.

The overall linearity of the function generator was tested by generating the linear function of Fig. 42. With this function mask in place a sinusoidal input was applied to the function generator. The function generator output was then applied to a subtracter and compared with the input signal. Fig. 43 is a triple exposure photograph of the results of this test. A 240 c.p.s. sine wave is the test signal. Fig. 43 gives plots of the input, output, and difference between input and output versus time for the function generator. The difference trace was photographed with an additional gain in the observing oscilloscope of ten times. One sees from this test that

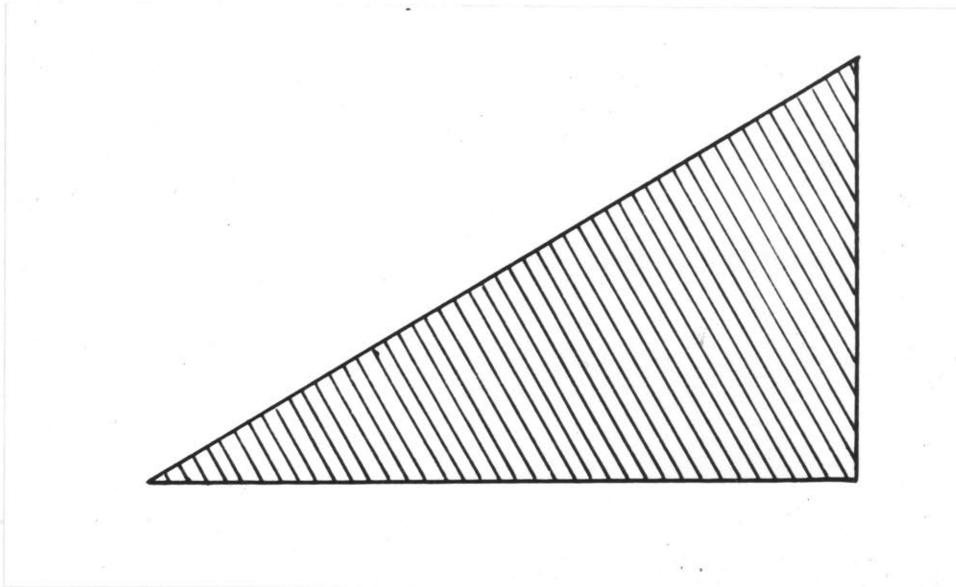


Fig. 42 Linearity test function

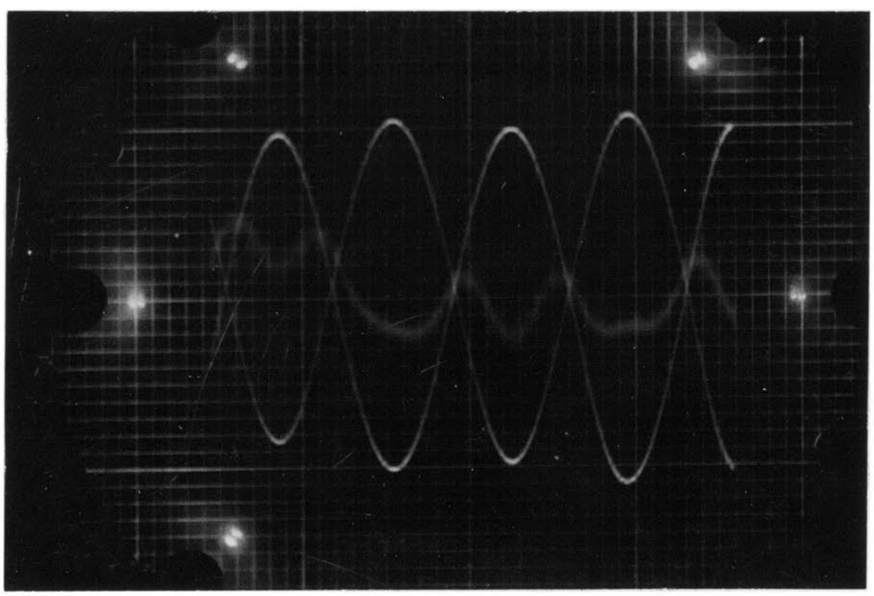


Fig. 43 Results of linearity test.

the overall non-linear distortion of the function generator is 2.5%.

The function generator described here has been utilized as a component of the electronic differential analyzer to solve a number of non-linear differential equations, see Section VI.

4.64 Possible Modifications of Function Generator

The carrier type of feedback loop described in connection with the crossed-fields multiplier can also be applied to the function generator. This has been done by the author and satisfactory results were obtained. As in the case of the multiplier, however, the increased complexity of the carrier system does not seem to be warranted. If the improved signal to noise characteristics of the carrier amplifier and phase detector system were required, this scheme could be employed. A rather minor advantage of the carrier system is that it removes the necessity for an elaborate light shield around the photocell pickup, since a carrier system is only sensitive to light modulated at the proper carrier frequency.

In addition to the obvious applications of the function generator to the solution of differential equations, the unit may be used as a component in either the logarithmic or difference-of-squares multiplying schemes. This application would require two or three function generators per multiplier, and would therefore be considerably more complicated than, for example, the crossed-fields multiplier.

Photographs of the photocell pickup and feedback amplifier for the function generator are shown in Fig. 44. Fig. 45 shows the complete function generator with the photocell unit in place before the face of the modified Dumont 208 oscilloscope.

Fig. 44 Photocell pick-up unit for feedback function generator.

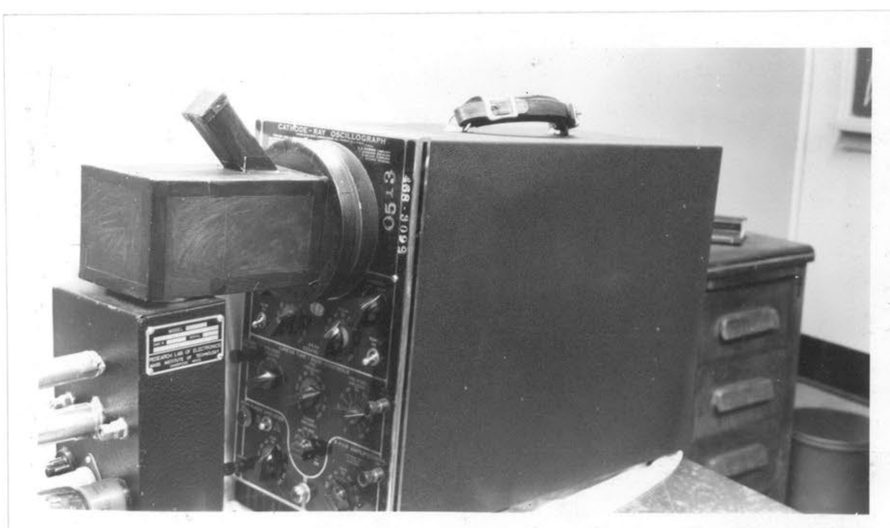


Fig. 45 Feedback function generator.

4.7 Common Adding Circuits

Three common means of obtaining the sum of two voltages are given in Fig. 46. Although this figure shows the connections for addition of only two voltages, any of these schemes can be generalized to sum any number of voltages. For the circuit of Fig. 46(a) the various voltages to be added are connected to the grids of a number of pentode tubes having a common plate load resistor R_L. If this load resistance is chosen to be much smaller than the parallelled plate resistances of all the tubes connected to it, then the output voltage is given by

$$E_{out} = (g_{m1}E_1 + g_{m2}E_2 + \dots + g_{mn}E_n)R_L \ , \qquad (43)$$

where n is the number of tubes with parallelled plates,

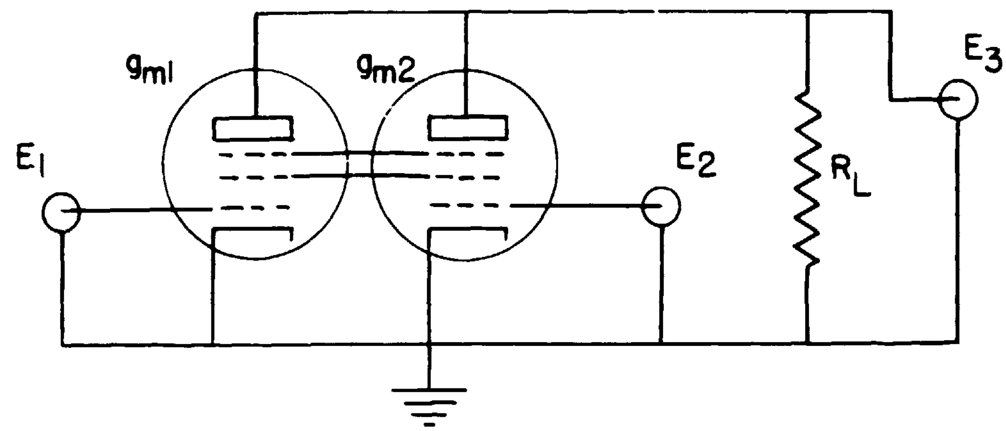
g_{mn} is the transconductance of the nth tube, and

E_n is the grid input voltage to the nth tube.

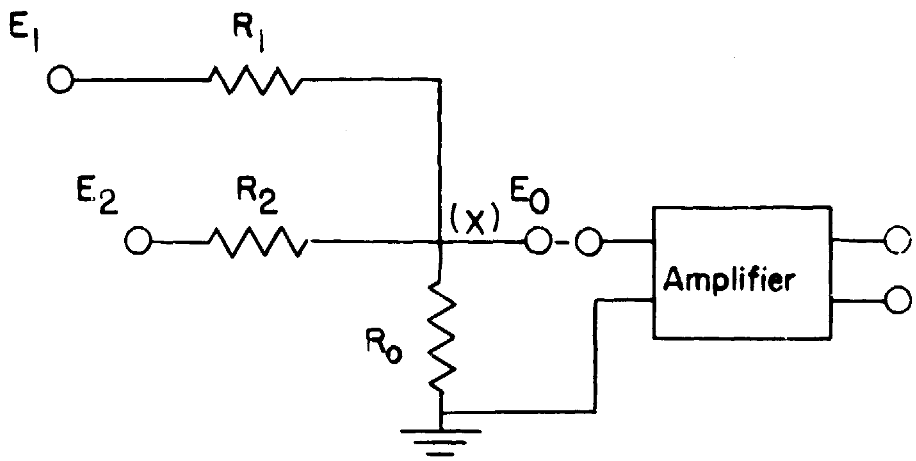
This adding circuit requires one tube per input. The multiplying constants for each term in the sum of Eq. (43) can be varied by adjusting the quiescent operating point and thus the transconductances of the various tubes. This dependence of the sum coefficients on the tube parameters is usually more of a drawback than an advantage, because of the difficulty in preventing these constants from varying with age, line voltage, and the like.

The passive adder of Fig. 46(b) forms the sum according to the relation

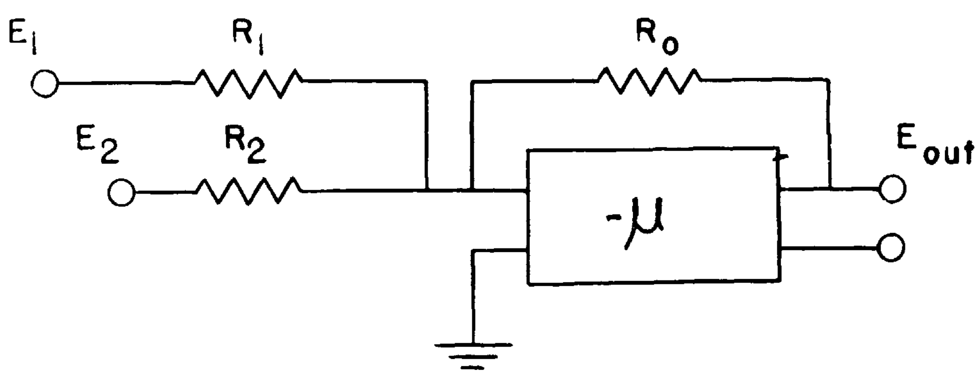
$$E_{out} = E_1\left[\frac{1}{1 + \frac{R_1}{R_o} + \frac{R_1}{R_2} + \frac{R_1}{R_3} + \dots + \frac{R_1}{R_n}}\right] + E_2\left[\frac{1}{1 + \frac{R_2}{R_o} + \frac{R_2}{R_1} + \frac{R_2}{R_3} + \dots + \frac{R_2}{R_n}}\right] + \dots + E_n\left[\frac{1}{1 + \frac{R_n}{R_o} + \frac{R_n}{R_1} + \frac{R_n}{R_2} + \dots + \frac{R_n}{R_{n-1}}}\right] , \qquad (44)$$



(a) Parallel Tube Adder



(b) Passive Adder



(c) Feedback Amplifier Adder

FIG. 46 SOME COMMON ADDING CIRCUITS

where n voltages E_1 through E_n are connected through resistors R_1 through R_n to the common point x on the output resistor R_o. If all the resistors R_o, $R_1, \ldots, R_n$ are made equal, then Eq. (44) simplifies to the expression

$$E_{out} = \frac{1}{(n + 1)} (E_1 + E_2 + E_3 + \ldots + E_n). \qquad (45)$$

This adder has the advantages of extreme simplicity, and dependence for its calibration on the relatively stable characteristics of a passive resistance. Its principle disadvantage lies in the fact that the gain of such an adder is always less than one. This can be overcome by amplifying the voltage E_{out} in an amplifier stabilized against changes in tube parameters, power supply hum, and the like, by negative feedback.

4.8 The Feedback Adder

The feedback amplifier adder of Fig. 46(c) combines the passive adder and stabilized amplifier into a single unit. This is the type of adder employed in the electronic differential analyzer of this thesis. To understand the behavior of this amplifier let us assume that the amplifier labelled $-\mu$ in the figure has a constant gain of $-\mu$ for all frequencies of interest, infinite input impedance, and zero output impedance. For this case one finds

$$E_{out} = - \frac{\mu}{\mu + 1 + R_o(\frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n})} \left[E_1\frac{R_o}{R_1} + E_2\frac{R_o}{R_2} + \ldots + E_n\frac{R_o}{R_n} \right], \qquad (46)$$

which reduces, for the special case of R_o, $R_1, \ldots, R_n$ all equal, to

$$E_{out} = - \frac{\mu}{(\mu + 1 + n)} \quad (E_1 + E_2 + E_3 + \ldots + E_n) . \qquad (47)$$

One sees that if μ is considerably greater than $(n + 1)$, the sum thus formed is substantially independent of changes in the amplifier gain μ.

Comparing this expression with Eq. (45) for the passive adder case, one observes that the passive adder plus an amplifier has exactly the same characteristic as that of Eq. (47) provided the amplifier gain is $-\frac{\mu(n + 1)}{\mu + n + 1}$. This is just the characteristic that one achieves if the amplifier were designed with an open loop gain of $-\mu$ and a negative feedback circuit to reduce the gain to the value $-\frac{\mu(n + 1)}{\mu + n + 1}$. Thus the feedback adder of Fig. 46(c) is exactly equivalent to the passive adder of Fig. 46(b) plus a stabilized feedback amplifier having the same open loop gain as the amplifier used in the feedback adder.

The use of the passive adder and separate feedback stabilized amplifier has perhaps a small advantage in that it isolates the two functions; thus there is no danger that changes in the adding resistors will influence the stability of the feedback unit. The feedback adding scheme is convenient, however, since with a very minor change it can be converted into a time integrator or differentiator. Its use therefore provides maximum flexibility in a small differential analyzer.

The feedback adder is basically a feedback amplifier and all the precautions normally employed in the design of amplifiers with large amounts of feedback must be observed to avoid poor transient response and oscillation. Some discussion of this point is given when the particular unit used in this electronic differential analyzer is described. The literature on the subject of stability in feedback amplifiers is very lengthy and the problem certainly cannot be treated in any detail here.[46,47,48,49] It should be noted that the feedback

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[46] Bode, H.W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand, New York, 1945.

[47] Radiation Laboratory Series, Vol. 25, Theory of Servomechanisms, McGraw-Hill, New York, 1947.

adder changes the algebraic signs of the voltages it adds due to the 180 degree phase shift in the feedback amplifier. An auxiliary use of such a unit is thus that of a sign-changer.

As indicated in Section III the present differential analyzer repeats the entire solution of the differential equation every 1/60 second. In order that these repeated solutions be identical, the output voltages of every unit in the differential analyzer must be identical at the beginning of every solution period. This condition requires the d-c output of each adder be uniquely determined by its input.

4.81 D-c Restoration

One means of accomplishing this is to use a direct-coupled amplifier in the feedback adder. It is desirable, however, to make the gain of that unit very high in order that considerable net gain be achieved with a reasonable amount of negative feedback stabilization. Further, since the d-c output of each adder influences the initial conditions of the differential equation being solved, this output should not vary with time because of extraneous factors such as temperature, tube age, power supply voltage, and the like. High gain d-c amplifiers are notoriously poor in this regard, and it was felt that they should be avoided wherever possible.

A-c coupled amplifiers are free from the difficulties of long-time instability and drift, but do not transmit the necessary zero

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[48] Hall, A.C., The Analysis and Synthesis of Linear Servomechanisms, Technology Press, Massachusetts Institute of Technology, May 1943.

[49] Brown, G.S., and Campbell, D.P., Principles of Servomechanisms, Wiley, 1948.

frequency component. Each RC coupling circuit in such an amplifier can be thought of as a time integrating circuit. In general the final value of the integral of the voltages being treated is not equal to the desired initial value; hence some means of forcing the condensers to assume the proper initial charges must be employed. There is fortunately a certain amount of time for this purpose, namely the off-time of the differential analyzer, approximately 1/120 second each 1/60 second. This problem is very analogous to one encountered in television where a zero-frequency component must also be transmitted and some unused time, the backsweep period for the television scan is available. To accomplish this result a number of pulsed d-c restoration or clamping schemes have been developed.

One such circuit which is particularly well adapted to the present problem is shown in Fig. 47.[50,51] When the diode in this circuit is not conducting, one has a normal a-c coupling consisting of the coupling condenser C_c and a shunt resistance to ground due to the input resistance of the amplifier tube and the back resistance of the diodes. During the clamping time gate signals from an external gate generator cause both diodes to conduct and thus form a low impedance path from the grid to a fixed potential, depending in the circuit shown on the setting of the potentiometer R_2. Provided the time constant C_oR_i is long compared to the clamping time, R_i being the internal impedance of the gate generator, the impedance of this clamping action is $2R_d + 2R_i$

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50 Roe, J.H., "New Television Field Pick-up Equipment Employing the Image Orthicon," Proc. I.R.E., v. 35, 1532-1546, Dec. 1947.

51 Wendt, K.R., "Television DC Component," RCA Review, v. 9, 85-111, March 1948.

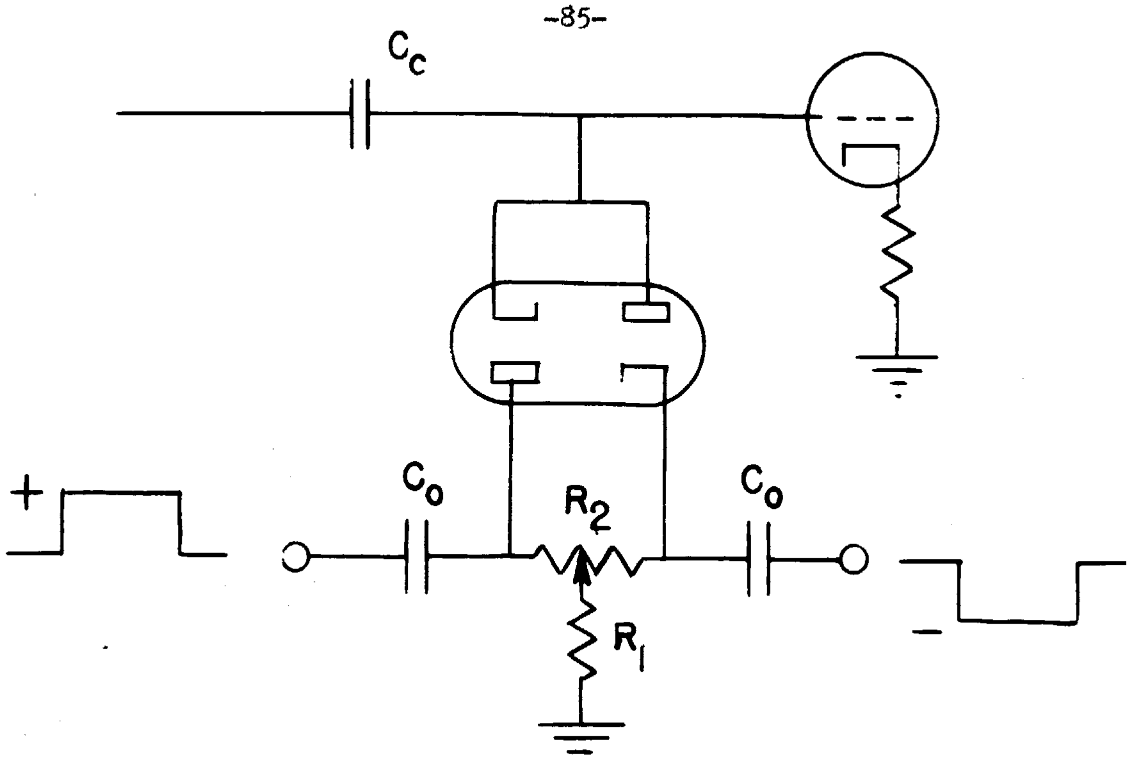

Fig. 47 RCA diode clamping circuit.

where R_d is the conducting resistance of the diodes. By using the circuit of Fig. 47 it is possible to obtain the necessary zero-frequency response over the solution period for the differential analyzer and still avoid sensitivity to changes in plate supply voltages, cathode emission, etc.

The resulting amplifier is more complicated than a simple a-c coupled amplifier since one of these clamping circuits must be employed for each interstage coupling condenser. The complexity is considerably less, however, than would be required if a d-c amplifier of comparable performance were employed.

The differential analyzer off-time is utilized for the clamping period and every clamping circuit in the entire analyzer is driven by a single gating pulse, thus insuring synchronous operation.

An ordinary relay could also be used instead of the diode clamping circuit if sufficiently precise operation could be obtained.

Brown Instrument Vibrators were used by the author for this purpose.[52] These relays gave good performance from the point of view of noise and contact chatter, but they were not easily adjusted for synchronous opening and closing times.

4.82 Practical Adding Unit

The circuit diagram of the complete adder used in the electronic differential analyzer is shown in Fig. 48. Miniature tubes are employed in this amplifier in the interest of small size and power consumption. The 6AG5 pentode first stage is followed by a cathode-coupled 6J6 amplifier. The cathode-coupled circuit is employed here to yield the 180 degree net phase shift necessary for stable negative feedback. The 6J6 output cathode follower gives a very low output impedance and prevents the output load from appreciably influencing the feedback stability.

The two a-c coupling circuits are d-c restored by the 6AL5 diodes shown. The first of these clamping circuits, between the 6AG5 and the first 6J6, is a d-c connected version of the circuit of Fig. 47. This saves two large coupling condensers. The gate pulses are equal and opposite in magnitude, so that the grid of the first 6J6 is clamped to zero volts by this circuit. The clamper on the output 6J6 grid employs a-c coupling of the gate pulses. This permits adjusting the potential to which the grid of the output tube is clamped with the potentiometer, and thus serves as an output d-c level adjustment.

The shunt feedback resistor R_o plugs into the terminal marked J_5 and the series resistors for the adding circuit plug into terminals

[52] Synchronous Converter No. 75828-1, Brown Instrument Co., Philadelphia, Pennsylvania.

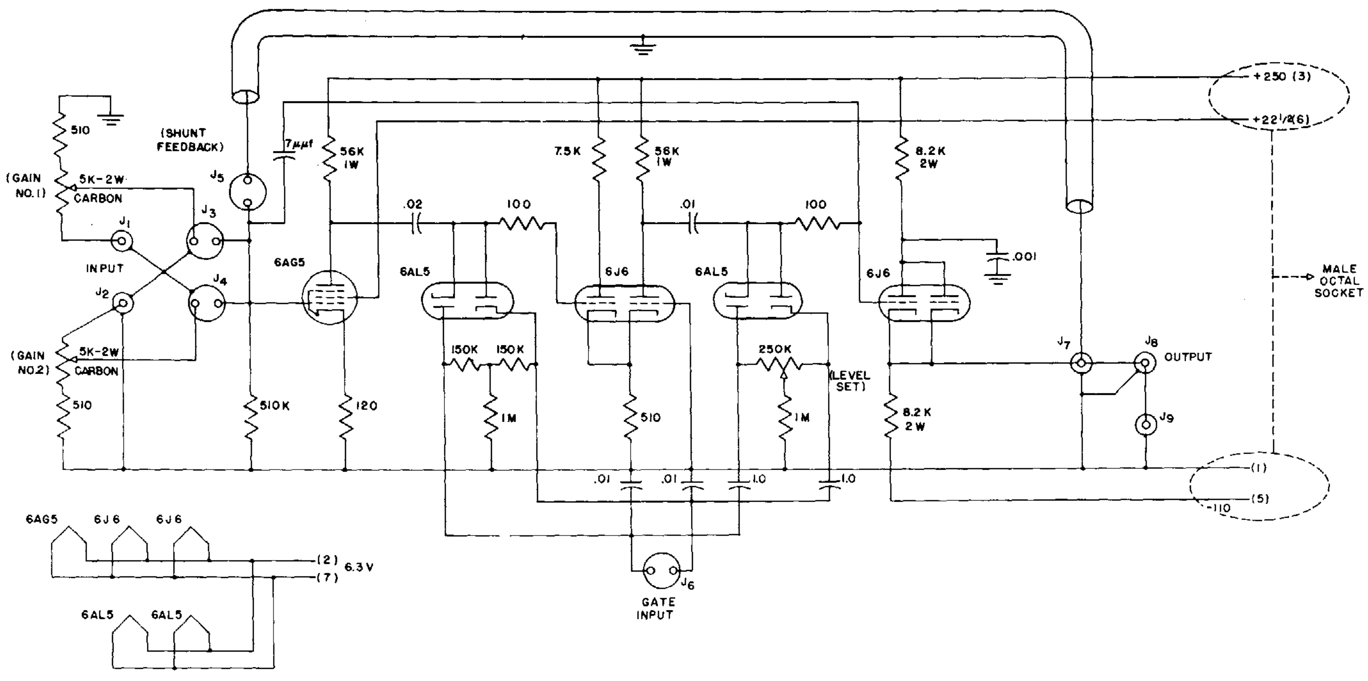

FIG. 48 FEEDBACK ADDER

J_3 and J_4. In normal operation R_o is 51,000 ohms. The adder gain for either input channel is then variable over a range of 10 : 1 by means of the input controls. Changes in gain greater than 10 : 1 are achieved by varying the values of the series input resistors at the terminals J_3 and J_4.

Equations (46) and (47) give the characteristics of a feedback adder built around an amplifier whose gain $-\mu$ is constant for all frequencies. The amplifier of Fig. 48 is not such an ideal amplifier; rather it has a characteristic at high frequencies of the form

$$-\mu = \frac{-\mu_o}{(1 + j\omega R_3C_3)(1 + j\omega R_4C_4)}. \qquad (48)$$

The measured amplitude and phase characteristics of the amplifier without feedback are plotted in Fig. 49(a) versus a logarithmic frequency scale. From this figure one finds the high frequency time-constants of the unit to be

$$R_3C_3 = 1.59 \ \mu\text{seconds}, \qquad (49)$$

$$R_4C_4 = .568 \ \mu\text{seconds}. \qquad (50)$$

For a two channel adder such as this, Eq. (46) becomes

$$E_{out} = -\frac{\mu}{\mu + 1 + R_o(\frac{1}{R_1} + \frac{1}{R_2})}\left[E_1\frac{R_o}{R_1} + E_2\frac{R_o}{R_2}\right]. \qquad (51)$$

In obtaining this expression stray capacities across the resistors were neglected. At high frequencies this assumption is no longer justified, and one must replace

$$R_o \text{ by } \frac{R_o}{1 + j\omega R_oC_o}, \qquad (52)$$

$$R_1 \text{ by } \frac{R_1}{1 + j\omega R_1C_1}, \text{ and} \qquad (53)$$

$$R_2 \text{ by } \frac{R_2}{1 + j\omega R_2C_2}, \qquad (54)$$

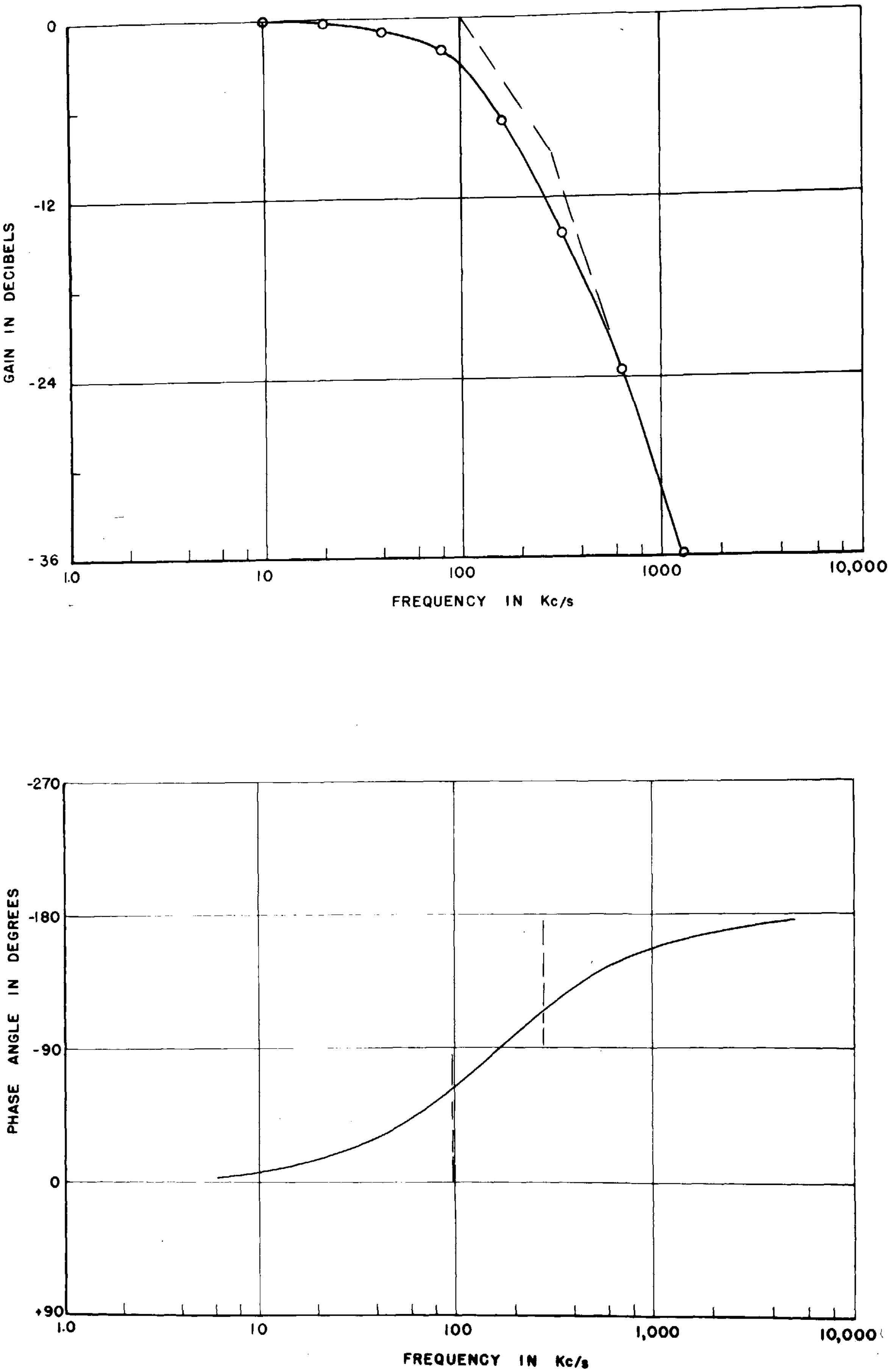

FIG. 49(a) ADDER AMPLIFIER CHARACTERISTICS where C_o, C_1, and C_2 are the shunt capacities associated with R_o, R_1, and R_2 respectively. Substituting Eqs. (48), (52), (53), and (54) in Eq. (51) gives

$$E_{out} = \frac{-\mu_o \left\{ E_1 \frac{R_o}{R_1} \left[\frac{1 + j\omega R_1 R_1}{1 + j\omega R_o C_o} \right] + E_2 \frac{R_o}{R_2} \left[\frac{1 + j\omega R_2 C_2}{1 + j\omega R_1 C_1} \right] \right\}}{\mu_o + 1 + \frac{R_o (1+j\omega R_3 C_3)(1+j\omega R_4 C_4) \left[R_2 (1+j\omega R_1 C_1) + R_1 (1+j\omega R_2 C_2) \right]}{R_1 R_2 (1 + j\omega R_o C_o)}} \tag{55}$$

as an accurate expression for the behavior of the feedback adder at high frequencies.

In this unit the time constants R_3C_3 and R_4C_4 are associated with load resistors and stray capacities within the amplifier and are not very easily varied. The external time constants R_oC_o, R_1C_1, and R_2C_2 are, however, easily varied. Through suitable adjustment of these external time constants, by analytic or experimental means, an optimum transient response for the unit can be achieved. The adder pulse response is shown in Fig. 49(b).

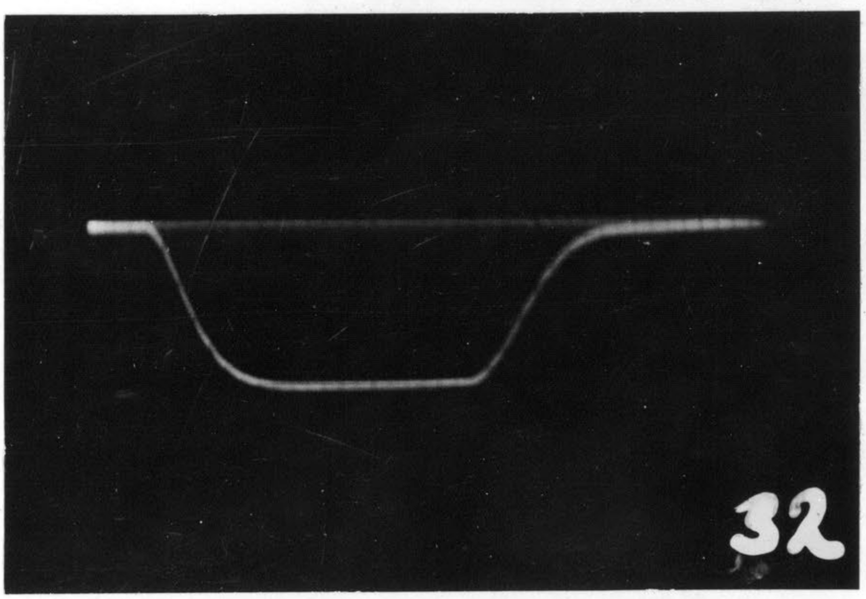

Fig. 49(b) Adder pulse response.

4.83 Measured Adder Characteristics

The open loop characteristics of this amplifier (no feedback) are as listed below:

Voltage gain - - - - - - - - - - - - - - - - - 1500 times or 63.5 db.

Upper half power frequency - - - - - - - - 100 Kc/s.

Lower half power frequency - - - - - - - - 0.16 c.p.s.

This adder can be operated at a variety of gains, according to the setting of the various controls and values of the plug-in resistors; it is never run with a gain for either channel of more than 150, or 43.5 db. The characteristics of the unit as normally operated are given below:

Voltage gain - - - - - - - - - - - - - 150 times or 43.5 db. maximum

Upper half-power frequency - - - - 650 Kc/s.

Lower half-power frequency - - - - 0.016 c.p.s.

Rise time - - - - - - - - - - - - - - 0.4 μsec. (10% to 90%)

Overshoot - - - - - - - - - - - - - - 2% maximum

Output noise - - - - - - - - - - - - - 1-10 millivolts

Maximum output - - - - - - - - - - - - ± 20 volts

Output impedance - - - - - - - - - - - 10 ohms maximum

The linearity of the adder unit was tested by applying a sinusoidal signal to the input; the gain was set to 100, and the output attenuated and subtracted from the input in a second adder. The results of this test are shown in Fig. 50 which is a triple exposure photograph of the input, output, and difference between the input and the output. The difference voltage is to a scale 100 times that of the input and output. From this figure one sees that the harmonic distortion is less than 0.2%. This data was taken for an output signal

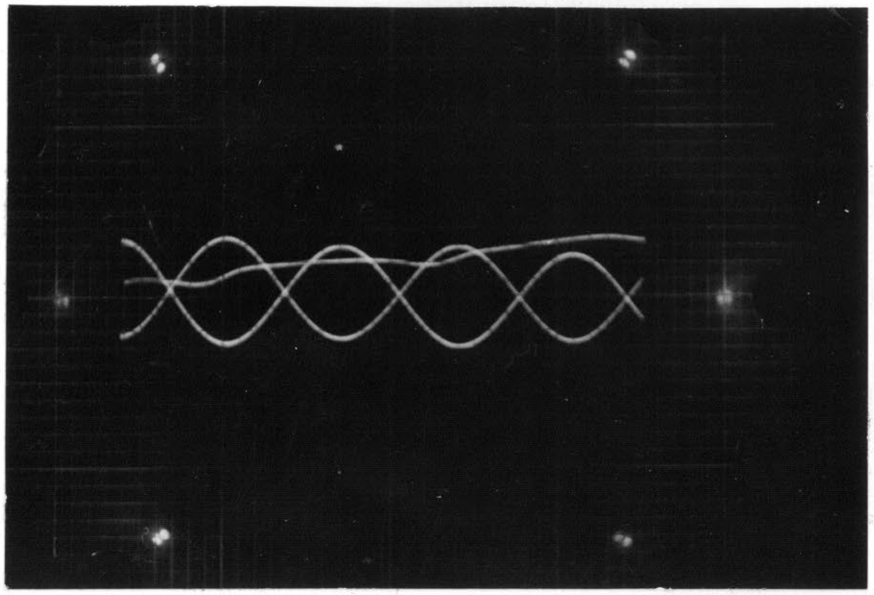

Fig. 50 Linearity test for adding and inverting unit.

having 30 volts peak to peak amplitude.

The application of adding units to the solution of differential equations on the electronic differential analyzer is described in Section VI. An investigation of the manner in which the finite bandwidth of the adding units can introduce errors into the solution of differential equations is given in Section V.

Integration

A large number of circuits designed to integrate a voltage with respect to time have been developed in recent years. This development has been done with two applications in mind (1) the generation of linear sweep voltages for television and radar and (2) computation in fire-control computers and electronic simulators.[53,54,55,56,57,58,59,60,61]

One of the simplest integrating circuits is the common RC low-pass filter shown in Fig. 51.

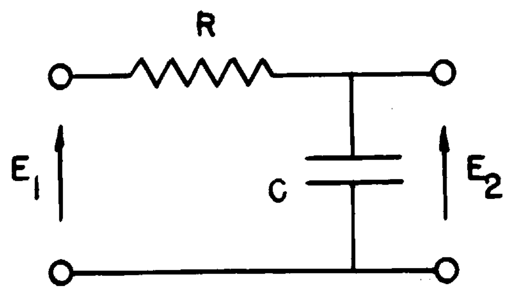

Fig. 51 RC Integrator.

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53 Bell Telephone Laboratories, op. cit., reference 13.

54 Ragazzini, et al., op. cit., reference 14.

55 Admiralty Computing, op. cit., reference 15.

56 Korn, op. cit., reference 16.

57 Mynall, op. cit., reference 22.

58 M.I.T. Radar School Staff, op. cit., reference 36.

59 Puckle, O.S., Time Bases, John Wiley & Sons, New York, 1943.

60 Radiation Laboratory Series, Electronic Instruments, Vol. 21.

61 Radiation Laboratory Series, Electronic Time Measurements, Vol. 20.

For this circuit

$$\frac{E_2}{E_1}(\omega) = \frac{1}{1 + j\omega RC} \quad . \tag{56}$$

For high frequencies, $\omega \gg \frac{1}{RC}$,

$$\frac{E_2}{E_1}(\omega) \approx \frac{1}{j\omega RC} \; . \tag{57}$$

This is the frequency response of an ideal time integrator, for which

$$E_2(t) = \frac{1}{RC} \int E_1(t)dt \; . \tag{58}$$

In order that this simple integrator perform accurately the time constant RC must be large. This makes the scale factor $\frac{1}{RC}$ in Eq. (58) very small. Most of the more complicated integrating circuits have as their objective increasing the effective time constant without this reduction in scale factor.

It is not possible here to go into a detailed discussion of all the various integrating circuits which have been proposed. Instead attention is focused on that circuit employed in the present electronic differential analyzer in order that its characteristics be well understood for future error analysis.

4.9 The Feedback Integrator

The basic feedback integrator employed in this work is indicated schematically in Fig. 52.

4.91 Ideal Characteristic

The similarity between this unit and the adder of Fig. 46(c) is evident. The only difference is the use of a condenser in the shunt feedback arm instead of a resistor. If one assumes the ideal case for which the amplifier has infinite gain and bandwidth together with zero output impedance and infinite input impedance, this circuit has

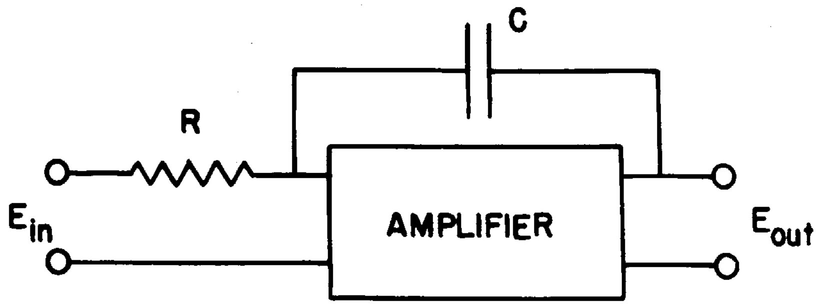


Fig. 52 Feedback integrator circuit.

the transfer characteristic

$$E_{out} = -\frac{E_{in}}{j\omega RC} . \qquad (59)$$

<u>4.92 Limitations of Realizable Integrators</u>

This is a characteristic that can only be approximated in practice, because practical amplifiers do not have infinite gain or bandwidth. The amplifiers used in this differential analyzer have at least one low- and one high-frequency time constant; they also have finite mid-band gain. Such an amplifier has a gain

$$-\mu = \frac{-(j\omega T_1)\mu_o}{(1 + j\omega T_1)(1 + j\omega\alpha)} , \qquad (60)$$

where μ_o = bandcenter gain of amplifier,

T_1 = low frequency time constant of amplifier,

α = high frequency time constant of amplifier,

$\omega = 2\pi f$ = frequency in radians per second.

Using this realizable gain characteristic in the circuit of Fig. 52 and assuming that $\mu \gg 1$ at all frequencies of interest one finds that the integrator transfer characteristic is

$$\frac{E_{out}}{E_{in}} = \frac{-1}{j\omega RC} \cdot \frac{j\omega\mu_o RC}{1 + j\omega\mu_o RC} \cdot \frac{j\omega T_1}{1 + j\omega T_1} \cdot \frac{1}{1 + j\omega \frac{\alpha}{\mu_o}} \cdot \quad (61)$$

The manner in which this characteristic differs from the ideal of Eq.(59) is best studied by considering a plot of the magnitude of Eq. (61) in decibels versus the logarithm of the frequency. Such a plot is given in Fig. 53.

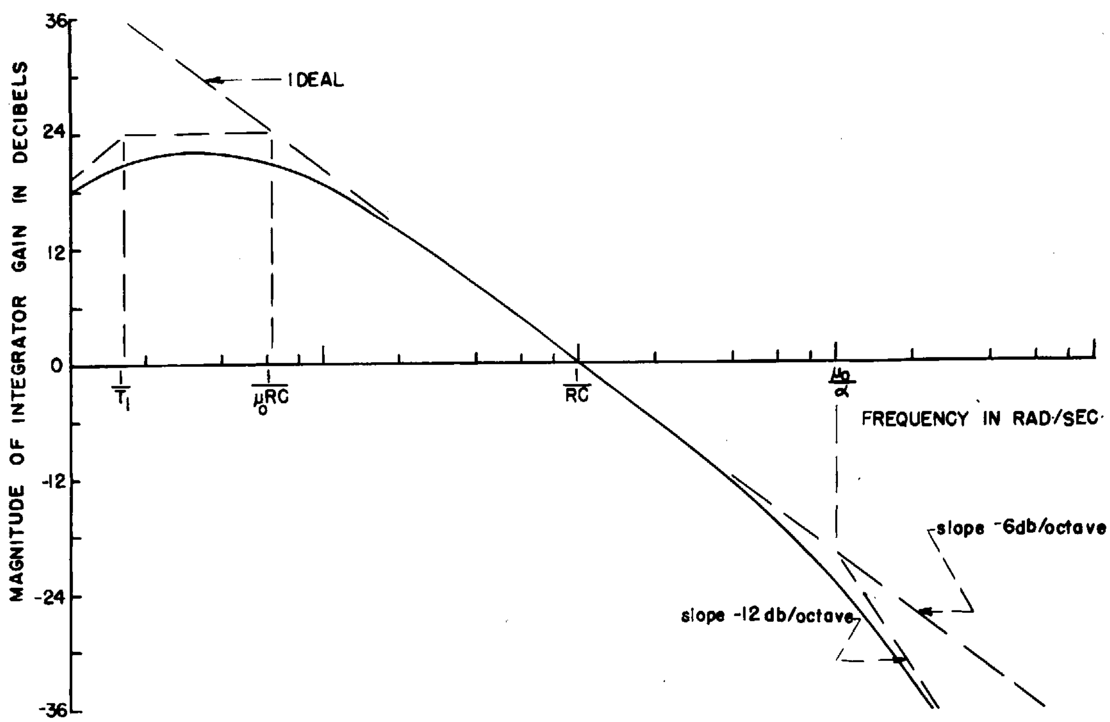


Fig. 53 Log-db plot for time integrator circuit.

The ideal integrator characteristic is plotted in dotted lines in this figure. It runs from infinity at zero frequency to zero at infinite frequency with a slope of -6 db per octave. The gain is zero db or unity at the frequency $\omega = \frac{1}{RC}$. The response in the time domain of this ideal integrator to a negatively applied unit step of voltage is plotted in Fig. 54 as a dotted line.

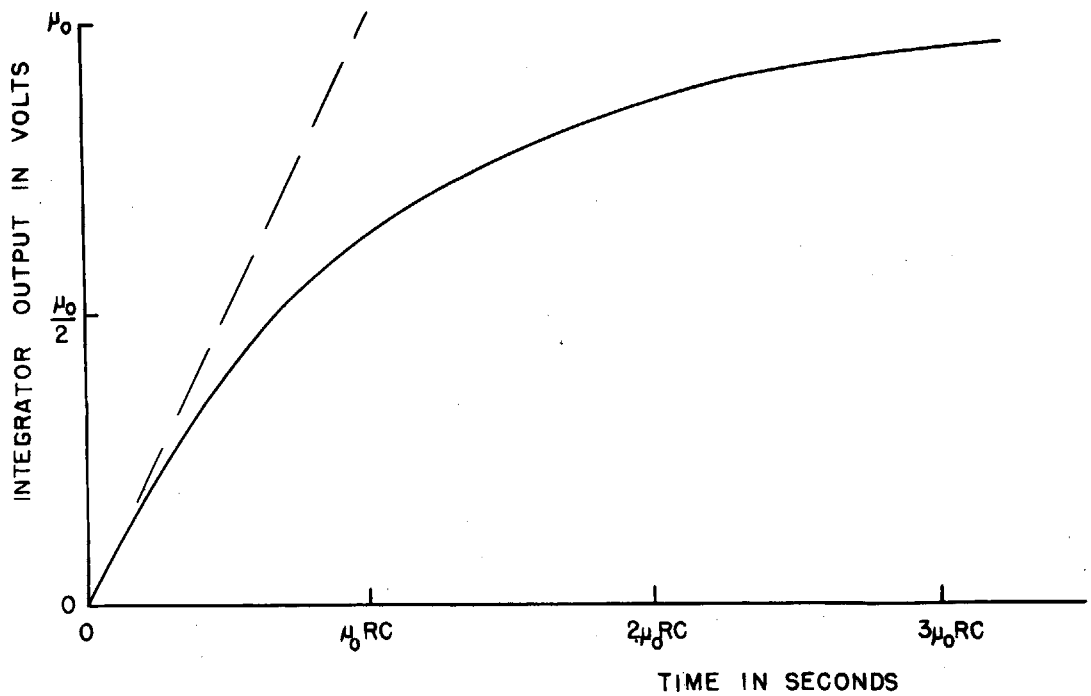


Fig. 54 Output voltage of time integrator versus time for a negative unit step input voltage.

Separately considering the deviations of the realizable characteristic of Eq. (61) from the ideal of Eq. (54), one observes first that because of the finite bandcenter gain μ_o of the amplifier the gain characteristic approaches the maximum value μ_o at the frequencies near $\frac{1}{\mu_o RC}$. The effect on the integrator step function response of this first limitation is shown by the solid line in Fig. 54. The output voltage, instead of rising linearly with time, is given by

$$E_{out} = \mu_o(1 - e^{\frac{-t}{\mu_o RC}}); \qquad (62)$$

this is exponential in character with a time constant of μ_oRC.

At high frequencies, because of the finite bandwidth of the amplifying unit, the characteristic also departs from the ideal. At the frequency $\frac{\mu_o}{\alpha}$ the slope of the amplitude characteristic increases from -6 to -12 db per octave. This high frequency effect influences the short time behavior of the integrator. This is illustrated for a step voltage input in Fig. 55.

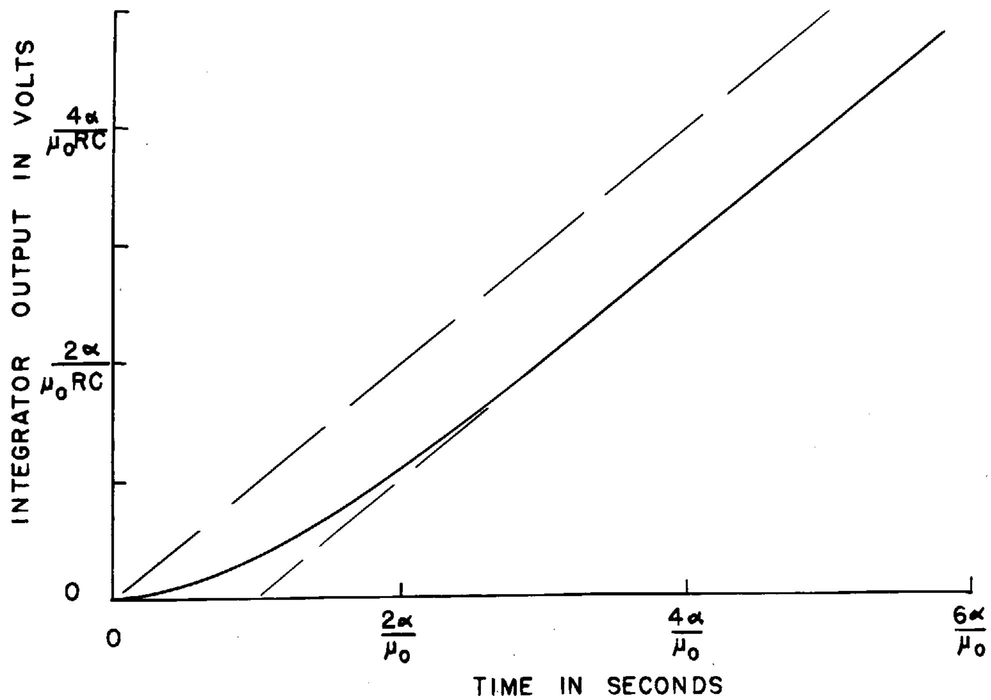


Fig. 55 Short-time behavior of time integrator.

One sees from this that at the beginning of the step response the unit behaves not as a single integrator, but as a cascade of two integrators. We can speak of the integrator as having a high-frequency cut-off at $\frac{\mu_o}{\alpha}$ or as having a rise-time $2.2 \frac{\alpha}{\mu_o}$. This high frequency point is, of course, analogous to the upper half-power frequency normally considered in an amplifier. The high frequency effects in integrators are equally as important as the bandwidth limitations in the adders.

In the case of ordinary amplifiers this high frequency cut-off is something that cannot be avoided, because of the stray circuit capacities which inevitably are encountered. For the integrator, however, there appear to be possibilities of removing this high frequency effect by suitable compensating schemes, since no gain-bandwidth limitation is violated by so doing. One such scheme is to shunt the series input resistor of Fig. 52 with a small capacity C_c. If this is done, the transfer characteristic of Eq. (61) becomes

$$\frac{E_{out}}{E_{in}} = - \frac{1}{j\omega RC} \cdot \frac{j\omega\mu_o RC}{1 + j\omega(\mu_o RC + RC_c)} \cdot \frac{j\omega T_1}{1 + j\omega T_1} \cdot \frac{1 + j\omega RC_c}{1 + j\omega \frac{\alpha}{\mu_o}} . \quad (63)$$

For this case, if C_c is chosen so that

$$RC_c = \frac{\alpha}{\mu_o} , \quad (64)$$

the high frequency response is exactly compensated. The limitation of this scheme lies in the assumption that the amplifier high frequency response can be represented by a single high frequency time-constant as was done in Eq. (60). It has been found possible in practice, however, to make the integrator high frequency response so much better than that of the adders and inverters that it is negligible in differential equation solutions.

The other effect which must be considered in practical integrators is the low frequency time constant T_1 of the amplifiers. In Fig. 53 it was assumed that this time constant

$$T_1 \approx 4\mu_o RC . \quad (65)$$

As long as T_1 is equal to or greater than this value, its effect is to cause another break in the gain-frequency characteristic at the very low frequency $\frac{1}{T_1}$, as shown in Fig. 53. At this point the slope changes from zero to +6 db per octave. Fig. 56 shows the influence of this effect on the step function response of an integrator. In the present differential analyzer the integrators are not permitted to operate longer than a short time interval T_o, as indicated in Fig. 56. When this is the case, the use of a-c coupled amplifiers is not a fundamental limitation on the integrator operation.

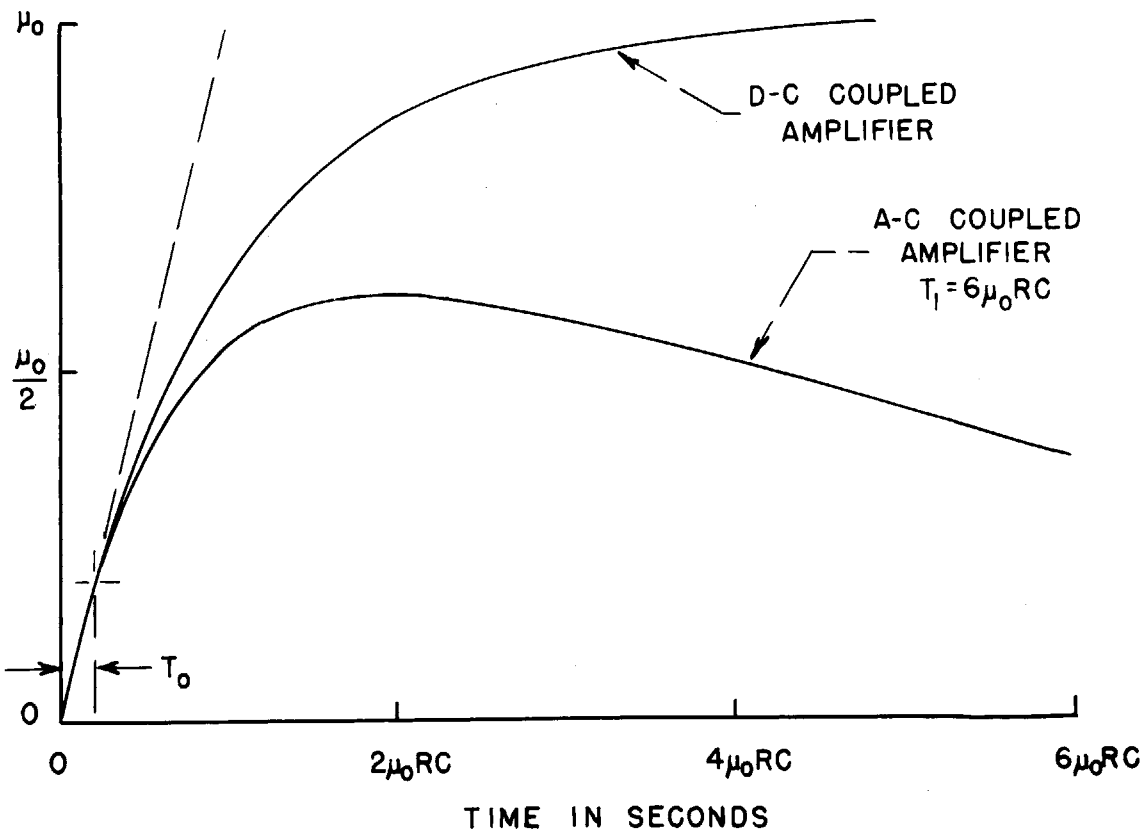


Fig. 56 Influence of a-c coupling on integrator step function response.

Two other practical limitations affecting integrator operation are leakage across the integrating condenser C and the finite input impedance of the amplifier. These effects both reduce the bandcenter gain μ_o of the amplifier. The new gain is

$$\mu_o' = \mu_o \frac{R_{in}}{R_{in} + R} \quad \frac{R_L}{\mu_o R + R_L}, \qquad (66)$$

where R_{in} = the amplifier input resistance,

R = the leakage resistance across the condenser C.

If Eqs. (61) and (63) are rewritten with μ_o everywhere replaced by μ_o', exact expressions are obtained.

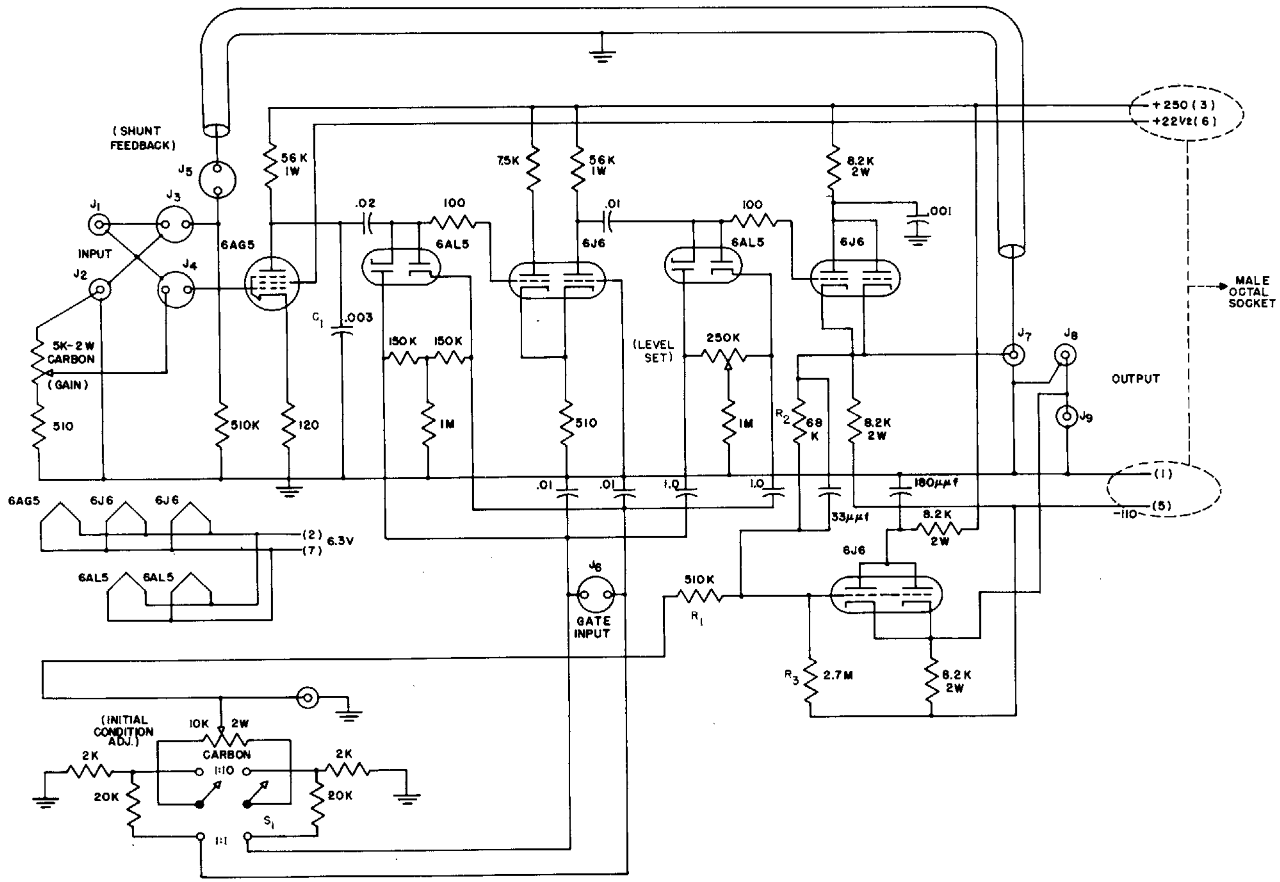
4.93 Practical Integrating Unit

The circuit diagram of the integrator used in the electronic differential analyzer is given in Fig. 57. Comparing this circuit with Fig. 48 one sees that both the adding and integrating units are built around the same amplifying unit. In the integrator the integrating condenser is inserted in the shunt feedback jack J_5, while the series resistor, R of Fig. 52, plugs into J_4. The scale factor of the integrator can be varied over a range of 10 to 1 with the input potentiometer; larger ranges than this are obtained by changing the integrating condenser at J_5 or the integrating resistor at J_4. The integrator input is at J_2, and the output appears at jacks J_8 and J_9.

4.94 Integrator Initial Conditions

The 6AL5 twin diodes are d-c clampers, as in the adding unit. The use of these clamping tubes is particularly convient in this integrator application. As pointed out earlier it is necessary to establish the proper initial conditions for each integrator at the beginning of each solution period. This can be done by bringing the charge on the integrating condenser to zero and connecting a battery in series with the output of the integrator at the beginning of each solution period. This scheme is indicated in Fig. 58. For this circuit the switches S_1 and S_2 should both be at position (1) during the differential analyzer off-time, and at position (2) during the on-time.

The d-c clamped amplifier automatically performs the operation of switch S_1 in Fig. 58. During the clamping period the input of the integrator is zero because it is driven either by an adder or another integrator. In addition the grid of the last 6J6 is clamped to a



FIG· 57 FEEDBACK INTEGRATOR

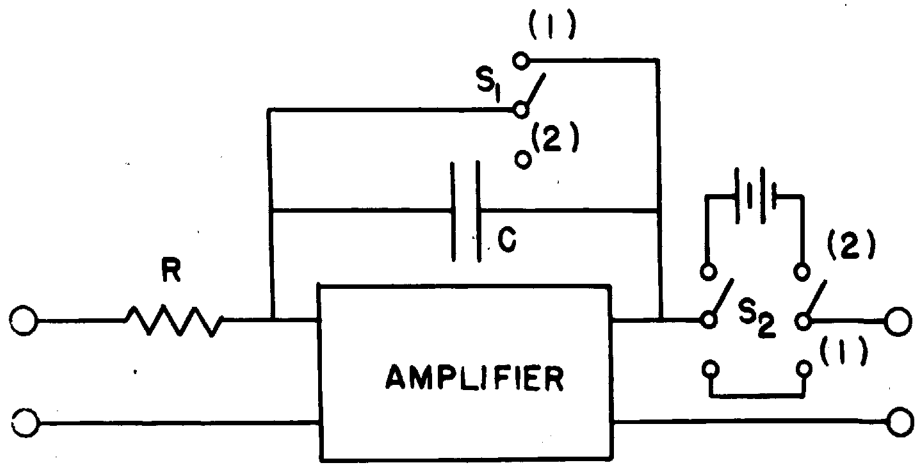


Fig. 58 Initial condition switching circuit.

potential forcing the cathode of that tube to zero volts. Thus during the off-time both the input and output of the integrator unit are forced to a potential of zero volts. Under this circumstance the integrating condenser discharges to zero with time constant RC. If RC and the duration of the off-time are properly chosen, the integrating condenser is almost completely discharged at the end of each off-time; hence the function of S_1 in Fig. 58 is automatically performed by the diode-clamped amplifiers.

It is still necessary to add an initial value to the output of the integrator at the beginning of every solution period. This is accomplished by adding an adjustable voltage step to the integrator output, as shown in Fig. 57. The adjustable height step is derived from the standard gating pulse by a potentiometer. A 10 to 1 change in the step height is also provided by S_1. The initial condition is added to the integrator output by the passive adding circuit consisting of R_1, R_2, and R_3. This sum is then applied through the output cathode follower to give a low output impedance for the unit. The 33 μμf condenser is inserted for high frequency compensation of the voltage divider.

4.95 Measured Integrator Characteristics

The integrator characteristic of Eq. (61) was calculated on the assumption that an amplifier described by Eq. (60) is employed. The gain-frequency characteristic of the amplifier of Fig. 57 is somewhat more complicated than this since it involves at least two high and two low frequency time constants of importance. The second high frequency time constant is particularly important since it can lead to oscillation in the integrator at high frequencies. A brute force method of preventing this is to make one of the high frequency time constants in the amplifier considerably longer than the other so that the amplifier does essentially have the high frequency behavior of Eq. (61) until the loop gain around the feedback loop is less than unity. This is the purpose of the condenser C_1 in Fig. 57.

A photograph of the pulse response of this integrating circuit is given in Fig. 59. The fuzziness of this photograph is caused by lack of synchronization between the test pulse and the integrator gating pulse.

Because of the finite gain of the integrator amplifier its step function response is that given by Eq. (62) rather than a linearly increasing wave with time, $E_{out} = \frac{t}{RC}$. Equation (62) can be rewritten

$$E_{out} = \frac{t}{RC}\left[1 - \frac{t}{2!\mu_o RC} + \frac{1}{3!}\cdot\frac{t^2}{(\mu_o RC)^2} - \cdots\right]. \qquad (67)$$

In this form one sees the way in which this integrator response differs from the response of an ideal integrator, $\frac{t}{RC}$. The error increases with time; thus for a given RC there is a maximum time over

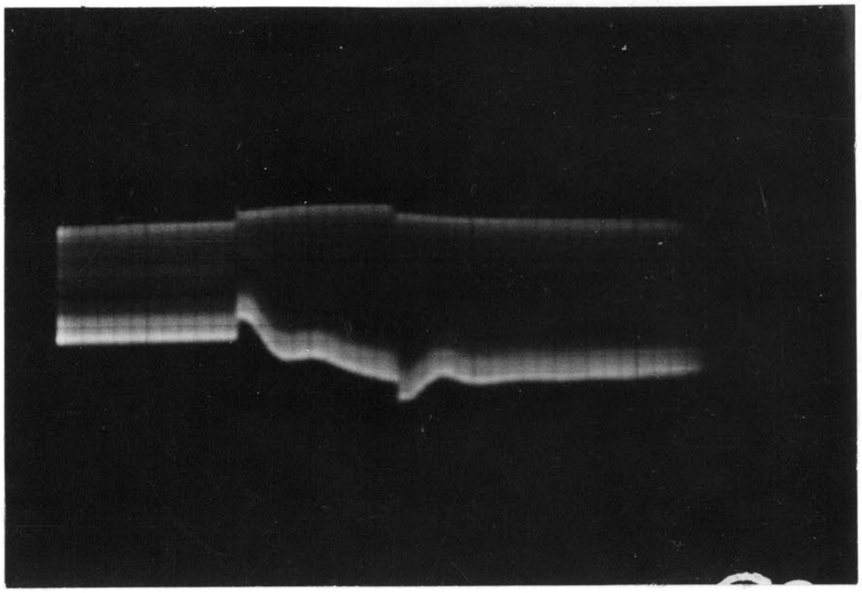


Fig. 59 Integrator pulse response.

which the integrator is accurate to within a given percentage error. For the electronic differential analyzer of this thesis the solution time is about $\frac{1}{120}$ second, and the gain of the integrating amplifier is 1500. For an error equal to or less than 1% one has from Eq. (67)

$$\frac{T}{2\mu_o RC} \leqq .01 \ , \tag{68}$$

which indicates that the minimum permissible RC for a 1% error is

$$RC = \frac{50\ T}{\mu_o} \ , \tag{69}$$

$$= 2.78 \times 10^{-4} \text{ seconds} = 278\ \mu\text{sec}.$$

For the differential equations discussed in Section VI a time constant at least twice this value was employed.

In order that condenser leakage be negligible, the leakage resistance R_L of the condenser must be

$$R_L \geqq 4\mu_o R \ . \tag{70}$$

For R = 50,000 ohms in the unit of Fig. 57 this requires a leakage resistance exceeding 300 megohms. It has been found necessary to use mica integrating condensers to achieve this condition in practice.

For the minimum value of RC determined by Eq. (69) the integrator time constant $\mu_o RC$ is 0.416 seconds. In order that the a-c coupling circuits of Fig. 57 not limit the integrator operation, their time constants must be at least four times larger, that is, 1.66 seconds. This large interstage time constant is realized with the relatively small coupling condensers shown in Fig. 57 by virtue of the fact that the only resistance from grid to ground of the d-c clamped stages during the on-time is the leakage of the insulation, the input impedance of the tubes operating at a negative bias voltage, and the back resistance of the clamping diodes. This, in practice, is of the order of hundreds of megohms. Occasionally after a spell of very damp weather some reduction in this resistance occurs, but it has always been the case that with a half hour's operation, the heat generated by the units themselves is sufficient to dry things out and bring the leakage resistance back to its normal high value.

Division

4.10 Dividing Circuits Employing a Multiplier

The operation of division is necessary for the electronic differential analyzer. Using the units already described, there are at least two possible methods of performing this operation. One is indicated in Fig. 60.

One voltage E_1 is applied to a function generator fitted with the reciprocal mask $\frac{1}{x}$. The output of this unit is then $\frac{1}{E_1}$; this is applied to a multiplier and there multiplied by the second input voltage E_2. The multiplier output is then the desired quotient $\frac{E_2}{E_1}$.

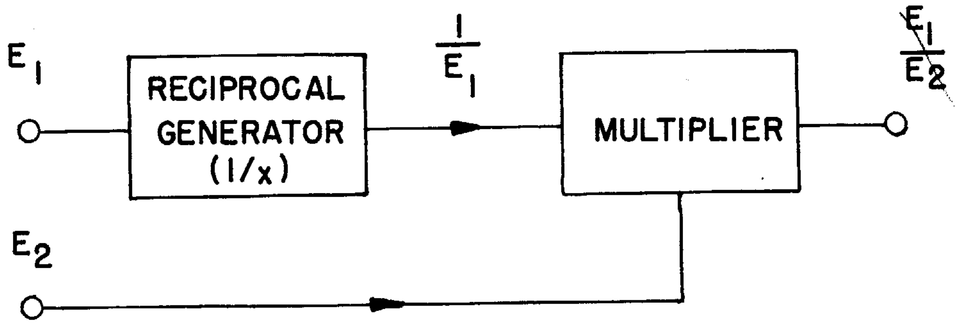


Fig. 60 Division with function generator and multiplier.

This method has the drawback of complexity in requiring the use of a function generator plus a multiplier. It is, however, very straightforward.

The dividing method actually used in the electronic differential analyzer is shown in Fig. 61.[62] If the multiplier has a gain constant k and zero output impedance, and the amplifier of gain $-\mu$ has zero output impedance and infinite input impedance, then this connection gives the characteristic

$$E_3 = \frac{E_2}{E_1} \cdot \frac{R_2}{kR_1} \cdot \frac{1}{1 + \frac{R_1 + R_2}{k\mu R_1 E_1}} , \qquad (71)$$

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[62] Radiation Laboratory Series, op. cit., reference 60.

and as long as

$$\frac{R_1 + R_2}{k\mu R_1 E_1} \ll 1, \tag{72}$$

we have

$$E_3 = \frac{E_2}{E_1} \cdot \frac{R_2}{kR_1}. \tag{73}$$

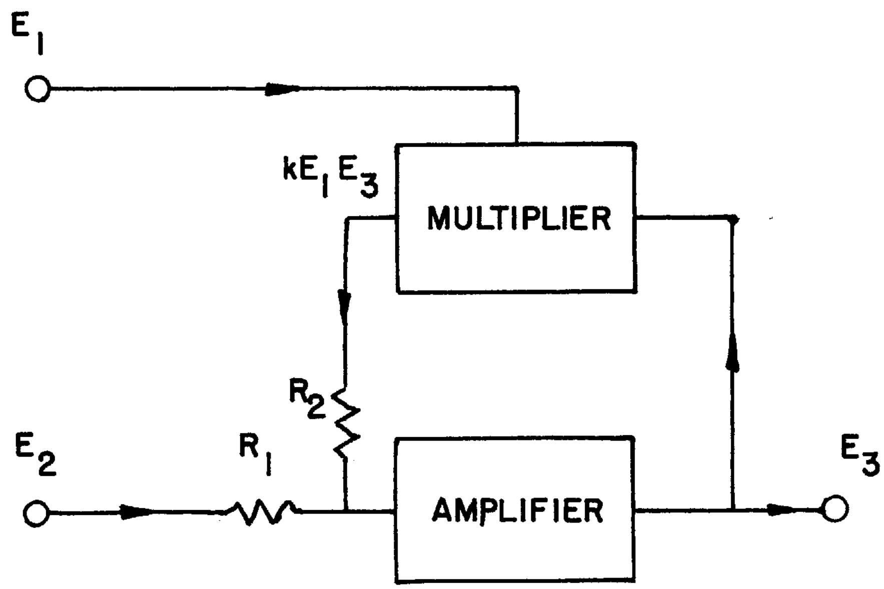

Fig. 61 Divider.

This connection was used for the solution of the Hill equation described in Section VI. It requires an adder plus a multiplier, but since the adder is a simpler unit than the function generator, it is less complicated than the method of Fig. 60. In practice it is somewhat more difficult to achieve since it involves a feedback loop within a feedback loop. The multiplier, which contains a feedback loop, is part of the negative feedback loop of the adder amplifier. For the multiplier of Fig. 19 mathematical analysis of this situation is complicated by the fact that the multiplier operates non-linearly at high frequencies, which is just the region where instability is encountered. Because of these difficulties an analytic investigation of the stability of this circuit has not been made. Experimentally, however, it is possible to operate this circuit over a dynamic range exceeding 50 to 1 in the output. The frequency response achieved is at least as good as that of the multiplying unit alone.

4.11 Modification of Crossed-Fields Multiplier for Division.

Mr. W. Green of the Massachusetts Institute of Technology Instrumentation Laboratory has pointed out to the author that the crossed-fields electron-beam multiplier can be made to perform division directly by the following modification. The amplified output from the photocells in Fig. 19 is applied to the y-deflecting plates of the cathode-ray tube instead of the x-deflecting plates. Then the voltage applied to the x-deflecting plates becomes one of the inputs of the unit. Now as long as the x-deflection is held to zero by the feedback circuit Eq. (41) requires

$$E_3 = k(E_1 \cdot E_2) \quad . \tag{41}$$

With the change of connections indicated, however, E_3 is now one of the input voltages and the output of the feedback unit is E_1; therefore for this connection

$$E_1 = \frac{E_3}{kE_2} \quad , \tag{74}$$

where E_1 is the feedback voltage to the y-plates,

E_2 is the input to the axial field amplifier, and

E_3 is the input applied to the x-plates.

The only change necessary to convert the circuit of Fig. 19 to a divider is to interchange the connections to the x and y deflecting plates of the cathode-ray tube. This dividing method has been checked experimentally and found satisfactory over a range of 10 to 1. The limitation on the range is the stray magnetic fields present in the unit employed.

This dividing method was conceived and tested after the completion of the experimental work of this thesis and has not therefore been fully investigated or applied to the solution of any differential equations. The method appears worthy of further investigation.

Gate Generation and Calibration

4.12 The Gate Generator

In the discussion of the adding and integrating units reference was made to the use of gate pulses for the purposes of d-c clamping and the insertion of initial conditions. As outlined in section III the basic system of the present differential analyzer solves differential equations periodically at a repetition frequency of 60 c.p.s. Two important switching functions must be performed, involving: (1) turning the various units of the differential analyzer off and on at the proper times and (2) removing the final conditions at the end of each solution period; this requires zero clamping or d-c restoring and inserting the proper initial values at the beginning of each new solution.

All these functions are performed by 60 c.p.s. square waves which are formed by the master gate generator and supplied to the various units by coaxial cables. A circuit diagram of the master gate generator unit is given in Fig. 62.

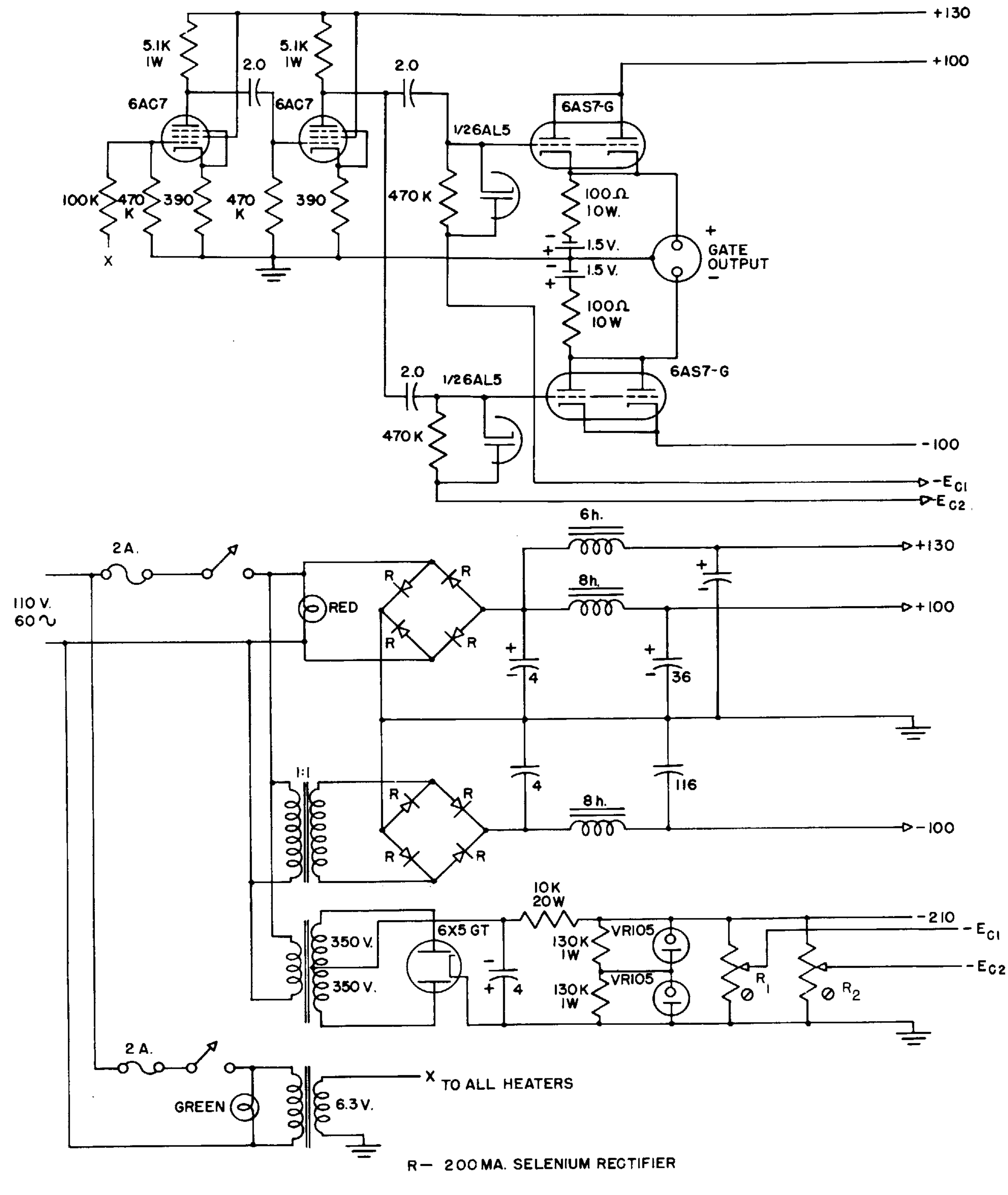

FIG. 62 MASTER GATE GENERATOR

A 60 c.p.s. sine wave, obtained from the power mains, is converted to a square wave by the two 6AC7 limiter stages. The last 6AC7 drives the grids of the two 6AS7 power output tubes. These output tubes operate as a phase inverter driving the two 100 ohm resistors.

The use of a very low resistance load assures a low output impedance for the unit. This is of importance since one of the functions of this unit is to provide gating pulses for the d-c clamping circuit of Fig. 47. In this circuit effective clamping action is achieved only if the internal impedance of the gate generator is at least as low as the impedances of the 6AL5's when conducting.

The two half 6AL5's of Fig. 62 are d-c restorers. By adjusting the setting of potentiometers R_1 and R_2 it is possible to control the level to which they restore and correspondingly the d-c level of the gate signals. This is required because the first clamper in the adding and integrating amplifiers of Figs. 48 and 57 is d-c coupled. It is desirable to have the amplitude of these gate pulses as great as possible since this amplitude limits the maximum output level of the adding and integrating units. If the signals in these units exceed the peak value of the gating pulses, the d-c clamping tubes conduct during the on-time and thus limit the output. For the unit of Fig. 62 the output pulses have a height of about 25 volts.

The gating pulses are used not only for d-c clamping and initial condition setting, but also to blank out the intensity of the viewing scope during the off-time of the differential analyzer.

4.13 Time and Amplitude Calibrating Circuits

The gating pulses can also be used to synchronize the time calibrating circuit. Since the two quantities utilized in this differential analyzer are time and voltage, a calibrating unit must be capable of measuring both of these quantities. The time calibrating circuit shown in Fig. 63 is a type frequently utilized in radar applications for this same purpose.[63]

The gate pulse turns on the cathode-coupled oscillator, in this circuit a 12AU7 twin triode. The feedback in the 12AU7 oscillator can be adjusted in this circuit by means of the regeneration control potentiometer R_1. In operation this control is adjusted to the point for which stable oscillation is just achieved over the time the gate pulse is applied. The frequency of this oscillation is adjustable by a tuned circuit in the grid of this tube. The output of the oscillator is supplied to a three stage triode limiter consisting of the three half-12AU7 tubes labelled V_3, V_4, and V_5. The square waves at the output of the last limiter are then differentiated by the RC circuit C_2 and R_2. One half of the resulting pips are limited by the 1N34 crystal at the grid of V_6, and the remaining positive pips are applied to cathode-follower V_6 to provide a low output impedance. The spacing of these pips is equal to the period of oscillation of the pulse oscillator V_1, V_2 and is controlled by the tuned circuit. In Fig. 63 three pip spacings, 50, 200, and 1000 μseconds, can be selected by the switch S_2.

Amplitude calibration is obtained with the very simple circuit shown in Fig. 64. A battery whose potential is measured with the d-c voltmeter M is connected to a Brown Instrument vibrator.[64] This

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63 M.I.T. Radar School Staff, op. cit., reference 36.

64 Brown Instrument Co., op. cit., reference 52.

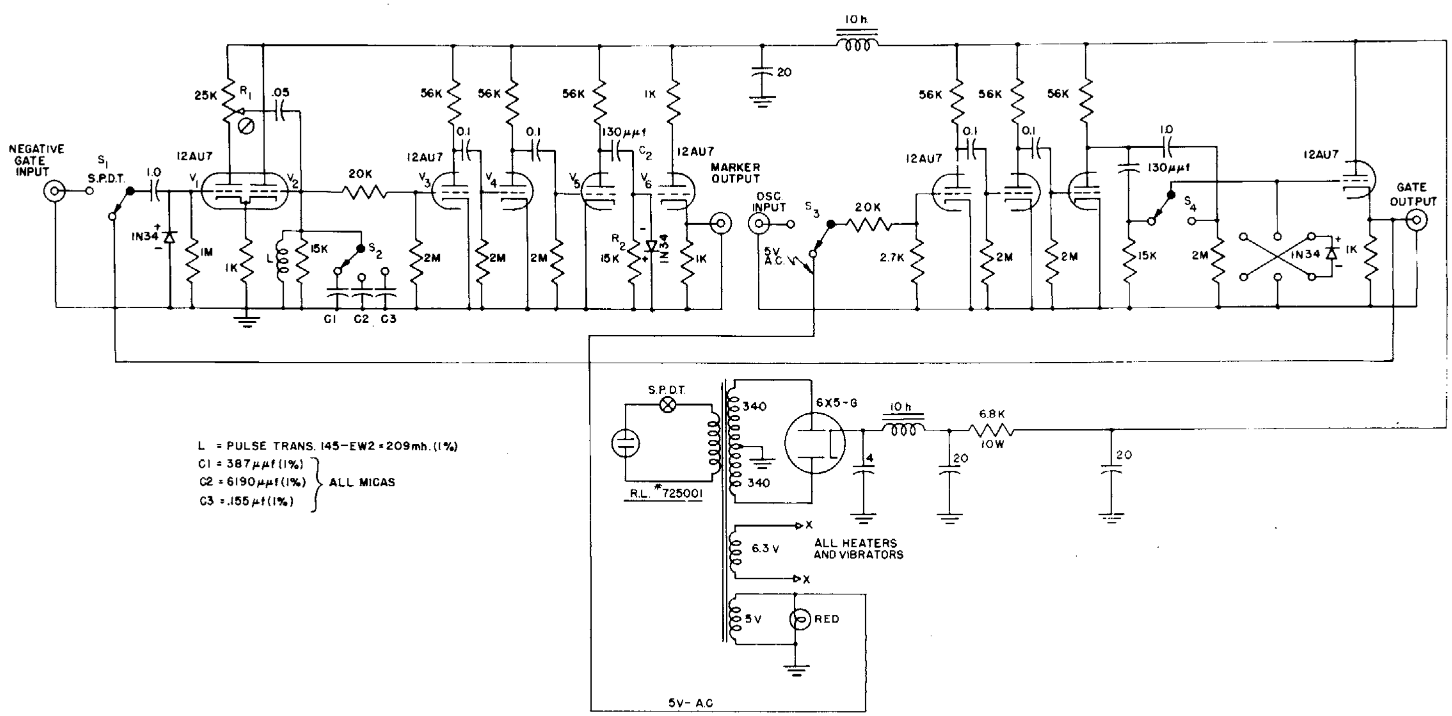

FIG. 63 TIME CALIBRATING CIRCUIT

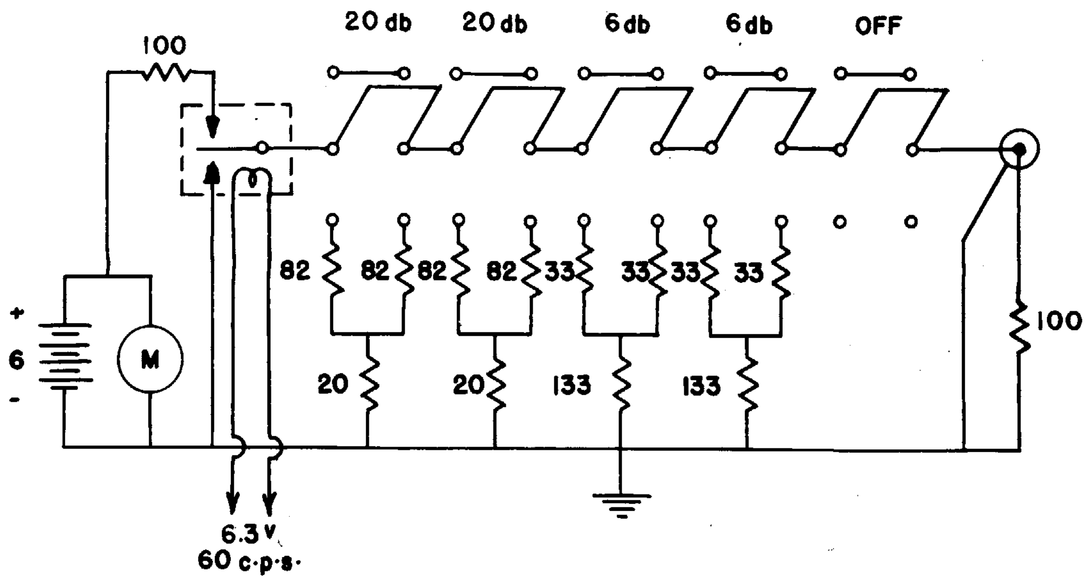

Fig. 64 Amplitude calibrating circuit.

unit 'chops' the d-c at a frequency of 60 c.p.s. in synchronism with the mains frequency. By suitable adjustment it is possible to make the vibrator perform this operation during the differential analyzer on-time. This chopped wave, which is a pulse of repetition frequency 60 c.p.s., is applied to a series of calibrated T-pad attenuators. The height of the pulse and the attenuation of the pads can all be measured with accurate d-c meters. The output of the attenuator chain is therefore a pulse, the amplitude of which is known to within 1%, which occurs during the on-time of the differential analyzer units. This calibrating pulse can be used to calibrate the deflection of the output viewing oscilloscope. The gains of the various adders, inverters, and multipliers can be measured by comparing the input and output levels on the oscilloscope, using this calibrating signal as input.

Calibration of the time-integrators is achieved by using a combination of the test pulse and the time calibrator. The test pulse is applied to the integrator and the output viewed on the oscilloscope. Time intervals are marked on the display by applying the calibrating pips to the intensity grid of the oscilloscope. In this manner the value of the integral of a known pulse amplitude after a known time can be measured, and the calibration obtained.

Photographs of the timing pips, the test pulse, and the output of an integrator with the test pulse applied are shown in Figs. 65, 66, and 67.

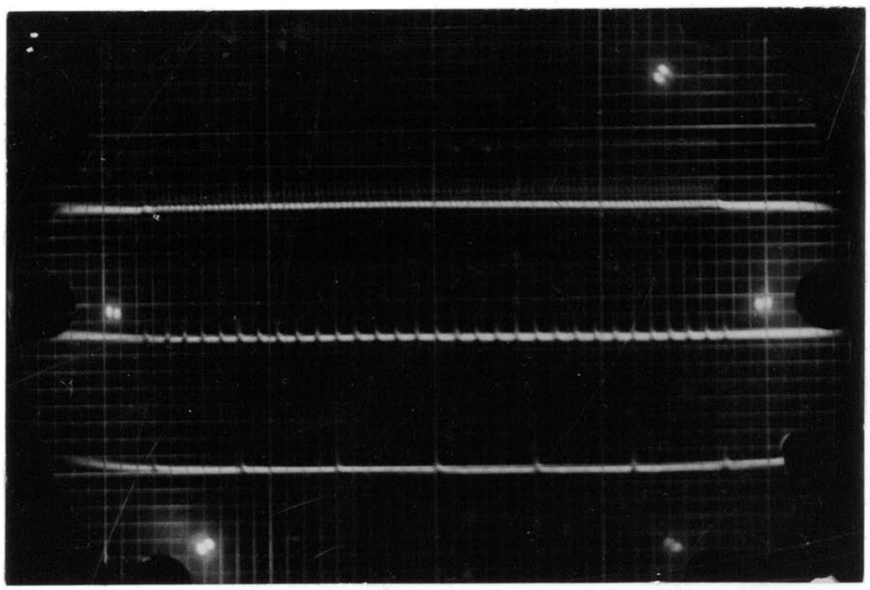

Fig. 65 Triple exposure photograph of timing pips.

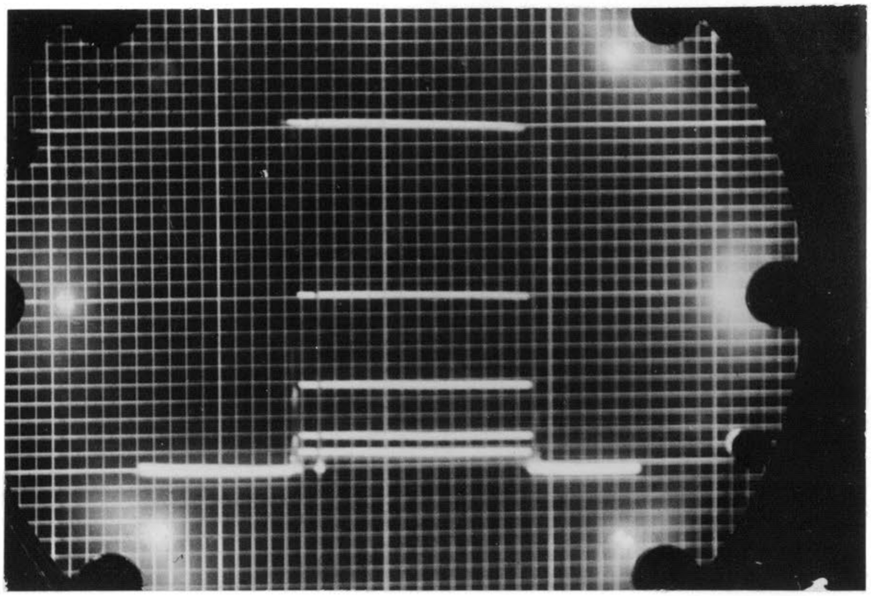

Fig. 66 Amplitude calibrating test pulse.

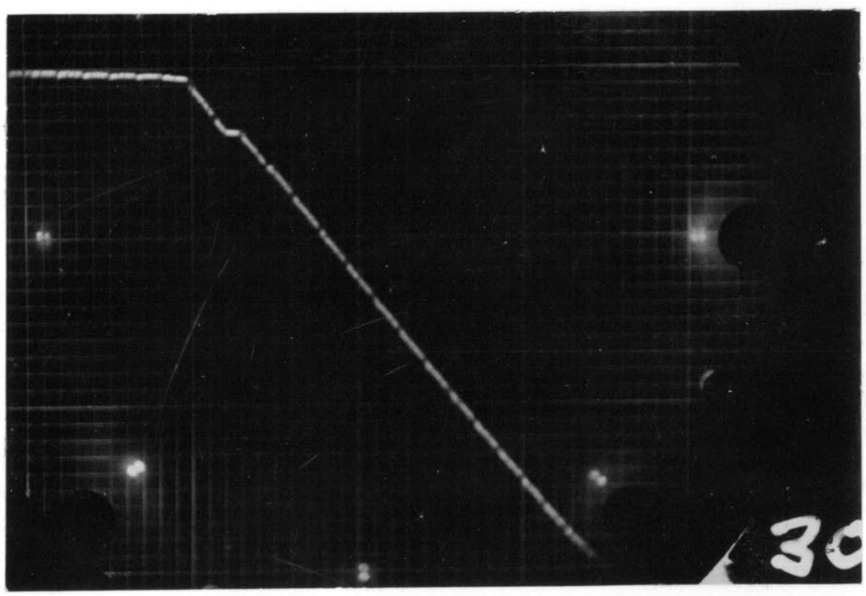

Fig. 67 Output of integrator with calibrating pulse applied to the input; 200 μsecond marker pips.

Power Supplies

All the tube filaments in the electronic differential analyzer are heated by 60 c.p.s. alternating current. One side of this filament supply is grounded to minimize hum in the amplifiers. The high voltage supplies are three in number, providing voltages of +250 volts, $+22\frac{1}{2}$ volts, and -110 volts. The $22\frac{1}{2}$ volt supply is for the screens of the 6AG5 tubes in the adding and integrating units. The current drain for these tubes is only 6ma and therefore this voltage is supplied by an ordinary heavy-duty B battery. The +250 volt and -110 volt supplies are electronically regulated. This provides a low output impedance thus preventing interaction between the various units of the differential analyzer.

Summary of Component Details

Multiplier

Maximum rate of change of output is 0.151 volts/μsec.

Maximum output voltage is ±2 volts.

Output normalized is $0.66(E_e E_m + .009E_e + .015E_m + .0005)$ volts,

where E_m is voltage controlling coil current,

E_e is voltage controlling y-deflection.

Non-linear distortion $\leq$ 2%.

Output impedance = 100 ohms.

Function Generator

Maximum output rate of change = .073 to .11 volts/μsec.

Maximum output voltage = ±2 volts.

Output impedance = 100 ohms.

Non-linear distortion $\leq$ 2.5%.

Amplifiers, Adders, and Inverters

without feedback ω_1 = 0.16 c.p.s.

ω_2 = 100 Kc/s

μ_o = -1500x, 63.5 db

as used in analyzer

ω_1 = .016 c.p.s.

ω_2 = 650 Kc/s

rise time = 0.4 μsec (10% to 90%)

overshoot = 2% or less

μ = -150x maximum

noise = 0.4 to 5 mv

maximum output = ±20 volts

output impedance = 10 ohms maximum

non-linear distortion $\leq$ 0.2%

<u>Integrators</u>

Low-frequency time constant = 0.75 sec. minimum.
(this makes error after 10,000 μsec = 1%)

High-frequency transient has 2 μsec duration.

Output impedance is 100 ohms.

SECTION V

ERRORS DUE TO COMPONENT LIMITATIONS

Precision and Calibration Accuracy

There are three principal types of error encountered in the solution of differential equations by electronic means; errors caused by

(1) lack of precision,

(2) loss of accuracy, due to

(a) lack of calibration accuracy, and

(b) limitations in the time (or frequency) domain of the differential analyzer components.

5.1 Precision

Precision is of great importance in this system of solving differential equations because of the high solution repetition rate employed. A lack of precision in setting initial conditions, for example, results in a slightly different solution for the differential equation every $\frac{1}{120}$ second. Because these solutions are superimposed on the output cathode-ray tube screen there results a 'jitter' or fuzziness of the displayed solution.

The precision of the differential analyzer limits its operation on equations whose solutions increase rapidly with the independent variable, in this case time. This is because the maximum output level of the analyzer components is fixed at about 20 volts peak to peak by the clamping circuits employed; if, for example, an equation solution increases by a factor of 2.5 during the solution time, then for a precision of 1% in the observed solution the initial value of the solution must be constant to .01 x $\frac{25}{2.5}$ = 0.1 volts. If on the other hand a solution increases by a factor of 25 times during the solution period then for the same precision of solution the initial value must be constant to within 0.01 volts.

The fact that the precision of setting the initial values limits equations which grow rapidly with time provides a convenient check. The photograph of such a test is shown in Fig. 70.

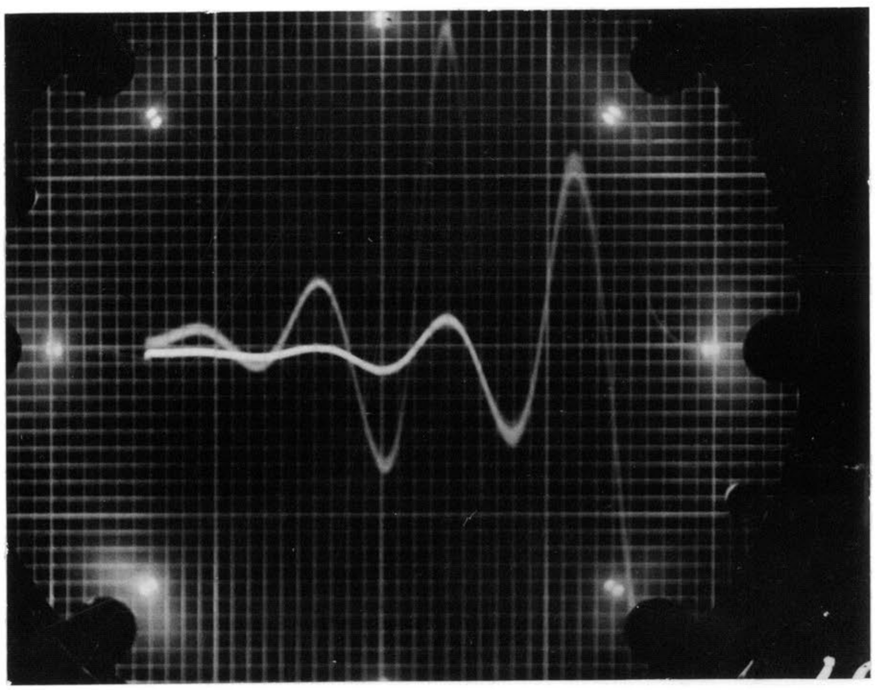

Fig. 70 Precision check of differential analyzer.

The equation being solved in this test is

$$\frac{d^2y}{dt^2} - 2\gamma\frac{dy}{dt} + \omega_0^2 y = 0 \,, \tag{75}$$

which has the solution

$$y = Ce^{\gamma t}\cos(\omega t + \theta) \,, \tag{76}$$

where

$$\omega_0 > \gamma \,, \tag{77}$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} . \tag{78}$$

Fig. 70 is a double exposure photograph of this sine wave with an exponentially increasing envelope. One exposure shows the entire solution, and the fuzziness at the end of this solution indicates that there is about 10% jitter in the initial value. The mean value of the last positive peak is 1.54 volts. The second exposure is taken with a ten-fold increase in gain of the output viewing oscilloscope. One can see that this solution increases in amplitude by a factor of 5.5 per cycle. The initial value for this solution is therefore $\frac{1.54}{390} = 3.95$ mv. Since the jitter is 10%, the variability in the initial value is about 0.4 mv. The maximum output level for this electronic differential analyzer is 20 volts; this shows that initial conditions are precise to within 0.002% of the maximum output level. For example with a more normal initial value of perhaps 1 volt the precision would be 0.04%.

It should be apparent that lack of precision can be due to causes other than hum or jitter in the initial condition pulses themselves. Other important effects are power supply hum in the output of the analyzer components and microphonics in the vacuum tubes employed. When originally built, the adding and integrating amplifiers, Figs. 48 and 57, used a 6AK5 tube as first stage. This tube was very microphonic in operation. Changing to 6AG5's reduced microphonism by a factor of about ten.

Precision of the differential analyzer is important in the solution of non-linear equations. Exploration of the regions between stable and unstable solutions, for example, requires extreme precision. An example of such a situation is the solution of the differential equation for a physical pendulum. In this case stable operation is that for which the pendulum oscillates back and forth while unstable operation corresponds to rotation of the pendulum. In the critical transition case the pendulum just balances in a position of unstable equilibrium. The degree to which one can approach this transitional case on the differential analyzer is limited by the precision of the operation.

Another situation which requires the utmost precision in the differential analyzer occurs in solving equations in which only a particular solution is desired. Such a situation is encountered, for example, in the equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0 , \qquad (79)$$

which has the general solution

$$y = C_1e^{-t} + C_2e^{5t} . \qquad (80)$$

If it is desired to observe only the first term of this solution, it is necessary to chose those initial values in the differential equation which make C_2 zero. The precision with which this adjustment can be made limits the range over which the particular solution

$$y = C_1e^{-t} \qquad (81)$$

can be observed, since the second, undesired solution will eventually mask the desired solution. The problem of isolating particular solutions can sometimes be made easier by a change of variable as discussed in Section VI, part 6.13.

The principal factor limiting precision in the present electronic differential analyzer is noise and microphonics in the outputs of the various units. The total output noise is between 0.4 and 10 millivolts, according to the unit considered. By improving the power supplies, etc., the precision could be improved by another factor of ten.

In order to obtain a good qualitative picture of the nature of the unknown solutions of a given differential equation, with regard to instability, periodicity, discontinuities, etc., the important requirement on the differential analyzer is that of precision rather than calibration accuracy. In this respect the present electronic differential analyzer is very satisfactory because of its high precision. For many engineering and physical applications this qualitative information is, alone, of the greatest value.

5.2 Calibration Accuracy

The calibration accuracy of the electronic differential analyzer is less than its precision, a desirable situation, and is limited by the calibrating system used to about 2%. In obtaining quantitative information two possibilities are open; one is to calibrate the various components, connect them to solve the equation, and observe the solution; the other is to connect the units to solve the equation, adjust the parameters by observing the solutions, and then to measure only those settings which give solutions of interest. Because of the extreme flexibility and ease of rapidly varying the equation coefficients this latter approach has been found most satisfactory for all equations investigated by the author. If the operator knows some of the physical background of the problem for which the differential equation is written, it is possible to adjust the equation parameters very quickly to give that range of solutions which is of interest. Quantitative information concerning this region can then be obtained by calibrating the units after they have been adjusted to give the desired solutions. Examples of calibration accuracy are included in Section VI.

Errors Due to Frequency or Time Limitations

The errors introduced into differential equation solutions on the electronic differential analyzer by the time or frequency limitations of its components are of the greatest importance, because, as will be shown, these limitations can cause the differential analyzer to solve an entirely wrong equation.

These limitations are most easily treated in two separate groups: (1) the short-time or high-frequency effects and (2) the long-time or low-frequency effects. For the case of ordinary differential equations with constant coefficients, analytic treatment of this problem is possible.

5.3 Errors due to Adder Finite Bandwidths

Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_o y = 0 , \qquad (82)$$

where n = order of the differential equation

a_m = the constant coefficient of the mth derivative in the equation ($m = 0, 1, \dots, n$).

The characteristic equation of this differential equation is obtained by setting $y = e^{st}$ as

$$F(s) = s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_o = 0 , \qquad (83)$$

which has roots $s_1, s_2, \dots, s_n$ in the complex s-plane. The solution of Eq. (82) is of the form

$$y = \sum_{m=1}^{n} A_m e^{s_m t} \qquad (84)$$

where the A_m's are constants depending upon the initial conditions of the particular solution desired. The electronic differential analyzer set-up for solving this equation is given in Fig. 71.

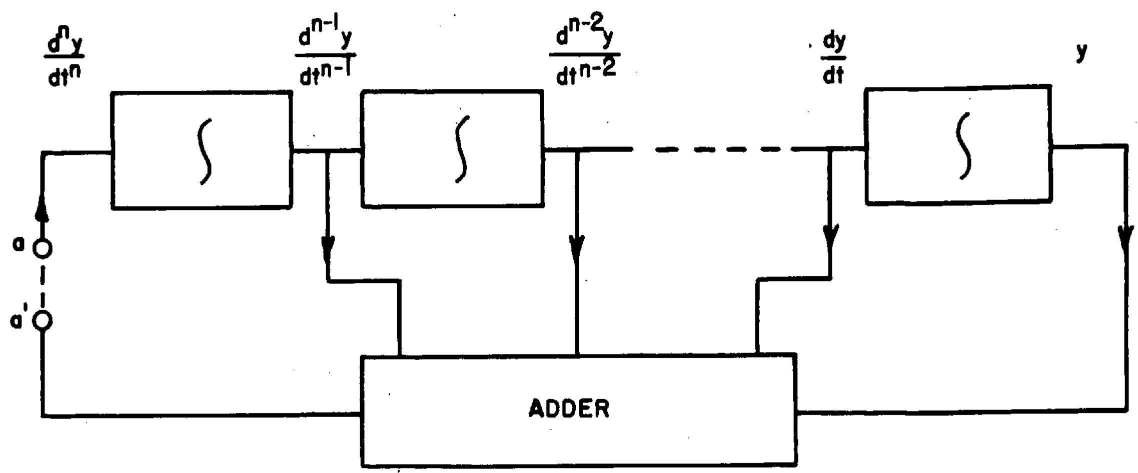

Fig. 71 Differential analyzer set-up for the solution of an arbitrary ordinary linear differential equation with constant coefficients.

5.31 Derivation of General Error Relation

There are only two types of components required by this set-up, adders and integrators. If these components are ideal, the correct solution Eq. (84) is obtained. As we know from Section IV, the components are not ideal. At high frequencies it is possible to make the integrator response considerably better than the adder response. Therefore the high frequency limitation in the set-up of Fig. 71 can be expected to come from the finite bandwidth of the adding unit. In order to investigate the effect of this bandwidth it will now be assumed that the Laplace transform of the adder characteristic is of the form

$$E_{out}(s) = \frac{c_1E_1 + c_2E_2 + \ldots + c_nE_n}{1 + \beta s} \quad . \tag{85}$$

The adder is thus assumed to have a single high frequency cut-off at a frequency $f = \frac{1}{2\pi\beta}$.

In the set-up of Fig. 71 the constants of the adder would be so adjusted as to make

$$\begin{aligned} c_1 &= a_{n-1} \, , \\ c_2 &= a_{n-2} \, , \\ &\ldots\ldots\ldots \, , \\ c_n &= a_o \quad . \end{aligned} \tag{86}$$

The characteristic equation of the differential equation actually solved by this differential analyzer set-up is therefore

$$s^n = - \frac{a_{n-1}s^{n-1} + \ldots + a_1s + a_o}{1 + \beta s} \, , \tag{87}$$

which can be written as

$$F(s) + \beta s^{n+1} = 0 \, , \tag{88}$$

where $F(s) = 0$ in Eq. (88) is the original characteristic equation of Eq. (83).

The equation actually solved therefore has n+1 roots, that is, it is one order higher than the equation one wishes to solve. If β is zero, the roots of Eq. (88) are the desired roots. When (as in practice) this is not the case, the equation solved has n+1 roots, n of which differ only slightly from the desired roots, and one new root at a very high frequency. By assuming that the changes in the values of the desired roots are very small (a necessary condition if the errors are to be small), a quantitative measure of the first order errors can be obtained.

Equation (88) has n roots $s'_1, s'_2, \ldots, s'_n$, which do not differ greatly from the roots $s_1, s_2, \ldots, s_n$, of Eq. (83). In addition there is the new root s'_{n+1}. This last root can easily be determined on the assumption of very small β, since then the second term of Eq. (88) becomes large only for very large values of s. For large values of s one can write approximately

$$F(s) \simeq s^n \, , \tag{89}$$

and therefore Eq. (88) becomes

$$s^n + \beta s^{n+1} = 0 \, , \tag{90}$$

which has the solution

$$s = -\frac{1}{\beta} \, . \tag{91}$$

This is the new root s'_{n+1}.

If the other new roots s'_m differ only slightly from the desired roots, s_m, one can write

$$\begin{aligned} s'_m &= s_m + e_m \, , \quad (m = 1, 2, \ldots, n) \\ &= s_m\left(1 + \frac{e_m}{s_m}\right) \, . \end{aligned} \tag{92}$$

Substituting Eq. (92) in Eq. (88) one obtains

$$F(s'_m) + \beta s'^{\,n+1}_m = 0 \, . \tag{93}$$

Now if

$$\frac{e_m}{s_m} \ll 1 \, , \tag{94}$$

one has

$$\left[s_m\left(1 + \frac{e_m}{s_m}\right)\right]^n \approx s_m^n\left(1 + \frac{ne_m}{s_m}\right); \tag{95}$$

also

$$F(s_m) = 0 \tag{96}$$

by definition.

Making use of Eqs. (95) and (96) in Eq. (93) one obtains

$$n e_m s_m^{n-1} + a_{n-1}(n-1) e_m s_m^{n-2} + \dots + a_1 e_m = -\beta s_m^{n+1} . \qquad (97)$$

This equation can be solved for the difference e_m between the desired roots and the roots actually obtained;

$$e_m = \frac{-\beta s_m^2}{n\,1 + a_{n-1}(\frac{n-1}{n}) s_m^{-1} + \dots + a_1(\frac{1}{n}) s_m^{-n+1}} . \qquad (98)$$

Equation (98) enables one to determine the errors caused by the finite bandwidth of the adder in Fig. 71 if that bandwidth, the coefficients of the differential equation, and the roots of the characteristic equation of the differential equation are known.

An experimental check on errors due to the finite bandwidth can always be made by changing the scale factor of the equation being solved as discussed in Section VI, part 6.1. This will change the values of the equation coefficients a_n and the characteristic roots s_m, but the value of β will remain constant; therefore the errors e_m will change value. If no change in the character of the solution is observable when the scale-factor is changed, then the errors e_m are negligible.

5.32 Examples

A rather good 'feel' for the situation can be obtained by applying this analysis to a few simple differential equations whose solutions are easily calculated. Consider the equation

$$\frac{d^2y}{dt^2} - a_o y = 0 \qquad (99)$$

The roots of the characteristic equation of this differential equation are

$$s_1 = +\sqrt{a_o} \text{ , and}$$
$$s_2 = -\sqrt{a_o} . \qquad (100)$$

When solving this equation on a differential analyzer whose adder bandwidth is $\frac{1}{2\pi\beta}$, the roots of the equation solved are from Eqs. (91), (92), and (98),

$$s_1' = +\sqrt{a_0} - \frac{\beta a_0}{2} , \qquad (101)$$

$$s_2' = -\sqrt{a_0} - \frac{\beta a_0}{2} , \text{ and} \qquad (102)$$

$$s_3' = -\frac{1}{\beta} \quad . \qquad (103)$$

If the difference between s_1, s_2 and s_1', s_2' is to be kept to 1%, for example, then one has for this equation the requirement

$$\frac{\beta \sqrt{a_0}}{2} = .01 , \qquad (104)$$

which sets the maximum value of β, or the minimum adder bandwidth once the a_0 of the equation to be solved is known. If the equality sign is taken in Eq. (104), the observed solution is of the form

$$y = C_0 e^{-50\sqrt{a_0}\ t} + C_1 e^{+.99\sqrt{a_0}\ t} + C_2 e^{-1.01\sqrt{a_0}\ t} \quad . \qquad (105)$$

The first term of this solution, which is an error, dies out very rapidly since its time constant is $\frac{1}{50}$ that of the other two terms. The second and third terms in this expression are the desired solution, with the exception that their time constants are in error by 1%. This is of slight importance, since the nature of the solution is not basically changed.

This first example was a differential equation whose characteristic equation had roots on the real axis of the complex s-plane. Its solutions therefore increase or decrease monotonically with time. As a second example consider the differential equation

$$\frac{d^2y}{dt^2} - \sqrt{2a_0}\frac{dy}{dt} + a_0 y = 0 . \qquad (106)$$

This equation has a characteristic equation whose roots are

$$s_1 = \sqrt{\frac{a_o}{2}}\,(1 + j) \ , \text{ and} \tag{107}$$

$$s_2 = \sqrt{\frac{a_o}{2}}\,(1 - j) \ , \tag{108}$$

which lie on 45 degree lines through the origin of the s-plane and in the right half plane. The solution of this equation therefore increases with time in an oscillatory manner with an exponential envelope. One form of the solution is

$$y = C_1 e^{\sqrt{\frac{a_o}{2}}\,t} \cos\left(\sqrt{\frac{a_o}{2}}\,t + \theta\right) \ . \tag{109}$$

If the adder time constant is again β, the roots of the equation the differential analyzer solves when it is set-up according to Eq. (106) are

$$s_1' = \sqrt{\frac{a_o}{2}}\left(1 - \beta\sqrt{\frac{a_o}{2}}\right)(1 + j) \ , \tag{110}$$

$$s_2' = \sqrt{\frac{a_o}{2}}\left(1 - \beta\sqrt{\frac{a_o}{2}}\right)(1 - j) \ , \tag{111}$$

$$s_3' = -\frac{1}{\beta} \ . \tag{112}$$

The observed solution is therefore

$$y = C_o e^{-\frac{t}{\beta}} + C_1 e^{\sqrt{\frac{a_o}{2}}\left(1-\beta\sqrt{\frac{a_o}{2}}\right)t} \cos\left(\sqrt{\frac{a_o}{2}}\left(1-\beta\sqrt{\frac{a_o}{2}}\right)t + \theta\right) \ . \tag{113}$$

If the adder bandwidth is made wide enough that

$$\beta\sqrt{\frac{a_o}{2}} \leq .01 \ , \tag{114}$$

then again the first term in Eq. (113), which is an error, damps out very rapidly and leaves a solution which differs from the desired solution of Eq. (109) only by the rather minor detail that the period of oscillation of the solution and the time constant of the exponential rise are in error by 1%.

A third simple case of considerable interest is that of a differential equation whose solution is an undamped sine wave. Such an equation is

$$\frac{d^2y}{dt^2} + a_o y = 0 \ , \tag{115}$$

which has the characteristic roots

$$s_1 = + j\sqrt{a_o}, \tag{116}$$

$$s_2 = - j\sqrt{a_o}; \tag{117}$$

these roots lie on the imaginary axis of the complex s-plane. For an adder of bandwidth $\frac{1}{2\pi\beta}$, the roots of the equation actually solved are, from Eq. (98),

$$s_1' = \frac{\beta a_o}{2} + j\sqrt{a_o} \ , \tag{118}$$

$$s_2' = \frac{\beta a_o}{2} - j\sqrt{a_o} \ , \tag{119}$$

$$s_3' = -\frac{1}{\beta} \ . \tag{120}$$

Therefore the equation solved has the solution

$$y = C_o e^{-\frac{t}{\beta}} + C_1 e^{+\frac{\beta a_o}{2}t} \cos(\sqrt{a_o}\, t + \theta) \tag{121}$$

instead of the desired solution, which is

$$y = C_1 \cos(\sqrt{a_o}\, t + \theta) \ . \tag{122}$$

For this case one sees that although making β small causes the first term to damp out rapidly and thus eliminates error from this source, there still is an error in the solution because the amplitude of the sine wave, instead of remaining constant, increases exponentially with time. The error from this effect is greatest at the end of the solution period T_o. If this error is to be held to 1%, then one can write as a necessary condition the inequality

$$\left(\frac{\beta a_o}{2}\right)T_o \leq .01 \; . \tag{123}$$

This can be put in a more useful form by writing

$f_o = \frac{\sqrt{a_o}}{2\pi}$ = frequency of oscillation of the solution in c.p.s.,

$\Delta f = \frac{1}{2\pi\beta}$ = bandwidth of adder unit in c.p.s.,

which converts Eq. (123) to

$$\Delta f = 100\pi f_o^2 T_o \; . \tag{124}$$

For the present differential analyzer the solution period T_o is $\frac{1}{120}$ second. If one wishes to solve Eq. (115) for f_o = 600 c.p.s. (for this frequency five complete cycles of the solution are displayed on the cathode-ray tube screen), the adder bandwidth must be

$$\Delta f = 942 \text{ Kc/s} \; , \tag{125}$$

if a one percent amplitude increase can be tolerated. This case of the undamped sine wave imposes the most stringent conditions on the necessary component bandwidths for the differential analyzer of any equation investigated. The corresponding condition, for example, for the solution of Eq. (106) to one percent error is found from Eq. (114) to be,

$$\Delta f = 60 \text{ Kc/s} \; , \tag{126}$$

for the same natural frequency of the solution.

For this reason the solution of Eq. (115) on the electronic differential analyzer affords a good test of one aspect of the analyzer operation. It is interesting to note that this is the same equation used to test mechanical differential analyzers in the well-known "circle-test."[65] Experimental verification of the validity of this

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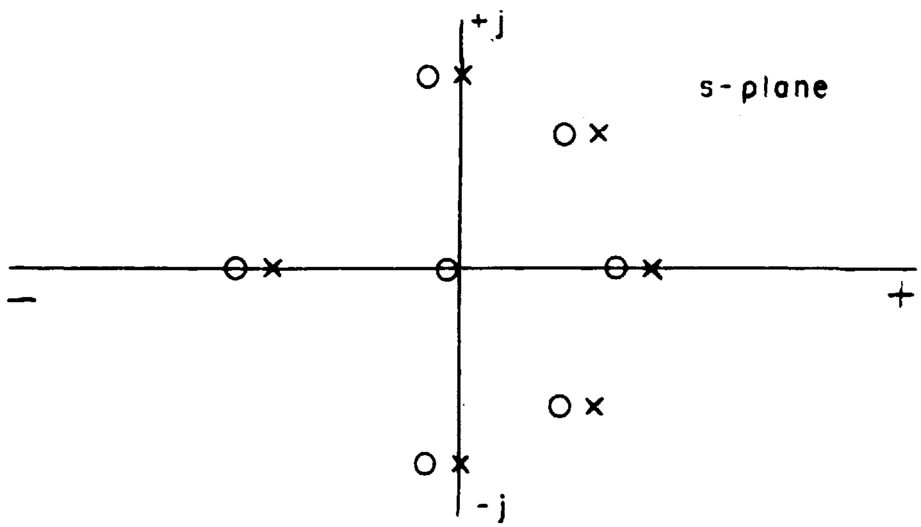
65 Bush, op. cit., reference 4, p. 469.

error analysis is included in Section VI, where observed solutions for Eq. (115) are given.

The results of the three examples considered here are displayed qualitatively in Fig. 72. This is a plot of the complex s-plane. The locations of the three original sets of roots are marked by crosses on the plane, while the new root positions resulting from the use of an adder with a finite bandwidth are marked by circles. The distance between the crosses and circles has been exaggerated so as to be readily apparent.



This is Fig. 73 -GBM

x- location of original equation roots
o- location of new roots due to finite adder bandwidth

Fig. 72 Plot of s-plane roots for Eqs. (99), (106), and (115) and the corresponding differential equations solved by a differential analyzer of finite bandwidth.

5.4 Errors Due to Integrator Low-frequency Limitation

The effect of low frequency limitations on the differential analyzer solution of differential equations can be handled analytically in a manner quite analogous to that employed for the high frequency case. As indicated by Eq. (61) each integrator has a low frequency time constant $\mu_o RC$ due to the finite gain of the feedback amplifier employed. For the a-c coupled units there is also a second time constant due to the coupling circuit. For simplicity of analysis, however, it is assumed that each integrator has a single low frequency time constant and that this time constant is the same for all integrators employed. To be precise, it is assumed that the Laplace transform of the transfer function of each integrator is

$$\frac{E_{out}}{E_{in}}(s) = \frac{1}{s} \cdot \frac{Ts}{1 + Ts} \quad . \tag{127}$$

5.41 Derivation of General Error Relation

If integrators having this characteristic are used to solve Eq. (82), according to the set-up of Fig. 71, the characteristic equation of the differential equation actually solved is

$$s^n + a_{n-1}s^{n-1}\left(\frac{Ts}{1+Ts}\right) + \ldots + a_1 s\left(\frac{Ts}{1+Ts}\right)^{n-1} + a_o\left(\frac{Ts}{1+Ts}\right)^n = 0 \quad . \tag{128}$$

This further assumes that the high-frequency effects can be neglected in this low-frequency analysis and that the low frequency time constant of the adder is much greater than T. Both of these assumptions are justified in normal practice with the electronic differential analyzer.

In order to calculate the first order effects of the low frequency time constants one can write

$$\frac{Ts}{1 + Ts} = \frac{1}{1 + \frac{1}{Ts}} \approx 1 - \frac{1}{Ts} \quad , \tag{129}$$

and similarly

$$\left(\frac{Ts}{1 + Ts}\right)^m \approx 1 - \frac{m}{Ts} \quad . \tag{130}$$

These approximations assume that $Ts \gg 1$, a condition which has to be met if the errors are to be small and which permits an evaluation of the first order errors. Making use of Eq. (130), Eq. (128) becomes

$$s^n + a_{n-1}s^{n-1}(1 - \frac{1}{Ts}) + \ldots + a_1 s(1 - \frac{n-1}{Ts}) + a_o(1 - \frac{n}{Ts}) = 0, \quad (131)$$

which is the characteristic equation of the differential analyzer set-up of Fig. 71. This can be rewritten in the form

$$F(s) = \frac{a_{n-1}s^{n-1} + 2a_{n-2}s^{n-2} + \ldots + (n-1)a_1 s + na_o}{Ts}, \quad (132)$$

where F(s) is the characteristic equation of Eq. (82) as given in Eq. (83).

For the case of perfect low frequency response, $Ts = \infty$, and Eq. (132) becomes

$$F(s) = 0, \quad (133)$$

which has the roots $s_1, s_2, \ldots, s_n$. When Ts is large but finite, Eq. (132) has n+1 roots, n of them differ only slightly from the roots of Eq. (133) and there is one new root for s very small (very low frequencies). This new root can be identified immediately. For s very small, Eq. (132) becomes approximately

$$a_o \approx \frac{na_o}{Ts}, \quad (134)$$

therefore

$$s'_{n+1} = \frac{n}{T}. \quad (135)$$

If the other n roots of Eq. (133) differ only slightly from the roots of Eq. (132), one can write

$$s'_m = s_m + e_m \qquad (m = 1,2,\ldots,n), \quad (136)$$

and since $e_m \ll s_m$,

$$(s'_m)^n = s_m^n(1 + \frac{ne_m}{s_m}). \quad (137)$$

Now substituting Eq. (137) in Eq. (132) and remembering that $F(s_m)$ is zero by definition, one can write

$$n e_m s_m^{n-1} + a_{n-1}(n-1) e_m s_m^{n-2} + \ldots + a_1 e_m = \tag{138}$$

$$\frac{a_{n-1}\left(1 + \frac{(n-1)e_m}{s_m}\right) s_m^{n-1} + 2a_{n-2}\left(1 + \frac{(n-2)e_m}{s_m}\right) s_m^{n-2} + \ldots + (n-1)a_1\left(1 + \frac{e_m}{s_m}\right) s_m + na_o}{Ts_m\left(1 + \frac{e_m}{s_m}\right)} .$$

If one now assumes

$$Ts_m \gg n - 1 , \tag{139}$$

then Eq. (138) can be solved for the difference e_m between the new and old roots,

$$e_m = \frac{a_{n-1}s_m^{n-1} + 2a_{n-2}s_m^{n-2} + \ldots + (n-1)a_1 s_m + na_o}{Ts_m(ns_m^{n-1} + (n-1)a_{n-1}s_m^{n-2} + \ldots + a_1)} . \tag{140}$$

This expression is completely analogous to Eq. (98) for the high frequency error, and it is used in the same manner to determine how the low-frequency time constants of the differential analyzer integrators perturb the differential equation solutions.

5.42 Examples

This result will be applied to the three particular equations considered in the high frequency analysis. For the differential equation (99) the new roots are

$$s_1' = +\sqrt{a_o} - \frac{1}{T} , \tag{141}$$

$$s_2' = -\sqrt{a_o} - \frac{1}{T} , \tag{142}$$

$$s_3' = +\frac{2}{T} . \tag{143}$$

The differential analyzer solution is therefore

$$y = C_o e^{+\frac{2t}{T}} + C_1 e^{+(\sqrt{a_o} - \frac{1}{T})t} + C_2 e^{-(\sqrt{a_o} + \frac{1}{T})t} . \tag{144}$$

The second and third terms in this solution differ from the desired solution only in their time constants. If this difference is to be kept to 1%, one should have

$$\frac{1}{T} \leqq .01\sqrt{a_o} \quad . \tag{145}$$

The first term in Eq. (144) is an error term which is most noticeable at the end of the solution period. If this term is essentially constant over the solution period, the error is not bothersome. This condition can be assured by requiring, for example, that the term be constant to within 1% over the solution period T_o. This condition is met provided

$$T \geqq 200\ T_o \quad . \tag{146}$$

For a solution period of $\frac{1}{120}$ second this indicates that the integrator time constant should be not less than 1.66 seconds.

Applying Eq. (140) to Eq. (106) for the x sine wave with exponentially growing amplitude, one finds the solution to be

$$y = C_o e^{+\frac{2t}{T}} + C_1 e^{+\left(\sqrt{\frac{a_o}{2}} - \frac{1}{T}\right)t} \cos\left(\sqrt{\frac{a_o}{2}}\ t + \theta\right) \quad . \tag{147}$$

For 1% error the conditions on T are again

$$T \geqq 100\sqrt{\frac{2}{a_o}} \quad , \tag{148}$$

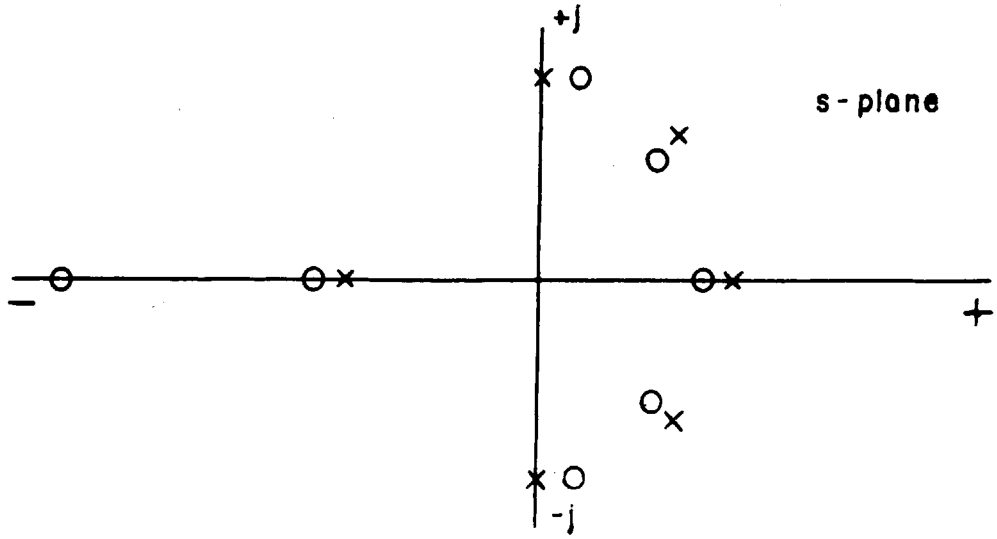
$$T \geqq 200\ T_o \quad . \tag{149}$$

Finally for the undamped sine wave of Eq. (115) one finds

$$y = C_o e^{+\frac{2t}{T}} + C_1 e^{-\frac{t}{T}} \cos(\sqrt{a_o}\ t + \theta) \quad , \tag{150}$$

and the condition for 1% error is again Eq. (149) or (146).

An s-plane picture of the root perturbation for these low frequency examples is given in Fig. 73.



This is Fig. 17

x- location of original equation roots
o- location of new roots due to low frequency time constant in integrators

Fig. 73 Plot of s-plane roots of Eqs. (99), (106), and (115) and the roots of the corresponding equations solved by a differential analyzer utilizing integrators with finite gains.

Summing up the above one sees that for ordinary linear differential equations with constant coefficients Eqs. (98) and (140) can be used to determine the errors caused by high- and low-frequency limitations of the differential analyzer components. These results are verified experimentally in Section VI. Experimental determination of the overall accuracy is also given there.

The situation for the equations with variable coefficients and the non-linear equations is not as clear. One can linearize, or hold the coefficients of such equations constant over limited ranges of the solutions and get some idea of the errors to be expected, but this does not permit the drawing of any general conclusions. There is little reason to suppose that the errors at low frequencies (for long times) differ appreciably from those observed for the linear equations. This is particularly true since we note in every example considered that the most stringent requirement is that given by Eq. (149), which does not involve equation parameters at all, but merely the ratio of integrator time constant to solution period.

Although the high frequency effects cannot easily be analyzed for these more difficult types of equations, one can experimentally check for high frequency errors by changing scale factors in the differential analyzer and observing whether this causes any change in the character of the solutions observed. If it does not, one is reasonably justified in assuming that frequency limitations are not causing difficulty.

SECTION VI

RESULTS

General Set-up Procedure

The general procedure used in setting up a differential equation on the differential analyzer can best be clarified by considering a particular example and going through the various steps required.

6.1 Typical Example

First equation (115)

$$\frac{d^2y}{dt^2} + a_o y = 0 \ , \qquad (115)$$

will be considered for the particular case of

$$a_o = 4\pi^2 \ , \qquad (151)$$

$$y(0) = y_o = 1 \ ,$$

$$\left.\frac{dy}{dt}\right|_{t=0} = \dot{y}_o = 0 \ ,$$

$$\text{for } 0 < t \leqq 4 \ .$$

Since this is a second order equation, two integrators are required. One starts out by assuming a voltage proportional to $\ddot{y}$ and integrating twice as shown in Fig. 74.

Since the constant multipliers to be used in these integrators have not yet been chosen, they are indicated by k_1 and k_2. The output of this cascade of two integrators is, as indicated, $+ k_1 k_2 y$. From the differential equation, however, one sees that $\ddot{y}$ is equal to $-a_o y$, thus a change of algebraic sign is required. An amplifying unit accomplishes this as shown in Fig. 75.

The output of the amplifying unit is $-k_1 k_2 k_3 y$; and by connecting the terminals a-a' this is made equal to the assumed $\ddot{y}$. The differen-

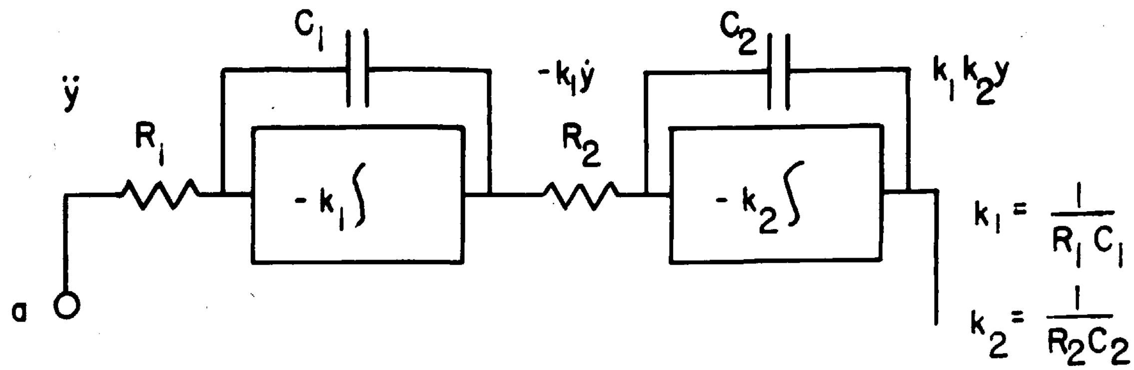


Fig. 74 First step in block diagram of differential analyzer set-up for Eq. (115).

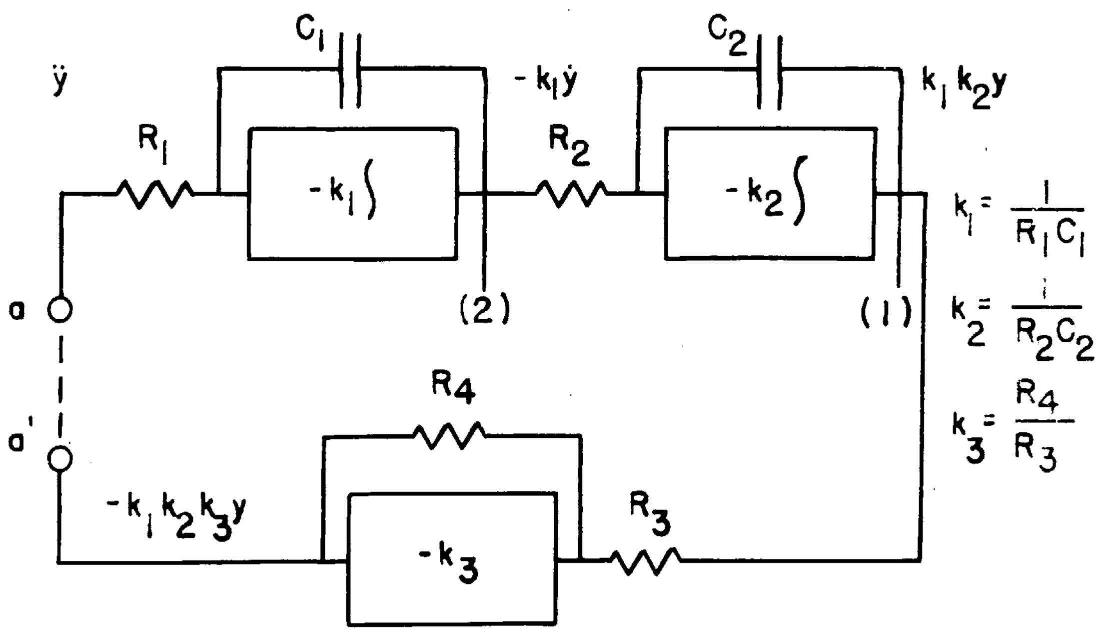


Fig. 75 Complete differential analyzer loop for Eq. (115).

tial equation solved by this set-up is

$$\frac{d^2y}{dt^2} + k_1k_2k_3y = 0 \ . \tag{152}$$

If $k_1k_2k_3$ is made equal to a_o, then Eq. (115) is solved. Because the differential analyzer solution period is only $\frac{1}{120}$ second, however, such an adjustment only gives the first $\frac{1}{120}$ second of the solution of Eq. (115), whereas the first four seconds of the solution are desired. This difficulty is overcome by making the change of variable

$$t' = \frac{t}{480} \ , \tag{153}$$

so that when the equation-time t = 4 seconds the differential analyzer-time $t' = \frac{1}{120}$ second. For this change of variable one has

$$\frac{dy}{dt} = \frac{dy}{dt'} \cdot \frac{1}{480} \ , \text{ and} \tag{154}$$

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dt'^2} \cdot \frac{1}{(480)^2} . \tag{155}$$

Therefore the transformed equation is

$$\frac{d^2y}{dt'^2} + (480)^2 a_o y = 0 \ , \tag{156}$$

where now

$$y_o = 1 \ , \tag{157}$$

$$\dot{y}_o = 0 \ , \tag{158}$$

$$0 < t' < \frac{1}{120} \ . \tag{159}$$

It was shown in Section IV, Eq. (69), that the maximum integrator coefficient is limited by the long time behavior of the unit to a maximum of about 4000 sec^{-1}. One might choose for this example to make the coefficients of both integrators $k_1 = k_2 = 1000 \ sec^{-1}$. If this is done, then on comparing Eqs. (151) and (156) one sees that k_3 must be adjusted so that

$$k_3 = \frac{(480)^2 a_o}{k_1 k_2} = 9.12 \; ; \qquad (160)$$

This is certainly a very modest gain requirement. The two integrators and the inverter are calibrated as described in Section IV and then connected according to the diagram of Fig. 75. Since the gating and initial condition circuits are built-in, no additional connections for these features are necessary. One merely sets the initial value knob on the output of the first integrator to zero to satisfy Eq. (158) and the initial value at the output of the second integrator to some arbitrary unit value for Eq. (157). A voltage of 10 volts might be chosen to correspond to $y = 1$.

If the differential analyzer is now turned on and a viewing scope is connected at (1) in Fig. 75, one observes y as a function of t, for values of t between zero and four. Connecting the oscilloscope at (2) will give a display of $\frac{dy}{dt} \cdot 480$ versus t, for the same range of time.

There is no need, as a matter of fact, to have the differential analyzer turned off while it is being set-up. All of the necessary connections can be made with the units running and the entire procedure outlined here can be accomplished in about 30 seconds. Fig. 76 is a photograph of a single relay rack panel on which two integrators and one inverter-adder have been built. In this panel no connections are made to the units. Fig. 77(a) shows the same panel connected to solve Eq. (156). The necessary interconnections are made with coaxial cables so that complete electrostatic shielding is preserved. Fig. 77(b) is a photograph of the same set-up together with the oscilloscope used

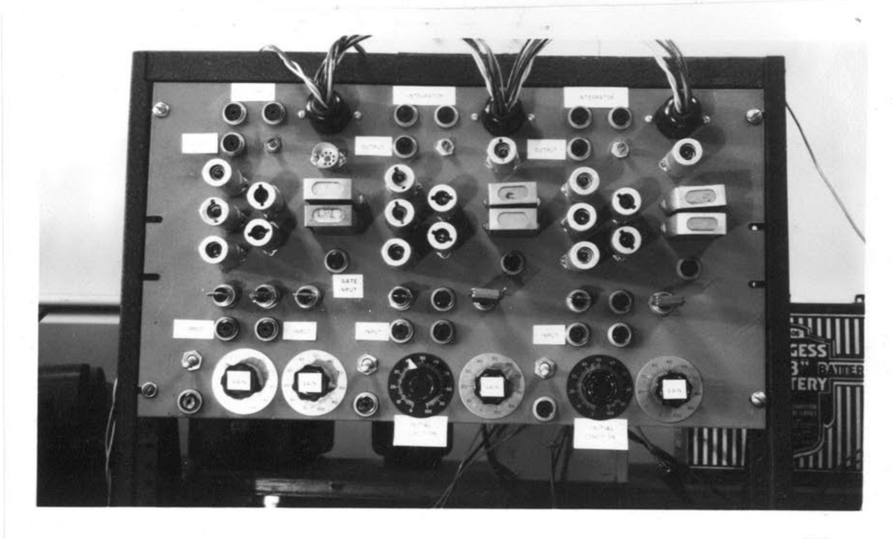


Fig. 76 Typical dielectronic differential analyzer panel showing two integrators and one adder-inverter.

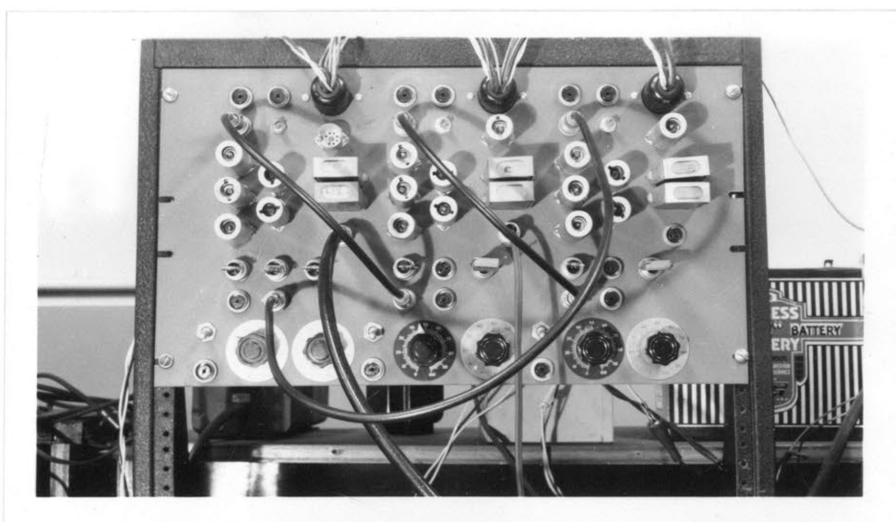


Fig. 77(a) Differential analyzer panel connected to solve the differential equation $\ddot{y} + 4\pi^2 y = 0$.

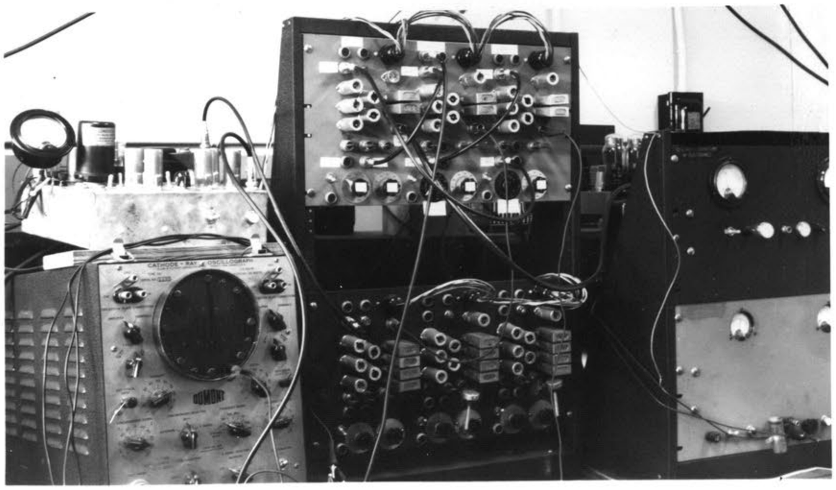


Fig. 77(b) Differential analyzer panel and viewing oscilloscope solving the equation $\ddot{y} + 4\pi^2 y = 0$ for $0 < t < 4$.

for viewing the differential analyzer solutions. The solution can be seen on the oscilloscope face.

6.2 Observed Solutions

6.21 Normal Solution Displays

Fig. 78 shows a photograph of the viewing oscilloscope. By means of a double exposure photograph y and $-480\dot{y}$ are recorded on the same photograph, between exposures the oscilloscope connection was moved from point (1) to point (2) in Fig. 75. More than four complete cycles of the solution are observed because the solution time is somewhat longer than $\frac{1}{120}$ second.

If the initial value of the first derivative given by Eqs. (151) had not been zero, then the initial value of the first integrator would have been set to 480 times the required initial value to take account of the time scale change.

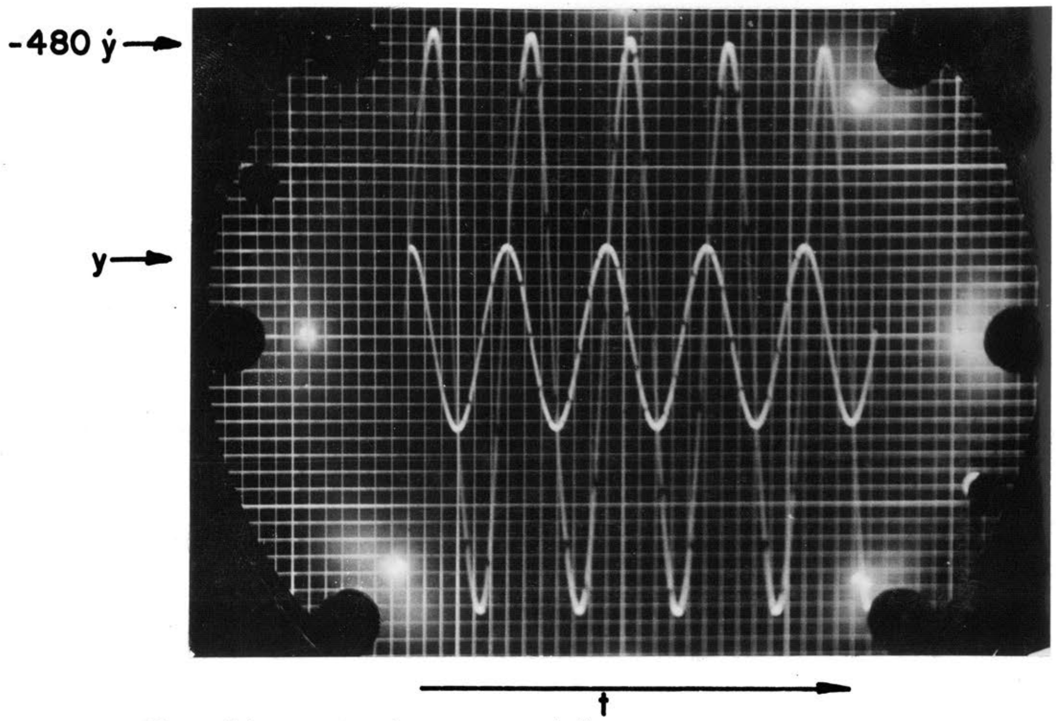


Fig. 78 Plot of y and $-480\dot{y}$ versus t for the differential equation $\ddot{y} + 4\pi^2 y = 0$.

The time scale in Fig. 78 is marked by the 200 μsecond markers applied to the intensity grid of the cathode-ray tube. These cause the reduction in intensity at points along the solution trace.

6.22 Effect of Poor Integrator Low-frequency Response

As was indicated in Section V, equation (115) is very useful for checking the frequency limitations of components such as the inverter and integrators. Poor low-frequency response in the integrators can, for example, easily be introduced by reducing the gain of the integrating amplifier. This was done experimentally by connecting a resistor across the integrating condenser of both integrators in the set-up of Fig. 75 equal to 100R; as indicated in Eq. (66) this has the effect of reducing the internal gain of these units to 100 or about $\frac{1}{15}$ the normal value. Fig. 79 shows the observed solution to Eq. (115) for this case. This verifies the error analysis result of Eq. (150).

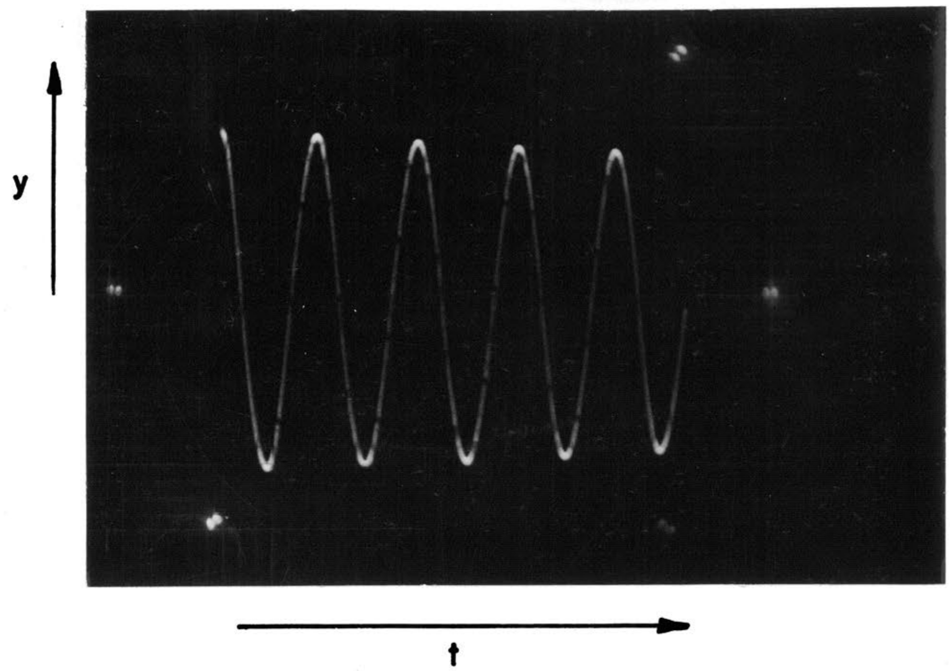


Fig. 79 Solution of $\ddot{y} + 4\pi^2 y = 0$ on electronic differential analyzer showing effect of poor integrator low-frequency response.

6.23 Effect of Inadequate Adder Bandwidth

The high frequency effects can be observed experimentally by reducing the adder bandwidth. This can be done, for example, by connecting a small shunt capacity across R_4 in Fig. 75. When this is done the resulting solution is as shown in Fig. 80.

This result verifies the high-frequency error analysis by conforming to Eq. (121) of Section V. The first term in that equation damps out too rapidly to be observable on the time scale of Fig. 80. Fig. 81 is an expanded photograph of the first 2000 μseconds of Fig. 80. At the very beginning of the solution a very small high frequency transient is observable.

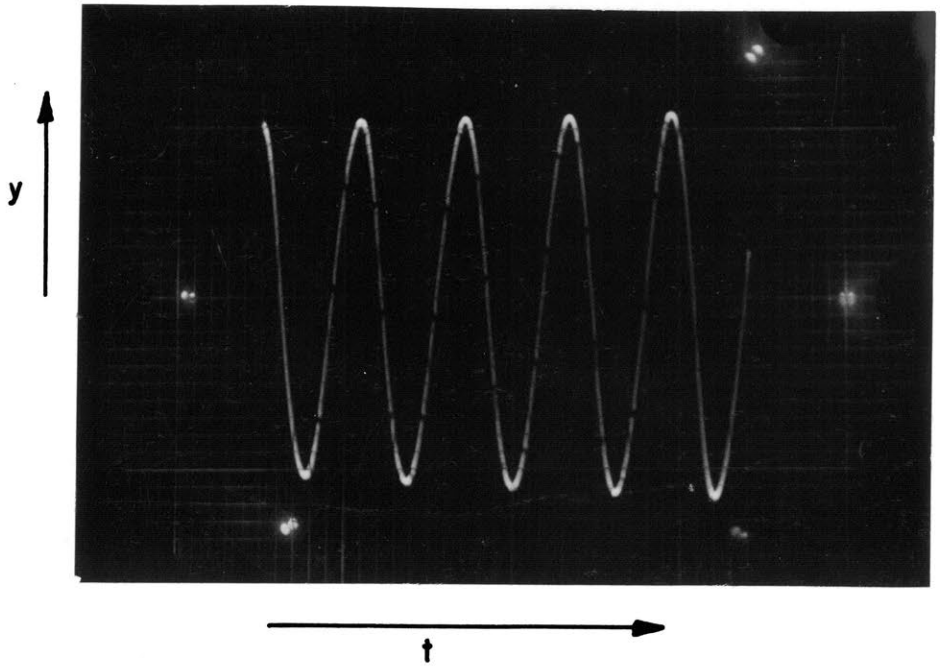


Fig. 80 Solution of $\ddot{y} + 4\pi^2 y = 0$ on the electronic differential analyzer showing the effect of inadequate high-frequency response.

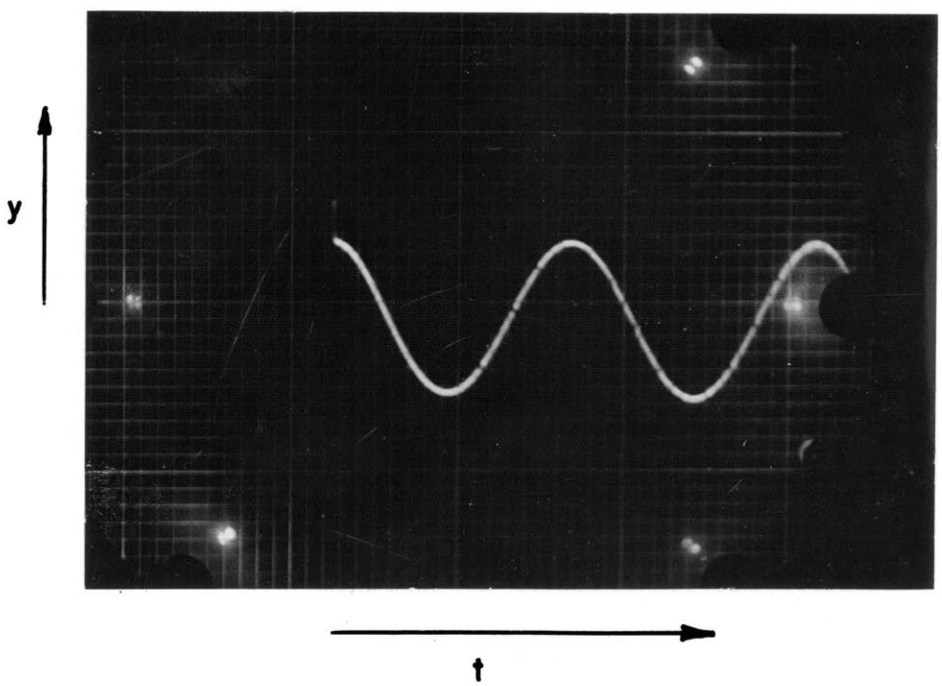


Fig. 81 Expanded photograph of first 2000 μseconds of Fig. 80.

6.24 Circle Test

It is sometimes desirable to obtain differential equation solutions plotted not only against the independent variable but also versus one or more of the other dependent variables. This can be done in the electronic differential analyzer very easily by connecting the two dependent variables to the vertical and horizontal deflecting systems of the viewing oscilloscope. If this is done for Eq. (115), the well-known differential analyzer "circle-test" is obtained.[66] Fig. 82 shows the results of such a connection; six revolutions around the circle are shown here. The slight flattening of the circle on the left side is due to non-linear distortion in the viewing oscilloscope amplifiers.

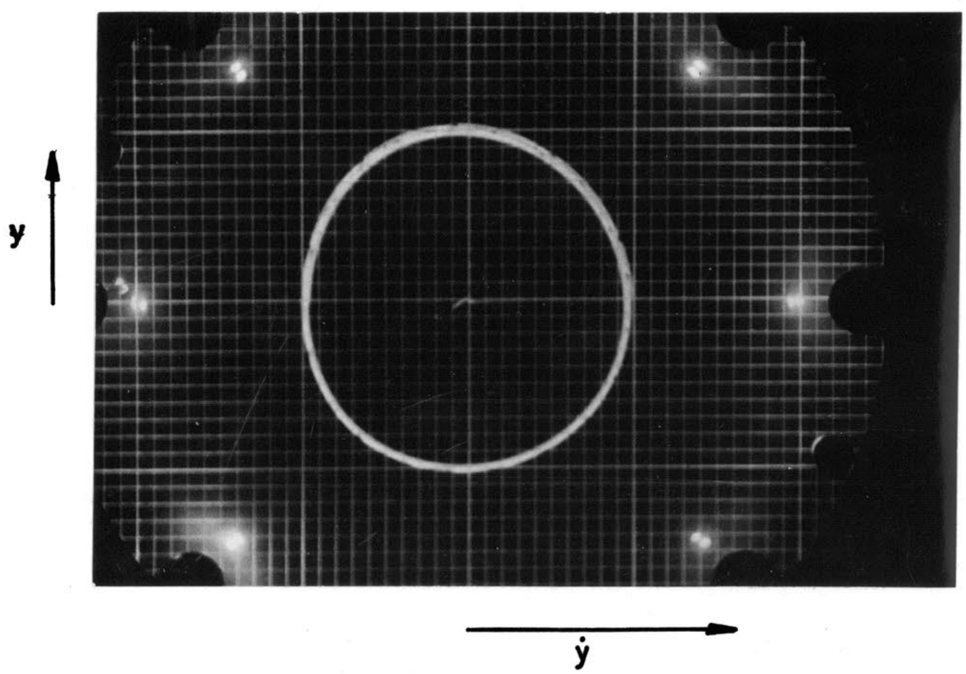


Fig. 82 Circle test of electronic differential analyzer.

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[66] Bush, op. cit., reference 4, p. 469.

6.3 Modification of the Differential Analyzer Set-up for the Solution of Some Other Second Order Differential Equations with Constant Coefficients

Some of the flexibility of the electronic differential analyzer is illustrated by considering the change in the set-up of Fig. 75 necessary to solve

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_oy = 0 \ , \tag{161}$$

for the case

$$a_1 = -0.2\sqrt{2a_o} \tag{162}$$

and the same conditions given by Eqs. (151).

Making the change of variable given by Eq. (153) one obtains a new transformed equation

$$\frac{d^2y}{dt'^2} - (480)0.2\sqrt{2a_o}\,\frac{dy}{dt'} + (480)^2a_oy = 0. \tag{163}$$

for the new range

$$0 < t' < \frac{1}{120} \ ,$$

and initial values

$$y_o = 1 \ , \tag{164}$$

$$\dot{y}_o = 0 \ . \tag{165}$$

Changing the set-up of Fig. 75 to solve this new equation requires only a single new connection. The resulting set-up is shown in Fig. 83.

The new constant k_4 gives an adjustment on the coefficient of the first derivative independent of the other coefficients in the differential equation. For Eq. (163) one sees that it should be

$$k_4 = \frac{96\sqrt{2a_o}}{k_1} = .853 \ . \tag{166}$$

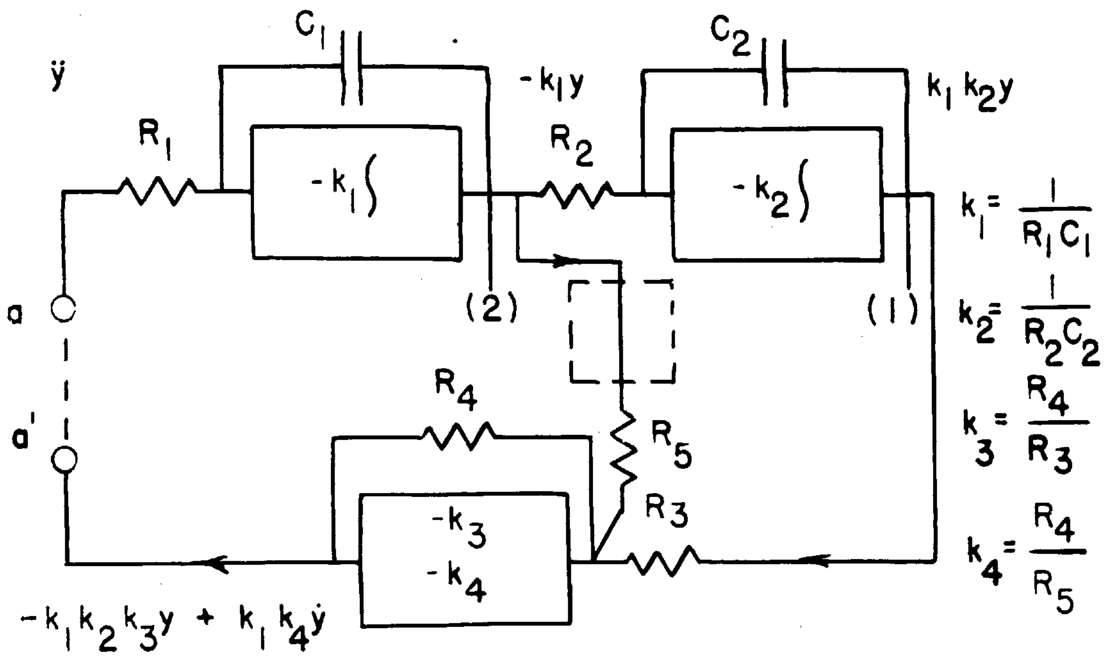

Fig. 83 Block diagram of differential analyzer set-up for Eq. (163).

The resulting differential analyzer solution is shown in Fig. 84. Since this solution increases rather rapidly with time, a gain setting on the viewing oscilloscope which shows the last half cycle of the solution reveals little about the behavior of the solution around $t = 0$; Fig. 84 therefore gives a double exposure photograph of the same solution with a change in gain of ten to one between exposures. The higher gain setting permits study of the short time behavior of the solution while the lower gain gives a better overall picture of the entire solution.

In the block diagrams of Figs. 75 and 83 the coefficients of the adding and indicating units are given in terms of their external feedback elements alone. As indicated in Section IV the units actually used in the electronic differential analyzer all have input potentiometers which

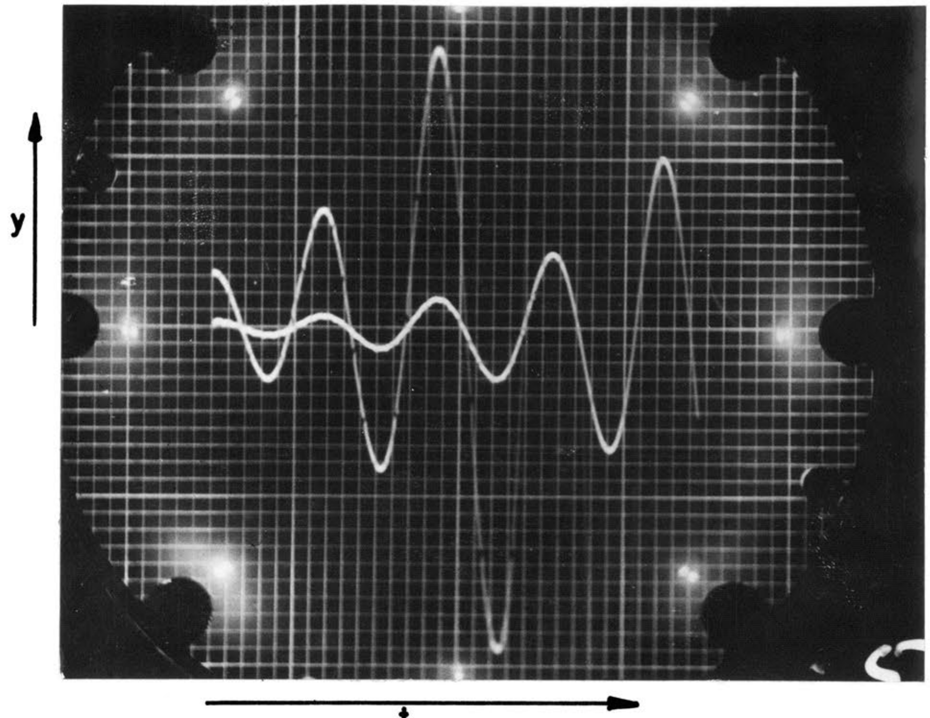

Fig. 84 Plot of y versus t for Eq. (163), $y_o = 1$, $\dot{y}_o = 0$. This is a double exposure record with a gain change of 10 to 1 between exposures.

permit a 10 to 1 change in scale factor without changing the feedback elements. Normally the coefficient adjustments are made with these potentiometers using the amplitude and time calibrators as indicated in Section IV.

Figure 84 will be recognized as the solution for the behavior of a parallel LC circuit shunted by a negative resistance. Since the differential equation considered is linear, it can impose no limit to the amplitude build-up. This is, for example, the operation encountered in a super-regenerative detector operating in the "linear mode." The case of a positive shunting resistance requires that the sign of the derivative term in Eqs. (161) and (163) be changed from negative to positive. This change can be easily accomplished on the differential analyzer by inserting an amplifier having a gain of minus one (-1) in the position indicated by a dotted box in Fig. 83. Solutions of Eq. (161) for the cases of positive, zero, and negative damping are shown in Fig. 85 in a triple exposure recording.

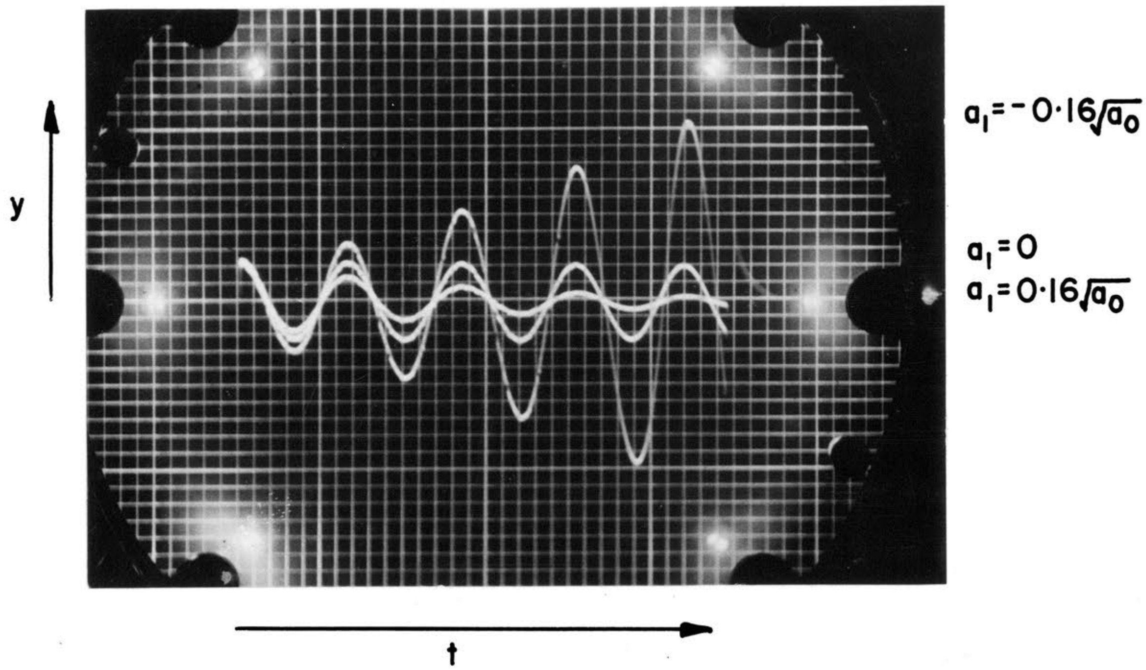

Fig. 85 Solutions, y versus t, for $\ddot{y} + a_1\dot{y} + a_0y = 0$.

Solution of Non-linear Differential Equations

6.4 Solution of van der Pol Equation

As has been indicated, Eq. (161) with a_1 negative describes the behavior of an oscillatory system having a negative damping term. This is a situation encountered in any physical self-excited oscillator for small amplitudes of oscillation. This equation therefore describes the manner in which oscillation begins to build up in a vacuum tube oscillator for example. Since this equation is linear, it provides no information concerning the ultimate amplitude to which the oscillation builds up or the steady state waveform of the oscillation. In order that these most important characteristics be studied, it is necessary to take account of the non-linearities which are inevitable in any physical oscillator.

The well-known van der Pol equation,[67]

$$\frac{d^2y}{dt^2} - (A-3By^2)\frac{dy}{dt} + y = 0 , \qquad (167)$$

is a non-linear equation pertaining to many types of oscillations, which has been very extensively studied. One sees that this equation describes a system for which the damping is negative for small amplitudes of oscillation and becomes positive for large amplitudes of oscillation.

6.41 Differential Analyzer Set-up

In order to solve Eq. (167) as it stands one would write it in the form

$$\frac{d^2y}{dt^2} = -y + A\frac{dy}{dt} - 3By^2\frac{dy}{dt} , \qquad (168)$$

from which the block diagram set-up of Fig. 86 can be determined.

This set-up is seen to require both a function generator to generate y^2 from y and a multiplier to form the product, $y^2\frac{dy}{dt}$. From a practical point of view it is desirable to keep the number of multipliers and function generators required by the differential analyzer set-up to a minimum, since as shown in Section IV these are the most complicated units of this electronic differential analyzer. It is worthwhile therefore to consider whether Eq. (167) can be simplified for machine solution by a change of variable. Such a change of

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[67] Minorsky, N., Introduction to Non-Linear Mechanics, J. W. Edwards, Ann Arbor, 1947.

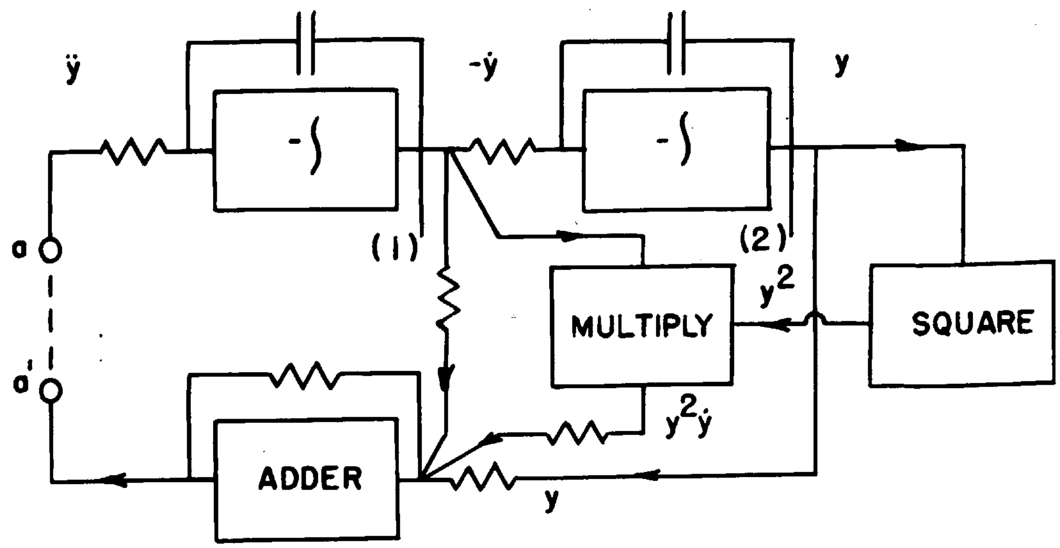

Fig. 86 Block diagram of differential analyzer set-up for the solution of van der Pol equation (167).

variable can in fact be made, namely

$$x = \int y dt \ . \tag{169}$$

Applying this to Eq. (167) one obtains

$$\frac{d^3x}{dt^3} - A\frac{d^2x}{dt^2} + 3B\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} + \frac{dx}{dt} = 0, \tag{170}$$

which can be integrated term by term with respect to time to give

$$\frac{d^2x}{dt^2} - \left(A - B\left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} + x = 0 \ . \tag{171}$$

This equation, discovered by Lord Rayleigh in connection with acoustic phenomena, is known as Rayleigh's equation.[68] On writing this equation in the form

$$\frac{d^2x}{dt^2} = -x + A\frac{dx}{dt} - B\left(\frac{dx}{dt}\right)^3 , \tag{172}$$

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[68] Minorsky, op. cit., reference 67, p. 178.

the block diagram of the differential analyzer set-up can easily be determined, as shown in Fig. 87.

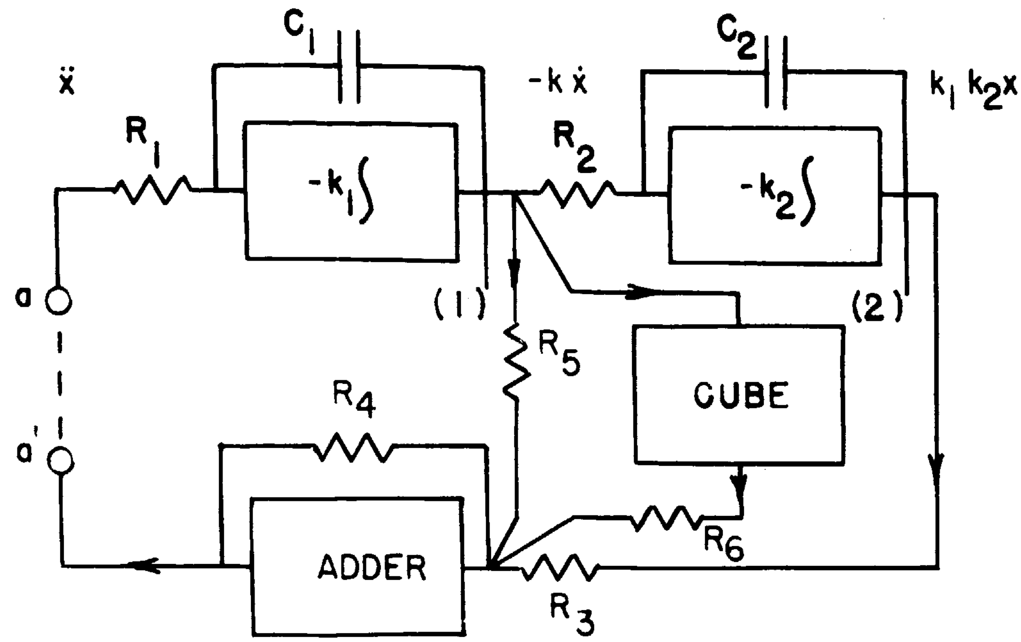

Fig. 87 Block diagram of differential analyzer set-up for the solution of Rayleigh equation (171).

The important difference between this set-up and that of Fig. 86 is the fact that now only a single function generator, to generate the cube of the first derivative, is required instead of a function generator plus a multiplier.

6.42 Typical Solutions

Rayleigh's equation has been solved on the electronic differential analyzer using the function generator described in Section IV. If the first derivative is displayed versus time, according to Eq. (169) the solution of the van der Pol equation is observed. A typical solution of this equation is photographed in Fig. 88.

The first few cycles of this solution are similar to the linear

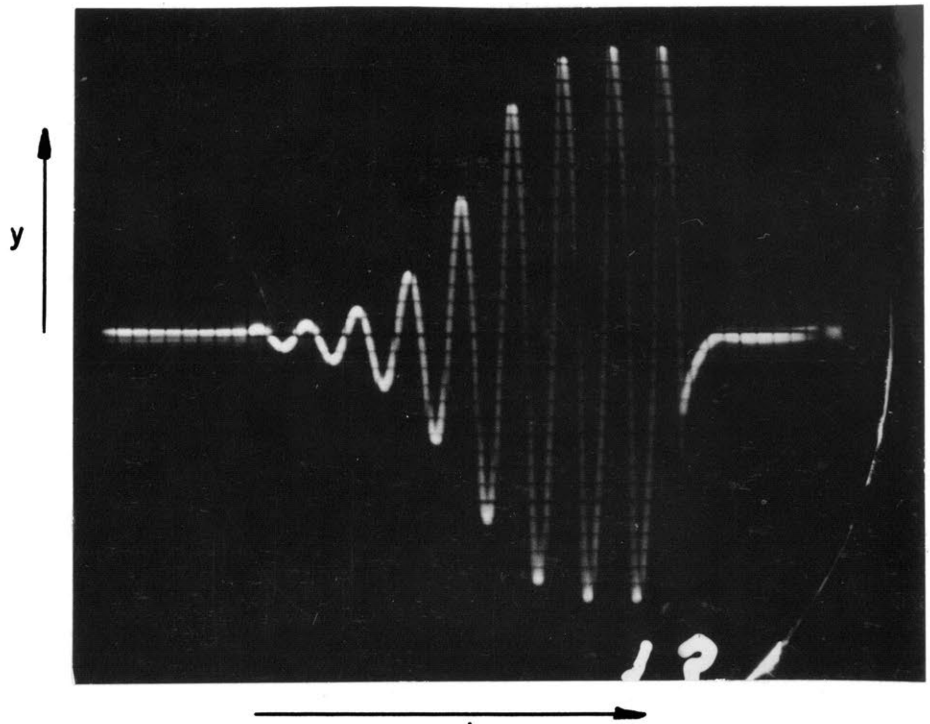

Fig. 88 Solution of van der Pol equation for high-Q case.

solution of Fig. 84, but for longer times the rate of amplitude rise soon drops, and the amplitude approaches a constant value because of the non-linear damping term. The case shown corresponds to what is normally referred to as the high-Q case in electrical engineering problems. It is the situation for which a large number of solution oscillations occur during the build up period. Mathematically this means that both A and B in Eq. (167) are small compared to unity.

Another solution display, which is of very great interest to the mathematician and engineer is the phase space plot, which is a plot of velocity versus displacement.[69] Such a plot is easily obtained on the electronic differential analyzer by connecting the vertical deflecting system of the output oscilloscope at (1) and the

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69 Minorsky, op. cit., reference 67, p. 7-124.

horizontal input at (2) in Fig. 87. A phase space plot for the Rayleigh equation is shown in Fig. 89. The build-up to the steady state limit-cycle is clearly shown by this photograph. For the engineer the shape of this limit-cycle is useful in determining the steady state waveform of the oscillator. If the oscillation were exactly sinusoidal, the limit-cycle would be exactly circular.

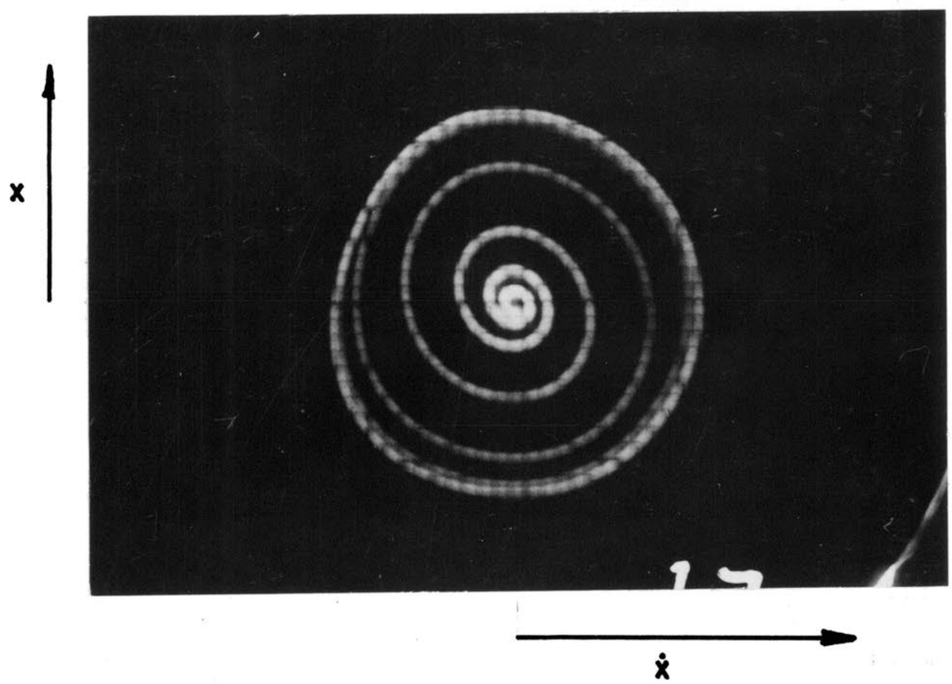

Fig. 89 Phase space plot for Rayleigh equation, high-Q case.

To obtain the low-Q solution, in which the oscillation very rapidly reaches its steady state, it is merely necessary to increase the adder gain on the inputs connected through R_5 and R_6 in Fig. 87. By decreasing the gain at R_3 at the same time the number of cycles of the solution displayed is reduced so that the details of the initial rapid build up for this case can be clearly seen. A typical low-Q solution is plotted versus time in Fig. 90. The corresponding phase space plot, with now much distorted limit-cycle, is shown in Fig. 91.

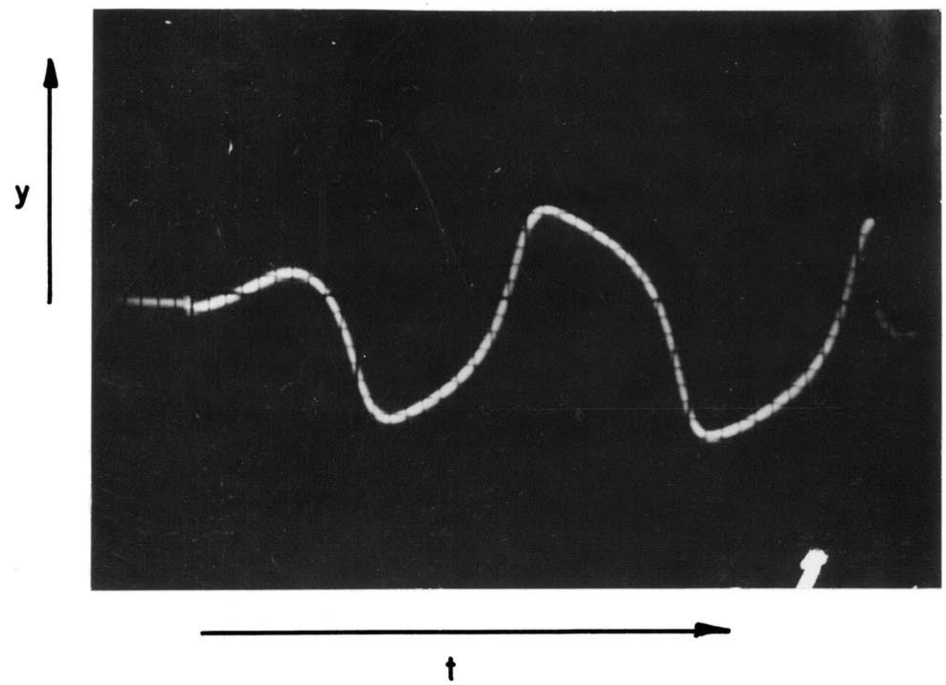

Fig. 90 Solution of van der Pol equation for low-Q case.

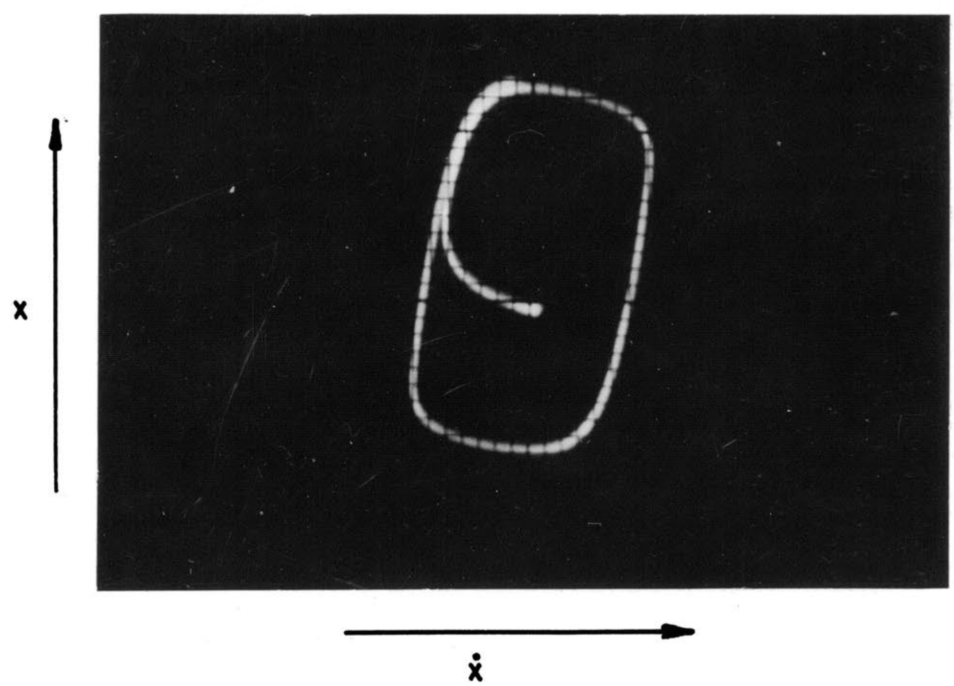

Fig. 91 Phase space plot for Rayleigh's equation, low-Q case.

It should be emphasized that the time necessary to shift between these two widely different solutions of Figs. 87 and 90 on the electronic differential analyzer is merely the time necessary to adjust two or three knobs. One can, for example, in a very short period explore the entire range of solutions existing between the two cases shown. If no record of the solutions is made, such an exploration takes the operator a matter of minutes; if photographic records are required, it is possible to obtain such recorded solutions at the rate of at least two or three per minute.

Two other photographs of the solutions of this equation are given in Figs. 92 and 93. Figure 92 is a plot of displacement versus time for three different initial values, showing the build up to the same steady state amplitude in each case. Fig. 93 is another phase space

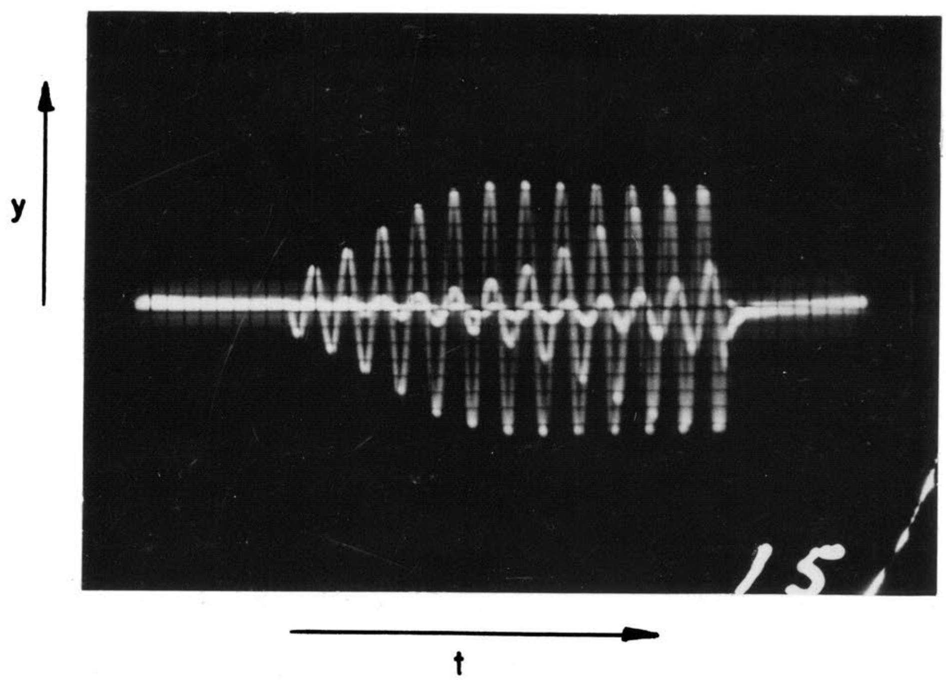

Fig. 92 Solutions of van der Pol equation for three different initial values.

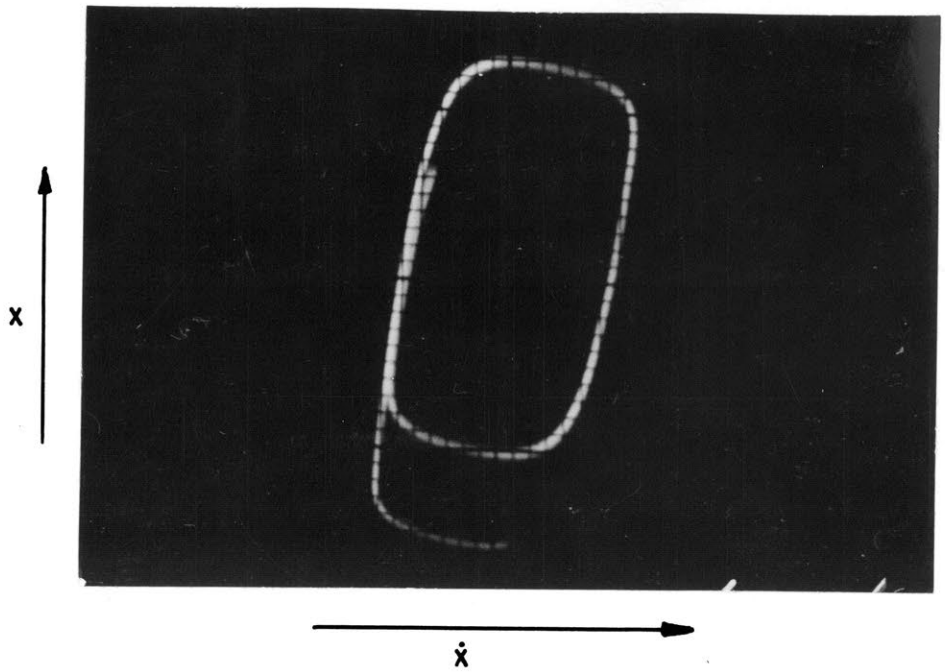

Fig. 93 Phase space plot of solution to Rayleigh equation for an initial amplitude exceeding the steady state amplitude of the stable limit cycle.

plot for the Rayleigh equation. For this case, however, the displacement was given an initial value exceeding the peak displacement of the steady state equilibrium motion. This figure therefore shows the solution dropping down to the limit cycle instead of building up to it as shown in Fig. 91.

6.5 Solution of Non-linear Force Equations

Another class of non-linear differential equations of considerable interest are the equations of the form

$$\frac{d^2y}{dt^2} = F(y). \tag{173}$$

These equations describe the one dimensional motion of a particle in a potential field

$$V(y) = -\int_0^y F(y)dy\,. \tag{174}$$

A block diagram of the electronic differential analyzer set-up necessary to solve this equation is given in Fig. 94.

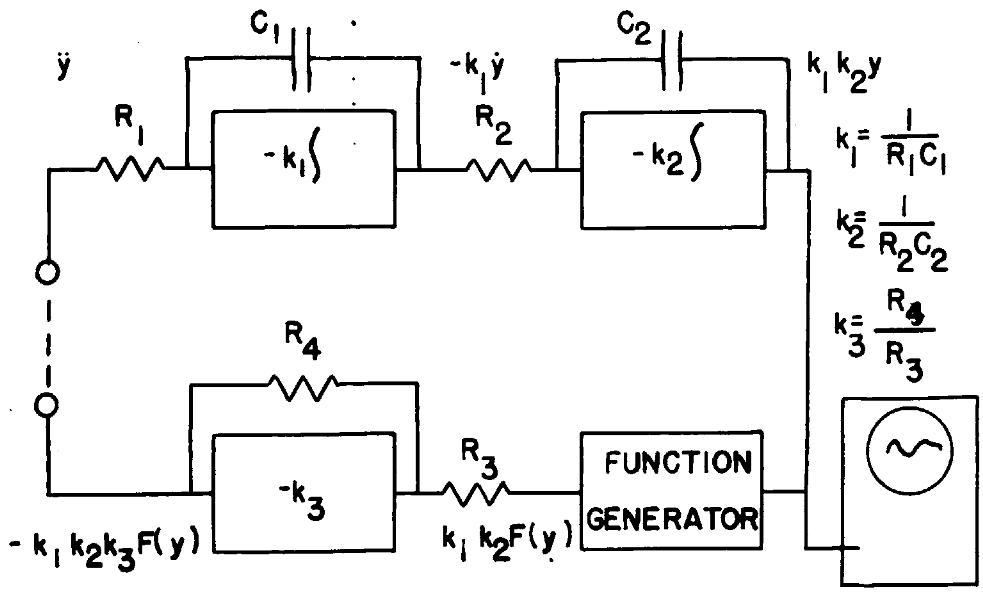

Fig. 94 Electronic differential analyzer set-up for the solution of the equation $\ddot{y} = -F(y)$.

The inverter is included after the function generator in this set-up for convenience in changing scale factors. This allows one to adjust the constant coefficients k_1, k_2, and k_3 so as always to operate the function generator over its most accurate range. This tends to minimize any error which may be introduced by the function generation unit.

6.6 Cubic Potential Case

A number of differential equations of this form have been solved. One interesting case is

$$F(y) = -(y^2 + y) \; . \tag{175}$$

From Eq. (174) one sees that this leads to a cubic potential curve

$$V(y) = \frac{y^3}{3} + \frac{y^2}{2} + \text{Const.} \tag{176}$$

A plot of this potential curve is shown in Fig. 95.

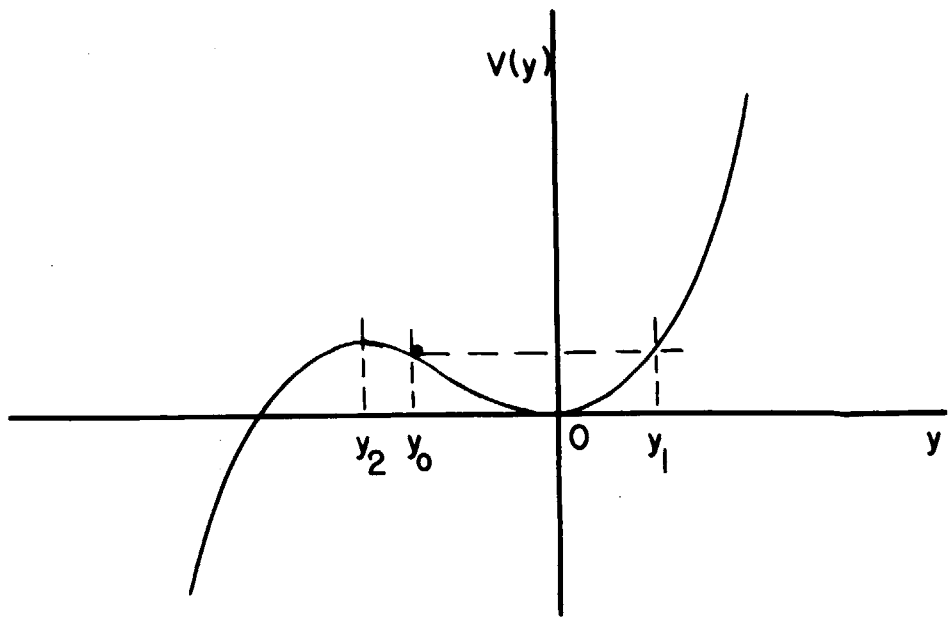

Fig. 95 Plot of cubic potential V(y) versus displacement y.

From the nature of this potential curve qualitative information concerning the differential equation solution can be obtained.[70,71] Physically, the differential equation describes the motion of a small ball released on this curve at $y = y_0$ with an arbitrary initial velocity. From the shape of the potential curve it is apparent that if the particle is released with zero initial velocity it simply oscillates stably between the limits y_0 and y_1 indicated in Fig. 95. If on the other hand the particle is given sufficient velocity to the left, it is possible for the particle to get over the potential hump at y_2, after which it rapidly continues its motion in that same direction to minus infinity. A third type of motion occurs if the particle is given

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[70] Slater, J.C., and Frank, N.H., Introduction to Theoretical Physics, Chap. IV, McGraw-Hill Book Co., New York, 1933.

[71] Minorsky, op. cit., reference 67, p. 24-39.

a sufficiently large initial velocity to the right in Fig. 95; for this case the particle traverses the potential minimum at 0 twice and then drops over the potential hump at y_2 to minus infinity.

These different modes of motion are easily observed on the electronic differential analyzer. Fig. 96 is a triple exposure photograph showing the three types of motion described qualitatively above.

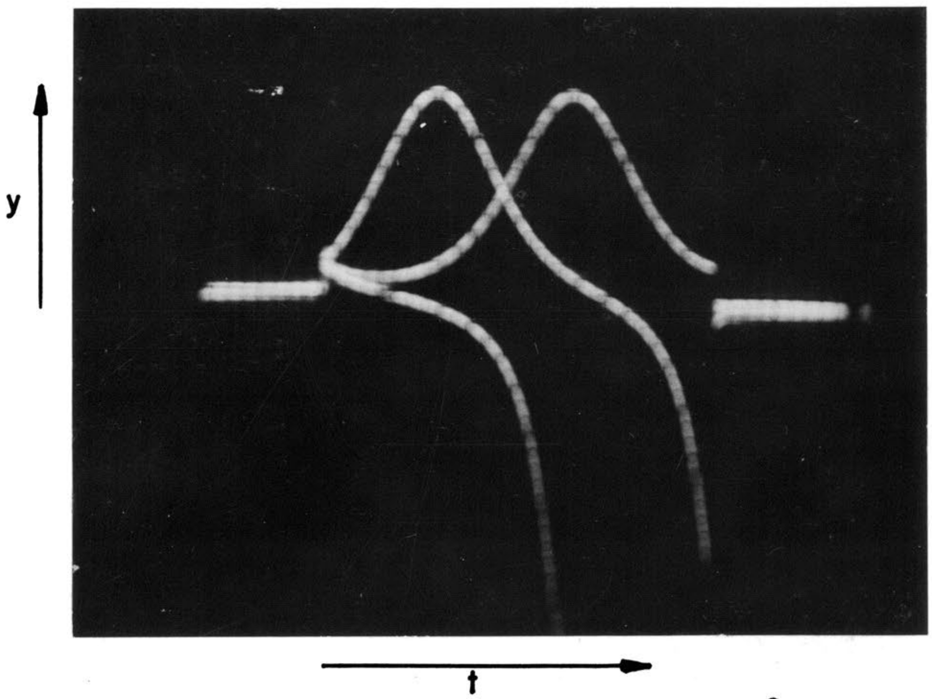

Fig. 96 Three solutions of the equation $\ddot{y} = -(y^2 + y)$ for y_0 fixed and $\dot{y}_0$ variable.

In solving this equation the general set-up of Fig. 94 was used. The function generator was made to generate the required square-function by using a parabolic function mask.

6.7 Physical Pendulum

If the F(y) of Eq. 173 is made equal to -siny, one obtains an equation which describes the motion of a physical pendulum under a gravitational restoring force as shown in Fig. 97. For this case the

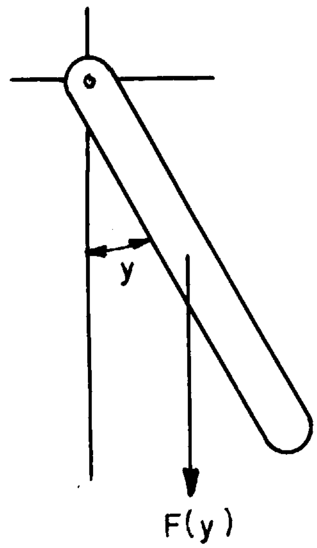

Fig. 97 Physical pendulum.

potential plot is

$$V(y) = -\cos y + \text{constant}, \tag{177}$$

as shown in Fig. 98.

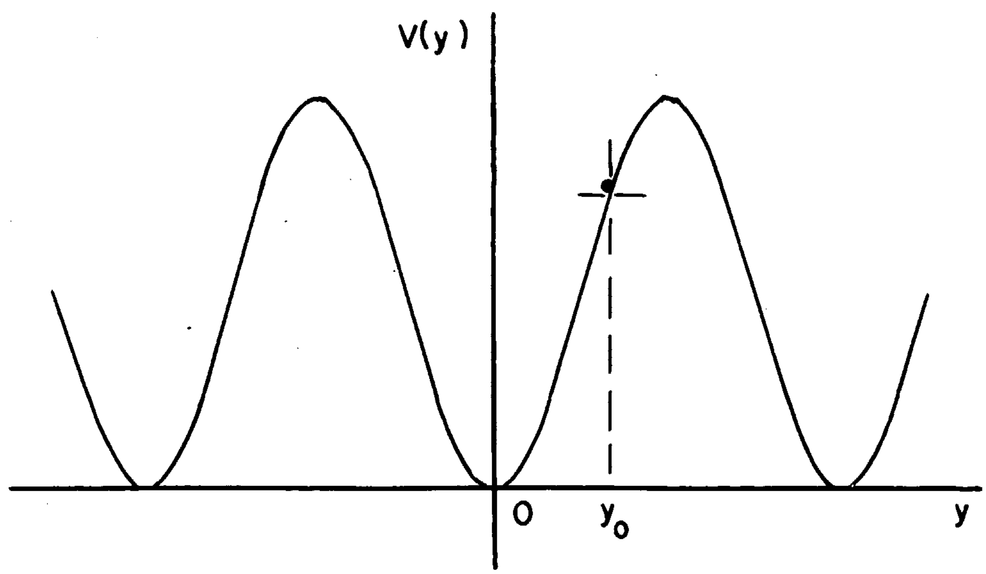

Fig. 98 Potential curve for physical pendulum.

If the pendulum is given any displacement and zero initial velocity, it oscillates about one of the potential minima of this curve. If on the other hand it is given some initial displacement, such as y_0 in Fig. 98, and sufficient velocity in either direction it is possible to obtain rotation in either direction. This is verified by the differential analyzer solutions. Fig. 99 shows three different solutions, for which the initial velocity is in every case zero, with different initial values. For small initial displacements siny is very nearly equal to y and sinusoidal motion of the pendulum results. As the initial amplitude is increased one notes that the most noticeable change is in the period of oscillation. In addition, as is noticeable to a lesser extent, the oscillation waveform is no longer sinusoidal but is flattened on the peaks.

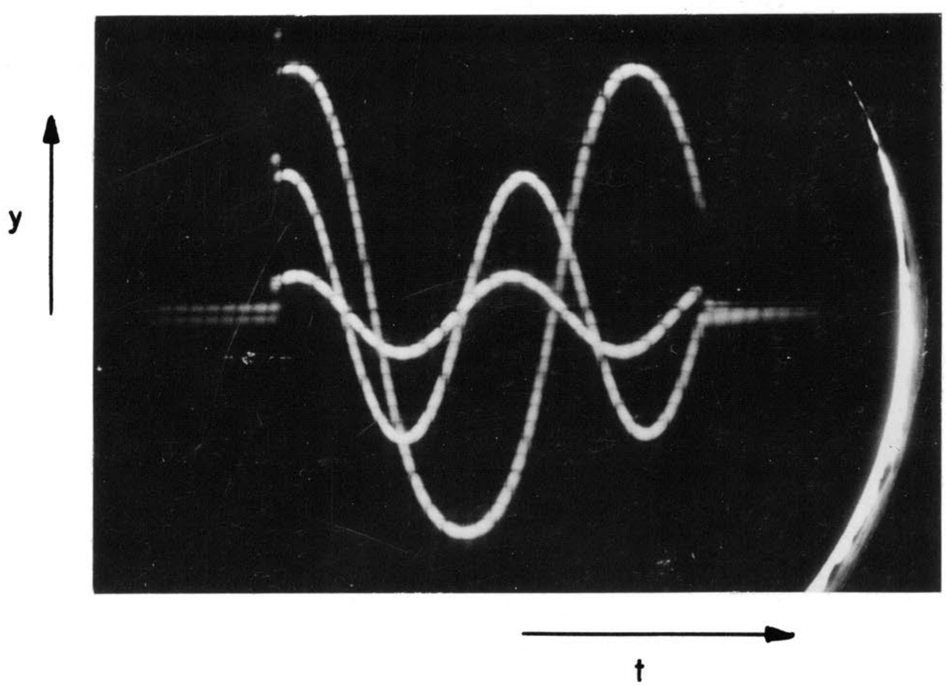

Fig. 99 Solutions of the equation $\ddot{y} = -\sin y$ for $\dot{y}_0$ zero and y_0 variable.

That the large amplitude motion is really non-sinusoidal is demonstrated by Fig. 100, which shows the amplitude, velocity, and acceleration for a large amplitude of swing. If the motion had been sinusoidal, these curves would all have been sine waves.

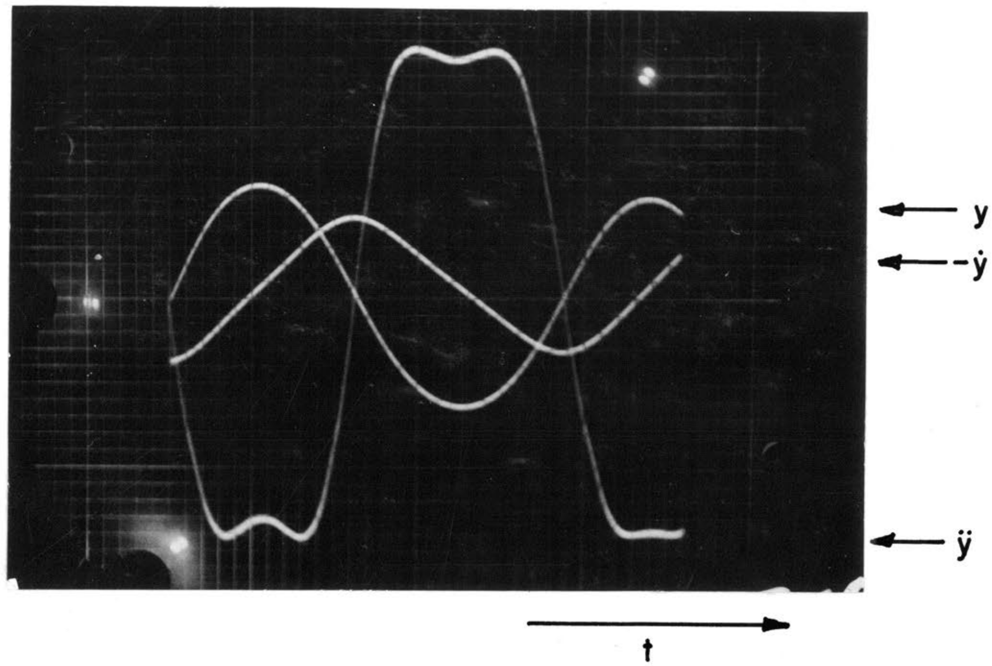

Fig. 100 Plots of $y, -\dot{y}$, and $\ddot{y}$ versus t for the equation $\ddot{y} = -\sin y$, $y_{max} \approx 120$ degrees.

From the acceleration curve in this plot it is apparent that the peak amplitude of swing in this case was about $\pm$ 120 degrees. It is interesting to note that the displacement curve still appears quite sinusoidal to the eye.

Fig. 101 gives the solutions obtained for the case of a fixed initial displacement of the pendulum and four values of initial velocity. The four curves show the cases of oscillation, rotation to the right, and rotation to the left. This verifies the expected behavior on the basis of the potential curve of Fig. 98.

These two examples show that the electronic differential analyzer

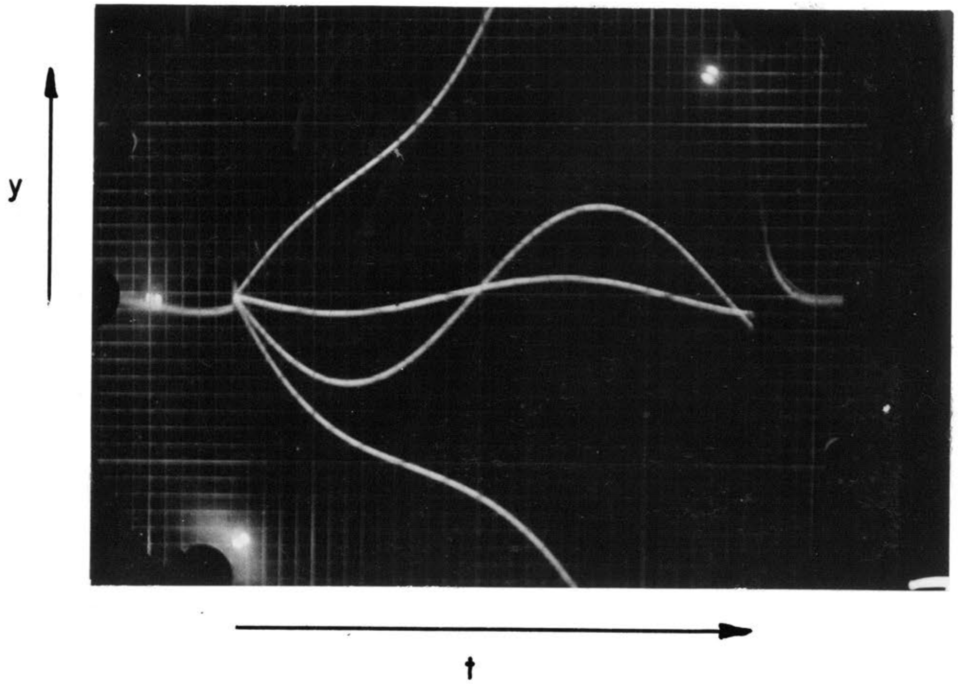

Fig. 101 Solutions of the equation $\ddot{y} = -\sin y$, for y_0 fixed and $\dot{y}_0$ variable.

is capable of very quickly and easily solving equations of the type given in Eq. (173). The only time required between solutions is that necessary to cut out a new function mask. The number of cases solved will be determined by the need of mathematicians, physicists and engineers to deal with new force functions F(y). The comparable mathematical difficulty of solving this general type of equation is indicated by the large numbers of published works on the subject.[72]

Higher Order Linear Differential Equations with Constant Coefficients

6.8 Solution of Simultaneous Second Order Differential Equations.

The solution of simultaneous linear differential equations with constant coefficients is of great practical importance to the engineer

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[72] Minorsky, op. cit., reference 67, Bibliography, p. 131.

and physicist. Although it is possible to handle such equations directly by analytic means, a considerable amount of time and labor is required by that approach. The electronic differential analyzer affords a means of obtaining solutions for such systems of equations quickly, easily, and inexpensively.

An example of a practical electric circuit requiring the solution of a pair of simultaneous differential equations is the coupled tuned circuit of Fig. 102.

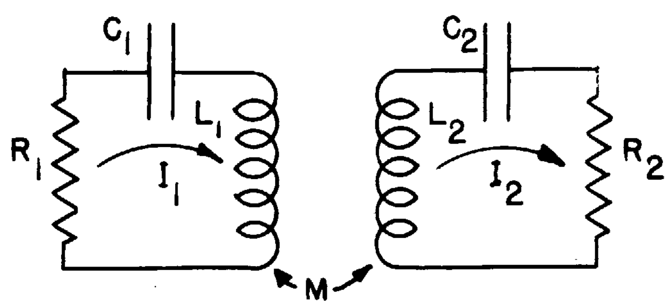


Fig. 102 Coupled tuned circuit.

For this circuit one can write the two differential equations

$$\frac{d^2I_1}{dt^2} + \frac{R_1}{L_1} \cdot \frac{dI_1}{dt} + \frac{1}{L_1C_1} I_1 - \frac{M}{L_1} \cdot \frac{d^2I_2}{dt^2} = 0 \,, \qquad (178)$$

$$\frac{d^2I_2}{dt^2} + \frac{R_2}{L_2} \cdot \frac{dI_2}{dt} + \frac{1}{L_2C_2} I_2 - \frac{M}{L_2} \cdot \frac{d^2I_1}{dt^2} = 0 \qquad (179)$$

The differential analyzer set-up necessary to solve either of these equations individually without the mutual coupling term was given in Fig. 83. The first step in solving the simultaneous equations is therefore to set-up two separate loops on the differential analyzer as indicated in the block diagram of Fig. 103 by solid lines.

Around these two loops all the dependent variables occurring in Eqs. (178) and (179) appear. In order to solve these simultaneous equations it is only necessary to interconnect these two loops as required by the differential equations. These interconnections are shown as dotted lines in Fig. 103. With these connections the set-up of Fig. 103 solves the equations

$$\frac{d^2I_1}{dt^2} = -k_4(k_1k_2I_1 + k_1k_3\frac{dI_1}{dt}) + k_5\frac{d^2I_2}{dt^2} , \tag{180}$$

$$\frac{d^2I_2}{dt^2} = -k_9(k_6k_7I_2 + k_6k_8\frac{dI_2}{dt}) + k_{10}\frac{d^2I_1}{dt^2} . \tag{181}$$

Comparing these equations with Eqs. (178) and (179) one observes that by satisfying the conditions

$$k_1k_2k_4 = \frac{1}{L_1C_1} , \tag{182}$$

$$k_1k_3k_4 = \frac{R_1}{L_1} , \tag{183}$$

$$k_5 = \frac{M}{L_1} , \tag{184}$$

$$k_6k_7k_9 = \frac{1}{L_2C_2} , \tag{185}$$

$$k_6k_8k_9 = \frac{R_2}{L_2} , \tag{186}$$

$$k_{10} = \frac{M}{L_2} , \tag{187}$$

the desired solutions are obtained. There are also maximum permissible values of k_1, k_2, k_6, and k_7, the integrator constants, set by the finite gain of the integrator amplifiers as discussed in Section IV.

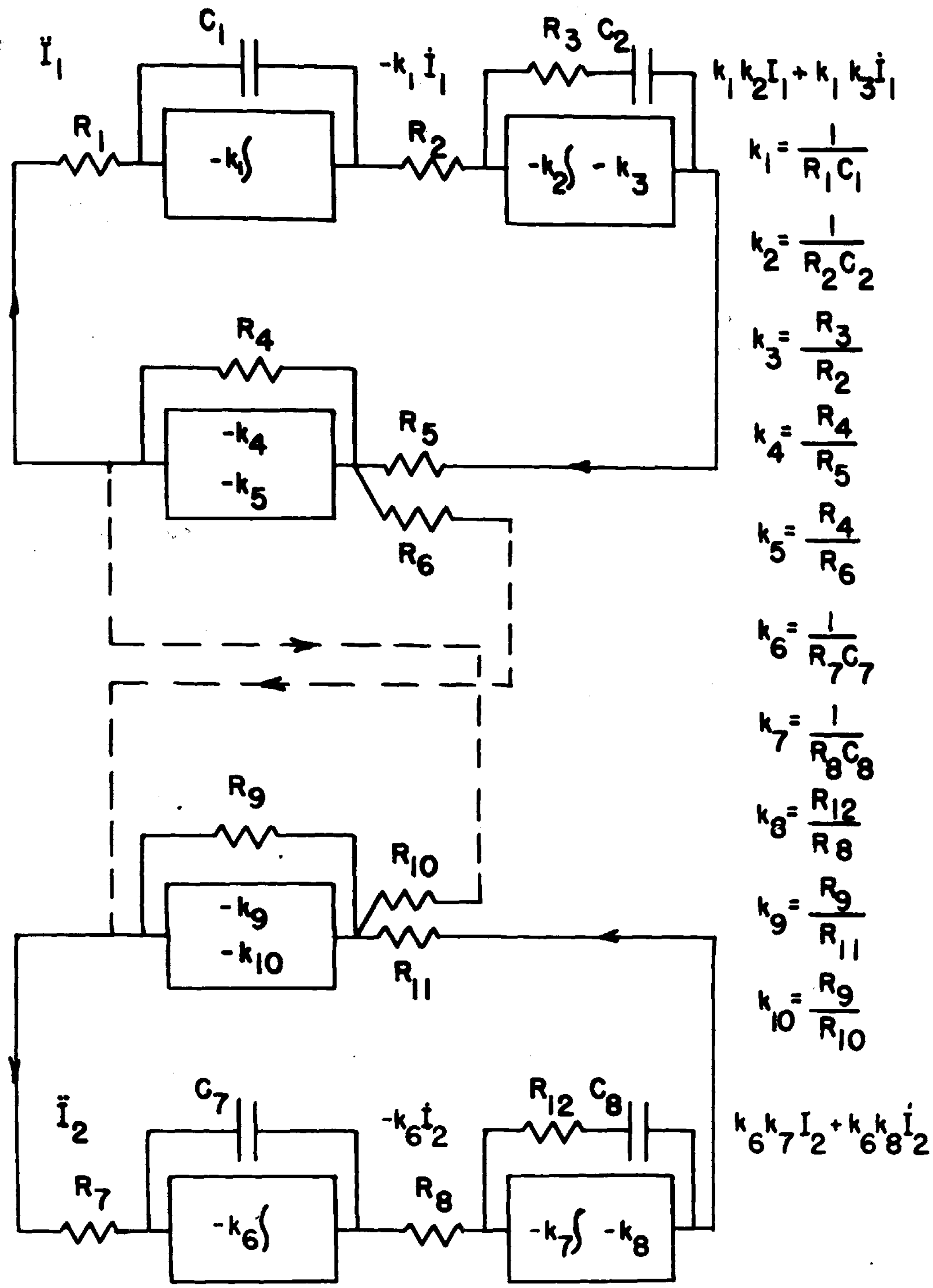
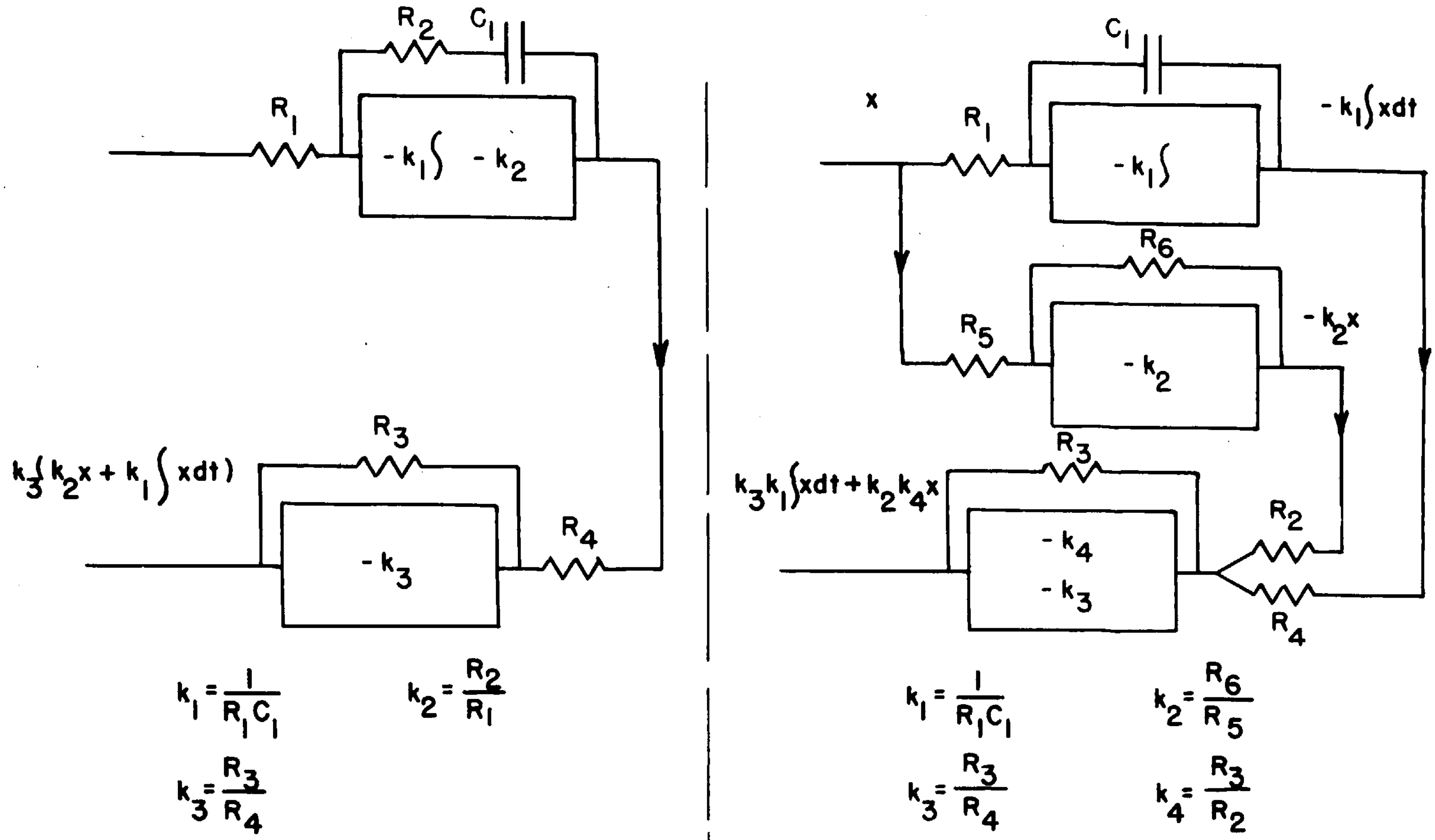


FIG. 103 SET-UP FOR SIMULTANEOUS 2ND ORDER DIFFERENTIAL EQUATIONS



FIG·104 TWO EQUIVALENT COMPUTING CIRCUITS

A practical procedure in setting up such a system of equations is to start by assuming that these constants have their maximum permissible value. This gives four additional constraints on the k's and permits the determination of all the constants uniquely from Eqs. (182) through (187). If for one reason or another the resulting k's are either extremely large or extremely small, a change in time scale factor is usually indicated.

It will be noticed in the set-up of Fig. 103 that the integrators in the upper and lower right hand corners have been modified slightly. For this connection the output of the unit is the integral plus a fraction of the input as indicated. This is simply a device to save an amplifying unit, as shown in Fig. 104. Both circuits of Fig. 104 form the integral of the input plus a fraction of the input. The circuit to the left has the advantage of requiring one less adding unit. It has the disadvantage that changing k_2 independently of the other constants requires changing the resistor R_2, which is inside a feedback loop. In the second circuit all constants can be varied by potentiometers at the input terminals of the various units without disturbing the feedback loops. For the higher order differential equations, where the number of computing units required may become very high, the circuit at the left and similar devices are very useful.

Some typical solutions of Eqs. (178) and (179) are shown in Figs. 105 and 106. The first figure shows the solution for the case of zero resistance in both loops of Fig. 102. The familiar beat phenomena is easily observed here. The photograph is a double exposure showing the primary and secondary current as a function of time.

Fig. 106 shows the same currents with damping in both the primary and secondary circuits.

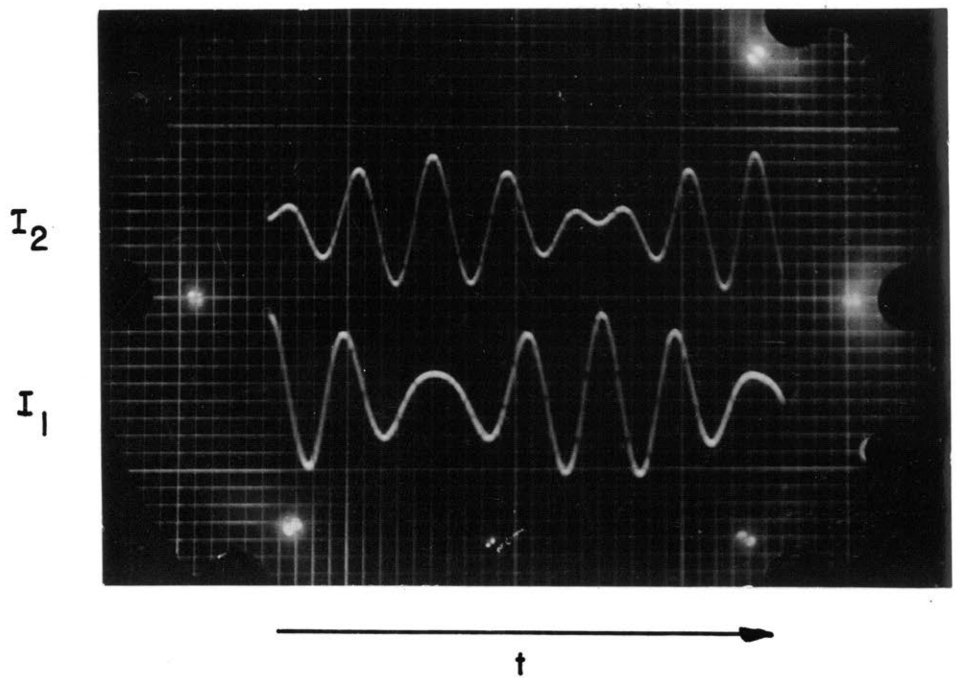


Fig. 105 Primary and secondary currents versus time for non-dissipative coupled circuits. $\dot{I}_{1o} = I_{2o} = \dot{I}_{2o} = 0$, $I_{1o} \neq 0$.

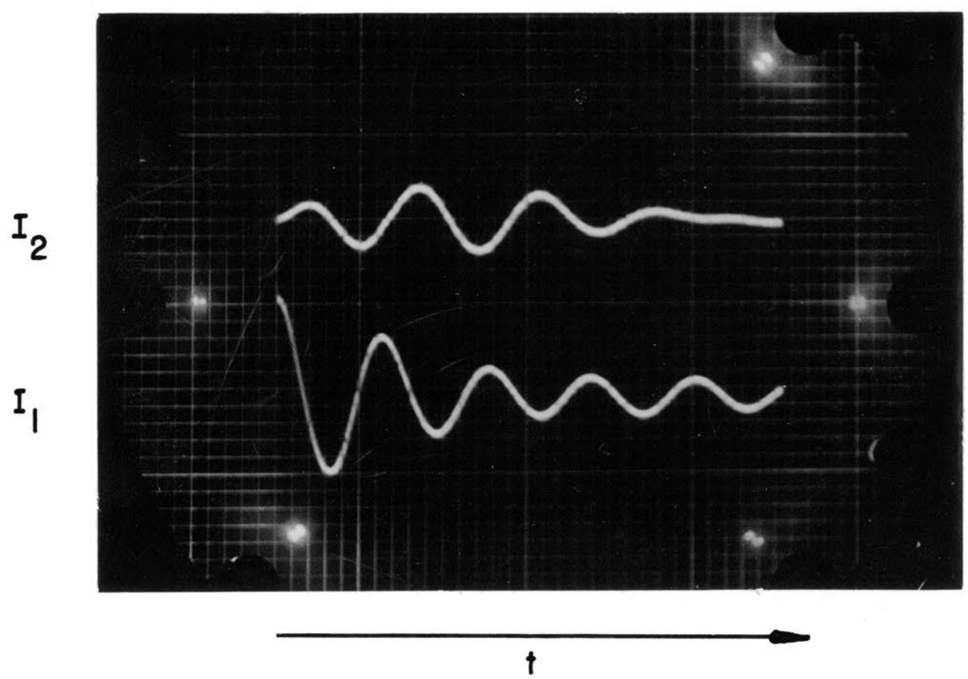


Fig. 106 Primary and secondary currents versus time for a pair of coupled circuits. $\dot{I}_{1o} = I_{2o} = \dot{I}_{2o} = 0$, $I_{1o} \neq 0$.

It will be recognized that these solutions of two simultaneous differential equations are equivalent from the differential analyzer point of view to the solution of a single fourth order differential equation. Differential equations of orders higher than this have not been solved on the present electronic differential analyzer because only four integrating units have been built. Since no unusual difficulties have been encountered in solving equations of third and fourth order, it is felt that the extension of the electronic differential analyzer to the solution of equations of even higher orders should not present any insurmountable difficulties.

Linear Differential Equations with Variable Coefficients

Linear differential equations with variable coefficients are of the utmost interest in engineering and physics and are in general not susceptible to analytic methods of solution. Special and relatively simple examples of such equations are Bessel's equation, the Mathieu equation, and the Hill equation. The electronic differential analyzer has been used to solve a number of equations of this type.

6.9 Gaussian Error Equation

One of the simplest equations with variable coefficients is

$$\frac{dy}{dt} + yt = 0 \; ; \qquad (188)$$

this equation is easily solved analytically, and its solution is the well-known error function

$$y = C_1 e^{-t^2} \; . \qquad (189)$$

The set-up of the differential analyzer is shown in Fig. 107.

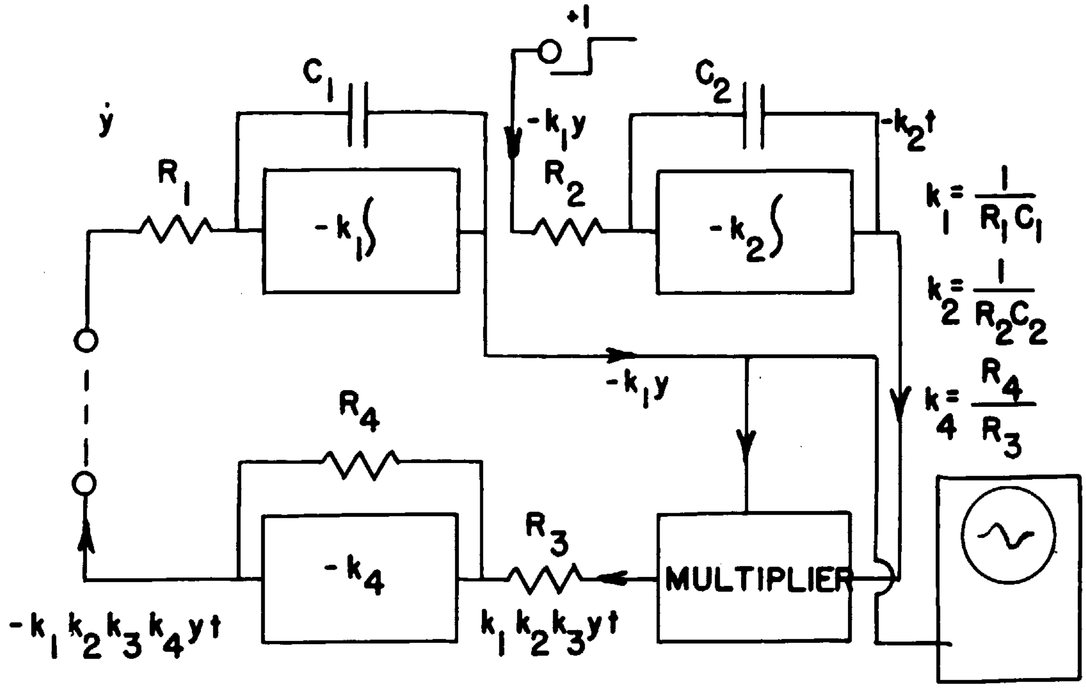


Fig. 107 Differential analyzer set-up for the solution of $\ddot{y} = -yt$.

As in the set-up of Fig. 94 the amplifying unit included here is superfluous; it is useful because it gives an additional degree of freedom in the adjustment of the various unit constants. This equation requires the use of a multiplying unit, as do all equations with variable coefficients.* Since the variable coefficient is a function of the independent variable t, an arbitrary function generator is not required for the solution of this equation; instead an auxiliary differential equation,

$$\frac{dx}{dt} = -k_2 , \tag{190}$$

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* The multiplier used in all results described in this section is the crossed-fields multiplier discussed in Section IV. The use of a multiplier for the solution of equations with variable coefficients illustrates a situation for which lack of speed in a single channel of the multiplier is frequently tolerable. The multiplier channel connected in the feedback loop must have the greatest possible speed to avoid high-frequency errors as discussed in Section V, but lack of speed in the channel to which $-k_2 t$ is applied in Fig. 107, for example, will not introduce a serious error.

is solved,

$$x = -k_2t + \text{constant}, \qquad (191)$$

to give the necessary voltage varying linearly with time. This technique, as in the mechanical differential analyzer, of using auxiliary differential equations to generate functions is applicable to many linear equations with variable coefficients, since only functions of the independent variable are involved in these equations.

The independent variable of the electronic differential analyzer t' always starts from zero at the beginning of every solution and increases. If one desires to view the solution of Eq. (188) for $-t_o < t < t_1$, it is necessary to make the change of variable

$$t' = t_o + t \; ; \qquad (192)$$

applied to Eq. (188) this gives the transformed equation

$$\frac{dy}{dt'} + y(t' - t_o) = 0. \qquad (193)$$

No change in the set-up of Fig. 107 is required other than the addition of a constant at the output of the function-generating integrator; that is, the constant in Eq. (191) is now non-zero.

A solution of Eq. (193) is given in Fig. 108; also plotted in this figure by means of a photographic double exposure is the negative of the derivative of y with respect to time.

6.10 Solution of Time Varying Force Equation

A more interesting group of linear differential equations with variable coefficients, which are not as susceptible to analytic treatment, are the equations of the form

$$\frac{d^2y}{dt^2} + F(t)y = 0. \qquad (194)$$

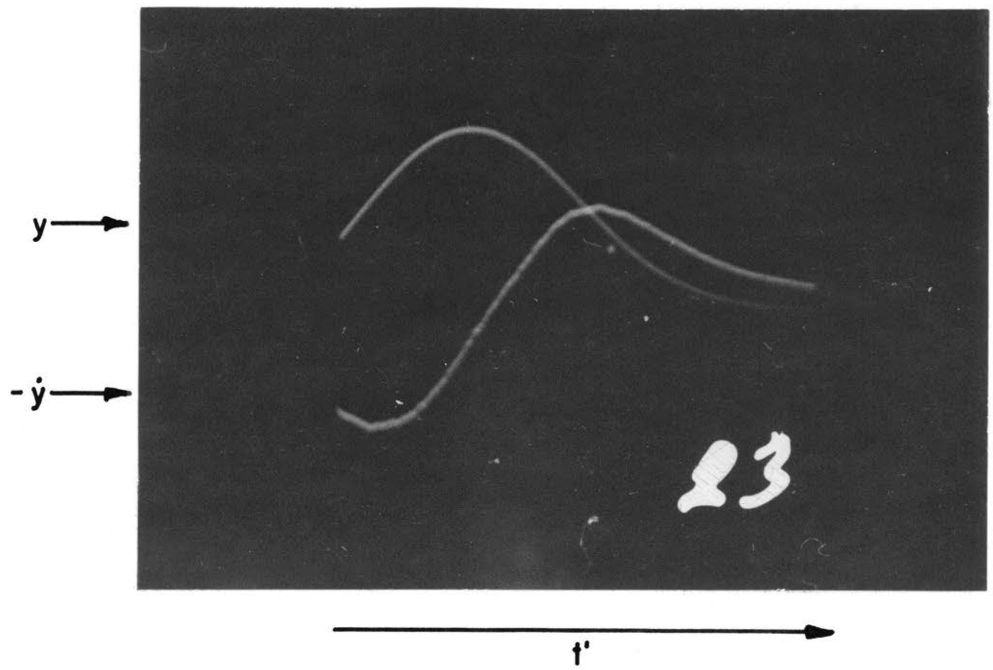

Fig. 108 y and $-\dot{y}$ versus t' for the equation $\dot{y} + y(t' - t_o) = 0$.

The differential analyzer set-up for the solution of equations of this form is given in Fig. 109.

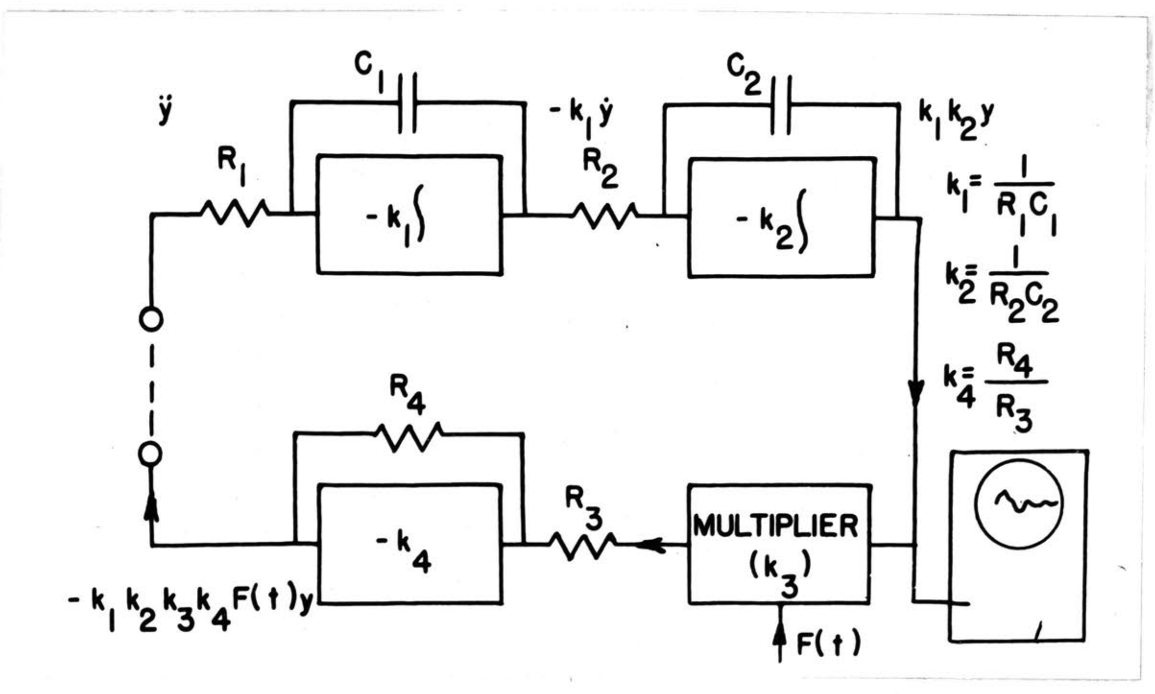

Fig. 109 Differential analyzer set-up for the solution of equation $\ddot{y} + F(t)y = 0$.

The only difference between this set-up and that of Fig. 107 is the addition of another integrator. A multiplier is again required and also a voltage proportional to F(t).

If one makes F(t) equal to bt^2 in Eq. (194), one obtains

$$\frac{d^2y}{dt^2} + bt^2y = 0. \qquad (195)$$

The solution of this equation is given by Jahnke and Emde, as[73]

$$y = t^{\frac{1}{2}} Z_{\frac{1}{4}}\left(\frac{\sqrt{b}}{2}t^2\right) , \qquad (196)$$

where

$$Z_{\frac{1}{4}}(x) = c_1 J_{\frac{1}{4}}(x) + c_2 N_{\frac{1}{4}}(x) \; ; \qquad (197)$$

$J_n(x)$ is the nth order Bessel function of the first kind, and $N_n(x)$ is the nth order Bessel function of the second kind.

To solve this equation with the set-up of Fig. 109 a voltage proportional to t^2 must be generated. This can be done by solving the auxiliary differential equation

$$\frac{d^2x}{dt^2} = b \; . \qquad (198)$$

The set-up for doing this is shown in Fig. 110.

A combination of Figs. 109 and 110 gives the complete set-up necessary to solve Eq. (195) on the electronic differential analyzer. A particular solution of this equation is shown in Fig. 111. The case chosen is the one for which c_2 is zero in Eq. (197) so that only the Bessel function of the first kind is observed. Fig. 112 gives the

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73 Jahnke, E., and Emde, F., Tables of Functions with Formulae and Curves, Dover Publications, New York, 1943, p. 147.

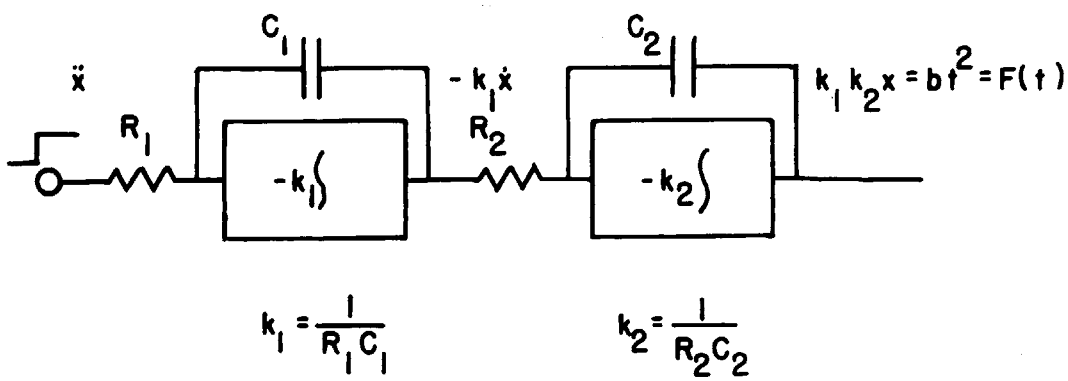

Fig. 110 Set-up of auxiliary differential equation to generate a voltage $x = bt^2$.

calculated solution for this special case, which was obtained by computing values for Eq. (196). One sees that the agreement between the two is very good.

Fig. 113 is a triple exposure photograph of another solution of this equation. This figure also shows the behavior of $\dot{y}$ and $\ddot{y}$ versus time.

It is interesting to note that this equation describes a motion in which, although the amplitude damps out with time, the velocity, and therefore the energy in the motion, continually increases with time. Motions of this type are encountered in oscillation of electrons about the stable orbit in high energy accelerators such as the betatron.[74]

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74 Rajchman, J. A., and Cherry, W. H., "The Electron Mechanics of Induction Acceleration," <u>Jour</u>. <u>Frank</u>. <u>Inst</u>., V. 243, p. 261, Apr. 1947.

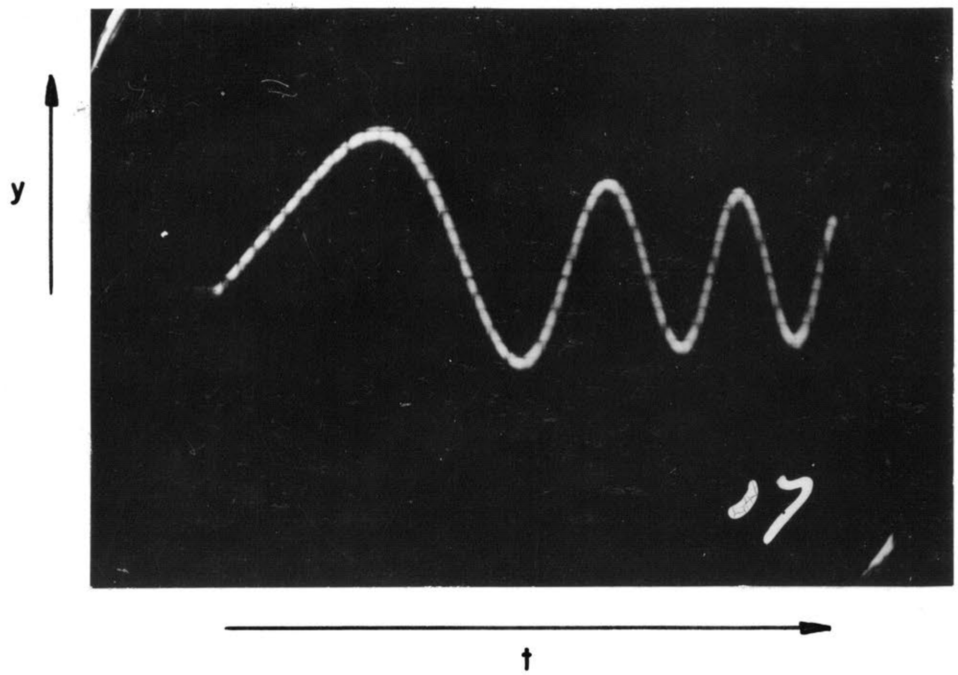

Fig. 111 Observed solution of equation, $\ddot{y} + bt^2y = 0$.

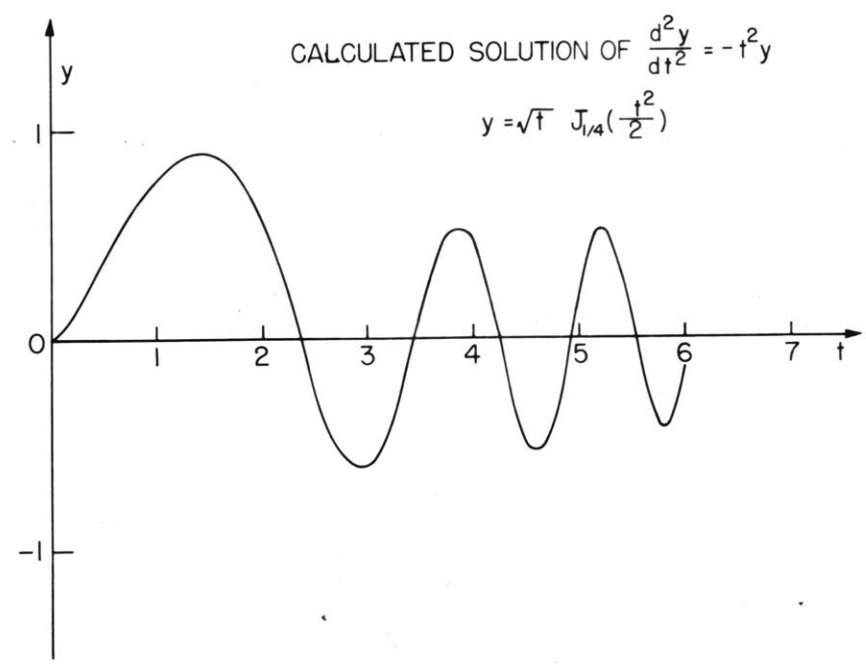

Fig. 112 Calculated solution of equation, $\ddot{y} + bt^2y = 0$.

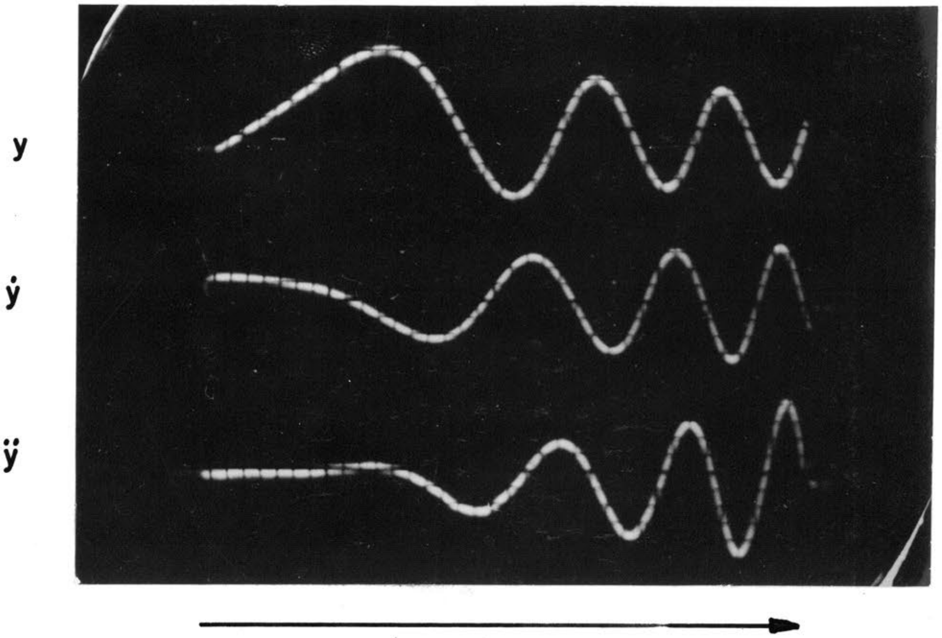

Fig. 113 Plot of y, $\dot{y}$, and $\ddot{y}$ versus t for the equation $\ddot{y} + bt^2y = 0$.

6.11 Solution of Mathieu Equation

If one chooses

$$F(t) = \omega_o^2(1 + \epsilon \cos\omega_m t) \tag{199}$$

in Eq. (194), one obtains the well-known Mathieu equation. This equation is encountered in the solution of Laplace's equation in elliptical coordinates and in connection with the following problem.

If for the non-dissipative circuit of Fig. 114 the capacity is made to vary according to the relation

$$C = C_o(1 - \epsilon \cos\omega_m t) , \tag{200}$$

one obtains for the differential equation describing the behavior of the current in this circuit the relation

$$\frac{d^2I}{dt^2} + \frac{I}{L_oC_o(1 - \epsilon \cos\omega_m t)} = 0. \tag{201}$$

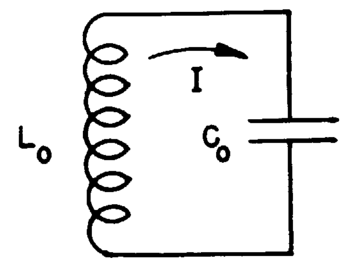

Fig. 114 Parallel LC circuit.

This physical situation has been of considerable interest in connection with the generation of acoustic warble-tones since Helmholtz and Rayleigh, and more recently, has been studied in connection with frequency modulation.[75] If $\epsilon \ll 1$ and $\omega_m \ll \omega_o$ (where $\omega_o = \frac{1}{L_o C_o}$), Eq. (201) corresponds to a typical frequency-modulated oscillator, as commercially employed.

Eq. (201), without the assumptions $\epsilon \ll 1$ and $\omega_m \ll \omega_o$, is of the Hill type and is very difficult to handle analytically. Barrow in studying this problem made the assumption that $\epsilon \ll 1$.[76] This enabled him to use the approximation

$$\frac{1}{1 - \epsilon \cos\omega_m t} \simeq 1 + \epsilon \cos\omega_m t \ . \qquad (202)$$

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75 van der Pol, B., "Frequency Modulation", Proc. I.R.E., V. 18, 1194-1206, July, 1930.

76 Barrow, W.L., "Frequency Modulation and the Effects of Periodic Capacity Variation in a Non-dissipative Oscillatory Circuit", Proc. I.R.E., V. 21, 1182-1203, Aug. 1933.

Substituting Eq. (202) in Eq. (201) one obtains

$$\frac{d^2I}{dt^2} + \omega_o^2(1 + \epsilon \cos\omega_m t)\, I = 0 \,, \tag{203}$$

which is the Mathieu equation. Both this equation and the original equation (201) of the Hill type can be handled with equal ease on the electronic differential analyzer, although analytically Barrow found it necessary to proceed immediately to the Mathieu equation from the original Hill equation because of "The formidable mathematical difficulties". Cambi has recently studied the Hill equation.[77]

The Mathieu equation is solved on the electronic differential analyzer with the set-up of Fig. 109. It is only necessary to generate the time function given by Eq. (199). Such a function is most easily generated by solving the auxiliary differential equation

$$\frac{d^2x}{dt^2} + \omega_m^2 x = 0 \,. \tag{204}$$

A detailed discussion of the solution of this equation was given at the beginning of this section and need not be repeated here. The differential analyzer set-up for its solution is given in Fig. 75.

The Mathieu functions are special solutions of the Mathieu equation which are periodic in behavior.[78] Fig. 115 shows the differential analyzer generation of the Mathieu function $ce_2(t)$. Fig. 116 gives the calculated solution for this case.[79]

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[77] Cambi, Enzo, "Trigonometric Components of a Frequency-modulated Wave", Proc. I.R.E., V. 36, 42-49, January 1948.

[78] McLachlan, N.W., Theory and Application of Mathieu Function, Oxford, 1947.

[79] Jahnke and Emde, op. cit., reference 73, p. 293.

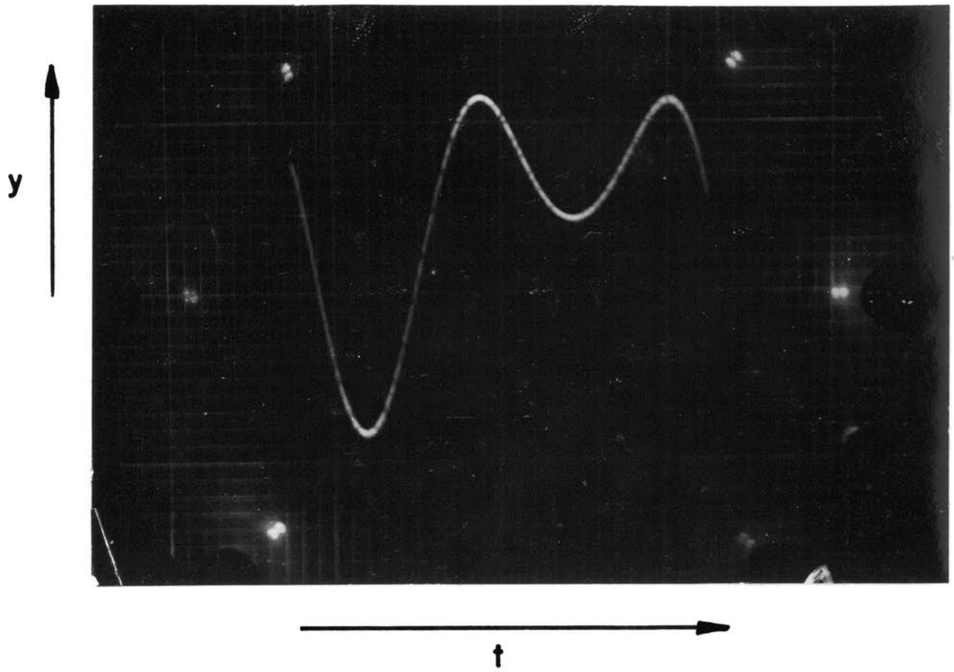

Fig. 115 Mathieu function, $ce_2(t)$, as observed on the differential analyzer.

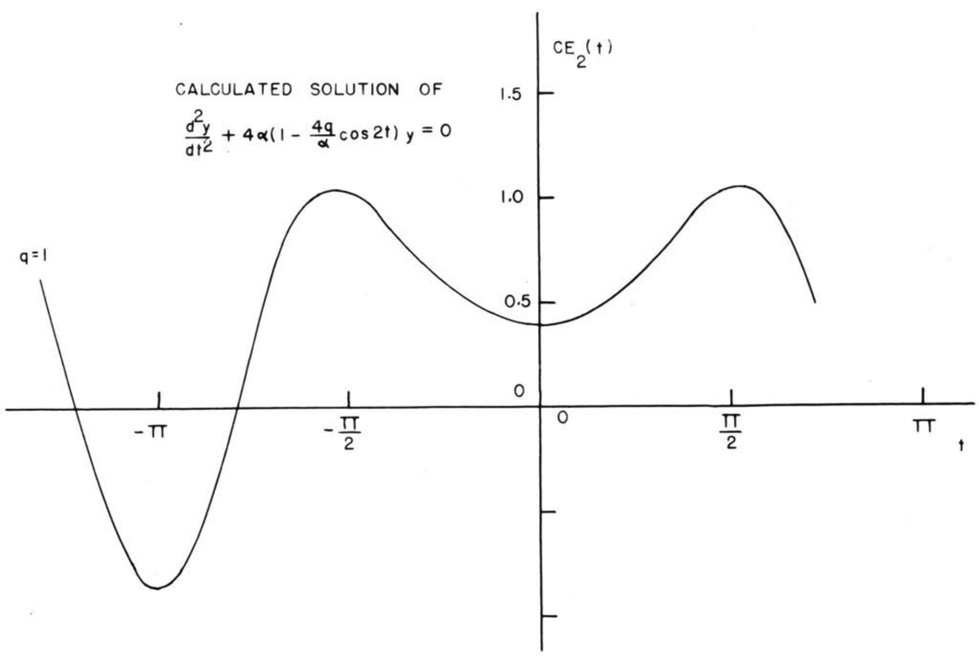

Fig. 116 Calculated Mathieu function, $ce_2(t)$.

Two other photographs of differential analyzer solutions of the Mathieu equation are shown in Figs. 117 and 118. Fig. 117 is a triple exposure showing (1) the solution of the equation with $\epsilon = 0$, (2) F(t) as a function of time, and (3) the solution of Eq. (203) for $\epsilon \approx 0.5$.

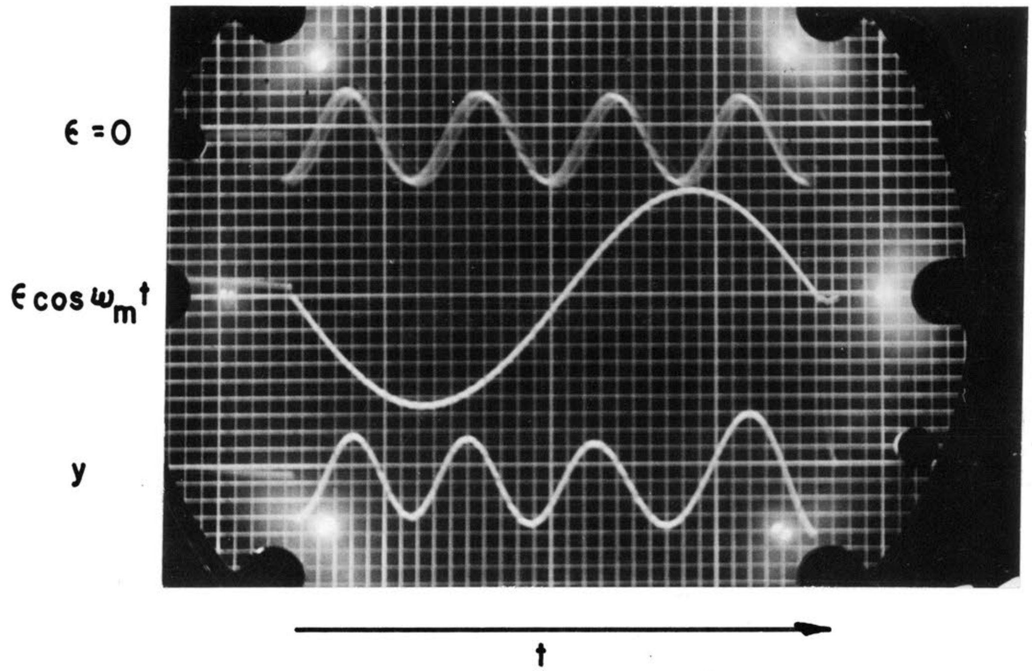

Fig. 117 Solution of Mathieu equation for $\frac{\omega_o}{\omega_m} = 4$ and $\epsilon \approx 0.5$.

It is apparent from this figure that although a modulation of the frequency of oscillation is certainly obtained, in this case the amplitude of the oscillation is not constant. Fig. 118 pertains to the same conditions; it shows the first and second derivatives of the circuit current as a function of time, in addition to the current itself.

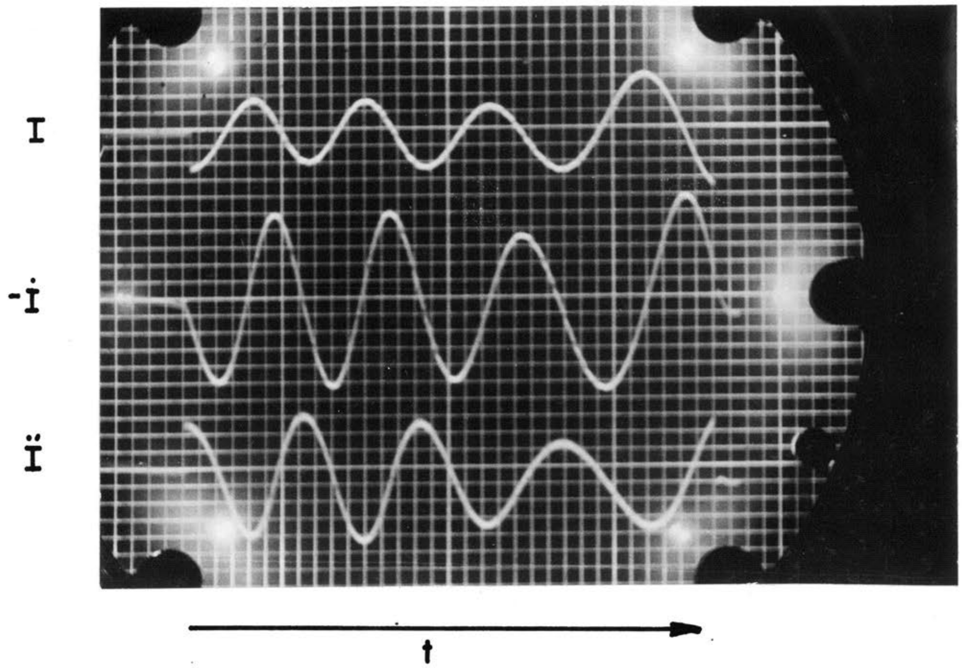

Fig. 118 I, $-\dot{I}$, and $\ddot{I}$ versus t for the equation $\ddot{I} + \omega_o^2(1 + \epsilon \cos\omega_m t)\ I = 0$ for $\frac{\omega_o}{\omega_m} = 4$, $\epsilon \approx 0.5$.

6.12 Solution of Equation of the Hill Type

Solving the, analytically more difficult, Hill equation requires only that the multiplier in the set-up of Fig. 109 be replaced by a divider. The new set-up is shown in Fig. 119. Comparing the two set-ups of Figs. 109 and 119 one sees that by the very simple change of moving one connection and adding one connection it is possible to shift the differential analyzer set-up from the Mathieu to the Hill equation. The analytic difficulty of the corresponding change is enormous. This difficulty is so great as to have prevented any considerable use of equations of this level of difficulty in normal engineering work. With a unit such as the electronic differential analyzer of this thesis available, this situation no longer need exist.

A typical solution of the Hill equation is shown in Figs. 120 and 121. The first of these shows the unmodulated current, the

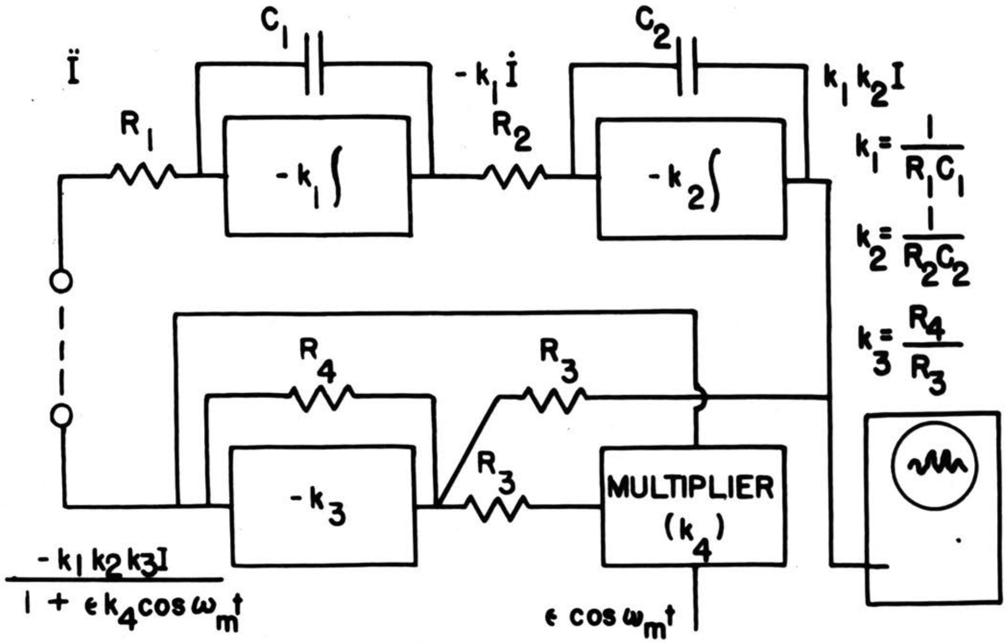

Fig. 119 Differential analyzer set-up for the solution of the Hill equation.

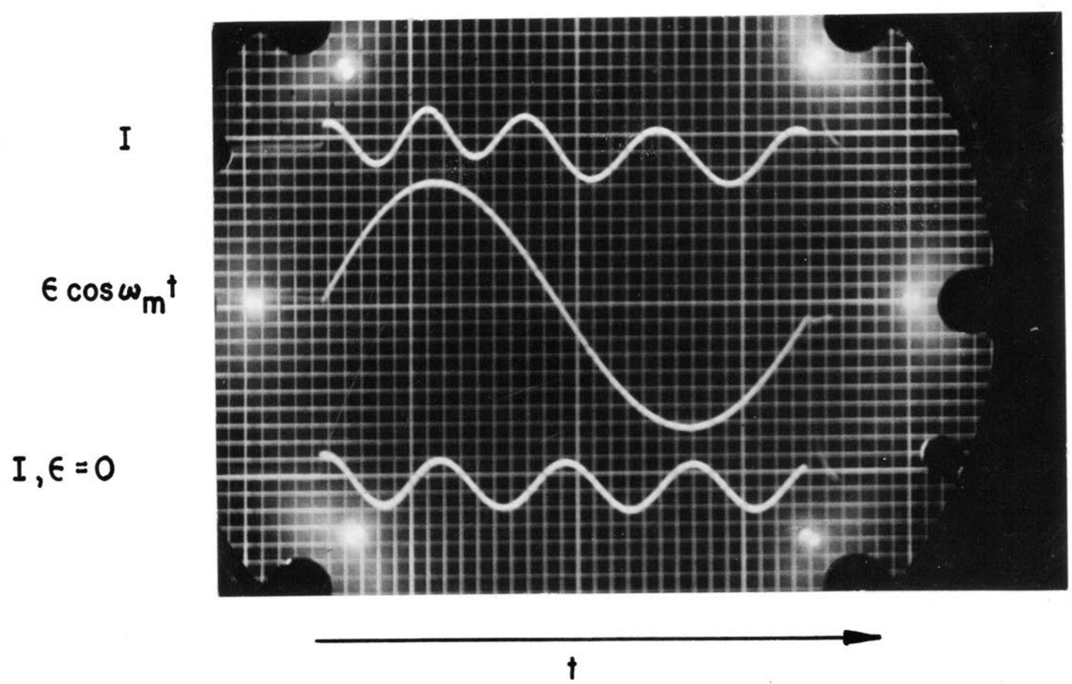

Fig. 120 Solution of the Hill equation, $\frac{d^2I}{dt^2} + \frac{\omega_o^2 I}{1 + \epsilon\cos\omega_m t} = 0$ for $\epsilon = 0$ and $\epsilon = 0.5$, together with a plot of $\epsilon\cos\omega_m t$ versus t.

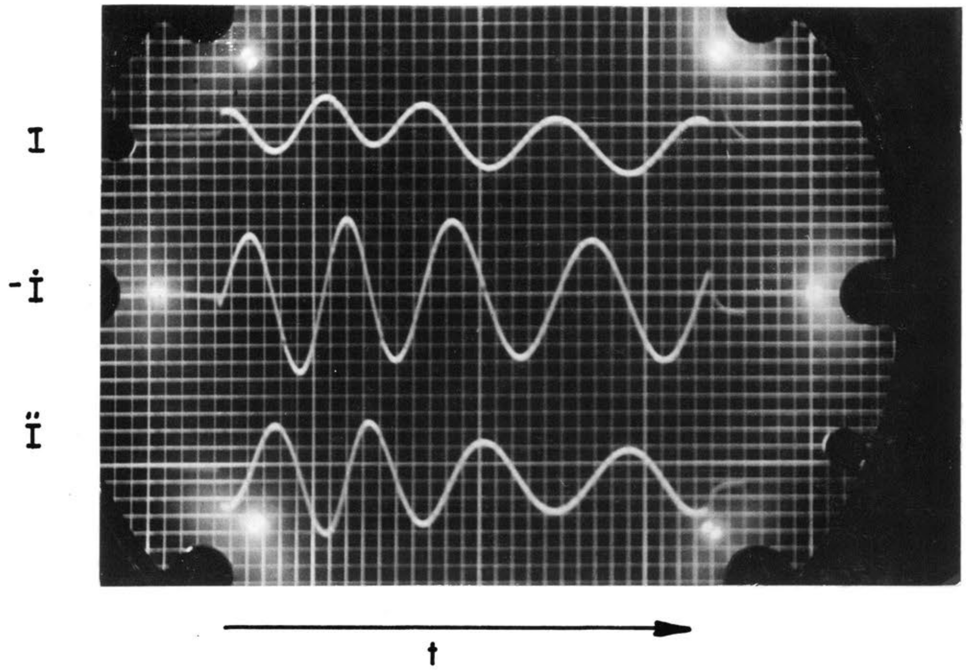

Fig. 121 Solution of the Hill equation showing I, $-\dot{I}$, and $\ddot{I}$ versus t.

modulating term, and the resulting modulated current, while the second shows the current and its derivatives as a function of time.

6.13 Third Order Linear Differential Equation with Variable Coefficients.

A problem which is of interest in connection with the transient behavior of some electrical circuits is the evaluation of the Fourier cosine transform of the frequency function $e^{-\omega^4}$,

$$y = f(t) = \int_{-\infty}^{\infty} e^{-\omega^4} \cos\omega t \, d\omega \tag{205}$$

This problem can be transformed to the solution of a differential equation as follows. Differentiating both sides of Eq. (205) with respect to time three times gives the relation

$$\frac{d^3y}{dt^3} = \int_{-\infty}^{\infty} \omega^3 e^{-\omega^4} \sin\omega t \, d\omega \ . \tag{206}$$

The right hand side of this equation can be integrated by parts with respect to ω and this gives

$$\frac{d^3y}{dt^3} = -\frac{t}{4}\int_{-\infty}^{\infty} e^{-\omega^4}\cos\omega t \, d\omega \ , \qquad (207)$$

or from Eq. (205)

$$\frac{d^3y}{dt^3} + \frac{yt}{4} = 0. \qquad (208)$$

Evidently Eq. (205) is a solution of this differential equation. The differential analyzer set-up for solving this equation is shown in Fig. 122.

There is nothing unusual about this set-up. When one observes the solutions obtained from this set-up, it is found that they tend to increase with increasing time rather than to approach zero. The solutions of the integral of Eq. (205) should approach zero in an oscillatory manner. This can be verified by a series of graphical integrations or by evaluating the series solution[80]

$$y = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Gamma\left(\frac{2n+1}{4}\right)t^{2n} \ . \qquad (209)$$

Investigation of Eq. (208) and its derivation reveals that although Eq. (205) is a solution of this differential equation it is not the only solution of the equation. While the desired solution approaches zero as time increases there is an extraneous solution which approaches infinity. Even though considerable care is taken in adjusting the initial conditions to suppress this growing solution, because of the speed with which it grows with time it soon masks the

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80 Titchmarsch, E.C., The Theory of Functions, 1st Edition, Oxford 1936, p. 262.

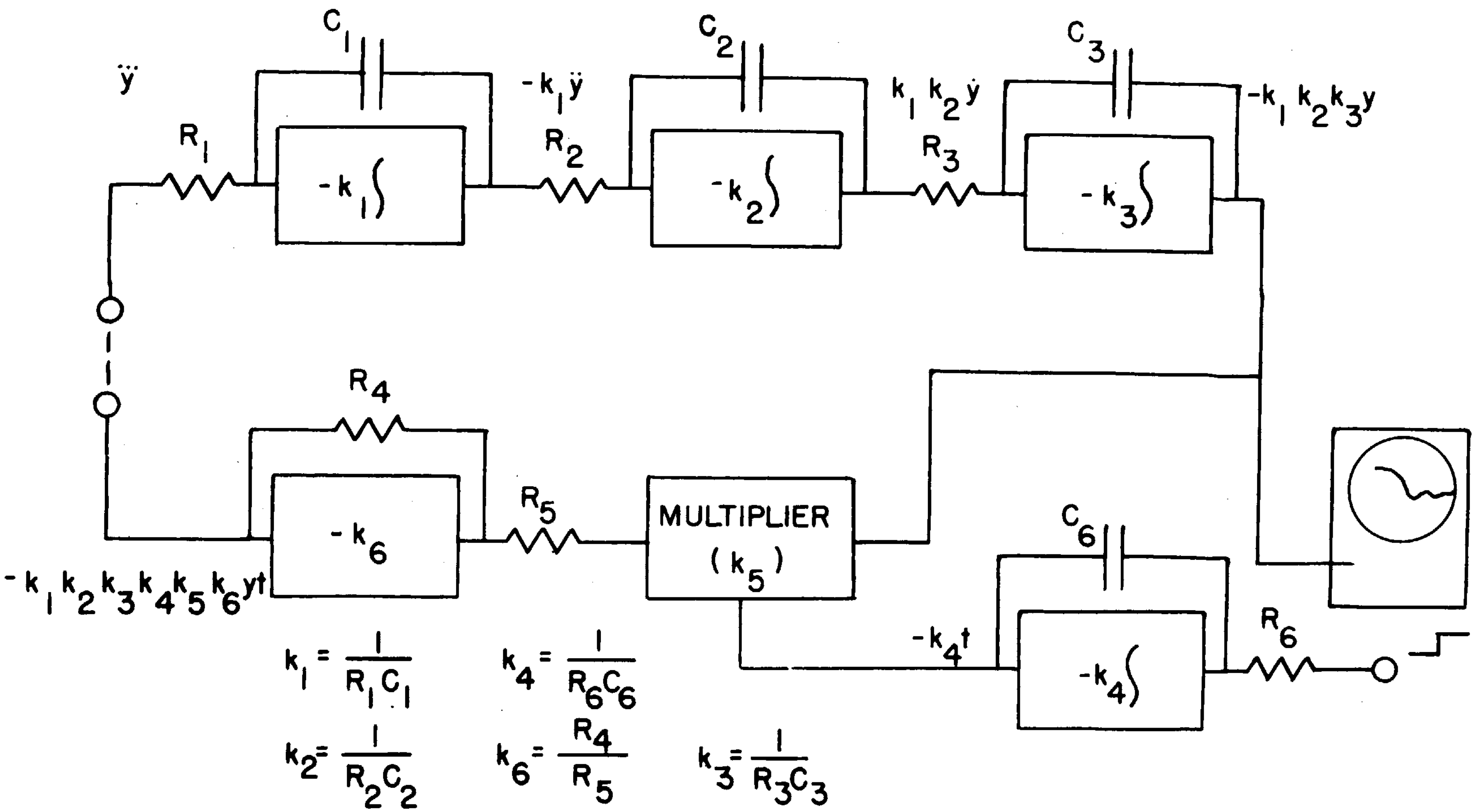

FIG. 122 DIFFERENTIAL ANALYZER SET-UP FOR $\ddot{Y} + YT/4 = 0$

desired solution.* If a mechanical differential analyzer were employed to solve this equation, this difficulty could be surmounted by reversing the direction of motion of the independent variable so that the desired solution would dominate as the machine solution progresses. This same effect can be achieved on the electronic differential analyzer by making the change of variable

$$t' = t_o - t \,, \tag{210}$$

which gives a transformed equation

$$\frac{d^3y}{dt'^3} + \frac{y}{4}(t_o - t') = 0 \,. \tag{211}$$

The solution of this transformed equation over the range $0 < t' < t_o$ corresponds to the solution of the original Eq. (208) for $t_o < t < 0$. This change of variable requires only minor changes of the set-up in Fig. 122. The unit step into the lowest integrator must have its sign reversed and a constant must be added at the output of this integrator. This is conveniently done with the integrator initial-condition control. A solution calculated from Eq. (205) directly by repeated integration is plotted in Fig. 124. Figs. 123 and 125 are observed solutions on the differential analyzer for the original and the transformed equation respectively.

In Fig. 123 the fuzziness toward the end of the solution results from the fact that the undesired term in the solution is beginning to dominate. The adjustment of the initial conditions to prevent this occurrence has been made to within the precision of the differential analyzer and jitter in this critical adjustment causes the

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* This is a similar situation to that described in Section V in connection with the solution of Eq. (79).

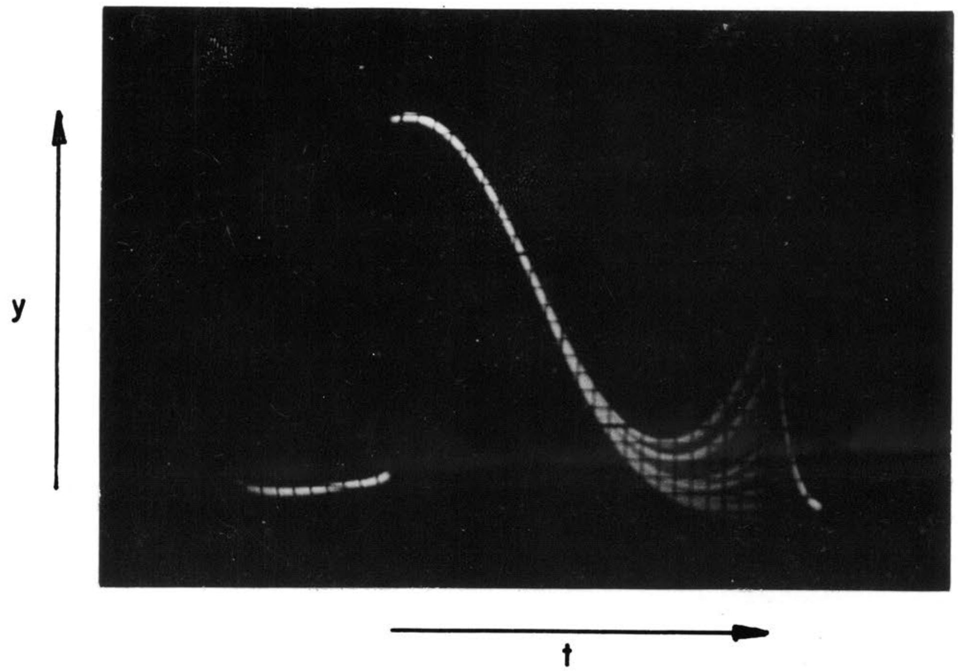

Fig. 123 A solution of the equation $\frac{d^3y}{dt^3} + \frac{yt}{4} = 0$ versus t.

fuzziness shown.

Fig. 125 gives a plot of y versus $t_o - t'$; it is printed backwards from the way that the solution is observed on the differential analyzer.

In obtaining this solution it is necessary to try different initial values in the transformed equation and observe the values obtained at $t' = t_o$. When these final values for the transformed equation match the initial values of the original equation, the desired solution is obtained. If it were not for the speed of operation and ease of varying the initial conditions on the electronic differential analyzer this process would be very time-consuming, since it involves the simultaneous adjustment of three different parameters. Even on the electronic differential analyzer the adjustment of parameters to obtain the solution shown in Fig. 125 required about one and a half hours.

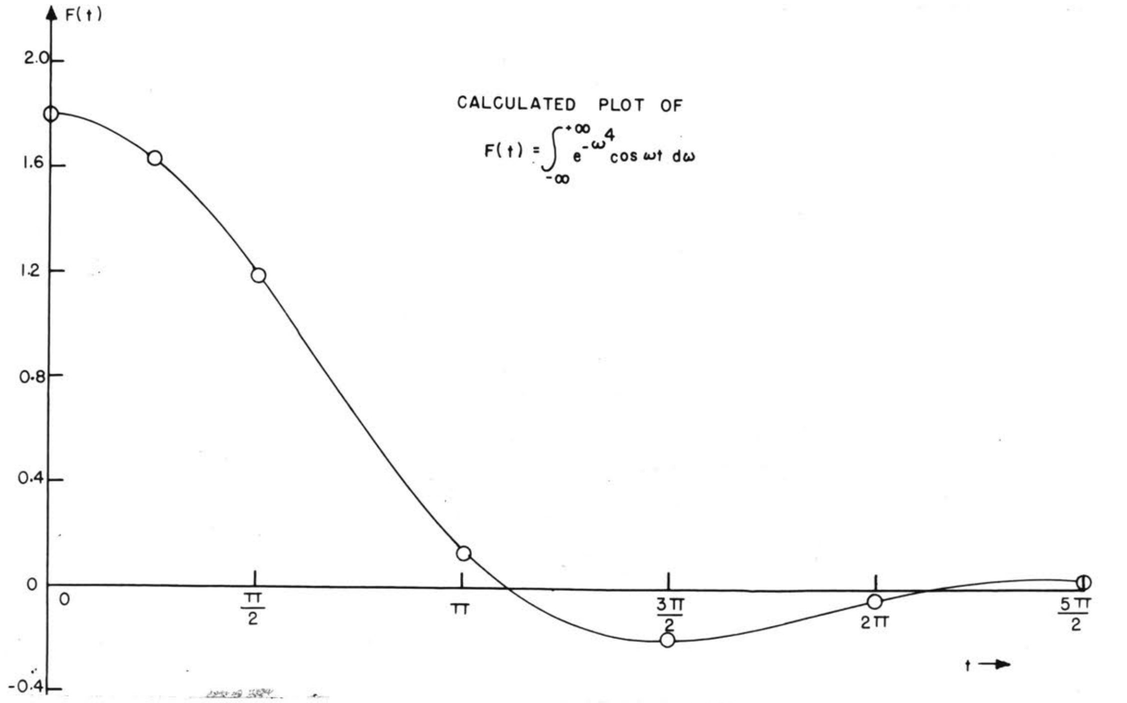

Fig. 124 Calculated curve of $f(t) = \int_{-\infty}^{\infty} e^{-\omega^4} \cos \omega t \, d\omega$.

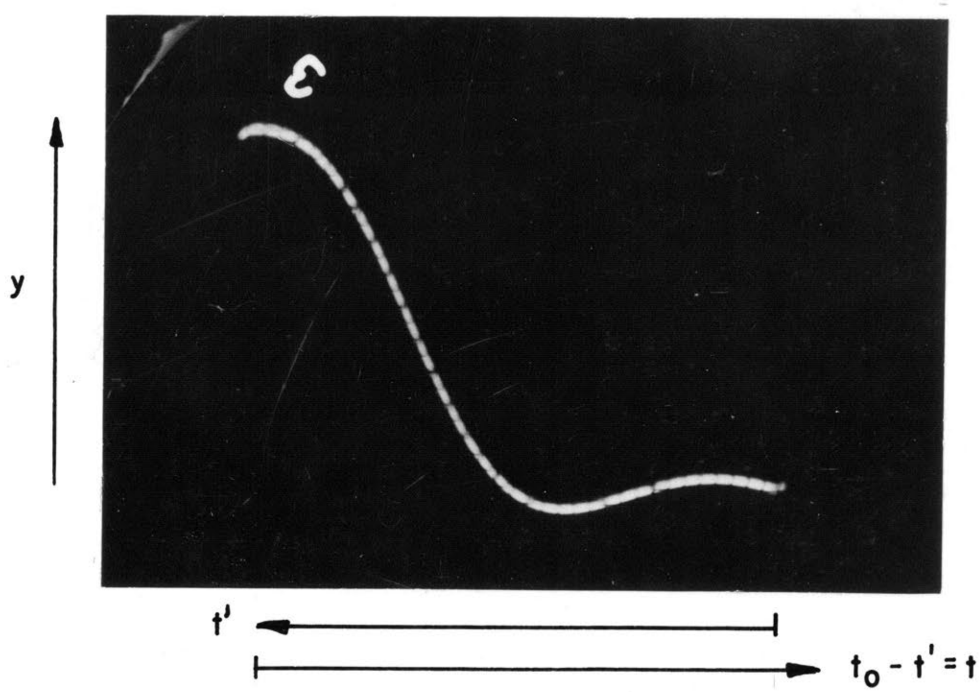

Fig. 125 A solution of the equation $\frac{d^3y}{dt'^3} + \frac{y}{4}(t_o - t') = 0$ versus $t_o - t'$.

SECTION VII

SUMMARY

An electronic differential analyzer has been developed and tested which can handle ordinary differential equations of orders through the fourth. Typical differential equations including examples of the non-linear and variable coefficient types have been solved. The observed precision of operation is extremely good, ranging from .002 to 0.1% depending upon the equation solved. The accuracy is between 1 and 5%, which is completely adequate for many applications in engineering, physics and mathematics.

The flexibility and speed of the electronic differential analyzer permits rapid investigation of wide ranges of equation parameters and solution initial conditions; this speed also renders feasible solution of equations for which the final rather than the initial values of the solution are known.

An analysis of the influence of high- and low-frequency limitations of the differential analyzer components has been made. This analysis permits quantitative determination of errors in the solution of ordinary differential equations with constant coefficients and has been verified experimentally.

The work on this electronic differential analyzer could be extended in a number of directions. (1) Additional components, to permit solution of more complicated differential equations, could be built. (2) Further development of the components for multiplication and division would appear worthwhile. In particular the modification of the crossed-fields multiplier for direct division appears worthy

of further investigation. (3) It would be desirable to apply the electronic differential analyzer to the solution of new problems in physics, engineering, and mathematics. The problems discussed in this thesis have all been solved by analytic means; these were chosen to permit verification of the operation of the analyzer.

Biography

Alan B. Macnee was born in New York City on September 19, 1920. He completed his secondary school education at the Holderness School, in 1938.

He received his S.B. and S.M. degrees from the Massachusetts Institute of Technology in February, 1943. While attending Massachusetts Institute of Technology he worked three terms in the Bell Telephone System as part of the Cooperative Course in Electrical Engineering.

From February 1943 to December 1945 the author worked as a Staff Member in the M.I.T. Radiation Laboratory. His work in the Receiver Group of that laboratory was concerned with the development of low-noise wide-band i-f amplifiers.

Since March 1946 the author has been enrolled in the Graduate School of Massachusetts Institute of Technology. In September 1946 the author joined the staff of the M.I.T. Research Laboratory of Electronics as a Research Assistant; more recently he has been advanced to the position of Research Associate.

The author wrote two chapters on VHF receivers for Volume 23 of the Radiation Laboratory Series, Microwave Receivers. He has also written an article on "A Low-noise Amplifier" together with Professor H. Wallman and C.P. Gadsden which was published in the June, 1948 issue of the Proceedings of the Institute of Radio Engineers.

Mr. Macnee is a member of Eta Kappa Nu and an associate member of Sigma Xi and the Institute of Radio Engineers.