SOME RESULTS ON PRINCIPAL SERIES FOR $GL(n,\mathbb{R})$

BY

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ABSTRACT

We prove reducibility criteria for generalized principal series representations of $GL(n,\mathbb{R})$ and give a procedure for computing the Langlands parameters of the constituents in the Jordan-Hölder series for principal series representations of $GL(n,\mathbb{R})$. Finally we give a description of the unitary dual of $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$.

Thesis Supervisor: Bertram Kostant
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CONTENTS

INTRODUCTION

In 1973, R. Langlands worked out a classification of all irreducible quasisimple representations for connected semisimple real algebraic groups G. The representations are uniquely determined by their "Langlands parameter" (P,ξ,ν), where P is a parabolic subgroup with Langlands decomposition $P = MAN$, ξ is an irreducible tempered representation of M, and $e^{-\nu}$ is a dormant character of A. Then the representation with Langlands parameter (P,ξ,ν) is equivalent to the restriction of the generalized principal series representation $\mathrm{ind}_P^G(\xi \otimes e^{-\nu})$ to the unique minimal invariant subspace. [12]

These results raise the following natural questions:

A) When are the generalized principal series representation reducible?

B) If they are reducible, what are the Langlands parameters of the representations occuring in the Jordan-Hölder series and what are the corresponding multiplicities?

In my thesis I give necessary and sufficient irreducibility criteria for generalized principal series of

GL(n,ℝ) [Theorem II.D.1]. Reducibility occurs iff the character is contained in a union of hyperplanes, which satisfy certain invariance conditions under the Weyl group of the parabolic. The proof is based on a reduction to computations for certain subgroups, which play a role similar to that of the rank 1 subgroups in the spherical case.

By refining the methods used to answer question A), I give a procedure to compute the Langlands parameters of the representations in the Jordan Hölder series for an arbitrary generalized principal series representation of GL(n,ℝ) [I.F].

This procedure applied to SL(4,ℝ) together with some multiplicity considerations for K-types allows us to give an explicit example of a principal series representation whose Jordan Hölder series contains a certain irreducible representation with multiplicity two. [Remarks: Page 112]. Furthermore using some results about unitarity which show that representations of GL(n,ℝ) whose continuous parameter is "too large" cannot be unitary [I.G.5], we obtain a classification of all unitary representations of GL(3,ℝ), GL(4,ℝ). [II.B.8][II.C.11]

The reducibility question was studied by B. Kostant in the spherical case [11] and by Bruhat [2] and Knapp-Stein [9,b,c] in the unitary case. N. Wallach generalized

Kostant's methods to arbitrary $\xi \in M$ for M compact and $G = SL(n,\mathbb{R})$ n odd, [19,a], whereas the complex case was solved by Parthasarthy, R. Ranga Rao and V. Varadarajan [13].

The composition series problem was solved recently by N. Wallach for all rank 1 groups and all principal series representations with nonsingular A character [19b]. J. A. Fomin obtained a complete answer for $SL(3,\mathbb{R})$ [6].

About unitarity not much is known. M. Duflo classified all irreducible representations for complex simple Lie groups of rank 2 [5], and I. Vakhutinskii classified the unitary dual of $SL(3,\mathbb{R})$ [17]. For a precise statement of the present knowledge see [10b].

The thesis is organized as follows: It is divided into two parts. In Chapter I we first present the relevant definitions and results (Sec A,B,C) which are then used to reduce the reducibility problem and the Jordan-Hölder series problem to the corresponding problems for $GL(2,\mathbb{R})$, $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$ (Sec D,E,F). The last section of Chapter I is devoted to deducing some results about unitarity.
The second part contains the calculations of Jordan Hölder

series for $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$ (Sec B,C). Using these results, we then give a classification of unitary representations for $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$ (Sec. B,C). Complete proofs are given even for the known results on $GL(3,\mathbb{R})$ since our methods are based on an entirely different approach. In the last section of Chapter II we derive a closed formula for reducibility of generalized principal series representations of $GL(n,\mathbb{R})$.

At this point, I want to acknowledge the influence and assistance of a number of people in the department of mathematics. I would especially like to thank my advisor Professor Kostant for calling my attention to reducibility questions. Professor Segal's seminar was a good training ground for many of the techniques and ideas used in this thesis. David Vogan's constant interest and encouragement during this work and the many long discussions with him were extremely helpful.

I also would like to thank Professor Harder from the University of Wuppertal (Germany) for introducing me to representation theory and supporting my plans to continue my research in the United States.

Finally I would like to thank Michael Forger for transforming my "German English" into English (and thus making the set of readers nonempty) and Marjorie Zabierek for her patient and excellent typing.

CHAPTER I

A. Notation and some technical Lemmata.

Let $G = GL(n,\mathbb{R})$, $G_{\mathbb{C}} = GL(n,\mathbb{C})$, and consider G as a subgroup of $G_{\mathbb{C}}$. Let $r \in \mathbb{N}$ and

$$C_r = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\theta_r & \sin\theta_r & & & \\ & & & -\sin\theta_r & \cos\theta_r & & & \\ & & & & & \epsilon_1 & & \\ & & & & & & \ddots & \\ & & & & & & & \epsilon_{n-2r} \end{pmatrix} \times \begin{pmatrix} a_1 & & & & & & & \\ & a_1 & & & & & & \\ & & \ddots & & & & & \\ & & & a_r & & & & \\ & & & & a_r & & & \\ & & & & & b_1 & & \\ & & & & & & \ddots & \\ & & & & & & & b_{n-2r} \end{pmatrix} \right.$$

$$\left| \theta_i \in [0,2\pi), I = 1,\ldots,r,\ \epsilon_i = \pm 1,\ i = 1,\ldots,n-2r,\ a_1, b_i \in \mathbb{R}^+ \right\}$$

C_r is a Cartan subgroup of G, and every Cartan subgroup of G is conjugate to some C_r.

Let $N(C_r)$ be the normalizer and $Z(C_r)$ be the centralizer of C_r in $G_{\mathbb{C}}$, and define $W_r^{\mathbb{C}} = N(C_r)/Z(C_r)$. Then

$$W_r^{\mathbb{C}} \cong \sigma_r \times (\mathbb{Z}_2)^r \times \sigma_{n-2r} .$$

The factor σ_r is the permutation group of the a_i's, the factor σ_{n-2r} is the permutation group of the b_i's , and $(\mathbb{Z}_2)^r$ describes the direction changes in the $SO(2,\mathbb{R})$ factors.

A character of C_r is called nonsingular if the restriction to each $SO(2,\mathbb{R})$ factor is nontrivial. The group $W_r^{\mathbb{C}}$ operates on the set $\hat{C}_r$ of all nonsingular characters as well.

The next paragraph will be devoted to explain the following theorem and some of its consequences.

<u>Theorem 1</u>. The set $\hat{G}$ of all irreducible quasisimple representations of G can be parametrized by $\hat{C}_r/W_r^{\mathbb{C}}$, $1 \leq r \leq n/2$.

This is essentially due to Langlands in a very different formulation, and in this formulation due to D. Vogan.

In order to explain this theorem, we have to establish some notations.

Let $A_r \cong \mathbb{R}^{n-r}$ be the vector part of C_r and $\mathfrak{t}_r, \mathfrak{a}_r$ be the corresponding Lie algebras. Let Σ be the roots of $(\mathfrak{t}_r \otimes \mathbb{C})'$ and Δ be a set of simple roots. The restriction of Σ to $\mathfrak{a}_r$ is denoted by Σ_r .

<u>Definition</u>. Let $\alpha \in \Sigma_r$. Then α has multiplicity n if there are n roots β in Σ s.t. $\beta|_{\mathfrak{a}_r} = id$. We write $mult(\alpha)$ for the multiplicity of α .

Let N_r be the normalizer of A_r and Z_r the centralizer of A_r in G. The Weyl group W_r of A_r in G is defined as N_r/Z_r. It is easy to see that

$$W_r \cong \sigma_r \times \sigma_{n-2r}$$

and

$$W_r \times (\mathbb{Z}_2)^r = W_r^{\mathbb{C}} .$$

The Weyl group W_r acts on Σ_r through the action induced by the adjoint representation.

Lemma 2. Each $\alpha \in \Sigma_r$ has multiplicity 1, 2, or 4. The Weyl group W_r operates transitively on the restricted roots of multiplicity 1, 4.

Proof. Let $\tilde{\mathfrak{a}}_r = \{H \in \mathfrak{a}_r \mid \operatorname{tr} H = 0\}$. We may assume that we have a set Δ of simple roots s.t. the restrictions of $\alpha_1, \alpha_3, \ldots, \alpha_{2r-1}$ to $\tilde{\mathfrak{a}}_r$ are zero. Then

$$\tilde{\mathfrak{a}}'_{\mathbb{R}} = \sum_{\alpha \in \Delta} \mathbb{R}\,\alpha\Big|_{\tilde{\mathfrak{a}}_r}$$

and the restrictions of the simple roots, if nonzero, are linearly independent.

Each positive root is of the form $\sum_{i_o}^{k_o} \alpha_i = \beta$, $\alpha_i \in \Delta$ and $i_o \leq k_o$. We write $\beta = \beta_I + \beta_{II}$, where

$$\beta_I = \sum_{io}^{2r} \alpha_i$$

$$\beta_{II} = \sum_{2r+1}^{\ell_o} \alpha_i$$

Hence $\beta_I|_{\sigma_r} \neq \beta_I$ and $\beta_{II}|_{\sigma_r} = \beta_{II}$. Furthermore if $\beta_I, \beta_{II} \neq 0$, then $\beta_I|_{\sigma_r}$ and $\beta_{II}|_{\sigma_r}$ are linearly independent.

Case 1: $i_o > 2r$

Claim: $\text{mult}(\beta) = 1$.

We have $\beta = \beta_{II} = \beta|_{\sigma_r}$. Let $\gamma = \gamma_I + \gamma_{II}$ be another positive root. Then γ_{II} and β are linearly independent unless $\gamma_{II} = \beta$. On the other hand, if $\gamma_I|_{\sigma_r} \neq 0$, then $\gamma_I|_{\sigma_r}$, $\gamma_{II}|_{\sigma_r}$, β are linearly independent. Hence γ and β are linearly independent. Thus $\gamma|_{\sigma_r}$ and $\beta|_{\sigma_r}$ are linearly independent unless $\gamma = \gamma_{II} = \beta$.

We notice that the roots β with $i_o > 2r$ can be considered as the positive roots for a subalgebra of type A_{n-2r-1} with Cartan subalgebra

$$\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & b_1 & & \\ & & & & \ddots & \\ & & & & & b_{n-2r} \end{pmatrix} \begin{matrix} \}\ 2r \\ \\ \}\ n-2r \end{matrix}$$

W_r permutes these roots transitively.

Case 2: $i_o > 2r$, $k_o < 2r$.

Claim: $\text{mult}(\beta) = 2$.

We have $\beta = \beta_I + \beta_{II}$, $\beta_I \neq 0$, $\beta_{II} \neq 0$ and β_I, β_{II} linearly independent. Let $\gamma = \gamma_I + \gamma_{II}$ be another positive root. Then $\gamma|_{\alpha_r} = \beta|_{\alpha_r}$ iff $\gamma_I|_{\alpha_r} = \beta_I|_{\alpha_r}$ and $\gamma_{II} = \beta_{II}$. But $\gamma_I|_{\alpha_r} = \beta_I|_{\alpha_r}$ iff $\gamma_I = \beta_I \pm \alpha$ with $\alpha \in \Delta$, $\alpha|_{\alpha_{\mathbb{R}}} = 0$. This leaves us with 2 possibilities for γ_I .

Case 3: $0 \leq i_o, k_o < 2r$.

Claim: $\text{mult}(\beta) = 4$.

In this case $\beta = \beta_I$. Let $\gamma = \gamma_I + \gamma_{II}$. Then $\gamma|_{\alpha_r} = \beta|_{\alpha_r}$ iff $\gamma_{II} = 0$ and $\gamma_I|_{\alpha_r} = \beta|_{\alpha_r}$. Vut this is only possible if $\gamma_I = \beta \pm \delta_1 \pm \delta_c$. Where $\delta_1, \delta_2 \in \Delta$ and $\delta_1|_{\alpha_r} = \delta_2|_{\alpha_r} = 0$. This leaves us with 4 possibilities for γ_I .

The roots with $0 < i_o, k_o < 2r$ restricted to α_r are multiples of roots for a subalgebra of type A_{r-1} with Cartan subalgebra

$$\begin{pmatrix} a_1 & & & & & & & \\ & a_1 & & & & & & \\ & & \ddots & & & & & \\ & & & a_r & & & & \\ & & & & a_r & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \begin{matrix} \left.\right\} 2r \\ \left.\right\} m-2r \end{matrix}$$

and are therefore permuted transitively by W_r . □

<u>Lemma 3</u>. There are $2r(n-2r)$ roots of multiplicity 2 in Σ_r .

<u>Proof</u>. Each such root is the restriction of a root of the form

$$\pm \sum_{io}^{k_o} \alpha_i \quad , \quad 1 \leq i_o < r \ , \quad k_o \geq 2r \ . \qquad \square$$

We showed furthermore that W_r operates transitively on the sets of $r(n-2r)$ roots of multiplicity 2, where exactly i roots are restrictions of negative roots. This implies

<u>Corollary 4</u>: Let $\mathcal{C}_r^i$ be a connected component of nonsingular elements in $\mathfrak{a}_r'$ (with respect to the restricted root system Σ_r) s.t. $\mathcal{C}_r^i$ is negative with respect to exactly i roots of multiplicity 2. Then W_r acts

transitively on the set of $\mathcal{C}_r^i$'s .

<u>Definition</u>. A Weyl chamber of $\mathfrak{a}_r'$ is a connected component of nonsingular elements of $\mathfrak{a}_r'$.

Hence we get the following

<u>Corollary 5</u>. $|\{\text{Weyl chambers}\}/W_r| = r(n-2r)$.

<u>Definition</u>. Let $\mathcal{C}_r$ be a Weyl chamber and $\Sigma_r^{\mathcal{C}}$ be the corresponding set of positive roots. The <u>Weyl chamber</u> $\mathcal{C}_r^o$ <u>opposite to</u> $\mathcal{C}_r$ is the Weyl chamber which is positive with respect to $-\Sigma_r^{\mathcal{C}}$.

<u>Remark</u>. $\mathcal{C}_r$ and $\mathcal{C}_r^o$ are usually not conjugate under W_r as one can easily see in the example $GL(3,\mathbb{R})$. In fact, if $r = 1$, then $|\Sigma_r| = 2$ and $W_r = \{id\}$.

One special and remarkable example of a restricted root system is the following: Let $G = SL(2n,\mathbb{R})$ and $r = n$. Then the restricted root system is of type A_{n-1} , and each root has multiplicity 4. Therefore all Weyl chambers are conjugate. If we take the scalar product on $\mathfrak{a}_r$ induced by the scalar product on $\mathfrak{a}_o$, the restricted roots have length 1.

Let γ_r be the Cayley transform $C_r \to C_o$. Then we have $\gamma_r(\mathfrak{a}_r) = \mathfrak{a}_r$, and we can consider W_r as a subgroup of W_o. Let $\mathcal{C}_o$ be the Weyl chamber corresponding to the upper triangular matrices, and let $\mathcal{C}(r)$ be the set of all connected boundary components of $\mathcal{C}_o$ which are singular with respect to exactly r strictly orthogonal roots. Using that $\overline{\mathcal{C}}$ is a fundamental domain for W_o on $\mathfrak{a}_o$, we deduce:

Lemma 6.

a) Each $\mathcal{C}_r^i$ is conjugate to an element in $\mathcal{C}(r)$ under W_o.

b) $\mathcal{C}_r^i$, $\mathcal{C}_r^j$ are conjugate under W_o iff $i = j$.

c) Let $a \in \mathfrak{a}_r$. There exist $w \in W_o$ and $\underline{\mathcal{C}}_r \in \mathcal{C}(r)$ s.t. $wa \in \overline{\underline{\mathcal{C}}}_r$.

Now let $\alpha \in \Sigma_r$. Then write $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} | \; [H,x] = \alpha(H)X$ for all $H \in \mathfrak{a}_o\}$, and for $\mathcal{C}_r \subset \mathfrak{a}_r$, define $\mathfrak{n}_r^{\mathcal{C}} = \sum_{\alpha \in \Sigma_r^{\mathcal{C}}} \mathbb{R}\mathfrak{g}^\alpha$ and $N_{\mathcal{C}_r} = \exp \mathfrak{n}_r^{\mathcal{C}}$.

Let L_r be the centralizer of A_r in G. The subgroup $\underline{P}_{\mathcal{C}_r} = L_r N_{\mathcal{C}_r}$ will be called the parabolic subgroup associated to $\mathcal{C}_r$. It is well known that in this way we get all parabolics which have A_r as a split component in the Langlands decomposition. We extend the Weyl group action from the Weyl chambers to the parabolics as follows: Let m_w be a representative of w in the normalizer of A_o, then $w \cdot \underline{\underline{P}}_{\mathcal{C}_r} = \underline{\underline{P}}_{w\mathcal{C}_r} = m_w \underline{\underline{P}}_{\mathcal{C}_r} m_w^{-1}$, and hence two parabolics are conjugate if the corresponding Weyl chambers are conjugate. It is not hard to show that this condition is necessary and sufficient [20].

Corollary 7. There are $r(n-2r)$ orbits of W_r on the set of parabolics with split component A_r.

Definition. We call a parabolic a standard parabolic if it contains the upper triangular matrices.

Corollary 8. Each parabolic $\underline{\underline{P}}_{\mathcal{C}_r}$ is conjugate to exactly one standard parabolic under W_o.

Examples.

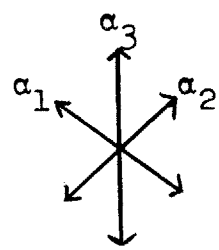
a) $G = GL(3,\mathbb{R})$, $r = 1$. The two standard parabolics are the matrices

$$P_1 = \begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & x \end{pmatrix} \qquad P_2 = \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix}$$

P_1 is the parabolic opposite to P_2, i.e. the parabolics are associated to opposite Weyl chambers.

b) $G = GL(4,\mathbb{R})$, $r = 1$. We have $|\Sigma_r| = 6$. There are four restricted roots with multiplicity 2 and two roots with multiplicity 1. But $W_1 \cong \sigma_2$. Therefore there are three conjugacy classes of Weyl chambers and hence three conjugacy classes of standard parabolics.

The graph of the restricted root system is as follows:



Let us consider $\alpha_1, \alpha_2, \alpha_3$ as positive roots. The corresponding parabolic is then conjugate to its opposite parabolic, since the corresponding Weyl chambers are conjugate under the Weyl group. The other two parabolics are not conjugate to their opposite parabolics.

For later use, the following convention is introduced: Let $\mathcal{C}_r$ be a Weyl chamber in $\mathfrak{a}'_r$ and $\underline{\underline{P}}_{\mathcal{C}_r}$ the parabolic associated to $\mathcal{C}_r$. We call the standard parabolic $P_{\mathcal{C}_r}$ conjugate to $\underline{\underline{P}}_{\mathcal{C}_r}$ the standard parabolic associate to $\mathcal{C}_r$.

Let $P_{\mathcal{C}_r} = M_pA_pN_p$ be the standard parabolic associated to a Weyl chamber $\mathcal{C}_r \subset \mathfrak{a}'_r$. Then there is a set Σ_r^p of r strictly orthogonal simple roots such that $\mathfrak{a}_p$ is just the intersection of their kernels. Now let $w \in W$ be such that $w(\Sigma_r^p)$ is again a set of simple strictly orthogonal roots. We associate to $w(\Sigma_r^p)$ a standard parabolic as follows:

Let $L_{w\Sigma_r^p}$ be the centralizer of $w\sigma_p$ and

$$\mathfrak{n}_{w\Sigma_r^p} = \bigoplus_{\substack{\alpha \in \Sigma^+ \\ \alpha > 0 \\ \alpha|_{w\sigma_p} \neq 0}} \mathfrak{g}^\alpha .$$

Define $N_{w\Sigma_r} = \exp \mathfrak{n}_{w\Sigma_r^p}$ and $P_{w\Sigma_r^p} = L_{w\Sigma_r^p} N_{w\Sigma_r^p}$. If $P_{\mathcal{C}_r} \neq P_{w\Sigma_r^p}$, then $P_{\mathcal{C}_r}$ and $P_{w\Sigma_r^p}$ are not associated to conjugate Weyl chambers.

To simplify notation, I introduce the convention

$$P_{w\Sigma_r^p} = wP_{\mathcal{C}_r} .$$

Now let $P_{\mathcal{C}_r}$ be a standard parabolic associated to a Weyl chamber $\mathcal{C}_r$. We construct a $w_{\mathcal{C}_r}$ s.t. $P_{\mathcal{C}_r}$ and $wP_{\mathcal{C}_r}$ are associated to opposite Weyl chambers. This construction will later be used in the definition of intertwining operators for generalized principal series.

Lemma 9. Let $\mathcal{C}_r \subset \overline{\mathcal{C}}_o$, $\mathcal{C}_r \in \mathcal{C}(r)$. There exists a $\tilde{w}_{\mathcal{C}_r} \in W_o$ s.t. $\tilde{w}_{\mathcal{C}_r}\mathcal{C}_r \subset -\overline{\mathcal{C}}_o$ and $\ell(\tilde{w}_{\mathcal{C}_r}) = \frac{1}{2}(n(n-1) - r)$, where ℓ denotes the length of w .

Proof. Let $\alpha_{i_1},\ldots,\alpha_{i_r}$ be the simple roots orthogonal to $\mathcal{C}_r$. If $\alpha_1 \notin \{\alpha_{i_1}\ldots\alpha_{i_r}\}$, define $\tilde{w}_1 = s_{\alpha_{n-1}}\ldots s_{\alpha_1}$; s_{α_i} is the reflection of the simple root α_i . Then $\ell(\tilde{w}_1) = n-1$, and $\tilde{w}_1\mathcal{C}_r$ is positive or zero with respect to $\alpha_1,\ldots,\alpha_{n-2}$, but negative or zero with respect to all positive roots not contained in the subsystem generated by $\alpha_1,\ldots,\alpha_{n-2}$. We have therefore reduced the problem to finding a suitable $\tilde{w}_{\mathcal{C}_r}$ in the Weyl group of $GL(n-1,\mathbb{R})$. If $\alpha_1 \in \{\alpha_{i_1},\ldots,\alpha_{i_r}\}$, define

$$\tilde{w}_1 = s_{\alpha_{n-1}}s_{\alpha_n}\ldots s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}s_{\alpha_2} .$$

Then $\ell(\tilde{w}_1) = 2(n-2) = (n-1) + (n-3)$, and $\tilde{w}_1\mathcal{C}_r$ is positive or zero with respect to all positive roots not contained in the subsystem generated by the first $n-3$ roots. This reduces the problem to solving the problem for $GL(n-2,\mathbb{R})$. Now we proceed by induction. □

Now rewrite $\tilde{w}_{\mathcal{C}_r}$ as follows:

a) Assume $\alpha_1 \notin \{\alpha_{i_1},\ldots,\alpha_{i_r}\}$. If $i \in \{i_1,\ldots,i_r\}$, write $w_i = s_{\alpha_i}s_{\alpha_{i-1}}$; if $i+1 \in \{i_1,\ldots,i_r\}$, write $w_i = id$; otherwise write $w_i = s_{\alpha_i}$. Then $\tilde{w}_1 = w_{n-1}\ldots w_1$ and $\ell(\tilde{w}_1) = \sum_1^{n-1} \ell(w_i)$.

b) Assume $\alpha_1 \in \{\alpha_{i_1},\ldots,\alpha_{i_r}\}$. If $i \in \{i_1,\ldots,i_r\}$, write $w_i = s_{\alpha_i} s_{\alpha_{i-1}} s_{\alpha_{i-2}} s_{\alpha_{i-1}}$; if $i, i+1 \notin \{i_1,\ldots,i_r\}$, write $w_i = s_{\alpha_i} s_{\alpha_{i-1}}$, and if $i+1 \in \{i_1,\ldots,i_r\}$, $w_i = \mathrm{id}$. Then $\tilde{w}_1 = w_{n-1},\ldots,w_1$ and $\ell(\tilde{w}_1) = \Sigma_1^{n-1} \ell(w_i)$.

Proceeding by induction on the rank of the root system, we thus construct a decomposition $\tilde{w}_{\mathcal{C}_r} = w_k \ldots w_1$ for $\tilde{w}_{\mathcal{C}_r}$ such that $\ell(\tilde{w}_{\mathcal{C}_r}) = \Sigma\, \ell(w_i)$, and for $k \geq i_o \geq 1$, $w_{i_o} \ldots w_1 \mathcal{C}_r$ is again singular with respect to r strictly orthogonal simple roots. □

Now let P be a standard parabolic associated to a Weyl chamber $\mathcal{C}_r \subset \mathfrak{a}_r'$, and let $\tilde{w} \in W$ s.t. $w\mathcal{C}_r \subset \mathcal{T}_o$. Define $w_{\mathcal{C}_r} = \tilde{w}\tilde{w}_{\mathcal{C}_r}$.

<u>Lemma 10</u>: $w_{\mathcal{C}_r} P$ is a standard parabolic associated to the Weyl chamber opposite to $\mathcal{C}_r$.

<u>Proof</u>. Let $\mathcal{C}_r' \subset \mathcal{T}_o \cap w_{\mathcal{C}_r} \mathfrak{a}_r'$ be the subset of elements singular with respect to exactly r simple roots. We have to show that $\tilde{w}^{-1} w_{\mathcal{C}_r}^{-1} \mathcal{C}_r'$ and $\mathcal{C}_r$ are opposite Weyl chambers. But this is equivalent to showing that $\tilde{w}\mathcal{C}_r$ and $w_{\mathcal{C}_r}^{-1} \mathcal{C}_r'$ are opposite, which is immediate from the definition of $W_{\mathcal{C}_r}$. □

Using the decomposition $\widetilde{w}_{\mathcal{C}_r} = w_k \cdots w_1$ given above, we can now construct a chain

$$P \ , \quad w_1 P \ , \quad w_2 w_1 P \ , \ldots , \ \widetilde{w}_{\mathcal{C}_r} P$$

of standard parabolics joining P and $w_{\mathcal{C}_r} P$.

B. Classification of representations.

The character group $\hat{C}_r$ of C_r is isomorphic to $(\mathbb{Z}\setminus 0)^r \times \mathbb{C}^{n-r} \times \mathbb{Z}_2^{n-2r}$. For each $\chi \in 1 \times \mathbb{C}^{n-r} \times 1 \subset \hat{C}_r$ we define a parabolic as follows: If $\mathrm{Re}(\log \chi)$ is contained in the interior of a Weyl chamber $\mathcal{C}_r$, define $\underline{\underline{P}}_\chi$ to be the parabolic associated to $\mathcal{C}_r$. Otherwise there is a minimal number of Weyl chambers $\mathcal{C}_r^1, \ldots, \mathcal{C}_r^\ell$ which contain χ in their closure. Then define $\underline{\underline{P}}_\chi$ to be the smallest parabolic which contains all the parabolics associated to these Weyl chambers. If χ is the trivial character, then of course $\underline{\underline{P}}_\chi = G$.

Let $\underline{\underline{P}}_\chi = \underline{\underline{M}}_\chi \underline{\underline{A}}_\chi \underline{\underline{N}}_\chi$ be the Langlands decomposition of $\underline{\underline{P}}_\chi$. We consider χ as a character on $A_\chi N_\chi$ by $\chi(an) = \chi(a)$, $a \in A_\chi$, $n \in N_\chi$.

Now assume $\mathrm{Re}(\log \chi)$ is nonsingular. Then the connected component of $\underline{\underline{M}}_\chi$ is just the product of r copies of $SL(2,\mathbb{R})$, i.e.

$$\underline{\underline{M}}_\chi^o = \left\{ \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_r & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \middle| \; A_i \in SL(2,\mathbb{R}),\ i = 1,\ldots,r \right\}.$$

Define

$$Z_r = \left\{ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \epsilon_{2r+1} & & \\ & & & & \ddots & \\ & & & & & \epsilon_n \end{pmatrix} \;\middle|\; \epsilon_i = \pm 1 \right\}$$

and $\underline{\underline{M}}_\chi^\# = \underline{\underline{M}}_\chi^0 \times Z_r$.

To each $\vec{r} \in (\mathbb{Z}\backslash 0)^r \times 1 \times 1 \subset \hat{C}_r$ we associate a representation $\underline{\underline{\Pi}}_{\vec{r}}$ of $\underline{\underline{M}}_\chi^0$ as follows: If $\vec{r} = (r_1, \ldots, r_r) \times 1 \times 1$, then

$$\underline{\underline{\Pi}}_{\vec{r}} = \Pi_{r_1} \otimes \ldots \otimes \Pi_{r_r}$$

where Π_{r_i} is the discrete series representation with Harish-Chandra parameter r_i . The representation Π_{r_i} is characterized by its minimal K-type (in the sense of Vogan-Schmid), which is

$$\begin{array}{ll} n+1 & \text{if } n > 0 \\ n-1 & \text{if } n < 0 \end{array} .$$

Let $\bar{\rho} = 1 \otimes 1 \otimes \rho \in 1 \otimes 1 \otimes \mathbb{Z}_2^{n-2r} \subset \hat{C}_r$ be a character of Z_r . We define a representation $\underline{\underline{\Pi}}^\#_{\vec{r},\bar{\rho}}$ of $\underline{\underline{M}}_\chi^\#$ by

$$\underline{\underline{\Pi}}^\#_{\vec{r},\bar{\rho}} = \underline{\underline{\Pi}}_{\vec{r}} \otimes \bar{\rho}$$

and a representation $\underline{\Pi}_{\vec{r},\rho}$ of $\underline{\underline{M}}_{\chi}$ by

$$\underline{\Pi}_{\vec{r},\rho} = \operatorname{ind}_{\underline{\underline{M}}_{\chi}^{\#}}^{\underline{\underline{M}}_{r}} \underline{\Pi}^{\#}_{\vec{r},\bar{\rho}} \quad .$$

Then $\underline{\Pi}_{\vec{r},\rho}$ is irreducible and equivalent to $\underline{\Pi}_{|\vec{r}|,\rho}$, where $|\vec{r}| = (|r_1|,\ldots,|r_r|) \times 1 \times 1$.

We now define representations of $\underline{\underline{P}}_{\chi}$ and G as follows: Let $\mu = (\vec{r},\chi,\rho) \in \hat{C}_r$, then

$$\underline{\Pi}_{\mu} = \underline{\Pi}_{\vec{r},\rho} \otimes \overline{\chi}^{-1}$$

and

$$\underline{U}_{\mu} = \operatorname{ind}_{\underline{\underline{P}}_{\mu}}^{G} \underline{\Pi}_{\mu} \quad .$$

The induced representation is defined in such a way that G acts on the left and that the representation is unitary if χ is unitary. $\underline{U}_{\mu}$ and $\underline{U}_{\mu'}$ are equivalent iff $\mu = w\mu'$ with $w \in W_r^{\mathbb{C}}$. Since if $\chi = w\chi'$ for $w \in W_r$, P_{χ} and $P_{\chi'}$ are conjugate to the same standard parabolic, we can get rid of the ambiguities by assuming that $\underline{U}_{\mu}$ is induced from this standard parabolic and that $\vec{r} = |\vec{r}|$.

Now assume $\mu = (\vec{r},\chi,\rho)$ with $\mathrm{Re}(\log \chi)$ singular. Let $\underline{\underline{P}}_{\chi} = \underline{\underline{M}}_{\chi}\underline{\underline{A}}_{\chi}\underline{\underline{N}}_{\chi}$ be the Langlands decomposition of $\underline{\underline{P}}_{\chi}$. Then $C_r \cap \underline{\underline{M}}_{\chi}$ is a Cartan subgroup of $\underline{\underline{M}}_{\chi}$. Now we define a representation of $\underline{\underline{M}}_{\chi}$ as follows:

Let P_r be the standard parabolic with Langlands decomposition

$$P_r = M_r A_r N_r \ .$$

Then $P_r \cap \underline{\underline{M}}_\chi$ is a parabolic of $\underline{\underline{M}}_\chi$ and $P \cap \underline{\underline{M}}_\chi = \underline{\underline{M}}_r(A_r \cap \underline{\underline{M}}_\chi)(N_r \cap \underline{\underline{M}}_\chi)$.

We consider χ as a unitary character on $(A_r \cap \underline{\underline{M}}_\chi)(N_r \cap \underline{\underline{M}}_\chi)$ by

$$\chi(an) = \chi(a) \qquad a \in \underline{\underline{M}}_\chi \cap A_r, \ N \in N_r \cap \underline{\underline{M}}_r \quad .$$

Now define

$$(\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi} = \underline{\Pi}_{\vec{r},\rho} \otimes \chi|_{A_r N_r \cap \underline{\underline{M}}_\chi}$$

and

$$(\underline{\underline{U}}_\mu)_{\underline{\underline{M}}_\chi} = \mathrm{ind}_{\underline{P}_r \cap \underline{\underline{M}}_\chi}^{\underline{\underline{M}}_\chi} (\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi} \quad .$$

Then $(\underline{\underline{U}}_\mu)_{\underline{\underline{M}}_\chi} \otimes \overline{\chi}^{-1}|_{\underline{\underline{A}}_\chi \underline{\underline{N}}_\chi}$ is an irreducible representation of $\underline{\underline{P}}_\chi$, and we define

$$\underline{\underline{U}}_\mu = \mathrm{ind}_{\underline{\underline{P}}_\chi}^{G} (\underline{\underline{U}}_\mu)_{\underline{\underline{M}}_\chi} \otimes \overline{\chi}^{-1}|_{\underline{\underline{A}}_\chi \underline{\underline{N}}_\chi} \quad .$$

Again $\underline{\underline{U}}_\mu$ and $\underline{\underline{U}}_{\mu'}$ are equivalent iff $\mu = w\mu'$ with $w \in W_r^{\mathbb{C}}$. We will always consider $\underline{\underline{U}}_\mu$ as induced from a cuspidal parabolic as follows: Let $\mathcal{C}_r$ be a Weyl chamber s.t. $\log \chi \in \overline{\mathcal{C}}_r$, and let $\underline{\underline{P}}_{\mathcal{C}_r}$ be the associate parabolic. Then $\underline{\underline{P}}_{\mathcal{C}_r} \cap \underline{\underline{M}}_\chi$ is a cuspidal parabolic of $\underline{\underline{M}}_\chi$. The parabolics $\underline{\underline{P}}_{\mathcal{C}_r} \cap \underline{\underline{M}}_\chi$ for different choices of $\mathcal{C}_r$ are usually not conjugate but nevertheless for each $(\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi}$, there is a representation $(\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi}^{\mathcal{C}_r}$ s.t.

$$\operatorname{ind}_{\underline{\underline{P}}_{\mathcal{C}_r} \cap \underline{\underline{M}}_\chi}^{\underline{\underline{M}}_\chi} (\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi}^{\mathcal{C}_r} \cong \operatorname{ind}_{P_r \cap \underline{\underline{M}}_\chi}^{\underline{\underline{M}}_\chi} (\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi} ,$$

and hence

$$\underline{\underline{U}}_\mu = \operatorname{ind}_{\underline{\underline{P}}_{\mathcal{C}_r}}^{G} (\underline{\underline{\Pi}}_\mu)_{\underline{\underline{M}}_\chi}^{\mathcal{C}_r} \otimes \overline{\chi}^{-1}_{\underline{\underline{A}}_\chi \underline{\underline{N}}_\chi} .$$

Again we consider $\underline{\underline{U}}_\mu$ as induced from a standard parabolic and assume $\vec{r} = |\vec{r}|$.

By a theorem of Langlands [12] the representation $\underline{\underline{U}}_\mu$ contains exactly one minimal invariant subspace.

<u>Definition</u>. $\underline{\underline{U}}_\mu$ restricted to this minimal invariant subspace will be called <u>the representation</u> J_μ <u>with Langlands parameter</u> μ .

A result of Langlands [12] now asserts that each irreducible quasisimple representation of G is isomorphic to one of the representations J_μ .

More generally, according to Langlands, all irreducible quasisimple representations of a connected semisimple matrix group are parametrized by 3 parameters:

a) a parabolic subgroup $P = M_p A_p N_p$ containing a fixed minimal parabolic subgroup P_o ,

b) an equivalence class of irreducible tempered representations of M_p with π as a representative,

c) a complex-valued linear function ν on the Lie algebra α_p such that Re ν is in the interior of the negative Weyl chamber.

The Langlands representation $J_p(\pi,\nu)$ is then the restriction of the representation $\mathrm{ind}_p^G(\pi \otimes \nu) = e^\nu$ to the unique minimal invariant subspace.

Theorem 1. Langlands [12].

The representations $J_p(\pi,\nu)$ are irreducible, quasisimple, infinitesimally inequivalent and exhaust the irreducible quasisimple representations of G .

The following theorem tells us what the tempered representations π appearing in the above theorem look like.

Theorem 2. Harish Chandra Trombi [16]:

Every irreducible tempered representation of G is infinitesimally equivalent with a constituent of some representation unitarily induced from a limit of discrete series representationsof a cuspidal parabolic.

For $SL(n,\mathbb{R})$, this theorem means that each tempered representation is equivalent to a constituent of the restriction of a representation J_μ, with $\mu = (\vec{r},\chi,\rho)$ and $Re(\log \chi) = 0$, to $SL(n,\mathbb{R})$.

Theorem 3. Knapp-Stein [$9a$].

a) The restriction of J_μ to $SL(n,\mathbb{R})$, is irreducible if n is odd.

b) The restriction of J_μ to $SL(n,\mathbb{R})$, n even, splits into at most two inequivalent pieces.

Studying case b) more closely and using the previous considerations, one can show that if the representation J_μ restricted to the connected component $GL_+(n,\mathbb{R})$ becomes reducible, there are at most two constituents and both are inequivalent. Moreover the representation of G is then uniquely characterized by the property that its restriction to $GL_+(n,\mathbb{R})$ contains one of them. However, if its restriction to $GL_+(n,\mathbb{R})$ is irreducible, then the representation is not uniquely determined by this restriction because there are two inequivalent extensions.

The following theorem is a special case of a theorem of Knapp-Zuckerman [10a] and determines the equivalence classes of tempered irreducible representations.

Theorem 4. [Knapp-Zuckerman]

Let $\mu = (\vec{r}, \chi, \rho)$ with χ unitary, then $U_\mu \cong U_{\mu'}$ iff $\mu = w\mu'$, $w \in W_r$.

This completes the rough sketch of the ideas involved in proving the classification theorem.

Finally we recall some definitions and theorems of [18].

Let V_λ be an irreducible representation of $K = O(n)$. If V_λ restricted to $SO(n)$ is an irreducible representation with highest weight λ, we define λ to be the highest weight of the representation V_λ of $O(n)$. If V_λ restricted to $SO(n)$ is the direct sum of two irreducible representations with highest weights λ_1, λ_2, we pick the highest weight of one component as follows: In this case $n = 2m$. We enumerate the simple roots α_i, $i = 1, \ldots, m$ as in Bourbaki [1]. Then

$$(\alpha_{m-1}, \lambda_1) = (\alpha_m, \lambda_2)$$

$$(\alpha_{m-1}, \lambda_2) = (\alpha_m, \lambda_1)$$

$$(\alpha_{m-1}, \lambda_1) \neq (\alpha_m, \lambda_1) .$$

We now choose $\lambda \in \{\lambda_1, \lambda_2\}$ such that $(\alpha_n, \lambda) > (\alpha_{m-1}, \lambda)$

and call λ the highest weight of V_λ .

Let now λ be the highest weight of an irreducible K-module, where $K = O(n)$, δ_K be half of the sum of the positive roots of $so(n,C)$, and

$$\|\lambda\| = \langle \lambda + 2\delta_K, \lambda + 2\delta_K \rangle$$

Now let X be an irreducible quasisimple representation of G and V_λ be a representation of K with highest weight λ . Let K_X be the set of all highest weights of $so(n,\mathbb{C})$ s.t.

$$\dim \operatorname{Hom}_K(V_\lambda, X) \neq 0 .$$

<u>Theorem 5</u> [Vogan].

There exists exactly one $\lambda_o \in K_X$ which is minimal with respect to $\| \ \|$. It satisfies

$$\dim \operatorname{Hom}_K(V_{\lambda_o}, X) = 1 . \qquad \square$$

We will call V_{λ_o} <u>the minimal</u> <u>K-type of</u> X . More generally we call an irreducible representation V of K <u>a K-type of</u> X if $\dim \operatorname{Hom}_K(V,X) \neq 0$.

For later use we now give a list of minimal K-types for the representations J_μ . We will notice immediately that unfortunately there are inequivalent representations with the same central character and the same minimal K-type.

For each simple root α we define an $sl(2,\mathbb{R})_\alpha$ subalgebra in the usual way: Let $H_\alpha \in \mathfrak{a}_o$, H_α coroot to α, $X_\alpha \in \mathfrak{g}^\alpha$, $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$ s.t. $[X_\alpha, X_{-\alpha}] = H_\alpha$, $[H_\alpha, X_\alpha] = 2X_\alpha$, $[H_\alpha, X_{-\alpha}] = 2X_{-\alpha}$, then $sl(2,\mathbb{R})_\alpha = \mathbb{R}X_\alpha \otimes \mathbb{R}X_{-\alpha} \otimes \mathbb{R}H_\alpha$.

It is easy to see that $m_\alpha = X_\alpha - X_{-\alpha} \in M_o$. We consider $\rho \in (1 \times 1 \times \hat{M}_r)$ as a character on M_r and define $|\rho|_r$ to be the maximal number of strictly orthogonal simple roots α, which are orthogonal to the first $2r-1$ roots and such that $\rho(m_\alpha) = -1$.

Now let n be even. We identify the imaginary part in the dual of the complexification of the maximal abelian subalgebra h_K of $so(n)$ with $\mathbb{R}^{n/2}$. Let $e_1, \ldots, e_{n/2}$ be an orthonormal basis. Then the simple roots of $so(n,\mathbb{C})$ are

$$\alpha_1 = e_1 - e_2 , \alpha_2 = e_2 - e_3, \ldots, \alpha_{\frac{n}{2}-1} = e_{\frac{n}{2}-1} - e_{\frac{n}{2}} ,$$

$$\alpha_{\frac{n}{2}} = e_{n-1} + e_n .$$

The minimal K-type of J_μ, $\mu = (\vec{r}, \chi, \rho)$ and $\vec{r} = (r_1, \ldots, r_r)$, $r_i > 0$, has highest weight $(r_{i_1}+1, \ldots, r_{i_r}+1, 1, \ldots, 1, 0, \ldots, 0)$ where $r_{i_j} \geq r_{i_{j+1}}$, and the number of 1's is $|\rho|_r$.

Now let n be odd. We identify the imaginary part in the dual of the complexification of the maximal abelian subalgebra h_k of $so(n)$ with $\mathbb{R}^{(n-1)/2}$. Let $e_1, \ldots, e_{\frac{n-1}{2}}$ be an orthonormal basis. The simple roots are

$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_1 - e_3, \ldots, \alpha_{\frac{n-1}{2}-1} = e_{\frac{n-1}{2}},\ \alpha_{\frac{n-1}{2}} = 2e_{\frac{n-1}{2}}$.

The minimal K-type of J_μ, $\mu = (\vec{r}, \chi, \rho)$ and $\vec{r} = (r_1, \ldots, r_r)$ has highest weight

$$(r_{i_1}+1, \ldots, r_{i_r}+1, 1, \ldots, 1, 0, \ldots, 0)\ ,\ \text{where}\quad r_{i_j} > r_{i_{j+1}},$$

and the number of 1's is $|\rho|_r$.

C. Intertwining operators I.

We now fix once and for all a maximal set of non-conjugate Weyl chambers $\mathcal{C}_r^1,\dots,\mathcal{C}_r^e$ and the corresponding parabolics $\underline{\underline{P}}_r^1,\dots,\underline{\underline{P}}_r^e$,

For each $\mu = (\vec{r},\chi,\rho) \in \hat{C}_r$ we define

$$U_\mu^{\underline{\underline{P}}_r^i} = \mathrm{ind}_{\underline{\underline{P}}_r^i}^G \,\underline{\underline{\Pi}}_{\vec{r},\rho} \otimes \chi \ .$$

Definition:

Let $r = 0$. The representation $U_\mu^{\underline{\underline{P}}_o}$ is called a principal series representation (p.s.r.) of G . Let $r \neq 0$. The representation $U_\mu^{\underline{\underline{P}}_r^i}$ is called a generalized principal series representation (g.p.s.r.) of G .

Let P_o be the Borel subgroup and P_r^i be the standard parabolic associated to the Weyl chamber $\mathcal{C}_r^i$. $w_p^i \in W$ s.t. $\underline{\underline{P}}_r^i = m_{w_p^i} P_r^i m_{w_p^i}^{-1}$, and define Π_μ by

$$\Pi_\mu(p) = \underline{\underline{\Pi}}_\mu(m_{w_p^i} P m_{w_p^i}^{-1}) \qquad p \in P_r^i \ .$$

Then $U_\mu^{\underline{\underline{P}}_r^i} = \mathrm{ind}_{P_r^i}^G \Pi_\mu$, and we shall always write $U_\mu^{P_r^i}$ instead of $U_\mu^{\underline{\underline{P}}_r^i}$.

Using step by step induction, we can always consider the g.p.s.r. as invariant subspaces of suitable p.s.r. as follows:

Let P_o be the Borel subgroup and P_r be a standard parabolic associated to a Weyl chamber in $\mathfrak{a}_r$. Then for $\mu \in \hat{\mathcal{C}}_o$ we have

$$U_\mu^{P_o} = \mathrm{ind}_{P_o}^G \underline{\Pi}_\mu = \mathrm{ind}_{P_r}^G (\mathrm{ind}_{P_o}^{P_r} \Pi_\mu) \quad .$$

But $P_r = M_r A_r N_r$, $M_r = K_M^r A_M^r N_M^r$ and $P_o \cap M_r = M_M^r A_M^r N_M^r$, where M_M^r is the normalizer of A_M^r in K_M^r, which is M_o. Then we can write

$$\begin{aligned} \mathrm{ind}_{P_o}^{P_r} \Pi_\mu &= \mathrm{ind}_{P_o}^{M_r A_r N_r} (\Pi_\rho \otimes \chi) \\ &= (\mathrm{ind}_{M_o A_M^r N_M^r}^{M_r} (\Pi_\rho \otimes \chi\big|_{A_M^r N_M^r})) \otimes \chi\big|_{A_r N_r} \quad . \end{aligned}$$

Since M_r is isomorphic to a product of a finite abelian group Z_r with r copies of $SL_{\pm}(2,\mathbb{R})$, we can choose a suitable pair $(\rho, \chi\big|_{A_M^r})$ for each $\mu' \in \hat{\mathcal{C}}_r$ s.t. $\Pi_{\mu'}|M_r$ is isomorphic to the restriction of $\mathrm{ind}_{M_o A_M^r N_M^r}^{M_r} (\Pi_\rho \otimes \chi\big|_{A_M^r N_M^r})$ to an invariant subspace.

<u>Lemma 1</u>:

$\chi\big|_{A_M^r}$ is uniquely determined by this requirement but we have 2^r choices of ρ.

Proof.

For a discrete series representation $SL_{\pm}(2,\mathbb{R})$ there are exactly 2 p.s.r. s.t. the discrete series representation is isomorphic to the restriction of this p.s.r. to an invariant subspace. Both these p.s.r. have the same continuous parameter but different $\hat{M}$ parameter [see II A]. □

Now assume that $r = 0$. For $w \in W_o$ let m_W be a representative of w in the normalizer of A. We define

$$N_W = m_W N_o m_W^{-1} \cap \exp(\theta(w_o))$$

where θ denotes the Cartan involution. Let dn be the euclidean measure on N_w and for $\mu = (\chi,\rho)$ define

$$H_\mu = \{f \in C^\infty(G) \mid f(gman) = \rho^{-1}(m)\chi(a^{-1})e^{\delta(\log a^{-1})} f(g)\} ,$$

where δ is the half sum of the positive roots $m \in M_o$, $a \in A_o$, $n \in N_o$. H_μ is the space of C^∞ vectors for $\mathrm{ind}_{P_o}^G \pi_\rho$, and for $w \in W$ we define

$$(A(\mu,w)f)(g) = \int_{N_W} f(gm_W n)dn .$$

The integral converges if $\mathrm{Re}(\log \chi)$ is strictly positive with respect to all positive roots, that are transformed into negative ones by w . In this case, it defines an intertwining operator from

$$U_{\mu}^{P_o} \quad \text{to} \quad U_{w\mu}^{P_o} .$$

If $A(\mu,w)$ is convergent and $w = w_1 w_2$ s.t. $\ell(w) = \ell(w_1) + \ell(w_2)$, then

$$A(w_2\mu,w_1)A(\mu,w_2) = A(\mu,w_1w_2)$$

and $A(w_2\mu,w_1), A(\mu,w_2)$ are convergent. [14] [9b]

To define the operator for g.p.s.r., we use the embedding in p.s.r.

Let $U_{\mu}^{P_r^i}$ be a g.p.s.r. induced from a standard parabolic which is defined by a set $\Sigma_r^i = \{\alpha_{i_1},\ldots,\alpha_{i_r}\}$ of simple orthogonal roots and let $w \in W$ s.t. $w\Sigma_r^p$ is contained in the set of simple roots. Now if $U_{\mu'}^{P_o}$ is a p.s.r. in which $U_{\mu}^{P_r^i}$ can be embedded by step by step induction, we define $A(P_r^i,\mu,w)$ to be the restriction of the formal integral operator $A(\mu',w)$ to the subspace $U_{\mu}^{P_r^i}$. In this case wP_r^i is again a standard parabolic for a Weyl chamber $\mathcal{C}_r^{w(i)} = \mathcal{C}_r^j$.

If the integral converges, it defines an intertwining operator from $U_\mu^{P_r^i}$ to $U_{w(\mu)}^{wP_r^i}$, where $w(\mu) = (w^j)^{-1} w w^i \mu$. Hence this operator, if defined, allows us to intertwine between representations induced from non-conjugate parabolics. Up to a nonzero scalar factor, this definition is independent of the embedding.

The integral converges if $(w^i)^{-1} \log \chi$ is strictly positive with respect to all positive restricted roots which w transforms into negative ones.

Theorem 2: Knapp-Stein [9b], Schiffmann [14].

For each $w \in W$ there is a function $\gamma_w : \mathfrak{a}_o' \otimes \mathbb{C} \to \mathbb{C}$ s.t.

$$\mathfrak{A}(\mu,w) = \gamma_w^{-1}(\log \chi) A(\mu,w)$$

is defined if $\mathrm{Re}(\log \chi, \alpha) \geq 0 \; \forall \alpha \in \Delta^+$ s.t. $w\alpha \in -\Delta^+$, and it defines an intertwining operator from $U_\mu^{P_o}$ to $U_{w\mu}^{P_o}$.

We can also extend the domain of definition of $A(P_r^i, \mu_1 w)$ to singular values of the continuous parameter.

Theorem: Knapp-Stein [9a].

Let $U_\mu^{P_i}$, $r > 0$, be a p.s.r. For each $w \in W$ s.t. $A(P_r^i, \mu, w)$ is defined, there is a function $\gamma_W^{P_r^i} : \mathfrak{a}_r' \otimes \mathbb{C} \to \mathbb{C}$ s.t.

$$\mathfrak{A}(P_r^i,\mu,w) = (\gamma_w^{P_r^i}(\log \chi))^{-1} A(P_r^i,\mu,w)$$

is defined if $\mathrm{Re}((w^i)^{-1}(\log \chi),\alpha) \geq 0$ for all positive roots α which are transformed into negative ones by w, and it defines an intertwining operator from $U_\mu^{P_r}$ to $U_{w(\mu)}^{P_r}$. □

If $w = w_1 w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$ then for μ in the domain of definition of $\mathfrak{A}(\mu,w_1w_2)$, $\mathfrak{A}(w_2(\mu),w_1)$ and $\mathfrak{A}(\mu,w_2)$ are defined and we have the product formula

$$\mathfrak{A}(\mu,w) = \mathfrak{A}(w_2(\mu),w_1)\,\mathfrak{A}(\mu,w_2) .$$

If $w\Sigma_r^P$ and $w_2\Sigma_r^P$ are even contained in the set of simple roots, then if $\mathfrak{A}(P_r,\mu,w)$ is defined, so are $\mathfrak{A}(w_2P_r,w_2(\mu),w_1)$ and $\mathfrak{A}(P_r,\mu,w_2)$, and we have the product formula

$$\mathfrak{A}(P_r,\mu,w) = \mathfrak{A}(w_2P_r,w_2(\mu),w_1)\,\mathfrak{A}(P_r,\mu,w_2) .$$

<u>Definition</u>. Assume $P_r = P_{\mathcal{C}_r}$ and let $w_{\mathcal{C}_r}$ be defined as in I.A. The operator

$$\mathfrak{A}(P_r,\mu,w_{\mathcal{C}_r}) \quad \text{for} \quad \mu = (r,\chi,\rho) \quad \text{of} \quad \mathrm{Re}(\log \chi) \in \overline{\mathcal{C}}_r$$

will be called the <u>long intertwining operator</u> for (P_r,μ).

The product formula for $w_{\mathcal{C}_r}$ implies a product formula for the long intertwining operator. Before we can deduce any results about reducibility or composition series, we have to study these intertwining operators more closely.

D. Intertwining operators II.

Let $\mu \in \hat{\mathcal{C}}_o$, $P \supset P_o$ standard parabolics. Using step by step induction, we identify $f \in \mathrm{ind}_{P_o}^{G} \Pi_\mu$, $f \in C^\infty(G)$, with an $\mathrm{ind}_{P_o}^{P} \Pi_\mu$-valued function $\tilde{f}$ on G by the formula

$$f(gp) = (\tilde{f}(g))(p) \qquad p \in P \text{ , } g \in G \text{ .}$$

Then

$$\tilde{f}(e) \in \mathrm{ind}_{P_o}^{P} \Pi_\mu \quad .$$

Now assume $w \in W$ is contained in the Weyl group of M_p , and define

$$[\tilde{A}(P,\mu,w)\tilde{f}(g)](p) = A(\mu,w)f(gp)$$

for μ s.t. $A(\mu,w)$ is convergent. $\tilde{A}(P,\mu,w)$ is well defined, and since $A(\mu,w)$ is an intertwining operator, so is $\tilde{A}(\mu,w)$. Hence if $P = MAN$,

$$\begin{aligned}[\tilde{A}(P,\mu,w)\tilde{f}(g)](p) &= U^{P_o}_{w(\mu)}(g^{-1})[\tilde{A}(P,\mu,w)\tilde{f}(e)](p) \\ &= U^{P_o}_{w(\mu)}(g^{-1})A(\mu,w)\, f(p)\end{aligned}$$

$$= U^{P_o}_{w(\mu)}(g^{-1}) \int_{N_w} f(p\, m_w\, n)dn$$
$$= U^{P_o}_{w(\mu)}(g^{-1}) \int_{N_w} f(m\, a\, n\, m_w\, n)dn$$
$$= U^{P_o}_{w(\mu)}(g^{-1})(\chi\ e^{\delta})(m_w a^{-1} m_w^{-1}) \int_{N_w} f(m m_w n)dn \ .$$

But $f\big|_{M_p} \in \mathrm{ind}^{M}_{P_o \cap M}(\Pi_\mu|M\cap P_o)$ and $f(m) \longmapsto \int_{N_w} f(m m_w n)dn$ is an intertwining operator

$$A_M(w,\mu):\ \mathrm{ind}^{M}_{P_o\cap M}(\Pi_\mu|_{M\cap P_o}) \to \mathrm{ind}^{M}_{P_o\cap M}(\Pi_w(\mu))\big|_{P_o\cap M}) \ .$$

Therefore

$$[\widetilde{A}(P,\mu,w)\widetilde{f}](e) \in \mathrm{ind}(A_M(w,\mu)\mathrm{ind}^{M}_{P_o\cap M}(\Pi_\mu|_{P_o\cap M}) \otimes \chi|_{AN} \ .$$

This proves

<u>Lemma 1</u>: Image of $\widetilde{A}(P,\mu,w)$

$$= \mathrm{ind}^{G}_{P}((A_M(w,\mu)\mathrm{ind}^{M}_{P_o\cap M}(\Pi_\mu|_{P_o\cap M})) \otimes \chi|_{AN}) \ .$$

□

<u>Corollary 2</u>: $A(\mu,w)$ is an isomorphism iff $A_M(w,\mu)$ is an isomorphism. □

Together with the product formula for intertwining operators, this lemma allows us to reduce the problem of determining the kernels of intertwining operators to lower dimensional groups.

Similar considerations apply to g.p.s.r.. Here we are in the situation $P_o \subset P_r \subset P = MAN$, and w is contained in the Weyl group of M . Let $\mu \in \hat{C}_r$, then by exactly the same arguments, we prove.

<u>Lemma 3</u>:

$$\text{Image of } \tilde{A}(P,\mu,w) = \text{Image of } A(P_r,\mu,w)$$

$$= \operatorname{ind}_P^G(A_M(P_r \cap M,\mu,w)\operatorname{ind}_{P_r \cap M}^M(\Pi_\mu\big|_{P_r \cap M}) \otimes \chi\big|_{AN}) . \quad \square$$

We will now study the operator $A_M(P_r \cap M,\mu,w)$ more closely. We have

$$M \cong SL_{\pm}(m_1,\mathbb{R}) \times SL_{\pm}(m_i,\mathbb{R}) \times \ldots \times SL_{\pm}(m_\ell,\mathbb{R}) \times Z_w \quad ,$$

where Z_w is a finite subgroup of Z_r and $\prod_i^\ell SL_{\pm}(m_i,\mathbb{R})$ is the subgroup whose Weyl group contains w . Then

$$P_r \cap M \cong P_r \cap SL_{\pm}(m_1,\mathbb{R}) \times \ldots \times P_r \cap SL_{\pm}(m_\ell,\mathbb{R}) \times Z_w$$

and

$$\mathrm{ind}_{M\cap P_r}^{M}(\Pi_\mu|_{P_r\cap M}) = \mathrm{ind}_{P_r\cap SL_\pm(m_1,\mathbb{R})}^{SL_\pm(m_1,\mathbb{R})}\Pi_\mu|_{P_r\cap SL_\pm(m_1,\mathbb{R})}$$

$$\otimes\ldots\otimes\ \mathrm{ind}_{P_r\cap SL_\pm(m_\ell,\mathbb{R})}^{SL_\pm(m_\ell,\mathbb{R})}\Pi_\mu|_{P_r\cap SL_\pm(m_\ell,\mathbb{R})}\otimes\Pi_\mu|_{Z_w} .$$

Since $P_r\cap SL_\pm(m_i,\mathbb{R})$, $i = 1,\ldots,\ell$, is a parabolic of $SL_\pm(m_i,\mathbb{R})$, each factor in the tensor product is a g.p.s.r. of $SL_\pm(m_i,\mathbb{R})$, and the intertwining operator $A_M(P_r\cap M_p,\mu,w)$ is the product of the intertwining operators for the corresponding g.p.s.r. of the $SL_\pm(m,\mathbb{R})$ factors. Similar results are of course valid for the normalized operators $\mathfrak{A}(P,\mu,w)$.

This result therefore allows us to reduce the computations for kernels and images of intertwining operators to the corresponding computations for lower dimensional groups, if the Weyl group element w is contained in a suitable subgroup.

If w does not satisfy these conditions, we usually can rewrite it as a product $w = w_1w_2 - -w_{\ell_o}$, s.t. $\ell(w) = \Sigma_{i=1}^{\ell_o}\ \ell(w_i)$ and such that each of the factors is contained in a suitable subgroup. In this way we can at least reduce the problem of proving injectivity of the intertwining operators to a similar problem for lower-dimensional subgroups.

E. Reducibility.

In our previous description of Langlands' classification we constructed representations $\underline{\underline{U}}_\mu$, $\mu \in \hat{\mathcal{C}}_r$, which contain the Langlands representation with parameter μ as the minimal invariant subspace. Now we go to the contragradient picture:

Let $\underline{\underline{U}}_\mu^o$ be the representation

$$\mathrm{ind}_{\underline{\underline{P}}_\chi}^{G}(\underline{\Pi}_{\vec{r},\rho} \otimes \chi) \qquad \text{if } \chi \text{ is nonsingular}$$

$$\mathrm{ind}_{\underline{\underline{P}}_\chi}^{G}(\underline{\Pi}^*_{\vec{r},\rho} \otimes \chi\big|_{\underline{\underline{P}}_\chi}) \quad \text{if } \chi \text{ is singular.}$$

As before, $\underline{\underline{U}}_\mu^o = U_\mu^{P_\chi}$. Then [12] the representation with Langlands parameter $(\vec{r},\chi,\rho)$ is a quotient of $\underline{\underline{U}}_\mu^o$ by the closure of the kernel of

$$(\mathring{A}(\underline{\underline{P}}_\chi,\mu)f)(x) = \int_{\underline{\underline{\bar{N}}}_\chi} f(x\bar{n})d\bar{n}$$

where f is a K-finite function in $\underline{\underline{U}}_\mu^o$ and $\underline{\underline{\bar{N}}}_\chi =$ $\exp(\theta(n_\chi))$.

If χ is nonsingular, then

$$A(P_\chi,\mu,w_{\mathcal{C}_r}) = R(m_{w_{\mathcal{C}_r}})A(\underline{\underline{P}}_\chi,\mu) ,$$

where $R(m_{w_{\mathcal{C}_r}})$ is the right regular representation of $m_{w_{\mathcal{C}_r}}$. Therefore

$$\ker \mathring{A}(\underline{\underline{P}}_\chi,\mu) = \ker A(P_\chi,\mu,w_{\mathcal{C}_r}) \quad .$$

This proves

<u>Lemma 1:</u> Let $\mu \in \hat{\mathcal{C}}_r$, $\mu \in (\vec{r},\chi,\rho)$ with χ nonsingular. Then $U_\mu^{P_\chi}$ is reducible if $A(P_\chi,\mu,w_{\mathcal{C}_r})$ has a nontrivial kernel.

We now assume that χ is singular. Then we choose $\mathcal{C}_r$ s.t. $\mathrm{Re}(\log\chi) \in \overline{\mathcal{C}}_r$ and rewrite $w_{\mathcal{C}_r}$ as a product $w^2_{\mathcal{C}_r} w^1_{\mathcal{C}_r}$ where $w^1_{\mathcal{C}_r}$ is contained in the M-part of the parabolic P_χ and $m^2_{w_{\mathcal{C}_r}} N_{P_\chi} (m^2_{w_{\mathcal{C}_r}})^{-1} = \overline{N}$. Using the product formula for $\mathcal{O}\!\mathcal{l}(P_{\mathcal{C}_r},\mu,w_{\mathcal{C}_r})$, we get

$$\mathcal{O}\!\mathcal{l}(P_{\mathcal{C}_r},\mu,w_{\mathcal{C}_r}) = \mathcal{O}\!\mathcal{l}(w^1_{\mathcal{C}_r} P_{\mathcal{C}_r},(w^1_{\mathcal{C}_r})(\mu),w^2_{\mathcal{C}_r})\,\mathcal{O}\!\mathcal{l}(P_{\mathcal{C}_r},\mu,w^1_{\mathcal{C}_r}) \quad .$$

The second factor in this product is an isomorphicm since $w^1_{\mathcal{C}_r}$ is contained in the Weyl group of $M_{P_{\mathcal{C}_r}}$ and the representation $\mathrm{ind}_{P_{\mathcal{C}_r}\cap M_{P_\chi}}^{M_{P_\chi}} \Pi_{\vec{r},\rho} \otimes \chi\Big|_{A_{M_{\mathcal{C}_r}} N_{M_{\mathcal{C}_r}}}$ is unitary.

On the other hand,

$$\mathcal{O}\!\mathcal{l}(w^1_{\mathcal{C}_r},P_{\mathcal{C}_r},(w^{-1}_{\mathcal{C}_1})(\mu),w^2_{\mathcal{C}_r}) = c_w(\chi)A(w^1_{\mathcal{C}_r}P_{\mathcal{C}_r},(w^1_{\mathcal{C}_r})\mu,w^2_{\mathcal{C}_r})$$

with $c_w(\chi)$ a nonzero constance, since χ is strictly positive with respect to all roots not contained in $M_{\mathcal{C}_r}$. Thus by the same argument,

$$A(w^1_{\mathcal{C}_r} P_{\mathcal{C}_r}, (w^1_{\mathcal{C}_r})^{-1}(\mu), w^2_{\mathcal{C}_r}) = \gamma R(m_{w^2_{\mathcal{C}_r}}) \mathring{A}(\underline{\underline{P}}_\chi, \mu) \ , \quad \gamma \in \mathbb{C}\setminus 0 \ .$$

This proves:

Theorem 2: Let $\mu \in C_r$. Then $\mathring{\underline{\underline{U}}}_\mu$ is reducible iff $A(P_{\mathcal{C}_r}, \mu, w_{\mathcal{C}_r})$ for $\mathcal{C}_r$ s.t. $\mathrm{Re}(\log \gamma) \in \overline{\mathcal{C}}_r$ has a nontrivial kernel.

If an intertwining operator has a nontrivial kernel and has a product representation, then at least one of the factors has a nontrivial kernel. Therefore the formula $w_{\mathcal{C}_r} = w_k, \ldots, w_1$ and the product formula for intertwining operators reduced the problem to showing that the intertwining operator for one of the w_i's has a nontrivial kernel. But for the w_i's we can always choose a smallest parabolic P_{w_i} containing $w_{i-1} \cdots w_1 P_{\mathcal{C}_r}$ and containing a representative of m_{w_i} , hence containing $w_i w_{i-1} \cdots w_1 P_{\mathcal{C}_r}$.

a) If $\mathcal{C}(w_i) = 1$, we can choose $P_{w_i} = M_i A_i N_i$ s.t. M_i is a product of $r+1$ $SL_{\pm}(2,\mathbb{R})$'s with a finite group, and m_w is contained in an $SL_{\pm}(2,\mathbb{R})$ factor. The representation

$$\operatorname{ind}_{M_i\cap(w_{i-1}\cdots w_1 P_{\mathcal{C}_r})}^{M_i}\left(\Pi(w_{i-1}\cdots w_1)(\mu)\Big|_{M_i\cap(w_{i-1}\cdots w_1)P_{\mathcal{C}_r}}\right)$$

is a tensor product of discrete series representations of r of the $SL_{\pm}(2,\mathbb{R})$ factors, a p.s.r. of the remaining $SL_{\pm}(2,\mathbb{R})$ factor, and a character of the finite group, whereas the operator $A_{M_i}(w_{i-1}\cdots w_1 P_{\mathcal{C}_r}\cap M_i, w_{i-1}\cdots w_1(\mu), w_i)$ is the product of a long intertwining operator of the p.s.r. factor and the identity on all other factors. Since $A(w_{i-1}\cdots w_1 P_{\mathcal{C}_r}, w_{i-1}\cdots w_1(\mu), w_i)$ has a nontrivial kernel iff $A_{M_i}(w_{i-1}\cdots w_1 P_{\mathcal{C}_r}\cap M_i, w_{i-1}\cdots w_1(\mu), w_i)$ has a nontrivial kernel, we have reduced the problem to finding necessary and sufficient conditions for reducibility of g.p.s.r. of $SL_{\pm}(2,\mathbb{R})$. These conditions have been known for quite a long time [see for example II.A.]

b) If $\ell(w_i) = 2$, we can choose $P_{w_i} = M_i A_i N_i$ such that M_i is a product of $r-1$ $SL_{\pm}(2,\mathbb{R})$ factors, an $SL_{\pm}(3,\mathbb{R})$ factor, and a finite group. The representation

$$\operatorname{ind}_{M_i\cap(w_{i-1}\cdots w_1 P_{\mathcal{C}_r})}^{M_i}\left(\Pi(w_{i-1}\cdots w_1)(\mu)\Big|_{M\cap(w_{i-1}\cdots w_1)P_{\mathcal{C}_r}}\right)$$

is a tensor product of discrete series representations of the $SL_{\pm}(2,\mathbb{R})$ factors, a g.p.s.r. representation of $SL_{\pm}(3,\mathbb{R})$ factor, and a character of the finite group. In this case $A_{M_i}((w_{i-1}\cdots w_1)P_{\mathcal{C}_r}\cap M_i, w_{i-1}\cdots w_1(\mu), w_i)$ is a product of a long intertwining operator of the g.p.s.r. and the identity on all other factors. By the same arguments as above, we reduce this case to finding necessary and sufficient conditions for reducibility of g.p.s.r. of $SL_{\pm}(3,\mathbb{R})$. This will be done in II.B.

c) If $\ell(w_i) = 4$, we can choose $P_{w_i} = M_i A_i N_i$, such that M_i is a product of r-2 $SL_{\pm}(2,\mathbb{R})$ factors, an $SL_{\pm}(4,\mathbb{R})$ factor, and a finite group. The representation

$$\mathrm{ind}_{M_i\cap(w_{i-1}\cdots w_1 P_{\mathcal{C}_r})}^{M_i}\left(\Pi(w_{i-1}\cdots w_1)(\mu)\Big|_{M\cap(w_{i-1}\cdots w_1)P_{\mathcal{C}_r}}\right)$$

is a tensor product of discrete series representations of the $SL_{\pm}(2,\mathbb{R})$ factors, a g.p.s.r. representation with $r = 2$ of the $SL_{\pm}(4,\mathbb{R})$ factor, and a character of the finite group. In this case, $A_{M_i}((w_{i-1}\cdots w_1)P_{\mathcal{C}_r}\cap M_i, w_{i-1}\cdots w_1(\mu), w_i)$ is a product of a long intertwining operator of the g.p.s.r. and the identity on all other factors. Again by the same arguments, we reduce this case to finding necessary and sufficient conditions for reducibility of g.p.s.r. of $SL_{\pm}(4,\mathbb{R})$. This will be done in II.C.

In part II.D. I will then use all this to derive explicit formulas for reducibility.

F. Jordan-Hölder series.

Let V be a Banach space, $\Pi : G \to \mathrm{End}\, V$ a quasi simple representations of G.

Definition. (π, v) is of finite length if we can find a family

$$0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{k-1} \subsetneq V_k = V \qquad (k < \infty)$$

of closed G-invariant subspaces such that G acts irreducibly on $V_i, /V_{i-1}$, $i = 0, \ldots, k$. Let us write π_i for the representation of G on V_i/V_{i-1}. The family $(\pi_i, V_i/V_{i-1})$, $i = 0, \ldots, k$ is called the Jordan-Hölder series of (π, V), and k is the length of the Jordan-Hölder series. We will say that an irreducible representation π' occurs in the Jordan-Hölder series of (π, V) with multiplicity ℓ if ℓ of the representations in the Jordan-Hölder series are equivalent to π'. To the representations in the J. H. series it will be referred to as composition factors of (V, π).

Now assume that $(\pi, V) = U_\mu^{P_r}$, $\mu \in \hat{\mathfrak{a}}_r$. By a theorem of Harish-Chandra [20], $U_\mu^{P_r}$ is of finite length.

Lemma 1: Let $\mu \in \hat{\mathfrak{a}}_r$, $U_\mu^{P_r}$ and $U_{w\mu}^{P_r}$ have the same

Jordan-Holder series.

Proof: If a representation π has a Jordan-Hölder series, then its character is the sum of the characters of the composition factors. On the other hand, the characters of the p s.r. and g.p.s.r. are invariant under the Weyl group W_r [20]. Hence if U_μ and $U_{w\mu}$ had different J.H. series, we could write the character of U_μ in two different ways as a sum of characters of irreducible representations. But since characters of irreducible representations are linearly independent [20], this is a contradiction. $\square$

Since $U_\mu^{P_r}$ and $U_{w\mu}^{P_r}$ $w \in W_r$ are in general not equivalent as representations, we will, except in some special cases, not try to determine the lattice of closed invariant subspaces of U_μ, but try to compute the J. H. series instead. To do this, we have to compute

a) the Langlands parameters of all the composition factors and

b) their multiplicities in the J. H. series.

Consider now the problem of determining the Langlands parameter of the composition factors.

Let $\mu \in \mathcal{E}_o$, $\mu = (\chi, \rho)$ with $\mathrm{Re}(\log \chi)$ dominant, and let $\mathcal{O}(\mu, w_o)$ be the long intertwining operator for $U_\mu^{P_o}$. By Langlands [12], $\mathcal{O}(\mu, w\ )U_\mu^{P_o} = J_\mu$ is irreducible, and therefore all composition factors except J_μ are contained in the kernel of $\mathfrak{A}(\mu, w_o)$. To analyse the kernel of $\mathfrak{A}(\mu, w_o)$, we use the product formula. Write

$$w_o = w_n, \ldots, w_1$$

with w_i a simple reflection and obtain

$$\mathfrak{A}(\mu, w_o) = \mathfrak{A}((w_{n-1}, \ldots, w_1)^{-1}(\mu), w_n), \ldots, \mathfrak{A}(\mu, w_1) .$$

Obviously a composition factor is contained in the kernel of $\mathfrak{A}(\mu, w_o)$ iff it is contained in the J. H. series of the kernel of at least one of the factors. To compute the kernel of such a factor, we use our results on intertwining operators. We choose the parabolic $P_{w_i} = M_i A_i N_i$ to be the smallest parabolic whose M-part contains m_{w_i}. Since w_i is a reflection of a simple root, the M-part of P_{w_i} is isomorphic to $SL_{\pm}(2, \mathbb{R}) \times Z_{w_i}$. Hence if $\mathfrak{A}_{M_i}((w_{i-1}, \ldots, w_1)(\mu), w_i)$ is not injective; its kernel is a discrete series representation of $SL_{\pm}(2, \mathbb{R})$. Thus the kernel of $\mathfrak{A}((w_{i-1}, \ldots, w_1)(\mu), w_i)$ is either 0 or a g.p.s.r., whose parameter can be computed in terms of the parameters of the $\mathfrak{A}_{M_i}(w_{i-1} \ldots w_1)^{-1}\mu, w_i)$ by the formula I.D.1.

This "reduces" the problem of computing composition series for $U_\mu^{P_o}$ to the problem of computing composition series for the g.p.s.r. $r = 1$ with these new parameters.

Now let $U^P_{\mu'}$ with $\mu' = (r',\chi',\rho') \in \hat{\mathcal{C}}_1$ be such a g.p.s.r. Applying the lemma above, we may assume that $P = P_{\mathcal{C}_1}$ for some $\mathcal{C}_1$ s.t. Re log $\chi' \in \overline{\mathcal{C}}_1$, i.e. that $U^P_{\mu'} = \underline{\underline{\mathring{U}}}_{\mu'}$. Now repeat the above argument for $\mathfrak{A}(P,\mu',W_{\mathcal{C}_1})$. Here

$$w_{\mathcal{C}_1} = w_m \cdots w_1$$

where the w_i's are either reflections of simple roots or contained in a $SL_{\pm}(3,\mathbb{R})$ subgroup which is associated to 2 simple non orthogonal roots.

In the first case, $\mathfrak{A}_{M_i}((w_{i-1}\cdots w_1)(\mu'),w_i)$ is a product of an intertwining operator for a p.s.r. of an $SL_{\pm}(2,\mathbb{R})$ factor and the identity on the other component. Hence the kernel of $\mathfrak{A}((w_{i-1}\cdots w_1)P_1,(w_{i-1}\cdots w_1)(\mu),w_i)$ is 0 or a generalized p.s.r. with $r = 2$.

In the second case, $\mathfrak{A}_{M_i}((w_{i-1}\cdots w_1)P,(w_{i-1}\cdots w_1)(\mu),w_i)$ long intertwining operator for a g.p.s.r. of the $SL_{\pm}(3,\mathbb{R})$ factor of M. We will later in part II show that the kernel of a long intertwining operator for a g.p.s.r. of $SL_{\pm}(3,\mathbb{R})$ is either 0 or again a g.p.s.r. of $SL_{\pm}(3,\mathbb{R})$. Hence the kernel of $\mathfrak{A}((w_{i-1}\cdots w_1)P,(w_{i-1}\cdots w_1)(\mu'),w_i)$ is 0 or a g.p.s.r. with $r = 1$.

In both cases the parameters of the kernels can be computed in terms of the parameters of the kernels of the $\mathfrak{A}_{M_i}((w_{i-1},\ldots,w_1)P_1(w_{i-1},\ldots,w_1)(\mu),w_i)$.

Now we start our procedure all over again. In the case $r = 2$, the above argument reduced the problem to intertwining operators for p.s.r. of $SL_{\pm}(2,\mathbb{R})$, long intertwining operators for g.p.s.r. with $r = 1$ for $SL_{\pm}(3,\mathbb{R})$, and long intertwining operators for $SL_{\pm}(4,\mathbb{R})$, with $r = 2$. It will turn out that for $SL_{\pm}(4,\mathbb{R})$ the kernel is in general not a g.p.s.r., but that we can find at most three g.p.s.r. such that each composition factor of the kernel is a composition factor for at least one of the three g.p.s.r. Again we compute the parameter of the generalized principal series in the kernels. We get g.p.s.r. with $r = 1$, $r = 2$ and $r = 3$. For $r > 2$ the above argument always reduced the computations to analogous computations for $SL_{\pm}(2,\mathbb{R})$, $SL_{\pm}(3,\mathbb{R})$ and $SL_{\pm}(4,\mathbb{R})$. We repeat this procedure until we come to an irreducible g.p.s.r.

Obviously this procedure yields the Langlands parameters of all representations in the J. H. series, but it fails to give us the multiplicities. In fact, the multiplicity of a composition factor for the kernel of the long intertwining operator can be strictly less than the multiplicity with which this composition factor appears in the J. H. series for the direct sum of the kernels of all its factors.

The computations involved in the above procedure can be simplified by using Gregg Zuckerman's results on tensor products:

Let M be an indecomposable H. Ch. module of $\mathfrak{g}$ with central character ν and V_λ be a finite-dimensional irreducible representation with highest weight λ. Then $M \otimes V_\lambda$ is again a H. Ch. module. Let $P_\nu^{\nu+\lambda}$ be the projection on the summand with central character $\nu + \lambda$. Let $\psi_\nu^{\nu+\lambda}$ be the functor

$$M \to P_\nu^{\nu+\lambda}(V \otimes M) \quad .$$

<u>Theorem 2</u>: (Zuckerman [21]).

Let W_ν and $W_{\nu+\lambda}$ be the stabilizer of ν and $\nu + \lambda$ in the Weyl group, respectively. If $W_\nu = W_{\nu+\lambda}$, then $\psi_\nu^{\nu+\lambda}$ is an exact functor in the category of H. Ch. modules.

For the rest of this paragraph, let the assumptions of the theorem be satisfied.

Let U_μ be a p.s.r., $\mu = (\chi, \rho)$ with $\mathrm{Re}(\log \chi) \in \overline{\mathcal{C}}_o$. Then U_μ has central character $\log \chi$ and

$$\begin{aligned} U_\mu \otimes V_\lambda &= \mathrm{ind}_{P_o}^G (\pi_\rho \otimes \chi) \otimes V_\lambda \\ &= \mathrm{ind}_{P_o}^G (\pi_\rho \otimes \chi \otimes V_{\lambda} |_{P_o}) \quad . \end{aligned}$$

The representation $\pi_\rho \otimes \chi \otimes V_\lambda|P_o$ has a composition series and as a minimal invariant, hence irreducible, subspace the space of weight $\log \chi + \lambda$. Let π_{ρ_λ} be the representation of M_o on the highest weight space of V_λ, then

$$\mathrm{ind}_{P_o}^{G}(\pi_\rho \cdot \pi_{\rho_\lambda}) \otimes (\chi, e^\lambda) = U_{(\rho, \rho_\lambda, \chi e^\lambda)}$$

is an invariant subspace of $U_\mu \otimes V_\lambda$ with central character $\log \chi + \lambda$.

Proposition 3: $\psi_{\log \chi}^{\log \chi + \lambda} U_\mu = U_{(\rho, \rho_\lambda, \chi e^\lambda)}$.

Proof: We need another result of Zuckerman:[21]
Let M be a H.Ch-module with central character ν' and $\widetilde{V}_\lambda$ be the contragradient module to V_λ. Let $\widetilde{P}_\nu^{\nu'-\lambda}$ be the projection from $M \otimes \widetilde{V}_r$ on the submodule with central character $\nu' - \lambda$, and define $\varphi_{\nu'}^{\nu'-\lambda}$ to be the functor

$$M \to \widetilde{P}_{\nu'}^{\nu'-\lambda}(M \otimes \widetilde{V}_\lambda) .$$

Then under the assumptions of the theorem

$$\varphi_{\nu+\lambda}^{\nu} \psi_\nu^{\nu+\lambda} M = M .$$

Now we want to show that $\varphi^{\log\chi}_{\log\chi+\lambda} U_{(\rho,\rho_\lambda,\chi e^\lambda)}$ contains the representation U_μ as a composition factor. We have

$$U_{(\rho,\rho_\lambda,\chi e^\lambda)} \otimes \widetilde{V}_\lambda = (U_{(\rho,\rho_\lambda,\overline{\chi}^{-1}e^{-\lambda})} \otimes V_\lambda)^* ,$$

where $*$ denotes the contragredient representation: Our previous considerations show therefore that $U_{(\rho^{-1},\bar{\chi}^{-1})} = U_{(\rho,\chi)}{}^*$ is contained as an invariant subspace in $U_{(\rho,\rho_\lambda,\bar{\chi}e^{-\lambda})} \otimes V_\lambda$. Going to the dual again and applying the projection, we see that $U_{(\rho,\chi)}$ is contained at least as a quotient in the image of the projection. Assume now that $\psi^{\log\chi+\lambda}_{\log\chi} U_\mu = M_{\mu+2} \supsetneq U_{(\rho,\rho_\lambda,\chi e^\lambda)}$ then

$$\psi^{\log\chi}_{\log\chi+\lambda}(U_{(\rho,\rho_\lambda,\chi e^\lambda)}) \supsetneq U_{(\rho,\chi)} .$$

On the other hand we showed that $U_{(\rho,\chi)}$ is a quotient of $\varphi^{\log\chi}_{\log\chi+\lambda} U_{(\rho,\rho_\lambda,\chi e^\lambda)}$, which yields a contradiction.

□

Applying these results to our composition series problem, we see

Proposition 5: To determine the length of the J. H. series and the multiplicities of the composition factors, it is enough to do so for the representations $U_{(\rho,\chi)}$, where $\rho \in \hat{M}_o$, $\chi = \sum_{i=1}^{n-1} \chi_i \delta_i$, $0 \geq \chi \geq 1$. The δ_i are here the fundamental weights. □

We will now show how the parameters of the composition factors change under the functor $\psi_{\chi}^{\chi+\lambda}$. Since this is an exact functor it is enough to compute

$$\psi_{\log\chi}^{\log\chi+\lambda} U^{P}_{(\vec{r},\chi',\rho)} ,$$

where χ' is dominant and $\log\chi = \log\chi' + \sum_{i=1}^{\ell} \frac{r_i}{2} \alpha_{2i-1}$. We have already proved that for $\mu \in \hat{C}_o$, χ dominant

$$P_{\log\chi}^{\log\chi+\lambda} (U_\mu \otimes V_\lambda) = \operatorname{ind}_{P_o}^{G} \theta_{\chi+\lambda}(\pi_\rho \otimes \chi \otimes V_{\lambda|P_o}) ,$$

where $\theta_{\chi+\lambda}$ is the projection on the invariant subspace of weight $\log\chi+\lambda$. But if $P_o \subset P_r$, we can also write this in the form

$$\begin{aligned} P_{\log\chi}^{\log\chi+\lambda} (U_\mu \otimes V_\lambda) &= \operatorname{ind}_{P_r}^{G}\left(\operatorname{ind}_{P_o}^{P_r}\theta_{\chi+\lambda}(\pi_\rho \otimes \chi \otimes V_{\lambda|P_o})\right) \\ &= \operatorname{ind}_{P_r}^{G}\left(\theta^{r}_{\chi+\lambda}\left[\operatorname{ind}_{P_o}^{P_r}(\pi_\rho \otimes \chi) \otimes V_{\lambda|P_r}\right]\right) , \end{aligned}$$

where

$$\theta^{r}_{\chi+\lambda} : (\operatorname{ind}_{P_o}^{P_r}(\pi_\rho \otimes \chi) \otimes V_{\lambda|P_r} \longrightarrow \operatorname{ind}_{P_o}^{P_r}\theta_{\chi+\lambda}(\pi_\rho \otimes \chi \otimes V_{\lambda|P_o}) .$$

Now let $\mu' = (\vec{r}', \chi', \rho') \in \hat{C}_r$ and assume that $U_{\mu'}$ can be embedded in some U_μ, $\mu = (\rho, \chi) \in \hat{C}_o$, with χ dominant by step-by-step induction, i.e. that we are given an embedding

$$\pi_{\mu'} \longrightarrow \mathrm{ind}_{P_o}^{P_r} \pi_\mu .$$

Then

$$\begin{aligned} P_{\log\chi}^{\log\chi+\lambda}(U_{\mu'} \otimes V_\lambda) &= P_{\log\chi}^{\log\chi+\lambda}\, \mathrm{ind}_{P_r}^{G}(\pi_{\vec{r}',\rho'} \otimes \chi' \otimes V_\lambda|P_r) \\ &= \mathrm{ind}_{P_r}^{G}\, \theta_{\chi+\lambda}^{r}\,(\pi_{\vec{r}',\rho'} \otimes \chi' \otimes V_\lambda|P_r) \\ &= \mathrm{ind}_{P_r}^{G}\, \theta_{\chi+\lambda}^{r}\,(\pi_{\mu'} \otimes V_\lambda|P_r) \end{aligned}$$

which reduces the problem to studying $\theta_{\chi+\lambda}^{r}$. The definition of $\theta_{\chi+\lambda}^{r}$ implies that

$$\begin{aligned} &\theta_{\chi+\lambda}^{r}\Big[\Big(\mathrm{ind}_{P_o}^{P_r}(\pi_\mu)\Big) \otimes V_\lambda|P_r\Big] \\ &\quad = \theta_{\chi+\lambda}^{r}\Big[\mathrm{ind}_{M_r \cap P_o}^{M_r} \pi_\mu|M_r \cap P_o \otimes \chi\big|_{A_r N_r} \otimes V_\lambda|P_r\Big] \\ &\quad = \mathrm{ind}_{M_r \cap P_o}^{M_r}(\pi_{\rho\rho_\lambda} \otimes \chi e^\lambda)\big|_{M_r \cap P_o} \otimes \chi e^\lambda\big|_{A_r N_r} \quad \text{by} \quad (F\,2) . \end{aligned}$$

Let V_λ^r be the representation of M_r on the minimal P_r invariant subspace of $V_\lambda|P_r$. Since

$$M_r \cong SL_{\pm}(2,\mathbb{R}) \times \ldots \times SL_{\pm}(2,\mathbb{R}) \times Z_r$$

the representation is characterized by an r-tupal $\vec{\lambda}_r = (\lambda_1^{\pm},\ldots,\lambda_r^{\pm})$ of highest weights, together with the representation of the representatives of the 2 connected components which are indicated by the +,- sign, and a representation ρ_λ of Z_r .

On the other hand, the representation $\mathrm{ind}_{M_r \cap P_o}^{M_r}(\pi_\rho \otimes \chi)\big|_{M_r \cap P_o}$ is a tensor product of p.s.r. U_{μ_i} of the $SL_{\pm}(2,\mathbb{R})$ factors, and a representation ρ_r of Z_r . The index μ_i is the restriction (χ_i,ρ_i) of (χ,ρ) to the corresponding $SL_{\pm}(2,\mathbb{R})$ factor. For each factor we define $P_{\log \chi_i}^{\log \chi_i + \lambda_i}$ as previously and define $P_M^{\log \chi + \lambda_r}$ to be the product of the $P_{\log \chi}^{\log \chi + \lambda_i}$'. Then

$$P_M^{\log \chi + \lambda_r}\left(\mathrm{ind}_{M_r \cap P_o}^{M_i}(\pi_\rho \otimes \chi)\big|_{M_r \cap P_o} \otimes V_\lambda^r\right)$$
$$= \mathrm{ind}_{M_r \cap P_o}^{M_r}(\pi_\rho \pi_{\rho_\lambda} \otimes \chi e^\lambda)\big|_{M_r \cap P_o}$$

and therefore

$$\theta_{\chi+\lambda}^r\left[\left(\left(\mathrm{ind}_{M_r \cap P_o}^{M_r}(\pi_\rho \otimes \chi)\big|_{M_r \cap P_o}\right) \otimes \chi\big|_{A_r N_r}\right) \otimes V_\lambda \big| P_r\right]$$
$$= P_{M_r}^{\log \chi + \lambda_r}\left(\mathrm{ind}_{M_r \cap P_o}^{M_r}(\pi_\mu\big|_{M_r \cap P_o} \otimes V_\lambda^r) \otimes \chi e^\lambda\right|_{A_r N_r} .$$

Hence

$$\theta^{r}_{\chi+\lambda}\left[\left(\pi_{\vec{r},\rho} \otimes \chi\big|_{A_rN_r}\right) \otimes V_{\lambda|P_r}\right]$$
$$= P_{M_r}^{\log \chi+\lambda}\eta(\pi_{\vec{r},\rho} \otimes V^{r}_{\lambda}) \otimes \chi e^{\lambda}\Big|_{A_rN_r} .$$

Thus we have shown

Proposition 5: The diagram

$$\begin{array}{ccc} \mathrm{ind}_{P_o}^{P_r}\pi_{\mu} & \longrightarrow & \theta^{r}_{\chi+\lambda}\left[(\mathrm{ind}_{P_o}^{P_r}\pi_{\mu}) \otimes V_{\lambda|P_r}\right] = \mathrm{ind}_{P_o}^{P_r}\pi_{(\rho\rho_{\lambda},\, \chi e^{\lambda})} \\ \uparrow & & \uparrow \\ \pi_{\mu'} & \longrightarrow & \theta^{r}_{\chi+\lambda}[\pi_{\mu'} \otimes V_{\lambda|P_r}] \end{array}$$

is commutative, where the inclusions are obtained by step by step induction. □

But $\psi^{\log \chi+\lambda}_{\log \chi}\, U^{P}_{(\vec{r},\chi',\rho)}$ is uniquely determined by the data of a p.s.r. and a standard cuspidal parabolic P_r such that it arises by step by step induction from P_o to P_r. Proposition 5 provides all the necessary information on these data.

Similar considerations apply of course to the functor $\varphi^{\log \chi+\lambda}_{\log \chi}$ under the same assumptions as before.

Now we drop the assumption that $\log \chi' + \sum_{i=1}^{\ell} \frac{r_i}{2} \alpha_{2i-1}$ is dominant.

First assume that $r = 0$. Let $\mu = (\rho, \chi)$, $\log \chi \in \overline{\mathcal{C}}$ and let $w \in W$ be such that $w\overline{\mathcal{C}}_o \subset \overline{\mathcal{C}}$. Now let $V^{w\lambda}$ be the subspace of $V_{\lambda|P_o}$ of weight $w\lambda$ and $\tilde{V}_{w\lambda}$ be $P_o(V^{w\lambda})$, i.e. the smallest P_o-invariant subspace containing $V_{w\lambda}$. $\tilde{V}_{w\lambda}$ is the direct sum of spaces of weight smaller or equal to $w\lambda$ and we can find a largest P_o invariant subspace $\tilde{\tilde{V}}_{w\lambda}$ of $\tilde{V}_{w\lambda}$ which does not contain $V_{w\lambda}$ and is such that $\tilde{V}_{w\lambda}/\tilde{\tilde{V}}_{w\lambda}$ is one dimensional and irreducible. We write $\pi_{w\lambda}$ for the representation of P on $\tilde{V}_{w\lambda}/\tilde{\tilde{V}}_{w\lambda}$.
Thus $(\mathrm{ind}_{P_o}^G \pi\mu) \otimes V_\lambda = \mathrm{ind}_{P_o}^G (\pi_\mu \otimes V_{\lambda|P_o})$ contains $\mathrm{ind}_{P_o}^G (\pi_\mu \otimes \pi_{w\lambda})$ as a composition factor. But $\mathrm{ind}_{P_o}^G (\pi_\mu \otimes \pi_{w\lambda})$ has the central character $w^{-1} \log \chi + \lambda$ and therefore is in the image $\psi_{\log \chi}^{\log \chi + w\lambda} = \psi_{w^{-1}\log \chi}^{w^{-1}\log \chi + \lambda}$. Exactly as before we show

$$\psi_{\log \chi}^{\log \chi + w\lambda} U_\mu^{P_o} = U_{(\rho\rho_{w\lambda}, \chi e^\lambda)}^{P_o} ,$$

where ρ_λ is the restriction of $\pi_{w\lambda}$ to M . Again we define $\mathcal{P}_{w^{-1}\chi+\lambda}$ to be the map which associates to $\pi \otimes V_{\lambda|P_o}$ the composition factor $\pi_\mu \otimes (\tilde{V}_{w\lambda}/\tilde{\tilde{V}}_{w\lambda})$. Then

$$P_{w^{-1} \log \chi}^{w^{-1} \log \chi + \lambda} (U_\mu^{P_o} \otimes V_\lambda) = \mathrm{ind}_{P_o}^G \mathcal{P}_{w^{-1}\chi+\lambda} (\pi_\mu \otimes \pi_{w\lambda}) .$$

Now if $r \neq 0$, let $P_r \supset P_o$ be a standard parabolic and define

$$\mathscr{P}^r_{w^{-1}\chi+\lambda} : (\mathrm{ind}_{P_o}^{P_r} \pi_\mu) \otimes V_{\lambda | P_r} \to \mathrm{ind}_{P_o}^{P_r}((\mathscr{P}_{w^{-1}\chi+\lambda}(\pi_\mu \otimes V_{\lambda | P_o})) .$$

If $\mu' \in \hat{C}_r$, then by the same arguments as before we get a commutative diagram

$$\begin{array}{ccc} \mathrm{ind}_{P_o}^{P_r}\pi_\mu & \to & \mathscr{P}^r_{w^{-1}\log\chi+\lambda}\left[(\mathrm{ind}_{P_o}^{P_r}\pi_\mu)\otimes V_{\lambda|P_r}\right] = \mathrm{ind}_{P_o}^{P_r}\pi_{(\rho\rho_{w\lambda},\, \chi e^{w\lambda})} \\ \uparrow & & \uparrow \\ \pi_{\mu'} & \to & \mathscr{P}^r_{w^{-1}\log\chi+\lambda}[\pi_{\mu'} \otimes V_{\lambda|P_r}] \end{array}$$

where all inclusions are obtained by step by step induction. Hence we can compute the parameter of $\psi^{w^{-1}\log\chi+\lambda}_{w^{-1}\log\chi} U_{\mu'}^{P_r}$ from the parameter of the parabolic P_r and the parameter of $\psi^{w^{-1}\log\chi+\lambda}_{w^{-1}\log\chi} U_\mu^P$.

Example: Assume r is maximal and $P_r = \underline{P}_r$. Then $\mu' = (\vec{r}, \chi)$ and $U_{\mu'}^{P_r} \hookrightarrow U_\mu^{P_o}$, where $U_\mu^{P_o}$ has the continuous parameter $\chi = \chi' e^{\Sigma r_i \alpha_{2i-1/2}}$. Then $\psi^{w^{-1}\log\chi+\lambda}_{w^{-1}\log\chi} U_{\mu'}^{P_r}$ can be embedded in $\psi^{w^{-1}\log\chi+\lambda}_{w^{-1}\log\chi} U_\mu^{P_o}$ which has the continuous parameter $\chi e^{w\lambda}$. This implies that the parameter $(\vec{r}_\lambda, \chi_\lambda)$ of $\psi^{w^{-1}\log\chi+\lambda}_{w^{-1}\log\chi} U^{P_r}$ is given by

$$(r_\lambda)_i = (\alpha_{2i-1}, \log \chi' + \Sigma_i\ r_i\ \alpha_{2i-1/2} + w\lambda)$$

and

$$\log \chi_\lambda' = \log \chi' + \Sigma_i\ r_i\, \alpha_{2i-1/2} + w\lambda - \Sigma_i (r_\lambda)_i \alpha_{2i-1/2}\ . \quad \square$$

G. Unitarity.

In this paragraph we establish some results concerning unitarity, which will be used later to classify the unitarity dual of $GL(n,\mathbb{R})$, $n \leq 4$.

Let U be an irreducible quasisimple representation of G on V and U_k the representation of the enveloping algebra $U(\mathfrak{g})$ on the subspace V_k of k-finite vectors. We call U <u>infinitesimally unitary</u> if there is a positive definite, $U(\mathfrak{g})$-invariant, hermitian form $\langle\ ,\ \rangle$ on V_k. Since there is a one-to-one correspondence between infinitesimally unitary representations and unitary representations, it is enough to classify the former. Furthermore, Dixmier's lemma [4] implies that a $U(\mathfrak{g})$-invariant form, if it exists, is unique up to a scalar multiple.

To classify infinitesimally unitary representations, we can therefore proceed as follows. In the first step we give necessary and sufficient conditions for the existence of a $U(\mathfrak{g})$-invariant hermitian form, and in the second step we find necessary and sufficient conditions for this form to be positive definite. Unfortunately, the second step is much harder than the first one. We call a representation U <u>hermitian</u> if U_k admits a $U(\mathfrak{g})$-invariant hermitian form, and we call a representation unitary if it is infinitesimally unitary.

<u>Theorem 1</u>: Let $\mu \in \hat{\mathcal{C}}_r$, $\mu = (\vec{r}, \chi, \rho)$, J_μ is hermitian iff there is a $w \in W_r$ s.t.

$$w^{-1}\mu = (\vec{r}, \bar{\chi}^{-1}, \rho) \quad .$$

<u>Proof</u>: This theorem is a reformulation in our notation of Theorem 7 in [10b].

The form is constructed as follows:
J_μ is the minimal invariant subspace in $U^P_{\bar{\mu}}$, where $\bar{\mu} = (\vec{r}, \bar{\chi}^{-1}, \rho)$ and $P = P_{\mathcal{C}_r}$ for some $\mathcal{C}_r$ such that $-\mathrm{Re}(\log \bar{\chi}) \in \bar{\mathcal{C}}_r$. Let $(U^P_{\bar{\mu}})^*$ be the representation contragradient to $U^P_{\bar{\mu}}$. Then

$$(U^P_{\bar{\mu}})^* = U^P_\mu \quad .$$

Let $\langle .,. \rangle$ be the pairing between $(U^P_{\bar{\mu}})_K$ and $(U^P_\mu)_K$. The theorem implies that there is a $\tilde{w} \in W$ s.t. $\mathfrak{A}(P,\mu,\tilde{w})(U^P_\mu)_K \supset (U^P_{\bar{\mu}})_K$ and in fact $\mathfrak{A}(P,\mu,\tilde{w})(U^P_\mu)_K = (J_\mu)_K$. Define

$$\langle .,. \rangle_\mu = \langle .,\mathfrak{A}(P,\mu,\tilde{w}).\rangle \quad .$$

Then $\langle .,. \rangle_\mu$ defines a degenerate hermitian $U(\mathfrak{g})$-invariant form on $(U^P_\mu)_K$ and a nondegenerate hermitian form on

$$(U_\mu)_k \,/\, \ker \mathcal{O}\!\mathcal{l}(P,\mu,\tilde{w})_k \;\cong\; (J_\mu)_k \quad .$$

This shows that the problem of classifying unitary representations of $GL(n,\mathbb{R})$ is equivalent to finding necessary and sufficient conditions for the operator $\mathcal{O}\!\mathcal{l}(P,\mu,\tilde{w})$ to be positive or negative semidefinite.

Now assume $w \in W_r$, order $w = 2$. We assume that there exists a unitarily induced irreducible representation $U^P_{\mu_o}$ with $\mu_o = (\vec{r}, \chi_o, \rho) \in \hat{C}_r$ s.t. $\mu_o = w\mu_o$. Then the operator $\mathcal{O}\!\mathcal{l}(P,\mu_o,\tilde{w})$ commutes with U_{μ_o} and is therefore a scalar. Let $\mu = (\vec{r}, \chi, \rho) \in \hat{C}_r$ such that $w\chi = \bar{\chi}^{-1}$, and let $\chi_{(t)}$, $0 \le t \le 1$, be a one-parameter family of characters s.t.

$$\chi_1 = \chi$$

$$\chi_o = w\chi_o$$

$$w\chi_t = \bar{\chi}_t^{-1} \; .$$

Let us also assume that $U^P_{\mu(t)}$ with $\mu(t) = (\vec{r}, \chi_t, \rho)$ is irreducible for all $t \in [0,1]$. By [9,6], this implies that $\langle \; , \; \rangle_{\mu(t)}$ is definite for all $t \in [0,1]$, and hence $U^P_{\mu(t)}$ is unitary.

<u>Definition 1</u>. The set of all irreducible representations U^P_μ with $\mu = (\vec{r}, \chi, \rho) \in \hat{C}_r$ and $P = P_{\mathcal{C}_r}$, for which we can

find $w \in W_r$ with $w^2 = \mathrm{id}$ and a one-parameter family $U^P_{\mu(t)}$ with $\mu(t) = (\vec{r},\chi(t),\rho) \in \hat{C}_r$, $0 \leq t \leq 1$, of irreducible representations s.t.

$$U^P_{\mu(t)} = U^P_{\mu}$$

$$U^P_{\mu(0)} = U^P_{\bar{\mu}(0)}$$

$$U^P_{\mu(t)} = U^P_{w\bar{\mu}(t)}$$

is called <u>the complementary series</u>.

Obviously we are able to determine all complementary series for $GL(n,\mathbb{R})$ as soon as we have explicit formulas for the reducibility of the representations $U^{P_r}_{\mu}$, $\mu \in \hat{C}_r$. An explicit classification of these complementary series by other methods is already contained in [9 a].

Let J_μ , $\mu \in \hat{C}_r$, be hermitian and $J_\mu \subsetneq U^P_\mu$. Assume furthermore that there is a $w \in W_r$ with $w^2 = \mathrm{id}$ s.t. $\mu = w\bar{\mu}$, and that we can find a sequence $U^P_{\mu_i}$ with $\mu_i = (\vec{r},\chi_i,\rho) \in C_r$, $i \in \mathbb{N}$, of complementary series representations s.t. $\mu_i \to \mu$, i.e. $\chi_i \to \chi$, and $w\bar{\mu}_i = \mu_i$. Then J_μ is unitary and is called <u>a limit of complementary series representation</u>.

<u>Lemma 2</u>. Let J_μ , $\mu \in \hat{C}_r$, $J_\mu \subsetneq U^P_{\bar{\mu}}$ be a limit of complementary series representation. Then $U^P_{\bar{\mu}}$ has at least 2

unitary composition factors.

Proof: Let $w \in W_r$ with $w^2 < \mathrm{id}$ s.t. $w\mu = \bar{\mu}$ and let $\mathfrak{A}(P,\rho,\tilde{w}): U_{\mu}^{P} \to U_{\bar{\mu}}^{P}$ be the intertwining operator corresponding to w. Now assume that

$$\mu(t) = (\vec{r},\chi(t),\rho) \in \hat{C}_r \quad \text{s.t.} \quad \begin{aligned} \mu(1) &= \mu \\ \mu(-1) &= \bar{\mu} \\ w\mu(t) &= \mu(-t) = \bar{\mu}(t) \end{aligned}$$

and that $U_{\mu(t)}^{P}$ is in the complementary series for $-1 < t < 1$. Then $\mathfrak{A}(P,\bar{\mu}(t),\tilde{w})$ is defined for $0 \le t \le 1$, and we can define

$$\langle .,. \rangle_{\bar{\mu}(t)} = \langle \;(P,\bar{\mu}(t),\tilde{w}).,.\rangle \qquad 0 \le t \le 1 .$$

Since $U_{\mu(t)}$ and $U_{\mu(-t)}$ are equivalent for $-1 < t < 1$, both forms are equivalent up to a scalar, i.e. both are definite. But this implies [9b] that $\langle .,. \rangle_{\bar{\mu}(t)} = \lim_{t \to 1} \langle .,. \rangle_{\bar{\mu}(t)}$ is definite or divergent. Choosing a regularization of $\mathfrak{A}(P,\bar{\mu}(1),\tilde{w})$, we get a definite hermitian $U(\mathfrak{g})$-invariant form on $U_{\bar{\mu}(1)}/\ker \tilde{\mathfrak{A}}(P,\bar{\mu}(1),\tilde{w})$. The representation $U_{\mu}/\ker \tilde{\mathfrak{A}}(P,\bar{\mu},\tilde{w})$ is not equivalent to J_{μ} (for example Vogan [18]), but a composition factor of $U_{\bar{\mu}}^{P}$. □

The unitary representations obtained in this way are in general very hard to identify in terms of their Langlands parameters. There are examples where a slight modification of this technique is the only "natural" way to show the positively of the hermitian form.

For some μ's in $\hat{\mathcal{C}}_r$ and $J_\mu \subsetneqq U'_\mu$ there is still another way of proving the definiteness of the form: Let $P = M_P A_P N_P$ be an arbitrary standard parabolic. Let π be a unitary representation of M_P , and χ_P a one-dimensional representation of $A_P N_P$. We will now find some conditions for $\mathrm{ind}_P^G \pi \otimes \chi_P = U(P, \pi \otimes \chi_P)$ to be unitary or at least to have a unitary invariant subspace.

a) It is obvious that $U(P, \pi \otimes \chi_P)$ is unitary if χ_P is unitary.

b) As before we can try to construct "complementary series".

Let $U_\mu^{P_r}$ and $U_{\mu'}^{P_r}$ be a g.p.s.r. or p.s.r. s.t.

$$U(P, \pi \otimes \chi_P) \hookrightarrow U_\mu^{P_r}$$

$$U(P, \pi \otimes \bar{\chi}_P^{-1}) \hookrightarrow U_{\mu'}^{P_r}$$

by step by step induction. Then a necessary condition for $U(P, \pi \otimes \chi_P)$ to be hermitian is that there is a $w \in W_r$ with $w^2 = \mathrm{id}$ s.t. $\mu = w\mu'$.

Now let $U(P, \pi \otimes \chi_P^o)$ be a unitarily induced irreducible representation. Assume that

$$U(P, \pi \otimes \chi_P^o) \hookrightarrow U_{\mu_o}^{P_r}$$

and that there exists a $w \in W_r$ such that $w^2 = \mathrm{id}$ and $w\mu_o = \mu_o$. Let $\chi(t)_P$, $t \in [0,1]$, be such that

1) $\chi(0)_P = \chi_P^o$

2) $U(P, \pi \otimes \chi(t)_P)$ is irreducible and $U(P, \pi \otimes \chi(t)_P) \subset U_{\mu(t)}^{P_r}$,

3) $U(P, \pi \otimes \bar{\chi}^{-1}(t)_P)$ is irreducible and $U(P, \pi \otimes \bar{\chi}^{-1}(t)_P) \subset U_{\mu'(t)}^{P_r}$, and $w\mu(t) = \mu'(t)$.

If we can again define an intertwining operator

$$B(P, \pi \otimes \chi(t)_P, w) : U(P, \pi \otimes \chi(t)_P) \hookrightarrow U(P, \pi \otimes \bar{\chi}^{-1}(t)_P)$$

such that $B(P, \pi \otimes \chi_P, w)$ is hermitian for unitary χ_P then by the same arguments as in [9b], we deduce the existence of new unitary representations. If π is unitary, or in the complementary series of M , then these representations can be shown to be in the complementary series. Otherwise we call these unitary representations <u>degenerate complementary series</u> or <u>limit of degenerate complementary series</u>.

After explaining the most important procedures to construct unitary representations we now state conditions under which hermitian representations are not unitary.

Let $\mu \in \hat{C}_r$, and $U_\mu^{P_r}, U_{\bar{\mu}}^{P_r}$ by g.p.s.r. Assume that there exists a $w \in W$ with $w^2 = 1$, such that $\mathfrak{A}(P_r, \bar{\mu}, w)$ is defined and that $\mathfrak{A}(P_r, \bar{\mu}, w) U_{\bar{\mu}}^P \subset U_\mu^P$. Assume furthermore that w is contained in the Weyl group of the M-part of a parabolic $P = MAN \supset P_r$. Then

$$\langle . , \mathfrak{A}(P_r, \bar{\mu}, w) . \rangle$$

is a hermitian form on $U_{\bar{\mu}}^{P_r}$ which we can rewrite as follows:

For $f_1, f_2 \in U_\mu^{P_r}$

$$\langle f_1, \mathfrak{A}(P_r, \bar{\mu}, w) f_2 \rangle = \int_{G/P_r} f_1(\dot{g}) \, \overline{\mathfrak{A}(P_r, \bar{\mu}, w) f_2(\dot{g})} \, d\dot{g}$$

$$= \int_{G/P} \langle \tilde{f}_1(\dot{g}), \mathfrak{A}_M(\bar{\mu}, w) \tilde{f}_2(\dot{g}) \rangle d\dot{g} \ .$$

<u>Claim 3</u>: Assume $\mathfrak{A}(P_r, \bar{\mu}, w) U_{\bar{\mu}}^{P_r}$ is irreducible, and the form $\langle . , \mathfrak{A}_M(\bar{\mu}, w) . \rangle$ is indefinite. Then $\langle . , \mathfrak{A}(P_r, \bar{\mu}, w) . \rangle$ defines an indefinite form in the irreducible representation

$$U_{\bar{\mu}}^P \, / \, \ker \mathfrak{A}(P_r, \bar{\mu}, w) \ .$$

<u>Proof</u>: We have to find f_1, f_2 K-finite s.t.

$$\langle \tilde{f}_1(g), \mathcal{O}\!\mathcal{l}_\mu(\mu,w)\tilde{f}_1(g)\rangle > 0 \quad \text{for all} \quad g \in G$$

$$\langle \tilde{f}_2(g), \mathcal{O}\!\mathcal{l}_\mu(\mu,w)\tilde{f}_2(g)\rangle < 0 \quad \text{for all} \quad g \in G$$

This is obvious if we formulate the question in terms of bundles. □

This result is not very hard, but extremely useful. To get more precise information, we need some detailed analysis: To get estimates on $\mathrm{Re}(\log \chi)$ for unitary J_μ, we use the asympotic expansion of H. Ch. together with the

Theorem (for example [8]).

The coefficients of unitary representations are bounded functions attaining their maximum at the identity.

More precise information about coefficients of unitary representations is in the following

Theorem 4: Let π be an irreducible unitary representation, π not one-dimensional, and let f_π be a coefficient s.t. $f_\pi(e) \neq 0$. Then $\lim f_\pi(a) = 0$, where a goes to infinity in $\exp(\mathcal{C}_o)$.

Proof: This theorem is a corollary to T. Sherman's thesis

He proved [15, Theorem 2.16]:

Let π be a unitary representation of $G = SL(n,\mathbb{R})$ on a Hilbert space H. Assume that the identity representation does not occur in π. Then (H,π) is unitarily equivalent to $(\widetilde{H},\widetilde{\pi})$, where

a) $\widetilde{H}$ is the Hilbert space of functions from $\mathfrak{a}_0'$ to a Hilbert space H_π' with norm

$$\|f\|^2 = \int_{\mathfrak{a}_0'} \|f(x)\|'^{,2} dx < \infty \quad ,$$

where $\|\ \|'$ is the norm of H_π' and dx is Lebesgue measure on $\mathfrak{a}_0'$.

b) For $x \in \mathfrak{a}_0$, $y \in \mathfrak{a}_0'$, $f \in \widetilde{H}$

$$(\widetilde{\pi}(\exp x)f)(x) = e^{iy(x)} f(y) . \qquad \square$$

Now let $g \in \widetilde{H}$ such that for $x \in \mathfrak{a}_0$

$$f_\pi(\exp x) = \langle \pi(\exp x)g,g\rangle$$

$$= \int_{\mathfrak{a}_0'} e^{iy(x)} \|g(y)\|'^{,2} dy \quad ,$$

i.e. f_π is the Fourier transform on an $\mathcal{L}_1$-function, and therefore by Riemann-Lebesgue tends to zero at infinity. $\square$

In paragraph A we constructed for each $\mu = (\vec{r}, \chi, \rho)$ a parabolic $\underline{\underline{P}}_\chi = \underline{\underline{M}}_\chi \underline{\underline{A}}_\chi \underline{\underline{N}}_\chi$. We now choose the positive roots in $\mathfrak{a}_o$ s.t. χ is contained in the closure of the corresponding dominant Weyl chamber $\mathcal{C}_o^\chi$. Let δ_χ be half the sum of the positive roots restricted to $\underline{\underline{\mathfrak{a}}}_\chi =$ Lie $\underline{\underline{A}}_\chi$. Then

<u>Theorem 5</u>: The coefficients of J_μ, $\mu = (\vec{r}, \chi, \rho)$ are unbounded if $(\mathrm{Re}(\log \chi - \delta_\chi, H)) > 0$ for all dominant $H \subset \mathfrak{a}_\chi$.

<u>Proof</u>: For the proof we use "the philosophy of leading coefficients". To simplify the notation we will always assume that we are dealing with $SL_{\pm}(n, \mathbb{R})$.

Let V_λ be an irreducible representation of K with highest weight λ, and let $J_\mu(\lambda)$, $J_\mu^*(\lambda)$ be the isotypic components of type V_λ in $(J_\mu)_K$ respectively $(J_\mu^*)_K = (J_\mu)_K{}^*$, the star denotes as usual the contragredient representation. Assume $J_\mu(\lambda) \neq 0$ and define

$$f_\mu(g) \colon J_\mu(\lambda) \otimes J_\mu^*(\lambda) \longrightarrow \mathbb{C}$$

$$u \otimes v \longrightarrow \langle J_\mu(g)u, v\rangle \quad .$$

By [20], there is a countable set $L(\mu, \lambda)$ in $(\mathfrak{a}_o \otimes \mathbb{C})'$ s.t.

$$f_\mu(a) = e^{-\langle \delta_o, H\rangle} \sum_{\nu \in L(\mu,\lambda)} p_\nu(H) e^{\nu(H)}$$

where $a = \exp H$ for an $H \in \mathfrak{a}_o$, s.t. $\alpha(H) > 0$ for all positive roots α, δ_o half the sum of the positive roots, and the p_ν's are $J_\mu(\lambda) \otimes J_\mu^*(\lambda)$-valued polynomial functions which do not vanish identically.

If $\nu_1, \nu_2 \in L(\mu,\lambda)$, we write $\nu_1 > \nu_2$ if

$$\mathrm{Re}\ \nu_1 = \mathrm{Re}\ \nu_2 + \sum_{\alpha_i \text{ simple}} x_i \alpha_i, \quad x_i \in \mathbb{R}^+ .$$

The maximal elements in $L(\mu,\lambda)$ are called leading coefficients and are the same for all V_λ s.t. $J_\mu(\lambda) \neq 0$ [12]

Let ν_o be a leading coefficient and let $\mathfrak{a}_{\nu_o}$ be the intersection of the kernels of all simple roots α which are orthogonal to $\mathrm{Re}\ \nu_o$. Then $\mathfrak{a}_o = \mathfrak{a}_{\nu_o} \oplus \mathfrak{a}_{\nu_o}^\perp$, and we have the following expansion for $a = \exp H_1 \cdot a_2 = \exp H$ with $H_1 \in \mathfrak{a}_{\nu_p}$ amd $a_2 \in \exp \mathfrak{a}_{\nu_o}$

$$f_\mu(a) = e^{-\langle \delta_o, H\rangle} \sum_{L_1(\mu,\lambda)} \tilde{P}_\nu(H_1, a_2) e^{\nu(H_1)} .$$

Here the $\tilde{P}_\nu$ are $J_\mu(\lambda) \otimes J_\mu^*(\lambda)$-valued functions which for fixed a_2 are polynomials in H_1 and for fixed H_1 are analytic functions in a_2, and $L_1(\mu,\lambda)$ is a countable set in $(\mathfrak{a}_{\nu_o} \otimes \mathbb{C})'$ [7].

Now if $\nu_o = \nu_o^1 + \nu_o^2$ with $\nu_o^1 \in (\mathfrak{a}_{\nu_o} \otimes \mathbb{C})'$ and $\nu_o^2 \in (\mathfrak{a}^1_{\nu_o} \otimes \mathbb{C})'$ then $\nu_o^1 \in L_1(\mu,\lambda)$, and furthermore we can find $u \otimes v \in J_\mu(\lambda) \otimes J^*_\mu(\lambda)$ s.t.

$$\tilde{P}_{\nu_o^1}(H_1, a_2)(u \otimes v) \text{ is not identically zero.} \qquad [12]$$

The uniqueness of the parabolic as "Langlands parameter" of J_μ implies that $\underline{\mathfrak{a}}_\chi = \mathfrak{a}_{\nu_o}$. On the other hand the uniqueness of the continuous parameter implies that $\nu_o^1 = \log \chi$, and

<u>Claim</u>: Let $\nu \in L_1(\mu,\lambda)$. Then there exist $x_i \in \mathbb{R}^+$ s.t.

$$\nu = \nu_o^1 - \sum_{\alpha_i \text{ simple}} x_i \, \alpha_i|\mathfrak{a}_{\nu_o} .$$

<u>Proof</u>: Let $\tilde{\nu}_o$ be another leading coefficient. Then since the whole construction was independent of the choice of the leading coefficient, $\mathfrak{a}_{\nu_o} = \mathfrak{a}_{\tilde{\nu}_o}$ and $\tilde{\nu}_o|\mathfrak{a}_{\nu_o} = \log \chi$.

On the other hand let $\nu \in L(\mu,\lambda)$ be arbitrary, then there is a leading coefficient $\tilde{\tilde{\nu}}_o$ s.t.

$$\nu = \tilde{\tilde{\nu}}_o - \sum_{\alpha_i \text{ simple}} x_i\alpha_i \, , \quad x_i \in \mathbb{R}^+ .$$

But then

$$\nu|_{\mathfrak{a}_{\nu_o}} = \tilde{\tilde{\nu}}_o|_{\mathfrak{a}_{\nu_o}} - \sum_{\alpha_i \text{ simple}} x_i\alpha_i|_{\mathfrak{a}_{\nu_o}}$$

$$= \log \chi - \sum x_i\alpha_i|_{\mathfrak{a}_{\nu_o}} \quad . \qquad \square$$

Now we consider the asymptotics of $f_\mu(a_1,a_2)(u\otimes v)$. We want to show that the exponential term $e^{\log \chi} = \chi$ determines the asymptotic behavior of

$$e^{\langle \delta_o, H\rangle} f_\mu(a)(u\otimes v)$$

in certain directions.

Choose H_1 and a_2 s.t. $\tilde{p}_{\nu_o^1}(H_1,a_2)(u\otimes v) \neq 0$. Now consider $\tilde{p}_{\nu_o}(tH_1,a_2)(u\otimes v)$ as a function of t, $t > 1$. Since it is a nonzero polynomial in t,

$$\lim_{t\to\infty} \text{Re}\, \tilde{p}_{\nu_o^1}(tH_1,a_2)(u\otimes v) \neq 0 \quad .$$

On the other hand, since $\alpha(H_1) > 0$ for all α s.t. $\alpha|_{\mathfrak{a}_{\nu_o}} \neq 0$, the term

$$\lim_{t\to\infty} |\text{Re}\, e^{(\delta_o, tH_1 + \log a_2)} f_\mu(\exp tH_1\, a_2)(u\otimes v)|$$

is dominated by

$$\lim_{t\to\infty} \text{Re}\, \tilde{p}_{\nu_o^1}(tH_1,a_2)(u\otimes v)e^{\log \chi(tH_1)} \quad .$$

In other words,

$$\lim_{t\to\infty} |\mathrm{Re}\ f_\mu(\exp tH_1 \cdot a_2)u \otimes v)|$$

is dominated by

$$\lim_{t\to\infty} |\mathrm{Re}\ \tilde{p}_{\nu_o^1}(tH_1, a_2)(u \otimes v)e^{-(\delta_o, \log a_2)} e^{+(\log \chi - \delta_o)H_1}|$$

and therefore

$$|\mathrm{Re}\ f_\mu((\exp tH_1)a_2)(u \otimes v)| \xrightarrow[t\to\infty]{} \infty \quad \text{if} \quad \mathrm{Re}(\log \chi - \delta_o)H_1 > 0$$

which implies the theorem. □

Corollary 6: The representations J_μ , $\mu = (\vec{r}, \chi, \rho)$, are not unitary if $\log \chi$ is real and $(\mathrm{Re} \log \chi - \delta_\chi)$ dominant in $\mathfrak{a}'_\chi$. □

This corollary was also proved by Knapp and Zuckerman for $r = 0$ with completely different methods.

We can prove an even stronger result:

Corollary 7: The representations J_μ , $\mu = (\vec{r}, \chi, \rho)$, are not unitary if $\chi = \delta_\chi$ and J_μ is not the one-dimensional representation.

Proof: We have shown above that for these parameters we can find a coefficient and on asymptotic direction in which this coefficient does not vanish at infinity, contradicting Theorem 4 . □

CHAPTER 2

A. GL(2,$\mathbb{R}$).

Let us recall some results about GL(2,$\mathbb{R}$).

In this case, A_o are just the diagonal matrices with nonzero entries, and we can identify the character group $\hat{A}_o$ of A_o with $\mathbb{C}^2$ by $(\nu_1,\nu_2) = \nu \longmapsto \chi_\nu$, $\chi_\nu\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \left(\frac{a}{b}\right)^{\nu_1/2} (a\cdot b)^{\nu_2/2}$.

We have $M_o = \left\{\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \middle| \ \epsilon_1,\epsilon_2 = \pm 1\right\}$ and define $\rho_i \in \hat{M}_o$ by

$$\rho_o(m) = 1 , \quad m \in M_o$$

$$\rho_1\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = \epsilon_1$$

$$\rho_2\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = \epsilon_2$$

$$\rho_3\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = \epsilon_1\epsilon_2 , \quad \text{i.e.} \quad \rho_3 = \rho_1\rho_2 .$$

Theorem: $U_\mu^{P_o}$, $\mu = (\rho,\chi_\nu)$, $\rho \in \hat{M}$, $\chi_\nu \in \hat{A}_o$ is reducible iff

$\rho = \rho_0$ or ρ_3 and $\nu_1/2 = 1 \bmod 2$

$\rho = \rho_1$ or ρ_2 and $\nu_1/2 = 0 \bmod 2$ and $\mu_1 \neq 0$. □

Let $w \in W_o$, w nontrivial and $\nu_1 > 0$. Then $A(\mu,w)$ is an isomorphism if $U_\mu^{P_o}$ is irreducible. Otherwise, the image is finite-dimensional and the kernel is a discrete series representation with parameter $(\nu_1/2,\nu_2/2)$.

Remark:

This implies: Let $\pi_{(\nu_1/2,\nu_2/2)}$ be a discrete series representation. Then there are two characters of M_o s.t. $\pi_{(\nu_1/2,\nu_2/2)}$ can be infinitesimally embedded in $U_{\rho,\chi_\nu}^{P_o}$, $\nu = (\nu_1/2,\nu_2/2)$.

Assume now that $\nu_1 < 0$. Then one can obtain an intertwining operator $\mathfrak{A}(\mu,w)$: $U_{(\rho,\chi_{\nu_1\nu_2})} \to U_{(\rho,\chi_{-\nu_1\nu_2})}$ by constructing a regularisation of $A(\mu,w)$. This operator is an isomorphism if $U_{(\rho,\chi_\nu)}$ is irreducible. Otherwise, the kernel is finite-dimensional and the image is a discrete series representation with parameter $(\nu_1/2,\nu_2/2)$.

$GL(2,\mathbb{R})$ has the following unitary representations

a) the unitarily induced p.s.r.

b) all discrete series representations

c) the complementary series representations for

$$\mu = (\rho_o,\chi_\nu) \qquad \nu_2 \in i\mathbb{R}\,, \quad 0 < |\nu_1/2| < 1$$

$$\mu = (\rho_3,\chi_\nu) \qquad \nu_2 \in i\mathbb{R}\,, \quad 0 < |\ {}_1/2| < 1$$

d) the one dimensional representation for purely imaginary μ_2 .

For proofs, see for example [20].

B. GL(3,ℝ) .

We will first compute the composition series for p.s.r., then use this information to settle the reducibility and composition series problem for g.p.s.r. with $r = 1$, and finally classify all unitary representations of GL(3,ℝ) .

The composition series problem for p.s.r of GL(3,ℝ) was recently solved by Fomin [6]. Since his methods are based on an entirely different set of ideas, we will present our proofs here.

We have $GL(3,\mathbb{R}) = SL(3,\mathbb{R}) \times \mathbb{R}^{x}$ as groups. Therefore a p.s.r of GL(3,ℝ) is a tensor product of a p.s.r. of SL(3,ℝ) and a one-dimensional representation of $\mathbb{R}^{x}$. Hence it is enough to determine the composition series and to classify the unitary representations for SL(3,ℝ) .

In SL(3,ℝ) we have

$$M_o = \left\{ \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix} \middle| \ \epsilon_i = \pm 1, \ \epsilon_1\epsilon_2\epsilon_3 = 1 \right\}$$

and we will use the notation

$$\rho_o : \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix} \to 1$$

$$\rho_i : \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix} \to \epsilon_i \quad , \; i = 1,2,3.$$

The character group $\hat{A}_o$ of

$$A_o = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{R} \, , \; a_1 a_2 a_3 = 1 \right\}$$

will be parametrized by $\mathbb{C}^2$ as follows:

$$\mathbb{C}^2 \ni \nu = (\nu_1, \nu_2) \to \chi_\nu = e^{\nu_1 \delta_1 + \nu_2 \delta_2} \; .$$

Here δ_1, δ_2 are the fundamental weights. As before we denote the minimal standard parabolic, i.e. the upper triangular matrices, by P_o. We identify the character group $\hat{A}_1$ of

$$A_1 = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & a^{-2} \end{pmatrix} \middle| a \in \mathbb{R}^x \right\}$$

with $\mathbb{C}$ by

$$\mathbb{C} \ni \nu \to \chi_\nu = e^{\nu \delta_2} \; .$$

Then the two Weyl chambers of $\mathfrak{a}_1'$ are

$$\mathcal{C}_1 = \{ \nu \in \mathbb{R} \mid \nu > 0 \}$$

$$\mathcal{C}_2 = \{ \nu \in \mathbb{R} \mid \nu < 0 \}$$

and

$$P_{\mathcal{C}_1} = \left\{ \begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & x \end{pmatrix} \right\}$$

$$P_{\mathcal{C}_2} = \left\{ \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix} \right\} .$$

We abbreviate $P_{\mathcal{C}_1} = P_1$ and $P_{\mathcal{C}_2} = P_2$, and if it simplifies the notation, we will write $\log \chi$ instead of χ without further mention.

In I.F, we have shown that to solve the composition series problem for p.s.r., it is enough to do so for $\mu = (\rho, \chi_\nu)$ with $\rho \in \hat{M}$ and $0 \leq \nu_1, \nu_2 \leq 1$. Hence in particular the reducibility question is settled by the following

<u>Lemma 1</u>. Let $0 \leq \nu_1, \nu_2 \leq 1$. Then $U^{P_o}_{(\rho_1, \chi_\nu)}$ is reducible iff

$\rho = \rho_o$ and $\nu_1 = 1$ or $\nu_2 = 1$ or $\nu_1 = \nu_2 - 1$

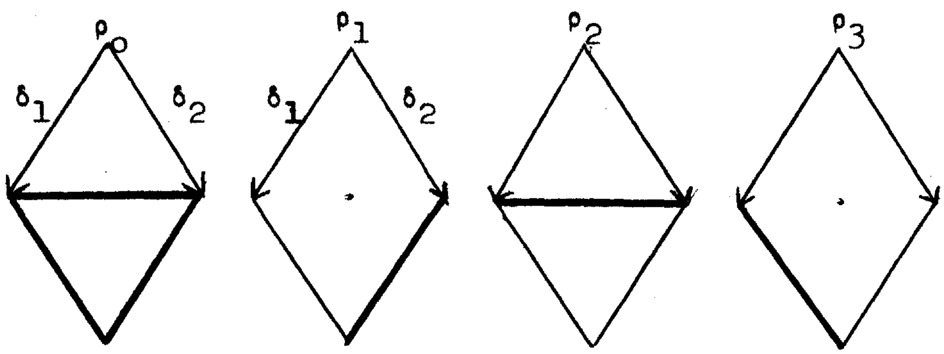
$\rho = \rho_1$ and $\nu_2 = 1$

$\rho = \rho_2$ and $\nu_1 = \nu_2 - 1$

$\rho = \rho_3$ and $\nu_1 = 1$

Proof. This lemma is a special case of theorem II.D , which will be proved later. It can also be derived directly from the results on $GL(2,\mathbb{R})$, together with those at the end of the paragraph on reducibility.

The following diagrams help to visualize the situation, with the bold lines denoting the domain on which reducibility occurs.



For the rest of this paragraph, we will make use of the formulas derived in I.D for computing kernels of intertwining operators without further mention.

Lemma 2. The following p.s.r. have J.H. series of length 2:

$$\rho = \rho_0 \quad \text{and} \quad \nu_1 = 1 \text{ , } 0 \leq \nu_2 < 1 \quad \text{or} \quad 0 \leq \nu_1 < 1 \text{ , } \nu_2 = 1 \quad \text{or} \quad \nu_1 = 1 - \nu_2 \text{ .}$$

$$\rho = \rho_1 \quad \text{and} \quad 0 \leq \nu_1 < 1 \text{ , } \nu_2 = 1$$

$$\rho = \rho_2 \quad \text{and} \quad \nu_1 = 1 - \nu_2$$

$$\rho = \rho_3 \quad \text{and} \quad \nu_1 = 1 \ , \ 0 \le \nu_2 < 1 \ .$$

One composition factor is a degenerate p.s.r. and the other one a g.p.s.r..

Proof. Unless we are in the case $\rho = \rho_0$ and $\nu_1 = 0, \nu_2 = 1$ or $\rho = \rho_0$ and $\nu_1 = 1, \nu_2 = 0$, only one of the factors of the long intertwining operator has a kernel. This kernel is a g.p.s.r.. Therefore we only have to show that the corresponding long intertwining operator for this g.p.s.r. has no kernel. This is easy and is left to the reader.

Now to the case $\rho = \rho_0$, $\nu_1 = 0$, $\nu_2 = 1$. We write $w_0 = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$, where s_{α_i} is a reflection of the simple root α_i. (Assume that we have again the standard ordering of the simple roots). Then $\mathfrak{A}(\mu, s_{\alpha_1}) U_\mu = U_\mu$, and by [9a], $\mathfrak{A}(\mu, s_{\alpha_1})$ is a scalar. Moreover,

$$\mathfrak{A}(\mu, s_{\alpha_1} s_{\alpha_2}) = \mathfrak{A}(s_{\alpha_2}(\mu), s_{\alpha_1}) \, \mathfrak{A}(\mu, s_{\alpha_2}) \quad ,$$

and $\mathfrak{A}(\mu, s_{\alpha_2})$ and $\mathfrak{A}(s_{\alpha_2}(\mu), s_{\alpha_1})$ have the same kernel, namely the g.p.s.r. with discrete series parameter $\vec{r} = (1)$ and $\chi_{P_2} = \chi_{-\frac{1}{2}}$. We have to prove two assertions:

a) $U^{P_2}_{((1), -\delta_2/2)}$ is irreducible.

b) $U^{P_2}_{((1), -\delta_2/2)}$ occurs with multiplicity one in the J. H. series of $U^{P_0}_{(\rho_0, \delta_2)}$.

We first prove a): We have to show that the long intertwining operator $\mathfrak{A}(P_2,((1),-\delta_2/2),s_{\alpha_2}s_{\alpha_1})$ is injective. To show this, we use the step by step embedding of $U^{P_2}_{((1),-\delta_2/2)}$ in $U^{P_o}_{(\rho_1,\delta_2)}$. Then $\mathfrak{A}(P_2,((1),-\delta_2/2),s_{\alpha_2}s_{\alpha_1})$ is the restriction of $\mathfrak{A}((\rho,\delta_2),s_{\alpha_2}s_{\alpha_1})$ to $U^{P_2}_{((1),-\delta_2/2)}$. But direct calculations based on the product formula prove the injectivity of $\mathfrak{A}((\rho_1,\delta_2),s_{\alpha_2}s_{\alpha_1})$, hence of $\mathfrak{A}(P_2((1),-\delta_2/2),s_{\alpha_2}s_{\alpha_1})$.

To prove b), we use the regularisation $\tilde{\mathfrak{A}}(\mu,w)$ of the intertwining operator $A(\mu,w)$ as computed in [9b], which allows us to define this operator for arbitrary χ. All formulas in the paragraphs I.C and I.D then continue to be valid for $\tilde{\mathfrak{A}}(\mu,w)$. Let us consider $U^{P_o}_{(\rho_o,-\delta_1)} = U^{P_o}_{(\rho_o,s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}\delta_2)}$. By I.D and the results on $GL(2,\mathbb{R})$ in the previous paragraph,

$$\tilde{\mathfrak{A}}((\rho_o,-\delta_1),s_{\alpha_2})U^{P_o}_{(\rho_o,-\delta_1)} = U^{P_o}_{((1),-\delta_2/2)}$$

and

$$\tilde{\mathfrak{A}}((\rho_o,s_{\alpha_1}(-\delta_1)),s_{\alpha_2})U^{P_o}_{(\rho_o,-s_{\alpha_1}\delta_1)} = U^{P_o}_{((1),-\delta_2/2)} .$$

Using the product formula for $\tilde{\mathfrak{A}}((\rho_o,-\delta_1),w_o)$, where $w_o = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, and the fact that

$$\tilde{\mathfrak{A}}((\rho_o,-\delta_1),w_o)U^{P_o}_{(\rho_o,-\delta_1)} \neq 0$$

we get

$$\widetilde{\mathcal{O}}((\rho_o,-\delta_1),w_o)U^{P_o}_{(\rho_o,-\delta_1)} = \widetilde{\mathcal{O}}((\rho_o,-\delta_1),s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})U^{P_o}_{(\rho_o,-\delta_1)}$$

$$= \widetilde{\mathcal{O}}((\rho_o,-s_{\alpha_1}\delta_1),s_{\alpha_1}s_{\alpha_2})\widetilde{\mathcal{O}}((\rho_o,-\delta_1),s_{\alpha_1})U^{P_o}_{(\rho_o,-\delta_1)}$$

$$= \widetilde{\mathcal{O}}((\rho_o,s_{\alpha_1}\delta_1),s_{\alpha_2})U^{P_1}_{((1),-\delta_2/2)}$$

$$= U^{P_2}_{((1),-\delta_2/2)} \quad .$$

Now assume that $U^{P_2}_{((1),-\delta_2/2)}$ occured with multiplicity 2, i.e. that

$$\ker \widetilde{\sigma}((\rho_o,s_{\alpha_2}\delta_2),s_{\alpha_1}) \subset \operatorname{im} \widetilde{\sigma}((\rho_o,\delta_2),s_{\alpha_2}) \quad .$$

Then

$$\operatorname{im} \widetilde{\sigma}((\rho_o,s_{\alpha_1}s_{\alpha_2}\delta_2),s_{\alpha_1}) = \ker \widetilde{\sigma}((\rho_o,s_{\alpha_2}\delta_2),s_{\alpha_1})$$

is contained in

$$\operatorname{im} \widetilde{\sigma}((\rho_o,\delta_2),s_{\alpha_2}) = \ker \widetilde{\sigma}((\rho_o,s_{\alpha_2}\delta_2),s_{\alpha_2})$$

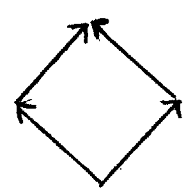
But this contradicts $\widetilde{\mathcal{O}}((\rho_1-\delta_1),w_o)U^{P_o}_{(p_o,-\delta_1)} \neq 0$. Therefore we have proved b).

Exactly the same considerations can be applied to the representation $U^{P_o}_{(\rho_o,\delta_1)}$, and hence the proof of the lemma is now complete. □

Therefore we now are left with computing the composition series for the p.s.r. $U^{P_o}_{(\rho_o,\delta_1+\delta_2)}$, $U^{P_o}_{(\rho_1,\delta_1+\delta_2)}$ and $U^{P_o}_{(\rho_3,\delta_1+\delta_2)}$.

Now introduce the following notation. Let V_1 , V_2 be closed invariant subspaces of a p.s.r. or g.p.s.r. If $V_1 \subset V_2$ and V_2/V_1 is irreducible, with $V_1 \to V_2$. Otherwise do not join these spaces by an arrow.

<u>Theorem 3</u>: The subspace diagram for $U^{P_o}_{(\rho_o,\delta_1+\delta_2)}$ is



and for $U^{P_o}_{(\rho_1,\delta_1+\delta_2)}$ and $U^{P_o}_{(\rho_3,\delta_1+\delta_2)}$ is

<u>Corollary 4</u>. The J. H. series for p.s.r. of $GL(3,\mathbb{R})$ have length 1, 2, 3 or 4. The J. H. series for g.p.s.r. of $GL(3,\mathbb{R})$ have either length 1 or length 2.

<u>Proof</u> of the proposition:

<u>Case 1</u>: $U^{P_o}_{(\rho,\delta_1+\delta_2)}$.

Using step by step induction, we see immediately that $U^{P_1}_{((1),\delta_2)}$ and $U^{P_2}_{((1),-\delta_2)}$ can be considered as invariant subspaces of $U^{P_o}_{(\rho_o,\delta_1+\delta_2)}$. We will show

a) $U^{P_1}_{((1),\delta_2)} \cap U^{P_2}_{((1),-\delta_2)} = U^{P_1}_{((2),0)} = U^{P_2}_{((2),0)}$.

b) $U^{P_1}_{((2),0)}$ is irreducible.

c) $U^{P_1}_{((1),\delta_2)}/U^{P_1}_{((2),0)}$ and $U^{P_2}_{(1,-\delta_2)}/U^{P_2}_{((2),0)}$ are irreducible.

Assertion b) is due to the fact that $U^{P_1}_{((2),0)}$ is unitarily induced.

Now let us assume a) and prove b): Let $\alpha(P_1,((1),\delta_2),s_{\alpha_2}s_{\alpha_1})$ be the long intertwining operator for $U^{P_1}_{((1),\delta_2)}$. By definition $\alpha(P_1,((1),\delta_2),s_{\alpha_2}s_{\alpha_1})$ is the restriction of $\alpha((\rho_o,\delta_1+\delta_2),s_{\alpha_2}s_{\alpha_1})$ to the subspace $U^{P_1}_{((1),\delta_2)}$. Since the kernel of $\alpha((\rho_o,\delta_1+\delta_2),s_{\alpha_2}s_{\alpha_1})$ is just the representation $U^{P_2}_{((1),-\delta_2)}$, the kernel of $\alpha(P_1,((1),\delta_2),s_{\alpha_2}s_{\alpha_1})$ is equal to $U^{P_1}_{((1,)\delta_2)} \cap U^{P_2}_{((1),-\delta_2)}$. Since the image of the long intertwining operator is irreducible, we have proved c).

Now we come to assertion a): First we show $U^{P_1}_{((1),\delta_2)} \cap U^{P_2}_{((1),-\delta_2)} \neq 0$. We have $\tilde{\alpha}((\rho_o,-\delta_1-\delta_2),s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})U^{P_o}_{(\rho_o,-\delta_1-\delta_2)} \neq 0$ and

$$\tilde{\alpha}((\rho_o, s_{\alpha_2} s_{\alpha_1}(-\delta_1-\delta_2)), s_{\alpha_1}) U^{P_o}_{(\rho_o, s_{\alpha_2} s_{\alpha_1}(-\delta_1-\delta_2))} = U^{P_1}_{((1),\delta_2)} .$$

Hence

$$\tilde{\alpha}((\rho_o, -\delta_1-\delta_2), s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}) U^{P_o}_{(\rho_o, -\delta_1-\delta_2)} \subset U^{P_1}_{((1),\delta_2)} .$$

On the other hand,

$$\tilde{\alpha}((\rho_o, -\delta_1-\delta_2), s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}) = \tilde{\alpha}((\rho_o, -\delta_1-\delta_2), s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}) ,$$

and therefore by the same argument

$$\tilde{\alpha}((\rho_1 \; -\delta_1-\delta_2), s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}) U^{P_o}_{(\rho_o, -\delta_1-\delta_2)} \subset U^{P_2}_{((1),-\delta_2)} .$$

Hence

$$\tilde{\alpha}((\rho_o, -\delta_1-\delta_2), s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}) U^{P_o}_{(\rho_o, -\delta_1-\delta_2)} \subseteq U^{P_2}_{((1),-\delta_2)} \cap U^{P_1}_{((1),\delta_2)} \neq 0 .$$

To compute this intersection we use that we can find another embedding of $U^{P_1}_{((1),\delta_2)}$ in a principal series representation. By step by step induction $U^{P_1}_{((1),\delta_2)}$ can also be considered as an invariant subspace of $U^{P_o}_{(\rho_3, \delta_1+\delta_2)}$. Hence the long intertwining operator $\alpha(P_1, ((1),\delta_2), s_{\alpha_1} s_{\alpha_2})$ is the restriction of

$\sigma((\rho_3, \delta_1+\delta_2), s_{\alpha_1} s_{\alpha_2})$ to this subspace. But

$$\sigma((\rho_3, \delta_1+\delta_2), s_{\alpha_1} s_{\alpha_2}) = \sigma((\rho_2, s_{\alpha_2}(\delta_1+\delta_2)), s_{\alpha_1}) \sigma((\rho_3, \delta_1+\delta_2), s_{\alpha_3}) .$$

Here $\sigma((\rho_3, \delta_1+\delta_2), s_{\alpha_2})$ is injective and

$$\ker \sigma((\rho_2, s_{\alpha_1}(\delta_1+\delta_2)), s_{\alpha_1}) = U^{P_1}_{((2),0)} .$$

Hence $$\ker \sigma((\rho_3, \delta_1+\delta_2), s_{\alpha_1} s_{\alpha_2}) = U^{P_1}_{((2),0)} ,$$

which implies that $\sigma((\rho_3, \delta_1+\delta_2), s_{\alpha_1} s_{\alpha_2})$ restricted to $U^{P_1}_{((1),\delta_2)}$ is either injective or by b) has kernel $U^{P_1}_{((2),0)}$. But we already proved that $U^{P_1}_{((2),0)}$ is reducible. Thus the long intertwining operator has kernel $U^{P_1}_{((2),0)}$. This completes the proof of case 1.

<u>Case 2</u>: $U^{P_o}_{(\rho_1, \delta_1+\delta_2)}, U^{P_o}_{(\rho_3, \delta_1+\delta_2)}$.

Because of the symmetry of the situation it is enough to deal with $U^{P_o}_{(\rho_3, \delta_1+\delta_2)}$. In the proof of case 1 we showed that we have the following chain of invariant subspaces

$$U^{P_o}_{(\rho_3, \delta_1+\delta_2)} \supset U^{P_1}_{((1),\delta_2)} \supset U^{P_2}_{((1),\delta_2)} .$$

We will show $U^{P_o}_{(\rho_3, \delta_1+\delta_2)} / U^{P_1}_{((1),\delta_2)}$ is irreducible.

To prove this, it is enough to show that the kernel of the long intertwining operator is just $U^{P_1}_{((1),\delta_2)}$. We proved already

$$\ker \sigma((\rho_3,\delta_1+\delta_2),s_{\alpha_1}s_{\alpha_2}) = U^{P_2}_{((2),\delta)}$$

$$\sigma((\rho_3,\delta_1+\delta_2),s_{\alpha_1}s_{\alpha_2})U^{P_1}_{((1),\delta_2)} \subset U^{P_2}_{((1),\delta_2)}$$

$$U^{P_2}_{((1),\delta_2)} \,/\, \sigma((\rho_3,\delta_1+\delta_2),s_{\alpha_1}s_{\alpha_2})U^{P_1}_{((1),\delta_2)} = U^{P_1}_{((2),\delta)}$$

But $$\ker \sigma((\rho_1,s_{\alpha_1}s_{\alpha_2}(\delta_1+\delta_2)),s_{\alpha_2}) = U^{P_2}_{((1),\delta_2)}$$

Thus

$$\ker \sigma((\rho_3,\delta_1+\delta_2),s_{\alpha_2}s_{\alpha_1}s_{\alpha_2})$$

$$= \ker \sigma((\rho_1,s_{\alpha_1}s_{\alpha_2}(\delta_1+\delta_2)),s_{\alpha_3})\sigma((\rho_3,\delta_1+\delta_2,s_{\alpha_1}s_{\alpha_1})$$

$$= U^{P_1}_{((1),\delta_2)} \quad .$$

This completes the proof of the theorem. □

Proof of the Corollary. We have shown that for $\rho \in \hat{M}$, $0 \leq \nu_1,\nu_2 \leq 1$, all J. H. series of $U^{P_2}_{(\rho,\nu_1\delta_1+\nu_2\delta_2)}$ have length 1, 2, 3, or 4. On the other hand by the proposition all p.s.r. have this property. □

Remarks:

a) Another proof of case 1 of the proposition is possible along the following lines:

Let V_n be the n-dimensional representation of $SO(3)$. It is not hard to prove that in $(U^{P_o}_{(\rho_o,\chi)})_K$, $\chi \in A_o$, V_5 occurs with multiplicity 2, V_7 occurs with multiplicity 1, and V_n, $n = 2,3,4,6$ do not occur at all. But V_5 is the minimal K-type of $U^{P_1}_{((1),\delta_2)}$ and $U^{P_2}_{((1),-\delta_2)}$, and V_7 occurs in $(U^{P_1}_{((1),\delta_2)})_K$ as well as in $(U^{P_2}_{((1),-\delta_2)})_K$ with multiplicity one. Hence V_7 is the minimal K-type of $U^{P_1}_{((1),\delta_2)} \cap U^{P_2}_{((1),-\delta_2)}$. According to our list of minimal K-types, V_7 is the minimal K-type of $U^{P_1}_{((2),0)}$ which is irreducible.

b) In the proof of the proposition above, we have actually derived the following additional statements.

Proposition 5. The J. H. series of $U^{P_o}_{(\rho_o,\delta_1+\delta_2)}$ consists of the representations $J_{(\rho_o,\delta_1+\delta_2)}$, $J_{((1),\delta_2)}$, $J_{((1),-\delta_2)}$ $J_{((2),0)}$, and each of these representations occurs with multiplicity one. The J. H. series of $U^{P_o}_{(\rho_1,\delta_1+\delta_2)}$ contains $J_{(\rho_1,\delta_1+\delta_2)}$, $J_{((1),\delta_2)}$ and $J_{((2),0)}$, the J.H.s of $U^{P_o}_{(\rho_3,\delta_1+\delta_2)}$ contains $J_{(\rho_3,\delta_1+\delta_2)}$, $J_{((1),\delta_2)}$ and $J_{((2),0)}$, and each of these representations occurs with multiplicity one. □

Now we come to the reducibility and composition series problem for g,p.s.r. In order not to confuse the notations, I will write (n) instead of (r) for the discrete series parameter.

Theorem 6. $U^{P_i}_{((n),\nu\delta_2)}$, i = 1,2 , is reducible iff $|\nu| - \frac{|n|}{2} \in \mathbb{N}$.

Proof. Without loss of generality assume $n \geq 0$. We first consider $U^{P_1}_{((n),\nu\delta_2)}$. By the results at the end of I.F, reducibility occurs if

$$n\frac{\alpha-1}{2} + \nu\delta_2 = n_1\delta_1 + n_2\delta_2 \quad \text{for some} \quad n_1 \;\; {}_2 \in \mathbb{N}$$

i.e. if $(n\frac{\alpha_1}{2} + \nu\delta_2, \alpha_1) = n_1 = n$

and $(n\frac{\alpha_1}{2} + \nu\delta_2, \alpha_2) = -\frac{n}{2} + \nu = n_2$.

For all other parameters ν s.t.

$$(n\frac{\alpha_1}{2} + \nu\delta_2, \alpha_2) > 0$$

the representation is irreducible. By the results at the end of I.F. it is also irreducible if $-\frac{n}{2} + \nu = 0$.

We assume $-\frac{n}{2}+\nu<0$ and $\nu>0$. We know already that the representation is irreducible if $n=2$ and $\nu=0$ or if $n=1$ and $-\frac{1}{2}<\nu<\frac{1}{2}$. Again by the results at the end of I.F., the proof of its irreducibility for the other parameters is equivalent to showing that either $\frac{n}{2}+\nu\delta_2 = \alpha_1 + n_1\delta_1 - n_2\delta_2$ for some $n_1, n_2 \in \mathbb{N}$ or $\frac{n}{2}+\nu\delta_2 = \alpha_1/2 + \tilde{\nu}\delta_2 + n_1\delta_1 - n_2\delta_2$ for some $n_1, n_2 \in \mathbb{N}$, $-\frac{1}{2} < \tilde{\nu} < \frac{1}{2}$.

In the first case we get the conditions

1) $2+n_1 = n$, $n_1 \in \mathbb{N}$ and $-\frac{n}{2}+\nu = -1-n_2$, $n_2 \in \mathbb{N}$

and in the second case.

2) $n = 1+n_1$, $n_1 \in \mathbb{N}$ and $-\frac{n}{2}+\nu = -\frac{1}{2}+\tilde{\nu}-n_2$, $n_2 \in \mathbb{N}$, $-\frac{1}{2} < \tilde{\nu} < \frac{1}{2}$.

It is easy to check that one can always find n_1, n_2, $\tilde{\nu}$ which satisfy either condition 1) or condition 2).

To prove the theorem for $\nu \leq 0$ we use that we have an intertwining operator from $U^{\bar{P}_2}_{((n),\nu\delta_2)}$ to $U^{P_1}_{((n),\nu\delta_2)}$ and that therefore $U^{P_1}_{((n),\nu\delta_2)}$ for $\nu \leq 0$ is reducible iff $U^{P_2}_{((n),\nu\delta_2)}$ is reducible. For $U^{\bar{P}_2}_{((n),\nu\delta_2)}$ $\nu \leq 0$ we repeat the above arguments with the central character $|\nu|\delta_1 + (|n|/2)\alpha_2$.

<u>Theorem 7</u>. Let $\mu = ((n,\nu\delta_2)$ s.t. $|\nu| - \frac{1}{2}|n| \in \mathbb{N}$. If $\nu > 0$, then

$\ker \alpha(P_1,((n),\nu\delta_2),s_{\alpha_1}s_{\alpha_2})$

$$= U^{P_1}_{((\frac{1}{2}|n|+\nu, s_{\alpha_2}(|n|\frac{\alpha_1}{2}+\nu\delta_2)-(\frac{1}{2}|n|+\nu)\frac{\alpha_1}{2})} \quad .$$

If $\nu < 0$, then

$\ker \alpha(P_2,((n),\nu\delta_2),s_{\alpha_2}s_{\alpha_1}) =$

$$= U^{P_2}_{((\frac{1}{2}|n|+\nu, s_{\alpha_2}(|n|\frac{\alpha_2}{2}+|\nu|\delta_1)-(\frac{1}{2}|n|+|\nu|\frac{\alpha_2}{2})} \quad .$$

Proof. We use the formulas in the example at the end of I.F. For $\nu > 0$, let $(m),\nu'\delta_2$ be the parameters of the kernel in the statement. Then

$$\frac{|m|}{2}\alpha_1 + \nu'\delta_2 = s_{\alpha_2}(\frac{|n|}{2}\alpha_1 + \nu\delta_2)$$

$$= \nu'\delta_2 = s_{\alpha_2}(\frac{|n|}{2}\alpha_1 + \nu\delta_2) - \frac{|m|}{2}\alpha_1 \ ,$$

$$|n| = \frac{|m|}{2}(\alpha, s_{\alpha_2}\alpha_1) + |\nu'|(\delta_2, s_{\alpha_2}\alpha_1) = \frac{|m|}{2} + |\nu'| \quad ,$$

$$|\nu| - \frac{|n|}{2} = \frac{|m|}{2} - |\nu'| = \frac{|m|}{2}(\alpha_1, s_{\alpha_2}\alpha_2) + |\nu'|(\delta_2, s_{\alpha_2}\alpha_2) \ ,$$

and therefore $|m| = \frac{1}{2}|n| + |\nu|$

$$|\nu'| = \frac{1}{2}(\frac{3}{2}|n| - \nu') \quad .$$

For $\nu < 0$ we argue in exactly the same way. □

Finally I give a classification of all unitary representations of $SL(3,\mathbb{R})$:

Theorem 8. The unitary dual of $SL(3,\mathbb{R})$ consists of the following representations:

a) unitarily induced p.s.r.

b) unitarily induced g.p.s.r.

c) complementary series representations

d) unitarily induced degenerate series representations $\supset$ limits of complementary series.

e) the one-dimensional representation.

Proof. First we classify all unitary irreducible representations with parameter $\mu \in \hat{G}_o$.

We may assume that $\mu = (\rho,\chi)$, $\rho \in \hat{M}$ and χ s.t. $\chi = e^{\nu_1\delta_1+\nu_2\delta_2}$ with $\mathrm{Re}\,\nu_1 \geq 0$, $\mathrm{Re}\,\nu_2 \geq 0$. Besides the unitarily induced representations, the representations with the following Langlands parameters are hermitian.

$$\rho = \rho_0 , \quad \log \chi = \nu(\delta_1 + \delta_2) + i\lambda s_{\alpha_1}\delta_1 , \quad \nu,\lambda \in \mathbb{R} ,$$

$$\rho = \rho_2 , \quad \log \chi = \nu(\delta_1 + \delta_2) + i\lambda s_{\alpha_1}\delta_1 , \quad \nu,\lambda \in \mathbb{R} .$$

The representations $U^{P_o}_{(\rho_o,\nu(\delta_1+\delta_2)+i\lambda s_{\alpha_1}\delta_1)}$ are reducible iff

$$\nu = 2m+1/2 \quad \text{and} \quad \lambda \neq 0 \qquad m \in \mathbb{N}$$

$$\nu = m \quad \text{and} \quad \lambda = 0 \qquad m \in \mathbb{N}\setminus 0$$

and the representations $U^{P_o}_{(\rho_2,\nu(\delta_1+\delta_2)+i\lambda\delta_1)}$ are reducible iff

$$\nu = 2m+1/2 \qquad \lambda \in \mathbb{R} \qquad m \in \mathbb{N} .$$

Hence we get complementary series representations for $\rho = \rho_0, \rho_2$, $\lambda \in \mathbb{R}$ and $0 < \nu < \frac{1}{2}$ and limits of complementary series for $\rho = \rho_0, \rho_2$, $\lambda \in \mathbb{R}$ and $\nu = \frac{1}{2}$.

If $\nu > 1$, then by theorem G7 the hermitian representations are not unitary. They are not unitary as well for

$\rho = \rho_0, \rho_2$	$\frac{1}{2} < \nu < 1$	λ arbitrary
$\rho = \rho_2$	$\nu = 1$	λ arbitrary
$\rho = \rho_0$	$\nu = 1$	$\lambda \neq 0$

as one can see as follows.

The representations with these Langland parameters are irreducible principal series representations and therefore equivalent to the p.s.r. $U^{P_o}_{(s_{\alpha_1}(\mu))}$. But $U^{P_o}_{(s_{\alpha_1}(\mu))}$ can be considered as induced from the parabolic P_1 with a non unitary representation on the M part and a unitary character on A_1 , and hence we can apply Lemma ( ).

For $\rho = \rho_0$, $\nu = 1$, $\lambda = 0$ the Langlands representation is the one-dimensional representation, hence it is unitary.

For $\rho = \rho_0, \rho_2$, $\nu = \frac{1}{2}$, λ arbitrary, we get the two series of representations, which are unitary induced form P_1 and one of the two onde-dimensional representation of M . This completes the list of all unitary representations with the Langlands parameter $\rho \subset \hat{C}_0$.

For $\rho \in \hat{C}$, the only hermitian representations are the unitary induced ones, which implies that our list is complete. □

This classification of unitary representations was also proved by I. Vakhutinskii [17].

C. GL(4, $\mathbb{R}$).

Again the composition series for p.s.r. is computed first, but only for nonsingular parameters. For singular parameters, the Langlands parameters for all composition factors are given, but their multiplicities or the composition series were not successfully computed. Next, the reducibility and composition series problem for g.p.s.r. with $r = 2$ is settled, and finally all unitary representations of $GL(4,\mathbb{R})$ are classified.

All results in this chapter are new and apparently have not been published in any of the literature consulted.

In $GL(4,\mathbb{R})$ we have

$$M_o = \left\{ \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_3 & \\ & & & \epsilon_4 \end{pmatrix} \mid \epsilon_i = \pm 1 \right\}$$

and write the elements of $\hat{M}_o$ as follows:

$$\rho_o: \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_3 & \\ & & & \epsilon_4 \end{pmatrix} \longrightarrow 1$$

$$\rho_i: \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_3 & \\ & & & \epsilon_4 \end{pmatrix} \longrightarrow \epsilon_i$$

$$\rho_{ij}: \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_3 & \\ & & & \epsilon_4 \end{pmatrix} \longrightarrow \epsilon_i \epsilon_j \ , \quad i < j$$

$$\epsilon_{ijk}: \begin{pmatrix}\epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_3 & \\ & & & \epsilon_4\end{pmatrix} \longrightarrow \epsilon_i\epsilon_j\epsilon_k \ , \quad i < j < k$$

$$\rho_{1234} = \det \qquad .$$

The characters of the group $Z_1 = \left\{\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \epsilon_1 & \\ & & & \epsilon_2\end{pmatrix} \,\middle|\, \epsilon_1 = \pm 1\right\}$ will be denoted by $\rho_o^1 = \rho_o\big|_{Z_1}$,

$\rho_3^1 = \rho_3\big|_{Z_1}$, $\rho_4^1 = \rho_4\big|_{Z_1}$ and $\rho_{34}^1 = \rho_{34}\big|_{Z_1}$.

We parametrize the charactergroup $\hat{A}_o$ of

$$A_o = \left\{\begin{pmatrix}a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4\end{pmatrix} \,\middle|\, a_i \in \mathbb{R}\setminus 0\right\}$$

by $\mathbb{C}^4$ as follows:

$$\mathbb{C}^4 \ni \nu = (\nu_1,\nu_2,\nu_3,\nu_4) \to \chi_\nu = e^{\nu_1\delta_1} + e^{\nu_2\delta_2} + e^{\nu_3\delta_3}\ (\det)^{\nu_4}$$

where δ_i , $i = 1,2,3$, are the fundamental weights for the standard ordering of the roots. The character groups $\hat{A}_1$ respectively $\hat{A}_2$ will be identified with the subgroups of $\hat{A}_o$ with $\nu_1 = 0$ respectively $\nu_1 = \nu_3 = 0$.

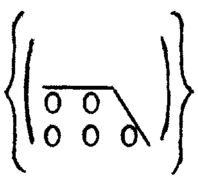
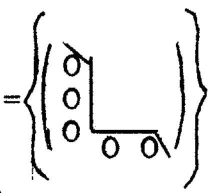
In $\mathfrak{a}_o'$ we choose the dominant Weyl chamber $\mathcal{C}_o$ and associate to it the standard minimal parabolic P_o , i.e. the upper triangular matrices. In $\mathfrak{a}_1'$ we choose the nonconjugate Weyl chambers

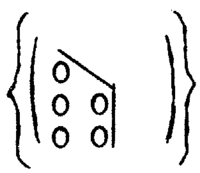
$$\mathcal{C}_1^1 = \{\nu_2\delta_2 + \nu_3\delta_3 \mid \nu_2, \nu_3 > 0\}$$

$$\mathcal{C}_1^2 = \{\nu_2\delta_2 + \nu_3\delta_3 \mid \nu_3 > 0,\ 0 > \nu_2 > -\nu_3\}$$

$$\mathcal{C}_1^3 = \{\nu_2\delta_2 + \nu_3\delta_3 \mid \nu_2 < 0,\ \nu_3 < 0\}$$

and set

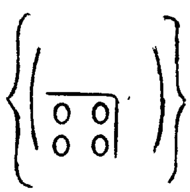
$P_{\mathcal{C}_1^1} = P_1^1 =$  , $P_{\mathcal{C}_1^2} = P_1^2 =$ 

$P_{\mathcal{C}_1^3} = P_1^3 =$  .

In $\mathfrak{a}_2'$ we have only one conjugacy class of Weyl chambers and choose the Weyl chamber

$$\mathcal{C}_2 = \{\nu\delta_2 \mid \nu > 0\}$$

and

$P_{\mathcal{C}_2} = P_2 =$  .

Again I will write $\log \chi$ instead of χ if it simplifies the notation, and assume $\nu_4 = 0$.

In I.F. we have shown that to solve the composition series problem for p.s.r. it is enough to do so for $u = (\rho, \chi_\nu)$ with $\rho \in \hat{M}$ and $0 \leq \nu_i \leq 1$, $i = 1,2,3.$ Hence in particular the reducibility question is settled by the following

Lemma 1: Let $0 \leq \nu_i \leq 1$, $i = 1,2,3$. Then $U^{P_o}_{(\rho,\chi_\nu)}$ is reducible iff

$\rho = \rho_o$ or ρ_{1234} and one of the ν_i's equal to 1 or $\nu_1+\nu_2 = 1$ or $\nu_2+\nu_3 = 1$ or $\nu_1+\nu_2+\nu_3 = 1$

$\rho = \rho_j$ or $\det \rho_j$ and one of the ν_i's with $i \neq j$, $j-1$ is equal to 1
or if $j = 1$: $\nu_1+\nu_2 = 1$ or $\nu_2+\nu_3 = 2$ or $\nu_1+\nu_2+\nu_3 = 2$
or if $j = 2$: $\nu_2+\nu_3 = 2$ or $\nu_1+\nu_2+\nu_3 = 1$
or if $j = 3$: $\nu_1+\nu_2 = 2$ or $\nu_1+\nu_2+\nu_3 = 1$
or if $j = 4$: $\nu_1+\nu_2 = 1$ or $\nu_2+\nu_3 = 2$ or $\nu_1+\nu_2+\nu_3 = 2$

$\rho = \rho_{ij}$ or $\det \rho_{ij}$ and if $i=1$, $j=2$: $\nu_1 = 1$ or $\nu_3 = 1$ or $\nu_1+\nu_2 = 2$ or $\nu_1+\nu_3 = 2$ or $\nu_1+\nu_2+\nu_3 = 2$
or if $i=1$, $j=3$: $\nu_1+\nu_2=1$ or $\nu_2+\nu_3 = 1$ or $\nu_1+\nu_2+\nu_3 = 2$

or if $i = 1$, $j = 4$: $\nu_2 = 1$ or $\nu_1 + \nu_2 + \nu_3 = 1$ or $\nu_1 + \nu_2 = 2$ or $\nu_2 + \nu_3 = 2$.

Proof: As in the case of $GL(3,\mathbb{R})$, this lemma is a special case of Theorem II.D.1, which will be proved later. It can also be derived directly from the results on $GL(2,\mathbb{R})$, together with those at the end of the paragraph on reducibility. □

For the rest of this paragraph, we will again make use of the formulas derived in I.D for computing kernels of intertwining operators without further mention.

Lemma 2: Let Σ be a subsystem of type A_2 of A_3 which contains two simple roots. Assume that $\log \chi$, for $\chi \in \hat{A}_o$, does not satisfy any integrbility condition with respect to roots not contained in Σ. Then we can compute the Jordan-Hölder series of $U_\mu^{P_o}$ for $\mu = (\rho,\chi)$ as follows:

Let P_Σ be the standard parabolic s.t. $\mathrm{Lie}(A_\Sigma)$ is the intersection of the kernels of the roots in Σ, and let $\{J_1,\dots,J_\ell\}$ be the Jordan-Hölder series of $\Pi_{M_\Sigma \cap P}^{M_\Sigma} \mu\big|_{M_\Sigma \cap P}$. Then the J.H.s. of $U_\mu^{P_o}$ is

$$\left\{\mathrm{ind}_{P_\Sigma}^G(J_1 \otimes \pi_\Sigma\big|_{A_\Sigma N_\Sigma}),\ldots,\mathrm{ind}_{P_\Sigma}^G(J_\ell \otimes \pi_\mu\big|_{A_\Sigma N_\Sigma})\right\} .$$

Proof: Step by step induction shows that for $J \in \{J_1,\ldots,J_\ell\}$ we can find closed invariant subspaces $V_1 \supset V_2$ s.t. $\mathrm{ind}_{P_\Sigma}^G(J \otimes \pi_\mu\big|_{A_\Sigma N_\Sigma}) \cong V_1/V_2$. Therefore all we have to show is that $\mathrm{ind}_{P_\Sigma}^G(J \otimes \pi_\mu\big|_{A_\Sigma N_\Sigma})$ is irreducible.

Let μ_J be the Langlands parameter of J_1 and let $U_{\mu_J}^{P_M} = \underline{U}_{\mu_J}$. Then we can find $\mu' \in C_o$, conjugate under the Weyl group to μ, s.t.

$$\mathrm{ind}_{P_\Sigma}^G\left(U_{\mu_J}^{P_M} \otimes \pi_\mu\big|_{A_\Sigma N_\Sigma}\right) \subset U_{\mu'}^{P_o}$$

as an invariant subspace. We may assume that μ' is dominant.

Then let $\mathcal{O}(P,\mu',w_o)$ be the long intertwining operator for $U_{\mu'}^P$, where $P = P_M A_\Sigma N_\Sigma$. We rearrange $w_o = w\tilde{w}_o$, where $\tilde{w}_o$ is contained in the Weyl group of Σ and is such that $\mathcal{O}_M(P_M,\mu_J,\tilde{w}_o)$ is the long intertwining operator for $U_{\mu_J}^{P_M}{}_o$. Since the image of $\mathcal{O}_M(P_M,\mu_J,\tilde{w}_o)$ is just J and since

$$\mathcal{O}(P,\mu',w_o) = \mathcal{O}(\tilde{w}_o P,\tilde{w}_o\mu',w)\mathcal{O}(P,\mu',\tilde{w}_o)$$

it suffices to prove that $\sigma(\tilde{w}_o P, \tilde{w}_o \mu', w)$ is an isomorphism.

But $\tilde{w}_o$ transforms all positive roots of Σ into negative ones. Hence if α is a positive root that w transforms into a negative one, then $\tilde{w}_o^{-1}\alpha$ is not in Σ. This implies that for $\mu' = (\rho', \chi')$, $(w_o\chi', \alpha) = (\chi', \tilde{w}_o^{-1}\alpha)$ is not an integer. Thus $\sigma(P, \mu', \tilde{w}_o)$ is an isomorphism.

$\square$

Let $\mu \in \hat{\mathcal{C}}_o$, and assume now that we have a subsystem Σ of type A_2, which does not contain two simple roots, but otherwise satisfies the conditions of the lemma. Then we can find $w \in W$ s.t. $w\mu$ and $w\Sigma$ satisfy all the conditions of the lemma. Since the J.H. series of $U_\mu^{P_o}$ and $U_{w\mu}^{P_o}$ are the same, we have an inductive procedure to calculate the J.H. series in this case too.

Assume now that we have a subsystem of type $A_1 \times A_1$ which satisfies the conditions of the lemma. Then we can compute the J.H. series for $U_\mu^{P_o}$ from the J.H. series of $\mathrm{ind}_{M_\Sigma \cap P_o}^{M_\Sigma}$ exactly as before.

These considerations show that we are left with computing the J.H. series for

a) $U_{(\rho, \delta_1+\delta_2+\delta_3)}^{P_o}$, $\rho \in \hat{M}$

b) $U_{(\rho, \delta_i+\delta_j)}^{P_o}$, $i,j \in \{1,2,3\}$, $i < j$, $\rho \in \hat{M}$

c) $U_{(\rho, \delta_i)}^{P_o}$, $i = 1,2,3$.

We will first deal with Case a):
Let $\rho = \rho_o$ and $\chi = \delta_1 + \delta_2 + \delta_3$. Using step by step induction, together with the formulas for $SL(3,\mathbb{R})$, we see that the representation with the following Langlands parameters are contained in the composition series:

$$\rho_o, \quad \delta_1 + \delta_2 + \delta_2$$

$$(1), \quad \tfrac{3}{2}\delta_2 + \delta_3 \ , \ \rho_o^1$$

$$(1), \quad -\tfrac{3}{2}\delta_2 - \delta_3 \ , \ \rho_o^1$$

$$(1), \quad -\tfrac{3}{2}\delta_2 + 3\delta_3, \ \rho_o^1$$

$$(2), \quad 2\delta_3 \ , \ \rho_3^1$$

$$(2), \quad -2\delta_3 \ , \ \rho_3^1$$

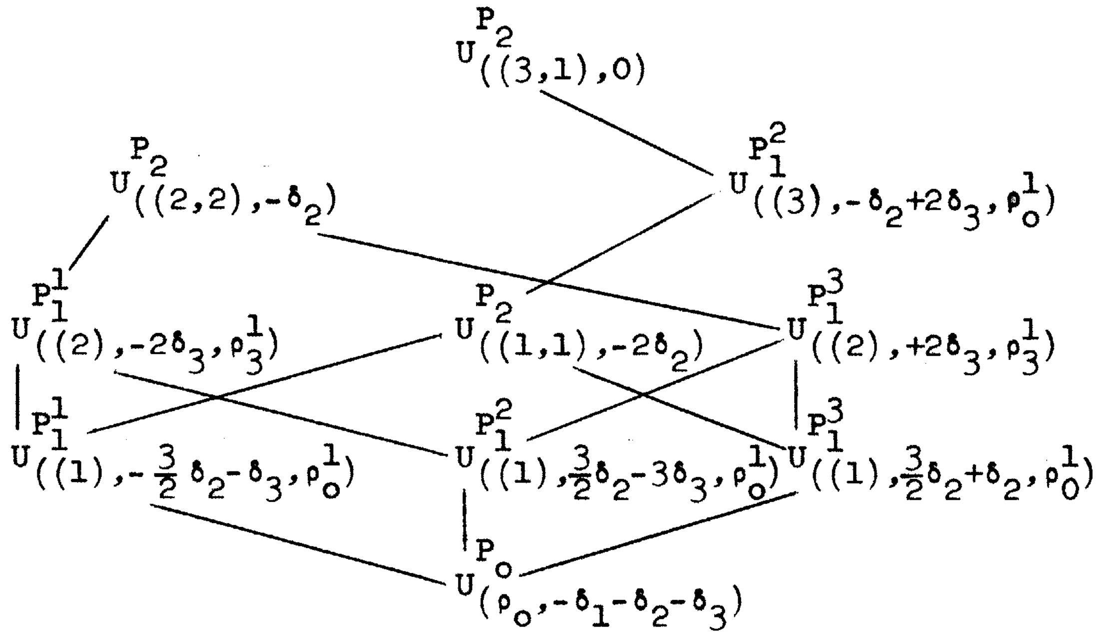
$$(3), \quad 2\delta_3 - \delta_2 \ , \ \rho_o^1$$

$$(2,2), \quad \delta_2$$

$$(3,1), \quad 0$$

$$(1,1), \quad 2\delta_2 \quad .$$

Using step by step induction we also derive the following relations between subspaces



Here we use the following convention:

Let V be a representation of G and V_1 a closed invariant subspace of V. Then we write

$$\begin{array}{c} V/V_1 \\ | \\ V \end{array}$$

Otherwise we do not join the two representations by a line.

<u>Claim</u>: The representation with Langlands parameter $((3),2\delta_3-\delta_2,\rho^1_{34})$ is contained in the J.H. series of $U^{P_o}_{(\rho,\delta_1+\delta_2+\delta_3)}$.

<u>Proof</u>: We use again the regularized intertwining operator

$\widetilde{\mathfrak{A}}(\mu,w)$ for $\mu \in G_o$, $w \in W_o$.

Step by step induction shows

$$U^{P_2}_{((1,1),2\delta_2)} \longrightarrow U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)} \quad .$$

We will show that the representations $J_{((3),2\delta_2-\delta_3,\rho^1_{3\,4})}$ occurs in the J.H. series of $U^{P_2}_{((1,1),2\delta_2)}$ or equivalently in the J.H. series of $U^{P_2}_{((1,1),-2\delta_2)}$. We have

$$U^{P_2}_{((1,1),-2\delta_2)} \longrightarrow U^{P_o}_{(\rho_{1\,2\,3\,4},-\delta_1-\delta_2-\delta_3+\alpha_1+\alpha_3)}$$

and

$$\widetilde{\mathfrak{A}}_{(\rho_{1\,2\,3\,4},-\delta_1-\delta_2-\delta_3,s_{\alpha_1}s_{\alpha_3})}U^{P_o}_{(\rho_{1\,2\,3\,4},-\delta_1-\delta_2-\delta_3)} = U^{P_2}_{((1,1),-2\delta_2)} \quad .$$

On the other hand,

$$\widetilde{\mathfrak{A}}_{(\rho_{1\,2\,3\,4},-\delta_1-\delta_2-\delta_3+\alpha_1+\alpha_3,s_{\alpha_2})}U^{P_o}_{(\rho_{1\,2\,3\,4},-\delta_1-\delta_2-\delta_3+\alpha_1+\alpha_3)} = U^{P^2_1}_{((3),-2\delta_3-\delta_2,\rho^1_{3\,4})} \quad .$$

Since $U^{P_o}_{((3),-2\delta_3-\delta_2,\rho^1_{3\,4})}$ contains only one minimal invariant subspace, namely the representation $J_{((3),2\delta_3-\delta_2,\rho^1_{3\,4})}$, we have only to show that

$$\widetilde{\mathcal{O}}_{(\rho_{1\,2\,3\,4},\,-\delta_4-\delta_2-\delta_3+\alpha_1+\alpha_3,\,s_{\alpha_2})} U^{P_2}_{((1,1),-2\delta_2)} \neq 0 \quad .$$

But

$$\widetilde{\mathcal{O}}_{(\rho_{1\,2\,3\,4},\,-\delta_1-\delta_2-\delta_3+\alpha_1+\alpha_2,\,s_{\alpha_2})} U^{P_2}_{((1,1),-2\delta_2)}$$

$$= \widetilde{\mathcal{O}}_{(\rho_{1234},-\delta_1-\delta_2-\delta_3+\alpha_1+\alpha_3,s_{\alpha_2})} \widetilde{\mathcal{O}}_{(\rho_{1234},-\delta_1-\delta_2-\delta_3,s_{\alpha_1}s_{\alpha_3})} U^{P_o}_{(\rho_{1234},-\delta_1-\delta_2-\delta_3)}$$

$$= \widetilde{\mathcal{O}}_{(\rho_{1234},-\delta_1-\delta_2-\delta_3,s_{\alpha_2}s_{\alpha_1}s_{\alpha_3})} U^{P_o}_{(\rho_{1234},-\delta_1-\delta_2-\delta_3)} \neq 0 \quad . \quad \Box$$

<u>Remark</u>: Since the restriction of $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$ to $SL(4,\mathbb{R})$ is irreducible and the restrictions of $J_{((3),2\delta_3-\delta_2,\rho^1_{34})}$ and $J_{((3),2\delta_3-\delta_2,\rho^1_o)}$ to $SL(4,\mathbb{R})$ are isomorphic, the restriction of $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$ to $SL(4,\mathbb{R})$ is an example of a p.s.r. of a connected semisimple Lie group, in which one composition factor occurs with multiplicity 2. The first example of a representation in which

one composition factor has a multiplicity larger than one, was given by Conze-Duflo in [3], but they could not determine the multiplicity exactly. We will later see that this phenomenon occurs quite frequently if the continuous parameter is singular.

<u>Theorem 3</u>: $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$ has the Jordan-Holder series

$$\{J_{(\rho_o,\delta_1+\delta_2+\delta_3)}, J_{((1),\frac{3}{2}\delta_2+\delta_3,\rho_o^1)}, J_{((1),-\frac{3}{2}\delta_2-\delta_3,\rho_o^1)},$$

$$J_{((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_o^1)}, J_{((2),2\delta_3,\rho_3^1)}, J_{((2),-2\delta_3.\rho_3^1)},$$

$$J_{((1,1),2\delta_2)}, J_{((3,1),0)}\} .$$

<u>Proof</u>: To prove the theorem we proceed as follows: First compute the minimal K-type for all irreducible representations with this central character. Then compute the multiplicities of these K-types in the representation $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta)}$. By comparing these two lists, we will see that for almost all representations which are not contained in the list of the theorem, the minimal K-type does not occur in $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$. On the other hand, it has already been proved that all representations occuring in the list are actually contained in the J.H. series of $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$. Therefore we are left with

computing the multiplicities with which these K-types occur in each representation of the list above. These tedious computations constitute the main, lengthy, part of the proof, and they will finally allow us to deduce that each of the representations above occurs with multiplicity one.

We use the formulas of I.B for computing the minimal K-types of an irreducible representation with a given Langlands parameter. We get the following list of minimal K-types for all irreducible representations with central character $\delta_1 + \delta_2 + \delta_3$:

Langlands parameter	Highest weight of minimal K-type
$(\rho_o,\delta_1+\delta_2+\delta_3),(\rho_{1234},\delta_1+\delta_2+\delta_3)$	(0,0)
$(\rho_i,\delta_1+\delta_2+\delta_3), i=1,2,3, (\rho_{ijk},\delta_1+\delta_2+\delta_3), i<j<k$	(1,0)
$(\rho_{ij},\delta_1+\delta_2+\delta_3),\ i<j$	(1,1)
$((1),\frac{3}{2}\delta_2+\delta_3,\rho_o^1),((1),\frac{3}{2}\delta_2+\delta_3,\rho_{34}^1)$	(2,0)
$((1),-\frac{3}{2}\delta_2-\delta_2,\rho_o^1),((1),-\frac{3}{2}\delta_2-\delta_3,\rho_{34}^1)$	(2,0)
$((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_o^1),((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_{34}^1)$	(2,0)
$((1),\frac{3}{2}\delta_2+\delta_3,\rho_3^1),((1),\frac{3}{2}\delta_2+\delta_3,\rho_4^1)$	(2,1)
$((1),-\frac{3}{2}\delta_2-\delta_3,\rho_3^1),((1),-\frac{3}{2}\delta_2-\delta_3,\rho_4^1)$	(2,1)
$((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_3^1),((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_4^1)$	(2,1)
$((2),2\delta_3,\rho_o^1),((2),2\delta_3,\rho_{34}^1)$	(3,0)
$((2),-2\delta_3,\rho_o^1),((2),-2\delta_3,\rho_4^1)$	(3,0)
$((2),2\delta_3,\rho_3^1),((2),2\delta_3,\rho_4^1)$	(3,1)
$((2),-2\delta_3,\rho_3^1),((2),-2\delta_3,\rho_4^1)$	(3,1)
$((3),2\delta_3-\delta_2,\rho_o^1)\ ((3),2\delta_3-\delta_2,\rho_{34}^1)$	(4,0)
$((3),2\delta_3-\delta_2,\rho_3^1)\ ((3),2\delta_3-\delta_2,\rho_4^1)$	(4,1)
$((1,1),2\delta_2)$	(2,2)
$((2,2),\delta_2)$	(3,3)
$((3,1),0)$	(4,2)

<u>Lemma 4:</u> We have the following multiplicity for K-types in $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$:

Highest weight	Multiplicity
(0,0)	1
(1,0)	0
(1,1)	0
(2,0)	3
(2,1)	0
(2,2)	2
(3,0)	0
(3,1)	3
(3,3)	0
(4,0)	1
(4,1)	0

<u>Proof</u>: Appendix to this paragraph. □

If we compare those two lists we see immediately that the representations $J_{(\rho_o,\delta_1+\delta_2+\delta_3)}$, $J_{((1),\frac{3}{2}\delta_2+\delta_3,\rho_0^1)}$, $J_{((1),-\frac{3}{2}\delta_2-\delta_3,\rho_o^1)}$ and $J_{((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_o^1)}$ occur with multiplicity one. On the other hand we see that the only irreducible representations with this central character, whose minimal K-type occurs with nonzero multiplicity, are the representations in the list of the theorem, and of course all representations of this list tensored with the determinant representation.

One of the two K-types with highest weight (2,2) is the minimal K-type of the representation $J_{((1,1),2\delta_2)}$, and the other one is contained in the representation induced from the one-dimensional representation of the parabolic P_2, since this K-type has an M_o-invariant vector in the space of weight (0,0) [Appendix]. Hence this K-type is contained in $J_{((1),-\frac{3}{2}\delta_2+3\delta_3,\rho_o^1)}$.

Now to the K-type (3,1). This K-type occurs in the representations $J_{((2),2\delta_3,\rho_3^1)}$ and $J_{((2),-2\delta_3,\rho_3^1)}$ exactly once. On the other hand this K-type is contained in the representation $U^{P_o}_{((1,1),2\delta_2)}$. If this K-type is not contained in $J_{((1,1),2\delta_2)}$, then either $J_{((2),2\delta_3,\rho_3^1)}$ or $J_{((2),-2\delta_3,\rho_3^1)}$ would be contained in the J.H. series of $U^{P_2}_{((1,1),2\delta_2)}$. But we have an outer automorphism of $GL(4,\mathbb{R})$ which maps $J_{((2),2\delta_3,\rho_3^1)}$ to $J_{((2),-2\delta_3,\rho_3^1)}$ and leaves $U^{P_2}_{((1,1),2\delta_2)}$ invariant. Hence if $J_{((2),2\delta_3,\rho_3^1)}$ occured in the J.H. series of $U^{P_2}_{((1,1),2\delta_2)}$, then so would $J_{((2),-2\delta_3,\rho_3^1)}$. Since the K-type (3,1) has multiplicity one, $J_{((2),2\delta_3,\rho_3^1)}$ and $J_{((2),2\delta_3,\rho_3^1)}$ cannot both occur in the J.H. series.

This proves the multiplicity 1 statement for $J_{((2),-2\delta_3,\rho_3^1)}$ and $J_{((2),-2\delta_3,\rho_3^1)}$.

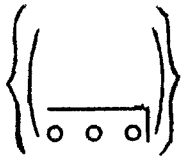
The K-type (4,0) is more complicated to deal with. This K-type is contained again in the representation induced from the one-dimensional representation of the parabolic P_2, since M_o operates trivially on the space of weight (0,0) [Appendix]. Hence one of the seven K-types is contained in $J_{((1),-\frac{3}{2}\delta_2+3\delta_3)}$.

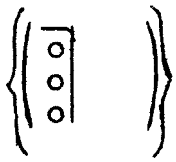
On the other hand the restriction of (4,0) to the subalgebra $so(3,\mathbb{C})$ contains the one-dimensional representation exactly once. Here we consider $so(3,\mathbb{C})$ as a subalgebra of $so(4,\mathbb{C})$ in either of the following ways:

$$so(3,\mathbb{C}) \cong \left\{ \begin{pmatrix} & A & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \ A \in so(3,\mathbb{C})$$

$$so(3,\mathbb{C}) \cong \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & A & \\ 0 & & & \end{pmatrix} \right\}, \ A \in so(3,\mathbb{C}) \ .$$

Hence this K-type is contained in the representations induced from the one-dimensional representations of the parabolics

$P_M^1 =$ 

$P_M^2 =$  .

Thus this K-type occurs in each of the representations $J_{((1),\frac{3}{2}\delta_2+\delta_3,\rho_o^1)}$ and $J_{((1),\frac{3}{2}\delta_2-\delta_3,\rho_o^1)}$ exactly once. The

The above arguments take care of three of the seven K-types with highest weight $(4,0)$. We already know that two of the remaining K-types are contained in the representations $J_{((3),2\delta_3-\delta_2,\rho_o^1)}$ and $J_{((3),2\delta_3-\delta_2,\rho_{34}^1)}$.

<u>Claim</u>: In each of the representations $J_{((2),2\delta_3,\rho_3^1)}$ and $J_{((2),-2\delta_3,\rho_3^1)}$, the K-type with highest weight $(4,0)$ is contained exactly once.

<u>Proof</u>: In each of the representations $U^{P_1^1}_{((2),2\delta_3,\rho_3^!)}$ and $U^{P_1^3}_{((2),2\delta_3,\rho_3^!)}$, this K-type is contained exactly twice. To prove the claim we have to show that the long intertwining operator has one of these two K-types in its kernel.

We rewrite $w_o^{\mathcal{C}_1^1} = w_1^{\mathcal{C}_1^1} w_2^{\mathcal{C}_1^1} w_3^{\mathcal{C}_1^1}$, where

$$w_1^{\mathcal{C}_1^1} = s_{\alpha_1}$$

$$w_2^{\mathcal{C}_1^1} = s_{\alpha_2} s_{\alpha_3}$$

$$w_3^{\mathcal{C}_1^1} = s_{\alpha_1} s_{\alpha_2} .$$

The results on $GL(3,\mathbb{R})$ imply that

a) $\mathcal{O}(P_1^1, ((2), 2\delta_3, \rho_3^1), w_3^{\mathcal{C}_1^1})$ is an isomorphism.

b) $\mathcal{O}(W_3^{\mathcal{C}_1^1} P_1^1, (W_3^{\mathcal{C}_1^1}((2), 2\delta_3, \rho_3^1), W_2^{\mathcal{C}_1^1}) = \mathcal{O}(P_1^2, ((2), 2\delta_3, \rho_3^1), W_2^{\mathcal{C}_1^1})$

has kernel $U^{P_1^2}_{((3), 2\delta_3 - \delta_2, \rho_{34}^1)}$.

c) $\mathcal{O}(w_2^{\mathcal{C}_1^1} w_3^{\mathcal{C}_1^1} P_1^1, w^{\mathcal{C}_1^1} w^{\mathcal{C}_1^1}((2), 2\delta_3, \rho_3^1), w_1^{\mathcal{C}_1^1}) =$

$\mathcal{O}(P_1^3, ((2), -2\delta_3, \rho_4^1), w_1^{\mathcal{C}_1^1})$ has kernel $U^{P_2}_{((2,2), \delta_2)}$.

Hence only one of the factors has the K-type $(4,0)$ in its kernel, and in this kernel it occurs with multiplicity one.

Similar considerations apply to $U^{P_1^3}_{((2), 2\delta_3, \rho_3^1)}$. This proves the claim. □

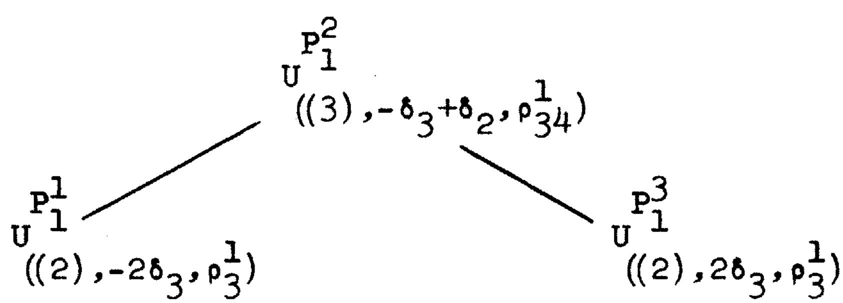
The above considerations show that the representations $J_{((3), 2\delta_3 - \delta_2, \rho_0^1)}$ and $J_{((3), 2\delta_3 - \delta_2, \rho_3^1)}$ occur with multiplicity one in the J.H. series of $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$.

The formulas for constructing minimal K-types for irreducible representations with a given Langlands parameter, together with the K-type multiplicities, show that no other representations than those in the theorem can occur in the J.H. series. Therefore to prove the theorem we are left with considering the representation $J_{((3,1),0)}$, and to prove multiplicity one for this representation. The multiplicity for the minimal K-type of $J_{((3,1),0)}$ in $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$ is very high, so that it is very complicated to apply similar considerations as above. We therefore use a different method.

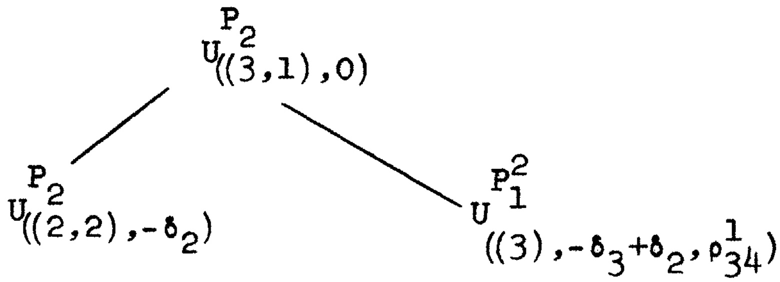
To this end, we first complete the diagrams on page 110

<u>Lemma 5</u>: We have

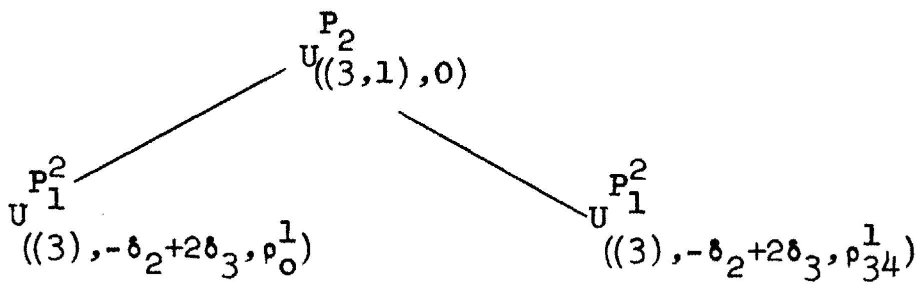
a)



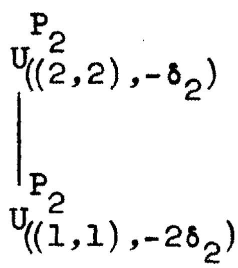
b)



c)

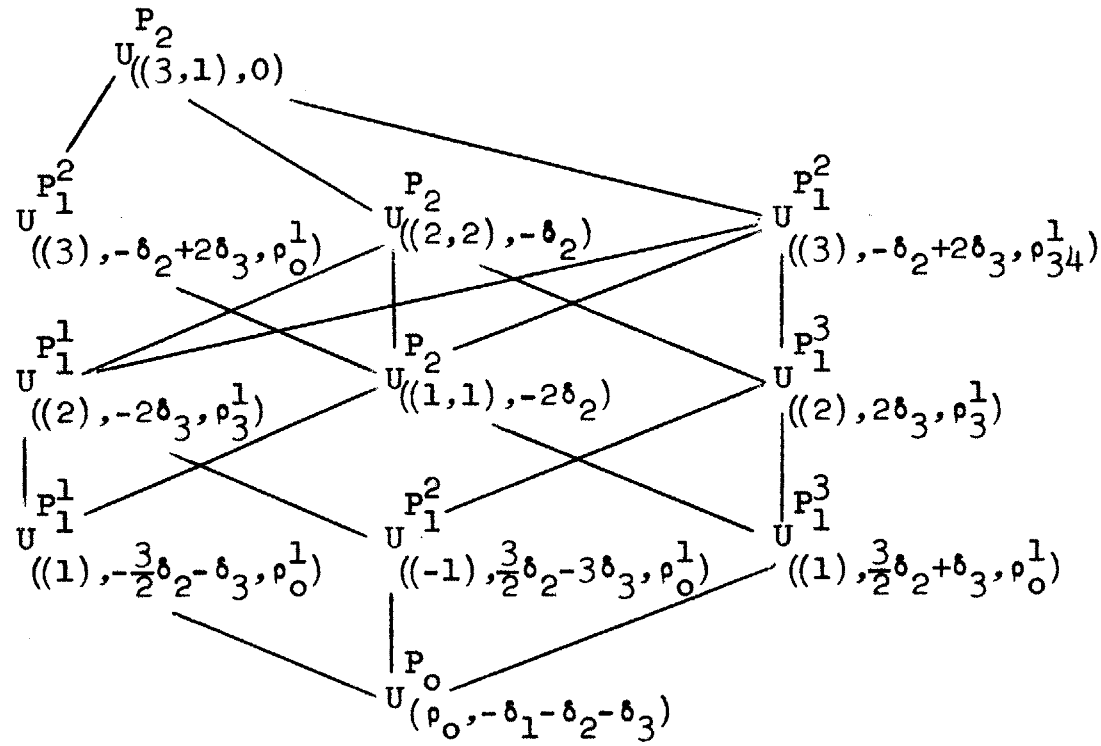


d)



We will prove this lemma later. □

Hence we get the following diagram :



Now consider the long intertwining operator for $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$. The Jordan-Hölder series of its kernel is contained in the union of the J.H. series of $U^{P_1^1}_{((1),-\frac{3}{2}\delta_2-\delta_3)}$, $U^{P_1^2}_{((1),\frac{3}{2}\delta_2-3\delta_3,\rho_o^1)}$ and $U^{P_1^3}_{((1),\frac{3}{2}\delta_2+\delta_3,\rho_o^1)}$.

Hence if V_1,V_2,V_3 are subspaces s.t.

$$U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}/V_1 = U^{P_1^1}_{((1),-\frac{3}{2}\delta_2-\delta_3,\rho_o^1)}$$

$$U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}/V_2 = U^{P_1^2}_{((1),\frac{3}{2}\delta_2-3\delta_3,\rho_o^1)}$$

$$U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}/V_3 = U^{P_1^3}_{((1),\frac{3}{2}\delta_2+\delta_3,\rho_o^1)} .$$

Then $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}\Big|_{V_1 \cap V_2 \cap V_3} = J_{(\rho_o,\delta_1+\delta_2+\delta_3)}$.

Let U be a g.p.s.r. contained in the diagram induced from a parabolic P_1^i , i = 1,2,3. Then direct calculations using I.F and the previous results on GL(2,$\mathbb{R}$) and GL(3,$\mathbb{R}$) show that the J.H. series of the kernel of the corresponding long intertwining operator is contained in the union of the J.H. series of these g.p.s.r. which lie above and are joint with U by a line. Now we can argue as before.

We will now show by case by case arguments that the above considerations are also true for the g.p.s.r. induced from P_2 .

<u>Case</u> a) $U^{P_2}_{((1,1),-2\delta_2)}$.

Here two of the factors of the long intertwining operator considered as an operator of the p.s.r. have a nontrivial kernel. These kernels have the same J.H. series as $U^{P_1^2}_{((1),\frac{3}{2}\delta_2-3\delta_3,\rho_o^1)}$ and $U^{P_1^2}_{((3),-\delta_2+2\delta_3,\rho_o^1)}$ respectively. Computing the intersection by using the diagram, we see that the image of the long intertwining operator for $U^{P_2}_{((1,1),-2\delta_2)}$ is the intersection of the spaces V_1,V_2,V_3 s.t.

$$U^{P_2}_{((1,1),-2\delta_2)}/V_1 \cong U^{P_1^2}_{((3),-\delta_2+2\delta_3,\rho_o^1)}$$

$$U^{P_2}_{((1,1),-2\delta_2)}/V_2 \cong U^{P_2}_{((2,2),-2\delta_2)}$$

$$U^{P_2}_{((1,1),-2\delta_2)}/V_3 \cong U^{P_1^2}_{((3),-\delta_2+2\delta_3,\rho_{34}^1)} .$$

Hence the lemma is true in this case too.

Case b) $U^{P_2}_{((2,2),-\delta_2)}$:

Here we use K-type multiplicities. The K-type with highest weight (4,2) is contained in the representation $U^{P_2}_{((2,2),-\delta_2)}$ exactly once. Hence $J_{((3,1),0)}$ is contained in $U^{P_2}_{((2,2),-\delta_2)}$ only once. Hence the lemma is true in this case too.

Now we can directly read off the multiplicity one for $U^{P_2}_{((3,1),0)} = J_{((3,1),0)}$ from the diagram. This completes the proof of the theorem, except for the

Proof of the lemma 5:

Case a)

Let us consider the product formula for the long intertwining operator for

$U^{P_1^1}_{((2),2\delta_3,\rho_3^1)}$. By I.D and the computations for $GL(3,\mathbb{R})$ and $GL(2,\mathbb{R})$, the kernels of the factors are

$$U^{P_2}_{((2,2),\delta_2)} \qquad \text{and} \qquad U^{P_1^2}_{((3),-\delta_2+2\delta_3,\rho_{34}^1)} .$$

The same considerations can also be applied to $U^{P_1^3}_{((2),-2\delta_3,\rho_3^1)}$. The multiplicity one statement for $U^{P_1^3}_{((3),-\delta_2+2\delta_3,\rho_{34}^1)}$ as a quotient of $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$ then implies the lemma.

<u>Case</u> d)

The minimal K-type of $U^{P_o}_{((2,2),-\delta_2)}$ has highest weight $(3,3)$ and is contained in $U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$ exactly once. But it is also contained in $U^{P_2}_{((1,1),-2\delta_2)}$. Hence the lemma.

<u>Case</u> b)

Let $P = P_1^i$, $i = 1,2,2$, and $\mu = ((r),\chi,-\rho) \in C_1$ s.t. $\mathrm{Re}(\log\chi) \in -\mathcal{C}_1^i$. We use the regularization of $\mathcal{O}\!\mathcal{I}(P,\mu,w_{\mathcal{C}_1^i\circ})$ to define the intertwining operator $\widetilde{\mathcal{O}\!\mathcal{I}}(P,\mu,w_{\mathcal{C}_1}^i)$. All formulas in the paragraphs I.C and I.D then continue to be valid for $\widetilde{\mathcal{O}\!\mathcal{I}}(P,\mu,w_{\mathcal{C}_1}^i)$.

Now take $P = P_1^2$, $(r) = (1)$, $\chi = \frac{3}{2}\delta_2 - 3\delta_3$, $\rho = \rho_o^1$.
We write

$$w_{\mathcal{C}_1^2} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} = w_3 w_2 w_1$$

where

$$w_1 = s_{\alpha_2} s_{\alpha_1}$$

$$w_2 = s_{\alpha_3}$$

$$w_3 = s_{\alpha_1} s_{\alpha_2} \quad .$$

Then

$$\widetilde{\mathfrak{A}}(P_1^2, ((1), \tfrac{3}{2}\delta_2 - 3\delta_3, \rho_o^1), w_{\mathcal{C}_1^2}) =$$

$$\widetilde{\mathfrak{A}}(P_1^1, w_2 w_1((1), \tfrac{3}{2}\delta_2 - 3\delta_3, \rho_o^1), w_3) \widetilde{\mathfrak{A}}(P_1^1, w_1((1), \tfrac{3}{2}\delta_2 - 3\delta_3, \rho_o^1), w_2)$$

$$\widetilde{\mathfrak{A}}(P_1^2, ((1), \tfrac{3}{2}\delta_2 - 3\delta_3, \rho_o^1), w_1)$$

and

$$\widetilde{\mathfrak{A}}(P_1^2, ((1), \tfrac{3}{2}\delta_2 - 3\delta_3, \rho_o^1), w_2 w_1) U^{P_1^2}_{((1), \frac{3}{2}\delta_2 - 3\delta_3, \rho_o^1)} = U^{P_2}_{((2,2), -\delta_2)} \quad .$$

Hence

$$\widetilde{\alpha}(\rho_o,-\delta_1-\delta_2-\delta_3,w_o)U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$$

$$= \widetilde{\alpha}(\rho_o,s_{\alpha_2}(-\delta_1-\delta_2-\delta_3),w_3w_2w_1)\widetilde{\alpha}(\rho_o,-\delta_1-\delta_2-\delta_3,s_{\alpha_2})$$

$$U^{P_o}_{(\rho_o,-\delta_1-\delta_2-\delta_3)}$$

$$= \widetilde{\alpha}(\rho_o,s_{\alpha_2}(-\delta_1-\delta_2-\delta_3),w_3w_2w_1)U^{P_1^2}_{((1),\frac{3}{2}\delta_2-3\delta_3,\rho_o^1)}$$

$$= \widetilde{\alpha}(P_1^2((1),\tfrac{3}{2}\delta_2-3\delta_3,\rho_o^1),w_3w_2w_1)U^{P_1^2}_{((1),\frac{3}{2}\delta_2-3\delta_3,\rho_o^1)}$$

$$= \widetilde{\alpha}(P_1^1,w_2w_1((1),\tfrac{3}{2}\delta_2-3\delta_3,\rho_o^1),w_3)U^{P_2}_{((2,2),-\delta_2)}$$

$$= U^{P_2}_{((3,1),0)} \quad .$$

This proves b).

<u>Case</u> c)

The following considerations imply the assertion:

Let $\pi\colon G \to \operatorname{End} V$ be a quasisimple representation of G on a Hilbert space V, s.t.

$$\pi \otimes \det \;\cong\; \Pi \qquad \text{(Naimark equivalent [ ])}.$$

Thus $\pi \otimes \det$ and Π has the same J.H. series. Hence if $W \subset V$ is an invariant subspace s.t.

$$\bar{\Pi}: G \to \text{End } V/W$$

is not equivalent to $\bar{\Pi} \otimes \det$, then we can find an invariant subspace W' s.t.

$$\bar{\Pi}': G \to \text{End } V/W'$$

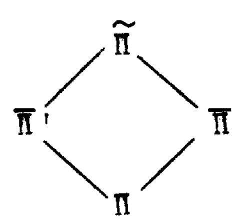
is equivalent to $\bar{\Pi} \otimes \det$.

Assume now that there is an invariant subspace $\widetilde{V}$ s.t.

$$\widetilde{\Pi}: G \to \text{End } V/\widetilde{V}$$

is irreducible and $\widetilde{\Pi} \otimes \det = \widetilde{\Pi}$.

But then $\widetilde{V} \supset W$ and $\widetilde{V} \supset W'$, hence we get the diagram



Now apply these considerations to

$$\Pi = U^{P_2}_{((1,1),2\delta_2)}$$

$$\bar{\Pi} = U^{P_1^2}_{((3),-\delta_2+2\delta_3,\rho_o^1)}$$

$$\Pi' = U^{P_1^2}_{((3),-\delta_2+2\delta_3,\rho^1_{34})}$$

$$\tilde{\Pi} = U^{P_2}_{((3,1),0)} \quad .$$

□

Without further comments, we now give the Jordan-Hölder series of all p.s.r. with central character $-\delta_1-\delta_2-\delta_3$. To compute these, we do not need any other tool than the results in I.F and those on $GL(4,\mathbb{R})$ and $GL(3,\mathbb{R})$. We also draw the diagrams but we replace the symbol for the corresponding g.p.s.r. by a dot.

Theorem 6:

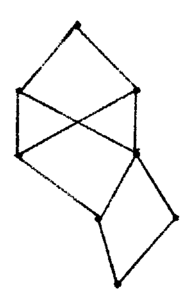
a) The Jordan-Hölder series of $U^{P_o}_{(\rho_1,-\delta_1-\delta_2-\delta_3)}$ is

$$\{J_{(\rho\ ,\delta_1+\delta_2+\delta_3)}, J_{((1),-\frac{3}{2}\delta_2-\delta_3,\rho^1_r)}, J_{((1),-\frac{3}{2}\delta_2+3\delta_3,\rho^1_3)},$$

$$J_{((2),-2\delta_3,\rho^1_{34})}, J_{((2),2\delta_3,\rho^1_o)}, J_{((2,2),\delta_2)},$$

$$J_{((3),\delta_2-2\delta_3,\rho^1_{34})}, J_{((3,1),0)}\}$$

and the diagram:



b) The Jordan-Hölder series of $U^{P}_{(\rho_2,-\delta_1-\delta_2-\delta_3)}$ is

$$\{J_{(\rho_2,\delta_1+\delta_2+\delta_3)}, J_{((1),-\frac{3}{2}\delta_2-\delta_3,\rho_3^1)}, J_{((2),-2\delta_3,\rho_{34}^1)}$$

$$J_{((3),\delta_2-2\delta_3,\rho_o^1)}, J_{((3,1),0)}\}$$

and the diagram

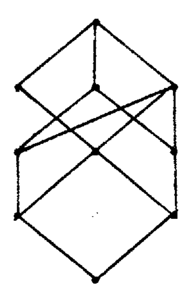
c) The Jordan-Hölder series of $U^{P_o}_{(\rho_{12},-\delta_1-\delta_2-\delta_3)}$ is

$$\{J_{(\rho_{12},\delta_1+\delta_2+\delta_3)}, J_{((1),\frac{3}{2}\delta_2-\delta_3,\rho_o^1)}, J_{((1),-\frac{3}{2}\delta_2-\delta_3,\rho_{34}^1)},$$

$$J_{((1,1),2\delta_2)}, J_{((3),\delta_2-2\delta_3,\rho_o^1)}, J_{((3),\delta_2-2\delta_3,\rho_{34}^1)}, J_{((3,1),0)},$$

$$J_{((1,2),2\delta_3,\rho_{34}^1)}, J_{((2),-2\delta_3,\rho_o^1)}\}$$

and the diagram



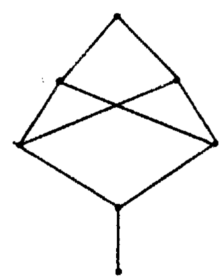
d) The Jordan-Holder series of $U^{P_o}_{(\rho_{13},-\delta_1-\delta_2-\delta_3)}$ is $\{J_{(\rho_{13},\delta_1+\delta_2+\delta_3)}\}$.

e) The Jordan-Holder series of $U^{P_o}_{(\rho_{14},-\delta_1-\delta_2-\delta_3)}$ is

$$\{J_{(\rho_{14},\delta_1+\delta_2+\delta_3)}, J_{((1),-\frac{3}{2}\delta_2-3\delta_3,\rho^1_{34})}, J_{((2),2\delta_3,\rho^1_4)},$$

$$J_{((2),-2\delta_3,\rho^1_4)}, J_{((3),\delta_2-2\delta_3,\rho^1_o)}, J_{((2,2),\delta_2)}, J_{((3,1),0)}\}$$

and the diagram



All other J.H. series for p.s.r. with parameter $\chi = e^{\delta_1+\delta_2+\delta_3}$ can be computed using the J.H. as above, either by applying symmetry considerations or by tensoring with the determinant.

<u>Remark</u>: Comparing these results with the corresponding p-adic results, we see that they are entirely different. In the p-adic case, there is a one-to-one correspondence between composition factors and parabolics. Nevertheless the J.H. series of $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$ contains exactly one

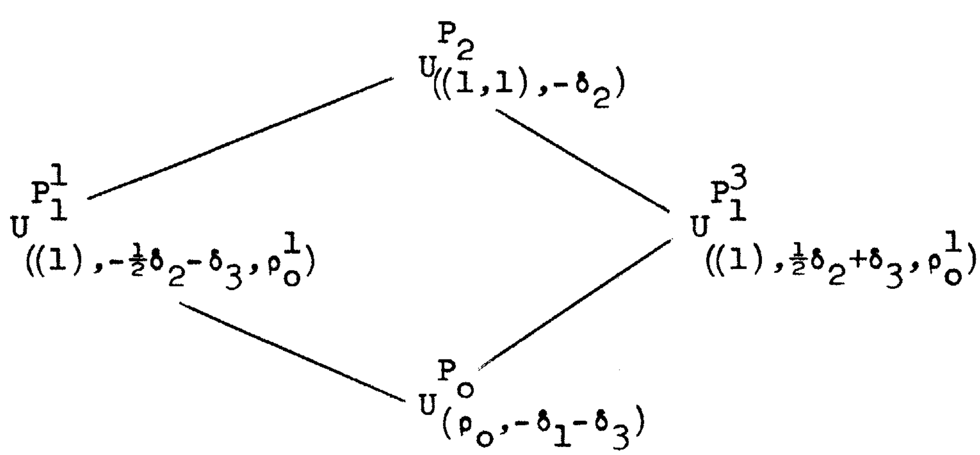
tempered representation, namely the representation $J_{((3,1),0)}$. This representation corresponds to the so-called "Steinberg representation".

This completes the computation of the Jordan-Hölder series of p.s.r. with non singular continuous parameter.

Before we can describe all J.H. series of p.s.r. we have to deal with two more special cases, namely the representations $U^{P_o}_{(\rho_o,\delta_1+\delta_3)}$ and $U^{P_o}_{(\rho_o,\delta_1+\delta_2)}$.

<u>Case 1</u> $U^{P_o}_{(\rho_o,\delta_1+\delta_2)}$

Using step by step induction, we see that $J_{(\rho_o,\delta_1+\delta_3)}, J_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)}, J_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}$ and $J_{((1,1),\delta_2)}$ are contained in its J.H. series and that we get the diagram



Claim 1.

$J_{((2),0,\rho_3^1)}$ is also contained in the J.H. series of $U^{P_o}_{(\rho_o,-\delta_1-\delta_3)}$ and we have the diagram

$$\begin{array}{l} U^{P_1^2}_{((2),0,\rho_o^1)} \\ \Big| \\ U^{P_2}_{((1,1),-\delta_2)} \end{array}$$

The proof is exactly analogous to the proof of the lemma on page 110. Therefore it will be omitted.

Claim 2.

$U^{P_2}_{((1,1),\delta_2)}$ has J.H. series $\{J_{((1,1),\delta_2)}, J_{((2),0,\rho_2^1)}\}$.

Proof: The minimal K-type of $J_{((2),0,\rho_3^1)}$ has highest weight (3,1). This K-type occurs with multiplicity three in $U^{P_o}_{(\rho_o,-\delta_1-\delta_3)}$, but only with multiplicity one in $U^{P_2}_{((1,1),-\delta_2)}$. Hence $J_{((2),0,\rho_3^1)}$ occurs with multiplicity one.

Using the formulas on page 31 for computing minimal K-types, we see that the minimal K-types of irreducible representations with central character $\delta_1+\delta_3$ have the following highest weights:

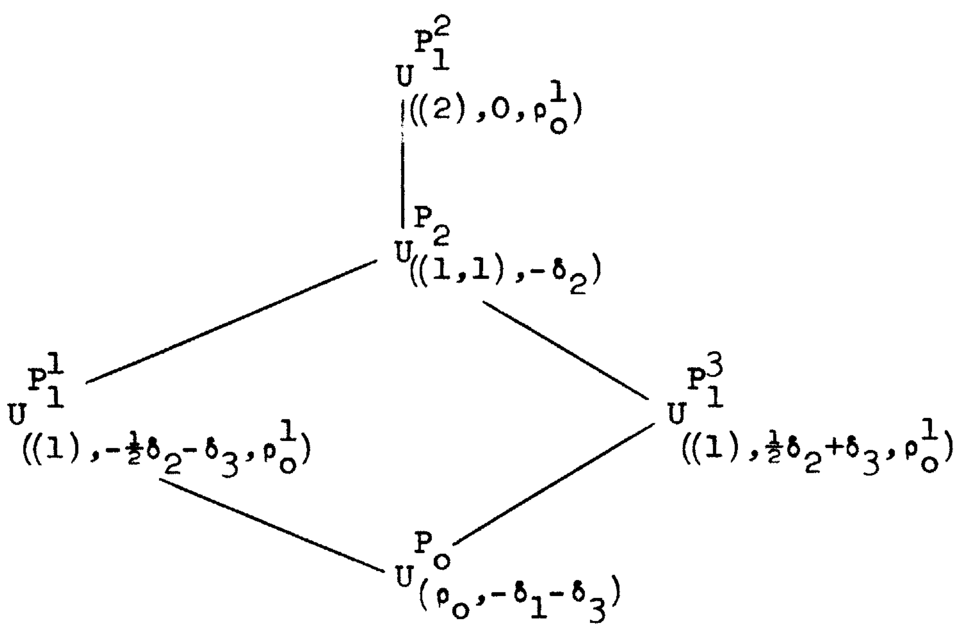
(0,0), (1,0), (1,1), (2,0), (2,1), (2,2), (3,0), (3,1). The minimal K-type of $U^{P_2}_{((1,1),\delta_2)}$ has highest weight (2,2), and the only representation whose minimal K-type has a highest weight larger or equal to (2,2) is the representation $J_{((2),0,\rho^1_3)}$. □

Thus we conclude:

<u>Proposition 7</u>. The J.H. series of $U^{P_o}_{(\rho_o,\delta_1+\delta_3)}$ is

$$\{J_{(\rho_o,\delta_1+\delta_3)}, J_{((1),\frac{1}{2}\delta_2+\delta_3,\rho^1_o)}, J_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho^1_o)}, J_{((1,1),\delta_2)}, J_{((2),0,\rho^1_3)}\}$$

and we have the diagram



<u>Proof</u>: We apply the considerations of I.F. of the factors of the long intertwining operator have a kernel, and the J.H. series of their kernel is equal to the J.H. series of $U^{P_1^1}_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}$ and of $U^{P_1^3}_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)}$ respectively.

Hence let V_1, V_2 be the subspaces s.t.

$$U^{P_o}_{(\rho_o,-\delta_1-\delta_3)}/V_1 \cong U^{P_1^1}_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}$$

$$U^{P_o}_{(\rho_o,-\delta_1-\delta_3)}/V_2 \cong U^{P_1^3}_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)} ,$$

then

$$J_{(\rho_o,\delta_1+\delta_2)} \cong U^{P_o}_{(\rho_o,-\delta_1-\delta_3)}/V_1 \cap V_2 .$$

The kernel of the long intertwining operator for $U^{P_1^1}_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}$ and $U^{P_1^3}_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)}$ is $U^{P_2}_{((1,1),-\delta_2)}$.

Hence if $W_1 \subset U^{P_1^1}_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}$ and $W_2 \subset U^{P_1^3}_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)}$ are s.t.

$$U^{P_1^1}_{((1),\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}/W_1 = U^{P_2}_{((1,1),-\delta_2)}$$

$$U^{P_1^3}_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)}/W_2 = U^{P_2}_{((1,1),-\delta_2)}$$

then

$$U^{P_1^1}_{((1),-\frac{1}{2}\delta_2-\delta_3,\rho_o^1)}/W_1 = J_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)}$$

$$U^{P_1^3}_{((1),\frac{1}{2}\delta_2+\delta_3,\rho_o^1)/W_2} = J_{((1,1),-\frac{1}{2}\delta_2+\delta_3,\rho_o^1)} .$$

Together with the previous claim, this proves the proposition. □

Case 2 $U^{P_o}_{(\rho_o,-\delta_1-\delta_2)}$

Using step by step induction techniques, we see that $J_{(\rho_o,\delta_1+\delta_2)}, J_{((1),\frac{3}{2}\delta_2)}, J_{((2),\delta_3)}$ and $J_{((2,1),\frac{1}{2}\delta_2)}$ are contained in the J.H. series of $U^{P_o}_{(\rho_o,-\delta_1-\delta_2)}$. The diagram is

$$\begin{array}{l} U^{P_2}_{((2,1),-\frac{1}{2}\delta_2)} \\ \big| \\ U^{P_1^1}_{((2),-\delta_3)} \\ \big| \\ U^{P_1^1}_{((1),-\frac{3}{2}\delta_2)} \\ \big| \\ U^{P_o}_{(\rho_o,-\delta_1-\delta_2)} \end{array}$$

Proposition 8: There are no other composition factors than those listed above.

Proof: Comparing minimal K-types we prove that $U^{P_2}_{((2,1),-\frac{1}{2}\delta_2)}$ is irreducible.

The kernel of the long intertwining operator for $U^{P_1^1}_{((2),\delta_3)}$ is $U^{P_2}_{((2,1),\frac{1}{2}\delta_2)}$. Thus

$$U^{P_1^1}_{((2),\delta_3)} / U^{P_2}_{((2,1),\frac{1}{2}\delta_2)} \cong J_{((2),\delta_3)} .$$

The kernel of the long intertwining operator for $U^{P_1^1}_{((1),\frac{3}{2}\delta_2)}$ is $U^{P_1^1}_{((2),\delta_3)}$. Thus

$$U^{P_1^1}_{((1),\frac{3}{2}\delta_2)} / U^{P_1^1}_{((2),\delta_3)} \cong J_{((1),\frac{3}{2}\delta_2)} .$$

Finally the kernel of the long intertwining operator for $U^{P_o}_{(\rho_o,\delta_2+\delta_3)}$ is $U^{P_1^1}_{((1),\frac{3}{2}\delta_2)}$. Thus

$$U^{P_o}_{(\rho_o,\delta_2+\delta_3)} / U^{P_1^1}_{((1),\frac{3}{2}\delta_2)} \cong J_{(\rho_o,\delta_1+\delta_2)} . \qquad \square$$

In the second appendix to this chapter, I will give a list of all J.H. series of p.s.r. with continuous parameter $\log \chi = \delta_1 + \delta_2, \delta_2 + \delta_3, \delta_1 + \delta_3, \delta_1, \delta_2, \delta_3$. Since the proofs are based on the previously employed ideas, they will be omitted.

Now we came to the reducibility question for g.p.s.r. induced from P_2 .

Theorem 9: Let $\mu = ((n_1,n_2),\nu\delta_2) \in \hat{\mathcal{C}}_2$. Then $U_\mu^{P_2}$ is reducible iff

a) $n_1 \geq n_2 > 0$, $\nu \in \mathbb{R}$ and $|\nu| - n_1/2 + n_2/2 \in \mathbb{N} \setminus 0$,

b) $n_2 \geq n_1 > 0$, $\nu \in \mathbb{R}$ and $|\nu| + n_1/2 - n_2/2 \in \mathbb{N} \setminus 0$.

Proof: $U^{P_2}_{((n_1,n_2),\nu\delta_2)}$ is reducible if $U^{P_2}_{((n_1,n_2),-\nu\delta_2)}$ is reducible. Hence we may assume $\nu \geq 0$, and furthermore using symmetry consideration we assume $n_1 \geq n_2 > 0$.

a) First assume that $(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2$ is contained in the interior of the positive Weyl chamber $\mathcal{C}_o$, i.e. that

$$-(n_1/2) - (n_2/2) + \nu > 0 \quad .$$

We can find $\lambda \in \overline{\mathcal{C}}_o$, λ integral s.t.

$$(n_1/2)\ {}_1 + (n_2/2)\ {}_3 + \nu\delta_2 - \lambda = (\alpha_1/2) + (\alpha_3/2) + \tilde{\nu}\delta_2$$

$$1 < \tilde{\nu} \leq 2$$

and s.t. the assumptions of Zuckerman's theorem are satisfied. Then

$$U_\mu^{P_2} = \Psi^{(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2}_{(\alpha_1/2) + (\alpha_2/2) + \tilde{\nu}\delta_2} = U^{P_2}_{((1,1),\tilde{\nu}\delta 2)} \quad .$$

Thus $U_{\mu}^{P_2}$ is reducible if $(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2$ is integral, i.e. iff $((n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2, \alpha_2) = -(n_1/2) - (n_2/2) + \nu \in \mathbb{N}\setminus 0$, which is equivalent to $\nu - (n_1/2) + (n_2/2) \in \mathbb{N}\setminus 0$, since by assumption $-(n_1/2) - (n_2/2) + \nu > 0$.

b) Now assume $(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2$ is contained in the wall of $\mathcal{C}_o$, i.e. that

$$-(n_1/2) - (n_2/2) + \nu = 0 .$$

We can find $\lambda \in \overline{\mathcal{C}}_o$, λ integral, s.t.

$$(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2 - \lambda = (\alpha_1/2) + (\alpha_3/2) + \delta_2$$

and s.t. the assumptions of Zuckerman's theorem are satisfied. Then

$$U_{\mu}^{P_2} = \Psi_{(\alpha_1/2) + (\alpha_3/2) + \delta_2}^{(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2} U_{((1,1),\delta_2)}^{P_2}$$

and thus by II.C.7, $U_{\mu}^{P_2}$ is reducible.

c) Next assume $(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2$ is contained in the interior of $s_{\alpha_2}\mathcal{C}_o$, i.e. that

$$-(n_1/2) - (n_2/2) + \nu < 0$$

$$-(n_1/2) + (n_2/2) + \nu > 0 \quad .$$

We can find $\lambda \in \bar{\mathcal{C}}_o$, λ integral, s.t.

$$(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2 - s_{\alpha_2}\lambda = \alpha_1 + \alpha_2 + \tilde{\nu}\delta_2$$

$$0 < \tilde{\nu} < 1$$

and s.t. the assumptions of Zuckerman's theorem are satisfied. Then

$$U_\mu^{P_2} = \Psi_{\alpha_1 + \alpha + \tilde{\nu}\delta_2}^{(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2} U_{((2,2),\tilde{\nu}\delta_2)}^{P_2} \quad .$$

Thus $U_\mu^{P_2}$ is reducible iff $s_{\alpha_2}((n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2)$ is integral i.e. if $-(n_1/2) + (n_2/2) + \nu \in \mathbb{N} \setminus 0$.

d) Now assume $(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2$ is contained in the wall of $s_{\alpha_2}\mathcal{C}_o$, i.e. that

$$-(n_1/2) + (n_2/2) + \nu = 0 \quad .$$

We can find $\lambda \in \bar{\mathcal{C}}_o$, λ integral, s.t.

$$(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2 - s_{\alpha_2}\lambda = (\alpha_1) + (\alpha_3/2) + \tfrac{1}{2}\delta_2$$

and s.t. the assumptions of Zuckerman's theorem are satisfied. Then

$$U_{\mu}^{P_2} = \Psi_{\alpha_1 + (\alpha_3/2) + \frac{1}{2}\delta_2}^{(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2} U^{P_2}_{((1,1),\delta_2)} .$$

Thus by II.C.8. the representation is irreducible.

e) Finally assume that $(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2$ is contained in the interior of $s_{\alpha_1}s_{\alpha_2}\mathcal{C}_o$, i.e. that

$$-(n_1/2) + (n_2/2) + \nu < 0 .$$

Assume first $-(n_1/2) + (n_2/2) + \nu \notin \mathbb{Z}$. Then we can find $\lambda \in \overline{\mathcal{C}}_o$, λ integral, s.t.

$$(n_1/2)\alpha_1 + (n_2/2)\alpha_3 + \nu\delta_2 - s_{\alpha_1}s_{\alpha_2}\lambda = \alpha_1 + (\alpha_3/2) + \tilde{\nu}\delta_2$$

$$0 < \tilde{\nu} < 1/2$$

and s.t. the assumptions of Zuckerman's theorem are satisfied. Then by the same arguments as above $U_{\mu}^{P_2}$ is irreducible.

Now assume $-(n_1/2) + (n_2/2) + \nu \in \mathbb{Z}$. Then we can find $\lambda \in \overline{\mathcal{C}}_o$, λ integral, s.t.

$$(n_1/2)\alpha_1 + (n_2/2)\alpha_2 + \nu\delta_2 - s_{\alpha_1}s_{\alpha_2}\lambda = (3/2)\alpha_1 + (1/2)\alpha_2$$

and s.t. the assumptions of Zuckerman's theorem are satisfied. Hence the representation $U_{\mu}^{P_2}$ is irreducible.

f) If $\nu = 0$ the representation is unitarily induced and hence irreducible.

This completes the proof of the theorem. □

<u>Theorem 10</u>: The J.H. series of the kernel of the long intertwining operator for $U^{P_2}_{((n_1,n_2),\nu\delta_2)}$, with $n_1 \geq n_2 > 0$, $\nu > 0$, is

a) if $-(n_1/2) - (n_2/2) + \nu \in \mathbb{N}\setminus 0$ contained in the union of the J.H. series of

$$U^{P_1^2}_{(((n_1/2)+(n_2/2)+\nu),s_{\alpha_2}((n_1/2)\alpha_1+(n_2/2)\alpha_3-\nu\delta_2) - ((n_1/2)+(n_2/2)+\nu)\alpha_2,\rho(n_1,n_2,\nu))}$$

$$U^{P_1^2}_{(((n_1/2)+(n_2/2)+\nu),s_{\alpha_2}((n_1/2)\alpha_1+(n_2/2)\alpha_3-\nu\delta_2) - ((n_1/2)+(n_2/2)+\nu)\alpha_2,\rho(n_1,n_2,\nu)\rho^1_{34})}$$

$$U^{P_2}_{(((n_1/2)-(n_2/2)+\nu,(n_2/2)-(n_1/2)+\nu),s_{\alpha_2}((n_1/2)\alpha_1+(n_2/2)\alpha_3+\nu\delta_2) -((n_1/2)-(n_2/2)+\nu)\alpha_1 - ((n_2/2)-(n_2/2)+\nu)\alpha_3)}$$

where

$$\rho(n_1,n_2,\nu) = \rho_3^1 \quad \text{if } \tfrac{3}{2}n_1 + \tfrac{3}{2}n_2 + \nu \text{ is even}$$

$$\rho(n_1,n_2,\nu) = \rho_o^1 \quad \text{if } \tfrac{3}{2}n_1 + \tfrac{3}{2}n_2 + \nu \text{ is odd.}$$

b) if $-(n_1/2) - (n_2/2) + \nu = 0$ is equal to the J.H. series of

$$U^{P_1^2}_{((n_1+n_2),((n_1/2)-(n_2/2))\delta_2,\rho_3^1)}$$

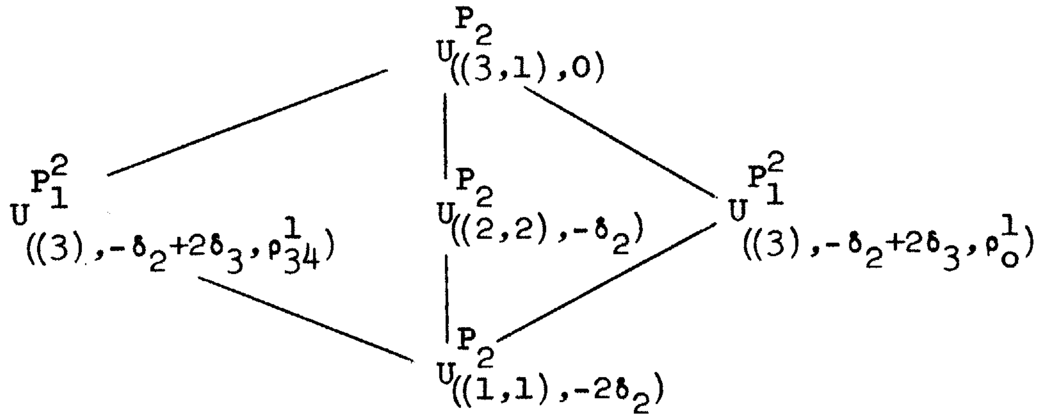
c) if $-(n_1/2) - (n_2/2) + \nu < 0$ and $-(n_1/2) + (n_2/2) + \nu > 0$ $\in \mathbb{N}\backslash 0$ is equal to the J.H. series of

$$U^{P_2}_{(((n_1/2)+(n_2/2)+\nu,(n_1/2)+(n_2/2)-\nu),}$$

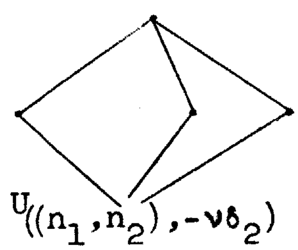
$$s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}((n_1/2)\alpha_1+(n_2/2)\alpha_3+\nu\delta_2) - ((n_1/2)+(n_2/2)+\nu)\alpha_1 - ((n_1/2)+(n_2/2)-\nu)\alpha_3) \quad .$$

<u>Proof</u>:

a) In Theorem we showed that $U^{P_2}_{((1,1),-2\delta_2)}$ has a J.H. series of length 5 and the diagram



Hence the arguments in the proof above show that $U^{P_2}_{((n_1,n_2),-\nu\delta_2)}$ also has a J.H. series of length 5 if $-(n_1/2) - (n_2/2) + |\nu| \in \mathbb{N}\setminus 0$, and calculations using the results in the last part of I.F. show that it has the diagram



,

where the dots in the middle row are substitutes for the representations listed under a) above. Since the calculations are exactly parallel to those in the proof of II.B.7. they will be omitted.

b) In Theorem   we proved that $U^{P_2}_{((1,1),\delta_2)}$ has a J.H. series of length 2, namely

$$\left\{J_{((1,1),\delta_2)}, J_{((2),0,\rho_3^1)}\right\} .$$

Hence by the arguments in the proof of the previous theorem, $U^{P_2}_{((n_1,n_2),\nu\delta_2)}$ with $-(n_1/2)-(n_2/2)+|\nu| = 0$ has a J.H. series of length 2. Then calculation exactly parallel to those in the proof of II.B.7 give the formulas of the theorem.

c) In Theorem 7 we proved that $U^{P_2}_{((2,2),\delta_2)}$ has a J.H. series of length 2, namely $\left\{J_{((2,2),\delta_2)}, J_{((3,1),\delta)}\right\}$. Hence by the arguments in the proof of the previous theorem $U^{P_2}_{((n_1,n_2),\nu\delta_2)}$ with $-(n_1/2)+(n_2/2)+\nu > 0$ and $-(n_1/2)-(n_2/2)+\nu < 0$ has a J.H. series of length 2. Again calculations exactly parallel to those in the proof of II.B.7 give the formulas of the theorem. □

Using this theorem on $GL(4,\mathbb{R})$ and the analogous theorems on $GL(3,\mathbb{R})$ and $GL(2,\mathbb{R})$ together with the reduction technique of I.F. we are now able to compute the Langlands parameters of all composition factors for p.s.r. of $GL(n,\mathbb{R})$.

In the last part of this paragraph, we will classify all unitary representations of $GL(4,\mathbb{R})$.

<u>Proposition 10</u>: The following non unitarily induced

representations J_μ are hermitian.

a) $\mu \in \hat{C}_2$: $\mu = ((n,n), a\delta_2)$ $a \in \mathbb{R}^+,\ n \in \mathbb{N}\setminus 0$

b) $\mu \in \hat{C}_1$: $\mu = ((n), -(a/2)\delta_2 + a\delta_3, \rho_0^1)$ $a \in \mathbb{R}^+,\ n \in \mathbb{N}\setminus 0$

$\mu = ((n), -(a/2)\delta_2 + a\delta_3, \rho_{34}^1)$ $a \in \mathbb{R}^+,\ n \in \mathbb{N}\setminus 0$

c) $\mu \in \hat{C}_0$: $\mu = (\rho_0, a(\delta_1+\delta_3) + ib\delta_1 + ic\delta_3 - i(c+b)\delta_2)$
$a \in \mathbb{R}^+,\ b,c \in \mathbb{R}$

$\mu = (\rho_0, a\delta_2)$ $a \in \mathbb{R}^+$

$\mu = (\rho_0, a(\delta_1+\delta_3) + b\delta_2 + ic(\delta_1 - \delta_3))$
$a,b \in \mathbb{R}^+,\ c \in \mathbb{R}$

$\mu = (\rho_{14}, a(\delta_1+\delta_3) + ib\delta_1 + ic\delta_3 - (c+b)\delta_2)$
$a \in \mathbb{R}^+,\ b,c \in \mathbb{R}$

$\mu = (\rho_{14}, a\delta_2)$ $a \in \mathbb{R}^+$

$\mu = (\rho_{23}, a(\delta_1+\delta_3) + ib\delta_1 + ic\delta_3 - i(c+b)\delta_2)$
$a \in \mathbb{R}^+,\ b,c \in \mathbb{R}$

$\mu = (\rho_{23}, a\delta_2)$ $a \in \mathbb{R}^+$

$\mu = (\rho_{1234}, a(\delta_1+\delta_3) + ib\delta_1 + ic\delta_3 - i(c+b)\delta_2)$
$a \in \mathbb{R}^+,\ b,c \in \mathbb{R}$

$\mu = (\rho_{1234}, a\delta_2)$ $a \in \mathbb{R}^+$

$\mu = (\rho_{1234}, a(\delta_1+\delta_3) + b\delta_2 + ic(\delta_1 - \delta_3))$
$a,b \in \mathbb{R}^+,\ c \in \mathbb{R}$.

Each hermitian irreducible quasisimple representation is, up to tensoring with a one dimensional unitary representation equivalent to one of these.

Proof: We have to check when the conditions of theorem I.G.1 are fulfilled. This is a straightforward computation and left to the reader. □

Using case by case arguments, we will now find out which of these hermitian representations are (infinitesimally) unitary.

Case 1 $u \in \hat{C}_2$, $u = ((n,n), a\delta_2)$, $a \in \mathbb{R}^+$, $n \in \mathbb{N}\setminus 0$.

By Theorem II.D.1 $U^{P_2}_{((n,n),a\delta_2)}$ is reducible if $a \in \mathbb{N}\setminus 0$. Hence $U^{P_2}_{((n,n),0)}$ being irreducible, we have complementary series representations for $0 < a < 1$ and a limit of complementary series representation for $a = 1$. On the other hand, Theorem I.G. shows that the representations $J_{((n,n),a\delta_2)}$ are not unitary if $a\delta_2 \geq \delta_{P_2} = 2\delta_2$, iff $a \geq 2$. For $1 < a < 2$, $J_{((n,n),a\delta_2)}$ is not unitary, since the nonunitary representation $J_{((n,n),2\delta_2)}$ is a "limit" of $J_{((n,n),a\delta_2)} = U^{P_2}_{((n,n),a\delta_2)}$ for $a \to 2$.

Case 2: $u \in \hat{C}_1$, $u = ((n), -(n/2)\delta_2 + a\delta_3, \rho^1_o)$, $a \in \mathbb{R}^+$, $n \in \mathbb{N}\setminus 0$.

By Theorem II.D.1. $U^{P_1^2}_{((n),-(a/2)\delta_2+a\delta_3,\rho_o^1)}$ is reducible iff

$$a = 2m + 1 \ , \quad m \in \mathbb{N} \ , \quad \text{if } n \text{ is odd}$$

$$a = 2m + 1 \ , \quad \text{and} \quad a = n + 2m \ , \quad m \in \mathbb{N} \quad \text{if } n \text{ is even.}$$

Hence $U^{P_1^2}_{((n),0,\rho_o^1)}$ being irreducible, we have complementary series representations for $0 < a < 1$ and a limit of complementary series representations for $a = 1$. On the other hand, Theorem I.G shows that the representations $J_{((n),-(a/2)\delta_2+a\delta_3,\rho_o^1)}$ are not unitary if $-(a/2)\delta_2 + a\delta_3 \geq \delta_{P_1^2} = -(3/2)\delta_2+3\delta_3$, i.e. if $a \geq 3$.

Now let $1 < a < 2$. Then $J_{((n),-(a/2)\delta_2+a\delta_3,\rho_o^1)}$ is not unitary since the nonunitary representation $J_{((n),-(3/2)\delta_2+3\delta_3,\rho_o^1)}$ is a "limit" of $J_{((n),-(a/2)\delta_2+a\delta_3,\rho_o^1)} = U^{P_1^2}_{((n),-(a/2)\delta_2+a\delta_3,\rho_o^1)}$ for $a \to 2$.

For $\mu = ((n),-(a/2)\delta_2+a\delta_3,\rho_{34}^1)$ exactly the same considerations can be applied.

Case 3 $\mu \in \hat{C}_o$

a) $\mu = (\rho_o, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_2)$, $a \in \mathbb{R}^+$, $b,c \in \mathbb{R}$:

By Theorem II.D.1., $U^{P_o}_{(\rho_o, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_3)}$ is irreducible iff

$$a = \frac{2m+1}{2} \quad \text{if} \quad b,c \neq 0$$

or

$$a = \frac{2m+1}{2} \quad \text{or} \quad a = 2(2m+1) \quad \text{if} \quad b = 0 \quad \text{or} \quad c = 0 , \quad m \in \mathbb{N} .$$

Since $U^{P_o}_{(\rho_o, ib_1+ic\delta_3-i(c+b)\delta_2)}$ with $b,c \in \mathbb{R}$ is irreducible and unitarly induced, we have complementary series representations for $0 < a < 1/2$ and a limit of complementary series representations for $a = 1/2$ and for all $b,c \in \mathbb{R}$. On the other hand, Theorem I.G5 shows that the representations $J_{((\rho_o, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_3)}$, for $a(\delta_1+\delta_3) \geq \delta_\chi = \frac{3}{2}\delta_1 + \frac{3}{2}\delta_3$, i.e. for $a \geq 3/2$, are not unitary.

Now assume $b,c \neq 0$. Then $U^{P_o}_{(\rho_o, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_2)}$ is equivalent to $U^{P_o}_{(\rho_o, 2a\delta_1-ib\delta_2+2ic\delta_3)}$, and the intertwining operator defining the hermitian form is $\mathfrak{A}(\rho_o, 2a\delta_1-ib\delta_2+2ic\delta_3, s_{\alpha_1})$. Let $P = P^1_1 = M_1A_1N_1$ and consider $\operatorname{ind}^{P^1_1}_{P_o}\pi(\rho_o, 2a\delta_1-ib\delta_2+2ic\delta_3)$. The representation $\operatorname{ind}^{M_1}_{M_1\cap P_o}(\pi(\rho_o, 2a\delta_1-ib\delta_2+i2c\delta_3)|_{M_1\cap P_o})$ is a tensor product of a p.s.r. of $SL_{\pm}(2,\mathbb{R})$ with parameter $(\rho_o, e^{2a\delta})$ and the trivial representation of the other Z_2-factors. Hence $\mathfrak{A}_{M_1}(P_o \cap M_1, \pi_{(\rho_o, 2a\delta_1 - ib\delta_2 + 2ic\delta_3)}|_{P_o \cap M_1}, s_{\alpha_1})$ does not define a positive definite scalar product on $\mathrm{ind}_{P_o \cap M_1}^{M_1}(\pi_{(\rho_o, 2a\delta_1 - ib\delta_2 + 2ic\delta_3)}|_{P_o \cap M_1})$ for $a > 1$. But if $\frac{1}{2} < a < \frac{3}{2}$, the p.s.r. of $GL(4,\mathbb{R})$ is irreducible, and hence the assumptions of lemma G3 are satisfied. We conclude that the representations $J_{(\rho_o, a(\delta_1+\delta_2)+ib\delta_1+ic\delta_3-i(c+b)\delta_2)}$, for $\frac{1}{2} < a < \frac{3}{2}$, $b,c \neq 0$, and not unitary.

Assume next $b \neq 0$, $c = 0$ or $b = 0$, $C \neq 0$. For $\frac{1}{2} < a < 1$ and $1 < a < \frac{3}{2}$, we use the above arguments to show that the representations are not unitary. Let $a = 1$, $b = 0$, $c \neq 0$. Then $U^{P_o}_{(\rho_o, \delta_1+\delta_3+ic\delta_3-ic\delta_2)}$ is reducible, and we can identify $J_{(\rho_o, \delta_1+\delta_2+ic\delta_3-ic\delta_2)}$, $c \neq 0$, with a representation unitarily induced from a one-dimensional representation of the parabolic

$$P = \left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right) .$$

Hence the representations for this parameter are unitary.

Similar considerations prove that the representations with parameter $a = 1$, $c = 0$, $b \neq 0$ are unitary.

Now assume $c = b = 0$. Again we apply the above proof to show that the representations are not unitary for $1/2 < a < 1$ and $1 < a < 3/2$. Hence let $a = 1$. Then $J_{(\rho_0, \delta_1+\delta_3)}$ is the representation unitarily induced from the trivial representation of P . Hence it is unitary.

a') In the case $\mu = (\rho_{1234}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(b+c)\delta_3)$ with $a \in \mathbb{R}^+$, $b,c \in \mathbb{R}$, we proceed exactly as in a).

b) $\mu = (\rho_{14}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3, i(b+c)\delta_2)$: By Theorem II.D.1., $U_{(\rho_{14}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(b+c)\delta_3)}$ is reducible iff

$$a = (2m+1)/2 \quad \text{if} \quad b,c \neq 0$$

or

$$a = (2m+1)/2 \quad \text{or} \quad 2 = 2m \quad \text{if} \quad b = 0 \quad \text{or} \quad c = 0 \ , \ m \in \mathbb{N} .$$

Thus we have complementary series representations for $0 < a < 1/2$ and a limit of complementary series representation for $a = 1/2$ and for all $b,c \in \mathbb{R}$.
On the other hand Theorem I.G.5 shows that the representation $J_{(\rho_{14}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_3)}$, for $a(\delta_1+\delta_3) \geq \delta_\chi = (3/2)\delta_1 + (3/2)\delta_3$, i.e. for $a \geq 3/2$,

are not unitary. For $1/2 < a < 3/2$, the argument in the first part of case a) can be applied to prove the representations are not unitary.

b') In the case $\mu = (\rho_{23}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(b+c)\delta_2)$: with $a \in \mathbb{R}^+$, $b,c \in \mathbb{R}$, we proceed exactly as in b).

c) $\mu = (\rho_o, a\delta_2)$:

By Theorem II.D.1., $U^{P_o}_{(\rho_o, a\delta_2)}$ is reducible iff $a = 2m+1$, $m \in \mathbb{N}$. Since $U^{P_o}_{(\rho_o, 0)}$ is irreducible, we have complementary series representations for $o < a < 1$ and a limit of complementary series representations for $a = 1$. Otherwise the representations are not unitary.

c') $\mu = (\rho_{1234}, a\delta_2)$: In this case we proceed as in c).

d) $\mu = (\rho_{14}, a\delta_2)$:

By Theorem II.D.1., $U^{P_o}_{(\rho_o, a\delta_2)}$ is reducible if $a = 2m+1$, $m \in \mathbb{N}$, and we have complementary series representations for $0 < a < 1$ and a limit of complementary series representations for $a = 1$. Otherwise the representations are not unitary.

d') $\mu = (\rho_{34}, a\delta_2)$: In this case we proceed as in d).

e) $\mu = (\rho_o, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3)$ $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, $c \in \mathbb{R}$.

By Theorem II.D.1. $U^{P_o}_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3)}$ is reducible iff

$$c \neq 0 \quad \text{and} \quad b = 2n+1 \quad \text{or} \quad 2a+b = 2m+1$$

or

$$c = 0 \quad \text{and} \quad b = 2n+1 \quad \text{or} \quad 2a+b = 2m+1 \quad \text{or} \quad a+b = 2m+1 \quad \text{or} \quad a = 2m+1 \qquad m \in \mathbb{N} .$$

Since $U^{P_o}_{(\rho_o, ic(\delta_1-\delta_3)}$ is irreducible and unitarily induced, we have complementary series for $0 < a+(b/2) < 1/2$, and limits of complementary series for $a+(b/2) = 1/2$, c arbitrary. On the other hand Theorem I.G.5 shows that the representations $J_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))}$, $a,b \geq 1$, are not unitary except if $a = b = 1$ and $c = 0$. In this later case $J_{(\rho_o, \delta_1+\delta_2+\delta_3)}$ is the one-dim. rep. and hence unitary.

Now assume $c \neq 0$. Then $U^{P_o}_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))}$ is isomorphic to $U^{P_o}_{(\rho_o, (2a+b)\delta_1+b\delta_3+ic\delta_3)}$ and the intertwining operator defining the hermitian form is $\mathfrak{A}(\rho_o, (2a+b)\delta_1+b\delta_3+ic\delta_2, s_{\alpha_1}s_{\alpha_2})$. Let $P_2 = M_2A_2N_2$ and and consider $\mathrm{ind}_{P_o}^{P_2}\pi(\rho_o, (2a+b)\delta_1+b\delta_3+ic\delta_2)$. The representation $\mathrm{ind}_{M_2\cap P_o}^{M_2}\pi(\rho_o, (2a+b)\delta_1+b\delta_3+ic\delta_2)\big|_{M_2\cap P_o}$ is is the tensor product of two p.s.r. of the two $SL_{\pm}(2,\mathbb{R})$ factors with parameters $(\rho_o, e^{(2a+b)\delta})$ and $(\rho_o, e^{b\delta})$

Hence $\mathcal{O}\!\mathcal{L}_{M_2}(P_o \cap M_2, \operatorname{ind}^{M}_{P_o\cap M_2} \pi(\rho_o,(2a+b)\delta_1+b\delta_3+ic\delta_2)\Big|_{M_2\cap P_o}, s_{\alpha_1}s_{\alpha_3})$ defines a positive semidefinite hermitian form iff $0 < 2a+b \leq 1$. But by lemma I.D.3 $J_{(\rho_o, a(\delta_1+\delta_3)+b\delta_1+ic(\delta_1-\delta_3))}$ is isomorphic to

$$\operatorname{ind}^G_{P_o}(\mathcal{O}\!\mathcal{L}((\rho_o, e^{(2a+b)\delta}), s_{\alpha_1})U^{P_o}_{(\rho_o,2a+b)} \otimes$$

$$\mathcal{O}\!\mathcal{L}((\rho_o, e^{b\delta}), s_{\alpha_3})U^{P_o}_{(\rho_o,b)} \otimes e^{ic\delta_2} .$$

Here $U^{P_o}_{(\rho_o,2a+b)}$ and $U^{P_o}_{(\rho_o+b)}$ are p.s.r. of $SL_{\pm}(2,\mathbb{R})$, and $\mathcal{O}\!\mathcal{L}((\rho_o, e^{(2a+b)\delta})s_{\alpha_1})$ and $\mathcal{O}\!\mathcal{L}(\rho_o, e^{b\delta}, s_{\alpha_3})$ are the corresponding intertwining operators, respectively.

Therefore the assumptions of lemma G3 are satisfied, and we conclude that $J_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))}$, with $c \neq 0$, are unitary iff $0 < 2a+b \leq 1$.

Next assume $c = 0$. For $a > 1$, $b < 1$ and $b > 1, a < 1$, we use the above argument to show that the representations for these parameters are not unitary except possible for $a+b = 2m+1$ and $a = 2m+1$, $m \in \mathbb{N}$. To treat these cases assume first $a = 2m+1$, $m \geq 1$, $b < 1$ and define $u(t) = (\rho_o, a(\delta_1+\delta_3)+tb\delta_2)$, $0 \leq t \leq 1$. Then $\langle\ ,\ \rangle_{u(t)}$, as defined in I.G., is a hermitian form. But $J_{u(t)}$, $t \in (0,1]$ is a representation induced from $P_2 = M_2A_2N_2$ with a finite-dimensional representation of M_2 and a a real character of A_2N_2 , and it follows from the composition series results that this representation is irreducible for $t \in (0,1]$. Thus if $\langle\ ,\ \rangle_{\mu(t)}$ is positive semidefinite for $t \in (0,1]$, it is also positive semidefinite for $t = 0$ by lemma 40 in [9.b]. But in b) we proved that this form is not positive semidefinite for $\mu = (\rho_o, a(\delta_1+\delta_3))$ if $a \geq 3/2$. Hence $\langle\ ,\ \rangle_{\mu(t)}$ is not positive semidefinite for all $t \in [0,1]$, and the representation $J_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2)}$ are not unitary for $b < 1$, $a = 2m+1$, $m \geq 1$.

For $c = 0$, $0 < b < 1$ and $a+b = 2m+1$, $m \in \mathbb{N}\setminus 0$ and $c = 0$, $0 < a < 1$ and $a+b = 2m+1$, $m \in \mathbb{N}\setminus 0$ a similar argumentation shows that the representations with these parameters are not unitary.

Thus we are left with the cases

$$c = 0\ , \quad a+b = 1 \qquad a,b \in (0,1)\ ,$$

once

$$c = 0\ , \quad a = 1\ , \quad 0 < b < 1\ .$$

Let $c = 0$, $a+b = 1$, $a,b \in (0,1)$. The J.H. series computations show that $J_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2)}$ is a representation induced from $P_2 = M_2A_2N_2$ with the trivial representation of M_2 and the character $\chi = e^{a\delta_2}$ on A_2N_2 and that $J_{(\rho_o,\delta_2)}$ is unitarily induced from the trivial representation of P_2 . Now define

$\mu(t) = (\rho_o, t(\delta_1+\delta_3)+(1-t)\delta_2)$ for $t \in [0,1)$ and consider $\langle .,. \rangle_{\mu(t)}$. This form is semidefinite for all $t \in [0,1)$ if it is semidefinite for one $t \in [0,1)$. But we already know that it is positive definite for $t = 0$. Thus the representations $J_{(\rho_o, a(\delta_1+\delta_3)+b\delta_2)}$ are unitary for $a+b = 1$, $a,b \in (0,1)$.

Finally, let $c = 0$, $a = 1$, $\delta < b < 1$. To show that in this case the representations $J_{(\rho_o, \delta_1+\delta_3+b\delta_3)}$ are not unitary we use lemma G2. These representations are induced from $P_2 = M_2A_2N_2$ with the trivial representation on M_2 and the character $\chi = e^{(1+a)\delta_2}$ on A_2N_2 . Now if they were unitary, then by lemma G2 the representation induced from P_2 with the trivial representation of M_2 and the character $\chi = e^{2\delta_2}$ on A_2N_2 would have 2 unitary composition factors. But checking in Theorem 3 , we see that this has only two composition factors at all, namely the representations $J_{(\rho_o, \delta_1+\delta_2+\delta_3)}$ and $J_{((1),(3/2)\delta_2-3\delta_3, \rho_o^1)}$, and by case 2 $J_{((1),(3/2)\delta_2-3\delta_3, \rho_o^1)}$ is not unitary, which is a contradiction.

This completes the proof of e).

e') $\mu = (\rho_{1234}, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))$, $a,b \in \mathbb{R}^+$, $c \in \mathbb{R}$.
In this case we proceed as in e).

f) $\mu = (\rho_{14}, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))$, $a,b \in \mathbb{R}^+$, $c \in \mathbb{R}$.

By Theorem II.D.1., $U^{P_o}_{(\rho_{14},a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))}$ is reducible iff

$$c \neq 0 \quad \text{and} \quad b = 2m+1 \quad \text{or} \quad 2a+b = 2m+1$$

or

$$c = 0 \quad \text{and} \quad b = 2m+1 \quad \text{or} \quad 2a+b = 2m+1 \quad \text{or} \quad a+b = 2m$$
$$\text{or} \quad a = 2m \ , \quad m \in \mathbb{N} \ .$$

Since $U^{P_o}_{(\rho_{14},ic(\delta_1-\delta_2))}$ is irreducible and unitarily induced, we have complementary series for $0 < 2a+b < 1$ and limits of complementary series for $2a+b = 1$, c arbitrary. Using the same techniques as in e), one proves that these are the only unitary representations for this family of parameters.

f') $u = (\rho_{23},a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))$, $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, $c \in \mathbb{R}$:

In this case we proceed as in f).

□

<u>Theorem 11</u>. The unitary dual of $GL(4,\mathbb{R})$ consists of the following representations:

a) unitarily induced p.s.r.

b) unitarily induced g.p.s.r.

c) complementary series representations for p.s.r. and g.p.s.r.

d) limits of complementary series representations,

e) unitarily induced degenerate series representations,

f) complementary series representations for degenerate series representations,

g) the one-dimensional unitary representations.

Compressing this result in terms of the parameters of the proposition, we proved that the following not unitarily induced p.s.r. or g.p.s.r. are unitary.

a) $\mu \in \hat{C}_2$, $\mu = ((n,n),a\delta_2)$, $0 < a \leq 1$, $n \in \mathbb{N}\backslash 0$,

b) $\mu \in \hat{C}_1$, $\mu = ((n),-(a/2)\delta_2+a\delta_3,\rho_o^1)$, $0 < a \leq 1$, $n \in \mathbb{N}\backslash 0$,

$\mu = ((n),-(a/2)+a\delta_3,\rho_{34}^1)$, $0 < a \leq 1$, $n \in \mathbb{N}\backslash 0$,

c) $\mu \in \hat{C}_o$, $\mu = (\rho_o,a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_2)$

$0 < a < 1/2$, $b,c \in \mathbb{R}$

or $a = 1$, $b = 0$, $c \in \mathbb{R}$,

$\mu = (\rho_o,a\delta_2)$, $0 < a \leq 1$

$\mu = (\rho_o,a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))$

$0 < a+(b/2) \leq 1/2$ and $0 < b \leq 1$ and $c \in \mathbb{R}$

or $a+b = 1$ and $0 < a < 1$, $0 < b < 1$, $c = 0$

or $a = b = 1$, $c = 0$,

$\mu = (\rho_{14},a(\delta_1+\delta_3)+ib\delta_1-ic\delta_3-i(c+b)\delta_2)$

$0 < a \leq 1/2$, $b,c \in \mathbb{R}$,

$$\mu = (\rho_{14}, a\delta_2) \ , \quad 0 < a \leq 1 \ ,$$

$$\mu = (\rho_{23}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_2)$$
$$0 < a \leq 1/2 \ , \quad b,c \in \mathbb{R} \ ,$$

$$\mu = (\rho_{23}, a\delta_2) \ , \quad 0 < a \leq 1 \ ,$$

$$\mu = (\rho_{1234}, a(\delta_1+\delta_3)+ib\delta_1+ic\delta_3-i(c+b)\delta_2)$$
$$0 < a \leq 1/2 \ , \quad b,c \in \mathbb{R}$$
$$\text{or} \quad a = 1 \ , \ b = 0 \ , \quad c \in \mathbb{R} \ ,$$

$$\mu = (\rho_{1234}, a\delta_2) \ , \quad 0 < a \leq 1 \ ,$$

$$\mu = (\rho_{1234}, a(\delta_1+\delta_3)+b\delta_2+ic(\delta_1-\delta_3))$$
$$0 < a+(b/2) \leq 1/2 \quad \text{and} \quad 0 < b \leq 1 \quad \text{and}$$
$$c \in \mathbb{R}$$
$$\text{or} \quad a+b = 1 \quad \text{and} \quad 0 < a < 1 \ , \ 0 < b < 1 \ ,$$
$$c = 0$$
$$\text{or} \quad a = b = 1 \ , \ c = 0 \ .$$

Appendix 1:

Before we can deal with $SO(4,\mathbb{R})$, we need some results on $SO(3,\mathbb{R})$:

We choose in $SO(3,\mathbb{C})$, the Cartan subalgebra

$$T_3 = \mathbb{C}\begin{pmatrix} & \frac{1}{2} & \\ -\frac{1}{2} & & \\ & & 0 \end{pmatrix} .$$

Let M_o^3 be the subgroup generalized by

$$m_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$m_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$m_3 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} = \exp 2\pi \begin{pmatrix} & \frac{1}{2} & \\ -\frac{1}{2} & & \\ & & 0 \end{pmatrix} .$$

We also have

$$m_1 \exp t \begin{pmatrix} & \frac{1}{2} & \\ -\frac{1}{2} & & \\ & & 0 \end{pmatrix} m_1 = \exp - t \begin{pmatrix} & \frac{1}{2} & \\ -\frac{1}{2} & & \\ & & 0 \end{pmatrix}$$

i.e. we can consider m_1 as a representative of the nontrivial Weyl group element $so(3,\mathbb{C})$. We will now compute the multiplicity of the M_o^3-fixed vectors in some low dimensional representation.

We denote the n-dimensional representation by V_n .

Case 1: V_3

There it is easy to see that $V_3|_{M_o^3}$ breaks up into the direct sum of 3 inequivalent nontrivial representations.

Case 2: V_5

We have $V_3 \otimes V_3 = V_5 \oplus V_3 \oplus V_1$, and there are 3 M_o^3-fixed vectors in $V_3 \otimes V_3$. Since one is contained in V_1 and none in V_3 , two are contained in V_5 .

Case 3: V_7

We have $V_5 \otimes V_3 = V_2 \oplus V_5 \oplus V_3$, and again there are 3 M_o^3-faixed vectors in $V_5 \otimes V_3$. Since 2 are contained in V_5 and none in V_3 , just 1 is contained in V_7 .

We will now use these results to compute multiplicities of K-types for $GL(4,\mathbb{R})$.

<u>Lemma 1:</u> We have the following multiplicities for K-types in $U^{P_o}_{(\rho_o,\delta_1+\delta_2+\delta_3)}$:

Highest Weight	Multiplicity
00	1
10	0
11	0
20	3
21	0
22	2
30	0
31	3
32	0
33	1
40	7

<u>Proof</u>: (Using Frobenius reciprocity.)

All the representations of $so(4,\mathbb{R})$ are tensor products of two representations of $so(3,\mathbb{R})$.
For $so(4,\mathbb{C})$ we choose the Cartan subalgebra

$$\mathbb{C}\begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & & \frac{1}{2} \\ & & -\frac{1}{2} & \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & & -\frac{1}{2} \\ & & \frac{1}{2} & \end{pmatrix} ,$$

where each of the two summands is a Cartan subalgebra of one $so(3,\mathbb{C})$ factor. M_o is generated by

$$m_1 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \exp 2\pi \begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & & \frac{1}{2} \\ & & -\frac{1}{2} & \end{pmatrix} = \exp 2\pi \begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & & \frac{1}{2} \\ & & -\frac{1}{2} & \end{pmatrix}$$

$$m_2 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} .$$

We have

$$m_3 \begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & & \frac{1}{2} \\ & & -\frac{1}{2} & \end{pmatrix} m_3 = \begin{pmatrix} & -\frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & & \frac{1}{2} \\ & & -\frac{1}{2} & \end{pmatrix} = m_4 \begin{pmatrix} & -\frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & & -\frac{1}{2} \\ & & \frac{1}{2} & \end{pmatrix} m_4$$

and

$$m_2 = \exp 2\pi \begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & 0 & \\ & & & 0 \end{pmatrix} .$$

We write $\widetilde{M}_o$ for $M_o \cap SO(4,\mathbb{R})$. It is enough to compute the multiplicities of $\widetilde{M}_o$ fixed vectors in representations of $SO(4,\mathbb{R})$ (by Frobenius reprocity).

The formulas imply that the $\widetilde{M}_o$-invariant vectors are contained in the direct sum of all subspaces of integral weight, and since the simple roots have the coordinates $(1,1)$ and $(1,-1)$, all K-types with highest weights (a,b), $a+b$ odd, do not occur in $(U^{P_2}_{(\rho_0,\delta_1+\delta_2+\delta_3)})_K$.

Hence the K-types (1,0), (2,1), (3,0), (3,2) and (4.1) occur with multiplicity zero.

Now the representation with highest weight (1,1) . This is the tensor product of the 3-dimensional representation of one $so(3,\mathbb{C})$ factor and the one-dimensional representation of the other factor. In this case we apply case 1 in the first part of the appendix to show that there are no $\widetilde{M}_0$-invariant vectors.

The representation with highest weight (2,0) is 2 dimensional and has the following weight

$$\begin{pmatrix} 2\ 0 & 1\,{-1} & 0\,{-2} \\ 1\ 1 & 0\ 0 & -1\,{-1} \\ 0\ 2 & -1\ 1 & -2\ 0 \end{pmatrix}$$

on the space with weight (1,-1) , m_2 operates as -1, hence there is no $\widetilde{M}_0$-invariant vector contained in the direct sum of the spaces of weights (1,-1), (1,1), (-1,1) and (-1,-1). The three orbits under $\widetilde{M}_0$ are $\{(2,0),(-2,0)\}$, $\{(0,2),(0,-2)\}$ and (0,0) . Hence we get at most three $\widetilde{M}_0$ fixed vectors. But since we know that there are three irreducible inequivalent composition factors which contain this K-type, it occurs with multiplicity three, and $\widetilde{M}_0$ operates trivially on the weight space (0,0) .

Now to the representation with highest weight $(2,2)$. This representation is the tensor product of a 5-dimensional representation of one $so(3,\mathbb{C})$ factor and the one-dimensional representation of the other factor. In this case, $m_3 m_4$ operates as the nontrivial element in the Weyl group of each $so(3,\mathbb{C})$ factor. Hence we can apply case 2 in the first part of the appendix, so this K-type occurs with multiplicity two.
In the case of the representation with highest weight $(3,3)$ we use the same arguments to reduce the problem to case 3 in the first part of the appendix and conclude that this K-type occurs with multiplicity one.

The representation with highest weight $(3,1)$ is 15 dimensional and has the following weights:

$$\begin{pmatrix} 3\ 1 & 2\ 0 & 1\,{-1} & 0\,{-2} & -1\,{-3} \\ 2\ 2 & 1\ 1 & 0\ 0 & -1\,{-1} & -2\,{-2} \\ 1\ 3 & 0\ 2 & -1\ 1 & -2\ 0 & -3\,{-1} \end{pmatrix}$$

Since m_2 operates nontrivially on the spaces of weights $(1,1)$, $(3,1)$ and $(1,3)$ we get four orbits under $\tilde{M}_0$, namely $\{(2,2),(-2,-2)\}$, $\{(2,0),(-2,0)\}$, $\{(0,2),(0,-2)\}$ and $(0,0)$. The representation of $m_1 m_3$ on the space of weight $(0,0)$ is nontrivial since it is the tensor product of a trivial representation with a nontrivial representation. Hence there are three $\widetilde{M}_o$ invariant vectors in this representation i.e. the K-type occurs with multiplicity three.

The representation with highest weight (4,0) is 25-dimensional and has the following weights

$$\begin{pmatrix} 4\ 0 & 3\ 1 & 2\ 2 & 1\ 3 & 0\ 4 \\ 3\ {-1} & 2\ 0 & 1\ 1 & 0\ 2 & -1\ 3 \\ 2\ {-2} & 1\ {-1} & 0\ 0 & -1\ 1 & -2\ 2 \\ 1\ {-3} & 0\ {-2} & -1\ {-1} & -2\ 0 & -3\ 1 \\ 0\ {-4} & -1\ {-3} & -2\ {-2} & -3\ {-1} & -4\ 0 \end{pmatrix}$$

Since m_2 operates nontrivially on the spaces of weights (3,1), (3,-1) and (1,1), we get exactly seven orbits under $\widetilde{M}_o$. The representation on the space of weight (0,0) of $\widetilde{M}_o$ is trivial since it is the tensor product of two trivial representations, i.e. we have multiplicity seven for this K-type.

Appendix 2:

We give a list of J.H. series without multiplicities for the representations with parameter

$\chi = e^{\delta_1+\delta_2}, e^{\delta_1+\delta_2}, e^{\delta_2+\delta_3}, e^{\delta_1}, e^{\delta_2}, e^{\delta_3}$.

Parameter of p.s.r.	Jordan-Hölder series
$e^{\delta_1+\delta_3}, \rho_o$	$J(\rho_o, \delta_1+\delta_3), J((1),(1/2)\delta_3+\delta_3, \rho_o^1)$, $J((1),-(1/2)\delta_2-\delta_3, \rho_o^1), J((1,1),\delta_2), J((2),0,\rho_2^1)$
$e^{\delta_1+\delta_3}, \rho_1$	$J(\rho_1, \delta_1+\delta_3), J((1),-(1/2)\delta_2-\delta_3, \rho_4^1)$, $J((2),0,\rho_o^1)$
$e^{\delta_1+\delta_3}, \rho_2$	$J(\rho_{12}, \delta_1+\delta_2), J((1),(1/2)\delta_2+\delta_3, \rho_{34}^1)$, $J((1,1),\delta_2), J((1),-(1/2)\delta_2-\delta_3, \rho_o^1), J((2),0,\rho_3^1)$
$e^{\delta_1+\delta_3}, \rho_{13}$	$J((1),(1/2)\delta_2+\delta_3, \rho_o^1), J(\rho_{12}, \delta_1+\delta_3)$, $J((2),0,\rho_o^1), J((1),-(1/2)\delta_2+\delta_3), \rho_{34}^1)$
$e^{\delta_1+\delta_3}, \rho_{14}$	$J(\rho_{14}, \delta_1+\delta_3)$

$e^{\delta_1+\delta_2}, \rho_o$	$J_{(\rho_o,\delta_1+\delta_2)}, J_{((1),(3/2)\delta_2,\rho_o^1)}, J_{((2),(1/2)\delta_2,\rho_3^1)}, J_{((1),-(3/2)\delta_2+2\delta_3,\rho_o^1)}$
$e^{\delta_1+\delta_2}, \rho_1$	$J_{(\rho_1,\rho_1+\rho_2)}, J_{((1),-(3/2)\rho_2+2\rho_3,\rho_3^1)}, J_{((2),(1/2)\rho_2,\rho_o^1)}, J_{((2,1),(1/2)\rho_2)}$
$e^{\delta_1+\delta_2}, \rho_2$	$J_{(\rho_2,\delta_1+\delta_2)}$
$e^{\delta_1+\delta_2}, \rho_{13}$	$J_{(\rho_{13},\delta_1+\delta_3)}, J_{((1),-(3/2)\delta_2+2\delta_3,\rho_{34}^1)}, J_{((2),(1/2)\delta_2,\rho_{34}^1)}$
$e^{\delta_1+\delta_2}, \rho_{14}$	$J_{(\rho_{14},\delta_1+\delta_2)}, J_{((1),-(3/2)\delta_2+2\delta_3,\rho_{34}^1)}, J_{((2),(1/2)\delta_2,\rho_3^1)}$
e^{δ_1}, ρ_o	$J_{(\rho_o,\delta_1)}, J_{((1),(1/2)\delta_2,\rho_o^1)}$
e^{δ_1}, ρ_1	$J_{(\rho_1,\delta_1)}$
e^{δ_1}, ρ_2	$J_{(\rho_2,\delta_1)}, J_{((1),(1/2)\delta_2,\rho_3^1)}$
e^{δ_1}, ρ_{13}	$J_{(\rho_{13},\delta_1)}, J_{((1),(1/2)\delta_2,\rho_{34}^1)}$
e^{δ_1}, ρ_{14}	$J_{(\rho_{14},\delta_1)}, J_{((1),(1/2)\delta_2,\rho_o^1)}$

e^{δ_2,ρ_0}	$J_{(\rho_0,\delta_2)}, J_{((1),\delta_2,\rho_0^1)}, J_{((1,1),0)}$
e^{δ_2,ρ_1}	$J_{(\rho_1,\delta_2)}, J_{((1),\delta_2,\rho_3^4)}$
$e^{\delta_2,\rho_{13}}$	$J_{(\rho_{13},\delta_2)}, J_{((1),\delta_2,\rho_0^1)}, J_{((1),\delta_2,\rho_{34}^1)}, J_{((1,1),0)}$
$e^{\delta_2,\rho_{14}}$	$J_{(\rho_{14},\delta_2)}, J_{((1),\delta,\rho_0^1)}, J_{((1),\delta_2,\rho_{34}^1)}, J_{((1,1),0)}$

The J.H. series of the other representations we get by symmetry considerations.

D. Reducibility

Let $\mu = (\vec{r},\chi,\rho) \in \hat{C}_r$ and $\beta \in \Sigma$, β positive with respect to the ordering defined by $\mathcal{C}_o$.

Assume $\mathrm{mult}(\beta|_{\mathfrak{a}_r}) = 1$. Then define m_α as in I.B.

Assume $\mathrm{mult}(\beta|_{\mathfrak{a}_r}) = 2$. Then by I.A. there is a simple root $\alpha_{2i(\alpha)-1}$ with $i(\alpha) \in \{1,\ldots,r\}$, s.r. either $\beta + \alpha_{2i(\alpha)-1} \in \Sigma$ or $\beta - \alpha_{2i(\alpha)-1} \in \Sigma$. Define $\vec{r}_\alpha = (r_{i(\alpha)})$.

Assume $\mathrm{mult}(\beta|_{\mathfrak{a}_r}) = 4$. Then by I.A. there are simple roots $\alpha_{2i(\alpha)-1}, \alpha_{2j(\alpha)-1}$ with $i(\alpha) < j(\alpha)$ and $i(\alpha), j(\alpha) \in \{1,\ldots,r\}$, s.t. either $\beta + \alpha_{2i(\alpha)-1} \in \Sigma$ or $\beta - \alpha_{2i(\alpha)-1} \in \Sigma$, and either $\beta + \alpha_{2j(\alpha)-1} \in \Sigma$ or $\beta - \alpha_{2j(\alpha)-1} \in \Sigma$. Define $\vec{r}_\alpha = (r_{i(\alpha)}, r_{j(\alpha)})$.

Now for $\vec{r} \in (\mathbb{Z}\setminus 0)^r$ and $\rho \in \hat{Z}_r$, define $H_\beta^{(\vec{r},\rho)} \subset \hat{A}_r$ as follows:

a) If $\mathrm{mult}(\beta) = 1$,

$$H_\beta^{(\vec{r},\rho)} = \left\{ \chi \;\middle|\; \begin{array}{ll} (\alpha,\log\chi) = 2m+1 , \ m \in \mathbb{N} , & \text{if } \rho(m_\alpha) = 1 \\ (\alpha,\log\chi) = 2m , \ m \in \mathbb{N} , & \text{otherwise} \end{array} \right\} .$$

b) If $\mathrm{mult}(\beta) = 2$,

$$H_\beta^{(\vec{r},\rho)} = \left\{ \chi \mid |(a,\log\chi)| - |r_{i(\alpha)}| \in \mathbb{N}\setminus 0 \text{ and } (\alpha,\log\chi) \in \mathbb{R} \right\}$$

c) If $\mathrm{mult}(\beta) = 4$,

$$H_\beta^{(\vec{r},\rho)} = \Big\{\chi \Big| \, |(\alpha,\log\chi)| - |r_{i(\alpha)}| + |r_{j(\alpha)}| \in \mathbb{N}\setminus 0 \text{ if } |r_{i(\alpha)}| \geq |r_{j(\alpha)}| \text{ and } (\alpha,\log\chi) \in \mathbb{R}$$

$$|(\alpha,\log\chi)| + |r_{i(\alpha)}| - |r_{j(\alpha)}| \in \mathbb{N}\setminus 0 \text{ if } |r_{j(\alpha)}| \geq |r_{i(\alpha)}| \text{ and } (\alpha,\log\chi) \in \mathbb{R}\Big\}$$

Let $\mathcal{C}_r$ be a Weyl chamber in $\mathfrak{a}_r'$ and let $(\Sigma_r)^+_{\mathcal{C}_r}$ be the positive roots system of Σ_r defined by $\mathcal{C}_r$. Then we can find an ordering $(\Sigma)^+_{\mathcal{C}_r}$ of Σ which is compatible with $(\Sigma_r)^+_{\mathcal{C}_r}$, i.e. the restriction of $\alpha \in (\Sigma)^+_{\mathcal{C}_r}$ to $\mathfrak{a}_r$ is either zero or contained in contained in $(\Sigma_r)^+_{\mathcal{C}_r}$. If $\alpha \in (\Sigma^+)_{\mathcal{C}_r}$, α simple, we write $\hat{\alpha}$ for $\{\alpha,-\alpha\} \cap \Sigma^+$ where Σ^+ is the ordering defined by $\mathcal{C}_0$. Define

$$\mathcal{H}_{\mathcal{C}_r}^{(\vec{r},\rho)} = \bigcup_\alpha H_{\hat{\alpha}}^{(\vec{r},\rho)}$$

where α runs through all simple roots in $(\Sigma)^+_{\mathcal{C}_r}$ s.t. $\alpha|_{\mathfrak{a}_r} \neq 0$, and

$$H_{\mathcal{C}_r}^{(\vec{r},\rho)} = \bigcup_{w\in W_r} \mathcal{H}_{w\mathcal{C}_r}^{(\vec{r},\rho)} .$$

<u>Theorem 1</u>. Let $P = \underline{P}_{\mathcal{C}_r}$ be parabolic associated to $\mathcal{C}_r$. Then $U^P_{(\vec{r},\chi,\rho)}$ is reducible iff $\chi \in H_{\mathcal{C}_r}^{(\vec{r},\rho)}$.

For the proof we need the

<u>Lemma</u>: Let P_1, P_2 be two parabolics associated to Weyl chambers in $\mathfrak{a}_r'$. Then $U^{P_1}_{(\vec{r},\rho,\chi)}$ is reducible, iff $U^{P_2}_{(\vec{r},\chi,\rho)}$ is reducible.

This lemma will be proved later.

<u>Proof</u> of the theorem. First I will show that $\chi \in H^{(\vec{r},\rho)}_{\mathcal{C}_r}$ implies $U^{P}_{(\vec{r},\chi,\rho)}$ is reducible.
Assume $\chi \in H^{(\vec{r},\rho)}_{\mathcal{C}_r}$. Then there is a $w \in W_r$ s.t. $\chi \in \varkappa^{(\vec{r},\rho)}_{w\mathcal{C}_r}$. The above lemma shows that it is enough to prove the assertion for $P = \underline{P}_{w\mathcal{C}_r}$. Now let β be a simple root in $(\Sigma)^+_{w\mathcal{C}_r}$ s.t. $\chi \in H^{(\vec{r},\rho)}_{\beta}$.
Assume $\mathrm{mult}(\beta|_{\mathfrak{a}_r}) = 1$. Then either $\mathfrak{A}(\underline{P}_{w\mathcal{C}_r},(\vec{r},\chi,\rho),s_\beta)$ or $\mathfrak{A}(\underline{P}_{w\mathcal{C}_r},s_\beta(\vec{r},\chi,\rho),s_\beta)$ is defined and by I.D. and II.A. has a nontrivial kernel. Hence the representation is reducible.
Assume $\mathrm{mult}(\beta|_{\mathfrak{a}_r}) = 2$. Then there is a $\beta' \in (\Sigma)^+_{\mathcal{C}_r}$ s.t. $\beta'|_{\mathfrak{a}_r} = \beta|_{\mathfrak{a}_r}$. Hence $\beta' - \beta \in (\Sigma)^+_{\mathcal{C}_r}$ and $\beta' - \beta$ is even simple. Then either $\mathfrak{A}(\underline{P}_{w\mathcal{C}_r},(\vec{r},\chi,\rho),s_{(\beta'-\beta)}s_\beta)$ or $\mathfrak{A}(s_{(\beta'-\beta)}s_\beta\,\underline{P}_{w\mathcal{C}_r},s_{(\beta'-\beta)}s_\beta(\vec{r},\chi,\rho),s_\beta s_{(\beta'-\beta)})$ is defined.
Using I.D., we reduce the problem to the reducibility problem for the representation of $GL(3,\mathbb{R})$ with parameter

$(\vec{r}_\beta, (\beta,\chi)\delta)$. By II.B., 6 this representation is reducible.

Assume $\text{mult}(\beta|_{\sigma_r}) = 4$. Then there are $\beta', \beta'' \in (\Sigma)^+_{\mathcal{C}_r}$ s.t. $\beta'|_{\sigma_r} = \beta''|_{\sigma_r} = \beta|_{\sigma_r}$ are s.t. $\beta'-\beta$ and $\beta''-\beta$ are simple roots. Then either $\mathcal{O}(\underline{P}_{w\mathcal{C}_r}, (\vec{r},\chi,\rho), s_\beta s_{\beta''-\beta} s_{\beta'-\beta} s_\beta)$ or $\mathcal{O}(\underline{P}_{w\mathcal{C}_r}, (\vec{r}, s_\beta s_{\beta''-\beta} s_{\beta'-\beta} s_\beta \chi, \rho), s_\beta s_{\beta'-\beta} s_{\beta''-\beta} s_\beta)$ is defined. Using I.D., we reduce the problem to the reducibility problem for the representation of $GL(4,\mathbb{R})$ with parameter $(\vec{r}_\beta, (\beta,\chi), \delta_2)$. By II.C.9 this representation is reducible.

Thus $U^P_{(\vec{r},\chi,\rho)}$ is reducible if $\chi_{\mathcal{C}_r}^{(\vec{r},\rho)}$.

Now we will show that $U^P_{(\vec{r},\chi,\rho)}$ reducible implies $\chi \in H_{\mathcal{C}_r}^{(\vec{r},\rho)}$.

Assume $U^P_{(r,\chi,\rho)}$ is reducible. By the above lemma we can assume that χ is dominant with respect to P. Now let $w_o = w_\ell, \ldots, w_1$ be a product decomposition of the Weyl group element defining the long intertwining operator, and let i_o be the first index such that the corresponding factor of the operator has a nontrivial kernel. Then we have $\beta \in \Sigma$ s.t. $\chi \in H_\beta^{(\vec{r},\rho)}$. The proof is complete if we can show

$$\beta = w\gamma$$

or

$$-\beta = w\gamma \quad \text{for } w \in W_r \text{ and } \gamma \text{ simple in } (\Sigma)^+_{\mathcal{C}_r} .$$

If $\operatorname{mult}(\beta) = 1$ or 4 this follows from I.A.2. and if $\operatorname{mult}(\beta) = 2$ from I.A.4. $\square$

<u>Proof</u> of the lemma.

In I.F.1. we showed that $U^P_{(\vec{r},\chi,\rho)}$ is reducible iff $U^P_{(\vec{r},w\chi,w\rho)}$ for $w \in W_r$ is reducible. But for $P = \underline{\underline{P}}_{\mathcal{C}_r}$ we have $U^{\underline{\underline{P}}_{\mathcal{C}_r}}_{(\vec{r},w\chi,w\rho)} \cong U^{\underline{\underline{P}}_{w^{-1}\mathcal{C}_r}}_{(\vec{r},\chi,\rho)}$. This proves the lemma for parabolics associated to conjugate Weyl chambers.

On the other hand in II.B. we showed that the lemma is true for $SL(3,\mathbb{R})$. Hence assume now P_1, P_2 are associated to Weyl chambers $\mathcal{C}_1, \mathcal{C}_2$ s.t. $(\Sigma_r)^+_{\mathcal{C}_1} \cap (\Sigma_r)^+_{\mathcal{C}_2} \cup \alpha = (\Sigma_r)^+_{\mathcal{C}_1}$ and $(\Sigma_r)^+_{\mathcal{C}_1} \cap (\Sigma_r)^+_{\mathcal{C}_2} \cup -\alpha = (\Sigma_r)^+_{\mathcal{C}_2}$ for $\alpha \in \Sigma_r$ with $\operatorname{mult}(\alpha) = 2$. But then by our results on $GL(3,\mathbb{R})$ we have an intertwining operator from

$$U^{P_1}_{(\vec{r},\chi,\rho)} \quad \text{to} \quad U^{P_2}_{(\vec{r},\chi,\rho)}$$

or from

$$U^{P_2}_{(\vec{r},\chi,\rho)} \quad \text{to} \quad U^{P_1}_{(\vec{r},\chi,\rho)}$$

Thus $U^{P_2}_{(\vec{r},\chi,\rho)}$ is reducible if $U^{P_1}_{(\vec{r},\chi,\rho)}$ is reducible and hence the lemma is true also for non conjugate parabolics. $\square$

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