14.452 Recitation Notes:
1. AK Model with Specialization and Trade
2. Directed Technical Change Model

Alp Simsek

MIT

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Trade, Specialization and World Income Distribution

- Based on Acemoglu and Ventura (2001). A model in which countries grow at the same rate because of international trade.
- Trade is Ricardian: each country specializes in certain goods. If a country grows more slowly, then its goods become scarce in the world, which increases their price. This in turn makes the country catch up with the frontier growth.
- Competitive trade model with AK features. Trade in intermediate goods, but not in the final good. I.e., international borrowing and lending is ruled out. This will simplify the analysis by enforcing trade balance in each period.

Model

- Total measure μ of intermediate goods. Each country can produce exactly μ_j of these goods (1 unit of capital -> one unit of intermediate). No two country can produce the same goods (Armington assumption, simplification for more general Ricardian forces).
- Also assume heterogeneity in savings behavior, i.e., different ρ_j.
- Suppose there is no labor, i.e., only factor of production is capital. Can do the Rebelo (1991) trick to introduce labor and to keep the AK structure.
- I will also consider a simpler version of the model in which consumption and investment goods are produced with the same technology, i.e., I assume $\zeta_j = 1$ for all j. (Having different technologies for consumption and investment would be important if we introduced labor. But since we are not doing that, not much loss of generality in this simplification).

- Consider and investment are the same good, and produced using capital and other intermediate goods according to:

$$C_j(t) + I_j(t) = \chi K_j^D(t)^{1-\tau} X_j(t)^{\tau}, \tag{1}$$

$$\text{where } X_j(t) = \left(\int_0^N x_j(t,\nu)^{\frac{\varepsilon-1}{\varepsilon}} d\nu \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{2}$$

  is the output of the tradable sector.
- Here, $K_j^D(t)$ denotes capital used in production of *domestic* goods. The remaining $K_j^{\mu}(t) = K_j(t) - K_j^D(t)$ of capital is used in production of export goods.
- Let $p_j^D(t)$ denote the price of the final good in country j. Do not normalize this to one (since we will choose a more convenient normalization). Assume $\delta = 0$, and let $r_j(t) = R_j(t)$ denote the rental rate of capital in country j. Note that the interest rate is $\frac{r_j(t)}{p_j^D(t)}$. Why?
- Define output from the production side as:

$$Y_j(t) = p_j^D(t)(C_j(t) + I_j(t)) = p_j^D(t)\chi K_j^D(t)^{1-\tau} X_j(t)^{\tau}. \tag{3}$$

  Note that, since there is no borrowing or lending between countries, output of a country is also equal to the income of a country:

$$Y_j(t) = r_j(t) K_j(t). \tag{4}$$

Line of Attack

- There are two essential trade-offs:
  1. Static trade-off: How much capital to allocate to domestic production and how much to exports (choice between $K_j^D(t)$ and $K_j^\mu(t)$)? What are the prices that clear the markets for each country's goods?
  2. Dynamic trade-off: How much output to allocate to consumption and investment (choice between $C_j(t)$ and $I_j(t)$), given the prices.

  Line of attack: Solve the static allocation (resolve trade-off 1, and solve for prices) for a given level of capital distributions $\{K_j(t)\}_j$ across countries. Then consider the dynamic allocation (resolve trade-off 2).

- How to resolve the static-trade-off:
  - The country's allocation $K_j^\mu(t)$ to the export sector determines its export revenue. But a country's export revenue is equivalent to all other countries' import expenditures on this country's goods. Calculate the latter (which will implicitly determine $K_j^\mu(t)$).
  - In addition, **trade balance** in each period requires (export revenues=import expenditures).

  These two steps will solve for the static allocation. They determine the prices and the allocation of capital between $K_j^D(t)$ and $K_j^\mu(t)$.

Static Allocation: Firms

- **Intermediate firms:** Linear technology, 1 unit of capital produces 1 unit of the intermediate good. Competitive sector. Thus, price of each intermediate goods is given by $p_j(t) = r_j(t)$.
- **Final good firms:** Produce final good (consumption or investment) according to technology in (1). At each t, firms in country j solve:

$$\max_{K_j^D(t), X_j(t)} p_j^D(t)\, \chi K_j^D(t)^{1-\tau} X_j(t)^{\tau} - r_j(t)\, K_j^D(t) - X_j(t).$$

We have normalized the price of the tradable good to 1 (which is the same in all countries, by free trade in intermediates). Solution to this problem implies two things:

1. Final good price in country j is given by the unit cost:

$$p_j^D(t) = r_j(t)^{1-\tau}.$$

(Here, we take $\chi = (1-\tau)^{-(1-\tau)}\,\tau^{-\tau}$, which cancels the constant terms and results in the above expression).

2. Total expenditure on imports is a constant fraction of output (recall the definition of output in (3)):

$$X_j(t) = \tau Y_j(t). \tag{5}$$

Static Allocation: Firms

- **Tradable good firms:** In each country i (I change notation for simplicity on next slide), a sector produces $X_i^D(t)$ according to the technology in (2). From Dixit-Stiglitz, we know that this is equivalent to two things:
  1. Price of tradable good (which was normalized to 1) is also equal to the ideal price index:
  $$1 = \left(\int_0^N p(t,\nu)^{1-\varepsilon}\, d\nu\right)^{1/(1-\varepsilon)} = \left(\sum_j \mu_j p_j(t)^{1-\varepsilon}\right)^{1/(1-\varepsilon)}.$$
  2. Country i's demand of a good by country j is given by:
  $$x_i(t,\nu(j)) = p_j(t)^{-\varepsilon} X_i(t). \tag{6}$$

Static Allocation: Trade Balance

- We are now in a position to calculate the trade balance equation.
  - Country j's total expenditure on imports $= \tau Y_j(t)$ (from Eq. (5)).
  - Country j's total export revenue $=$ all other country's expenditures on imports fromcountry $j =$

$$\mu_j p_j(t)\, p_j(t)^{-\varepsilon} \sum_{i=1}^{J} \tau Y_i(t),$$

  where the last equation uses Eq. (6) and substitutes for $X_i(t)$ from Eq. (5).

- Combining these two observations and using $r_j(t) = p_j(t)$ gives the trade balance equation in the lecture slides:

$$Y_j(t) = \mu_j r_j(t)^{1-\varepsilon} \sum_{i=1}^{J} Y_i(t).$$

- Use the expression for output in Eq. (3) in the previous equation to get:

$$r_j(t)\, K_j(t) = \mu_j r_j(t)^{1-\varepsilon} \sum_{i=1}^{J} r_i(t)\, K_i(t), \text{ for each } j. \tag{7}$$

- Given $\{K_j(t)\}_j$, the previous system has J equations in J unknowns. This can be solved.

Static Allocation: Solution

- The system in (7) solves for the static equilibrium prices $r_j(t)$.
  - This in turn solves for the remaining prices, $p_j(t) = r_j(t)$ and $p_j^D(t) = r_j(t)^{1-\tau}$.
  - This also solves for output levels $Y_j(t) = r_j(t) K_j(t)$. This further solves for import expenditures $X_j(t) = \tau Y_j(t)$ (and thus export revenues). This further solves for $K_j^{\mu}(t) = \frac{X_j(t)}{r_j(t)}$, the allocation of capital to the export sector.
- Eq. (7) captures the most essential intuition in this model: What happens to the prices of a country's goods (to the prices $p_j(t) = r_j(t)$) if $K_j(t)$ increases? Why?
- Our next goal is to characterize the dynamic trade-off, and solve for the evolution of capital stocks.

- Consumer solves the Euler equation:

$$\max_{[K_j(t),C_j(t)]_t} \int_0^\infty \exp(-\rho_j t) \log C_j(t) dt\,, \tag{8}$$

$$\text{s.t. } p_j^D(t)\left(\dot{K}_j(t) + C_j(t)\right) = r_j(t) K_j(t)\,. \tag{9}$$

  Log preferences (simplification). We do not work with $A_j(t)$, but instead directly work with $K_j(t)$ (simplification). Note that the interest rate in this model is $\frac{r_j(t)}{p_j^D(t)}$.

- The solution to Eq. (8) is characterized by the Euler equation and the transversality condition (along with the budget constraint):

$$\frac{\dot{C}_j(t)}{C_j(t)} = \frac{r_j(t)}{p_j^D(t)} - \rho_j, \tag{10}$$

$$\lim_{t\to\infty} \exp\left(-\int_0^t \frac{r_j(s)}{p_j^D(s)} ds\right) K_j(t) = 0. \tag{11}$$

- WIth log utility, these equations can be solved in closed form.

Closed form Solution with Log Utility

- First, note that the budget constraint in (9) can be rewritten as:

$$\exp\left(-\int_0^t \frac{r_j(s)}{p_j^D(s)}ds\right) C_j(t) = -\frac{d}{dt}\left(\exp\left(-\int_0^t \frac{r_j(s)}{p_j^D(s)}ds\right) K_j(t)\right).$$

  Integrate both sides from 0 to ∞, and use the transversality condition to get the lifetime budget constraint:

$$\int_0^\infty \exp\left(-\int_0^t \frac{r_j(s)}{p_j^D(s)}ds\right) C_j(t)\,dt = K_j(0). \quad (12)$$

- Second, solve for $C_j(t)$ from the Euler equation (10):

$$C_j(t) = C_j(0)\int_0^t \exp\left(\frac{r_j(s)}{p_j^D(s)} - \rho_j\right) ds. \quad (13)$$

- Third, plug this into the lifetime budget constraint and solve for $C_j(0)$ as:

$$C_j(0) = \rho_j K_j(0).$$

- Can repeat the same steps starting at any t (instead of 0). Thus,

$$C_j(t) = \rho_j K_j(t). \quad (14)$$

Dynamic Allocation: Solution

- With log preferences, income and substitution effects cancel. Consume a constant fraction of wealth (regardless of the return structure).
- This also characterizes consumer's investment (which is the residual of consumption). In particular, plugging Eq. (14) back into the Euler equation (10) gives

$$\frac{\dot{K}_j(t)}{K_j(t)} = \frac{r_j(t)}{p_j^D(t)} - \rho_j.$$

- Recall that $p_j^D(t) = r_j(t)^{1-\tau}$ (from the static equilibrium characterization). Thus,

$$\frac{\dot{K}_j(t)}{K_j(t)} = r_j(t)^{\tau} - \rho_j, \text{ for each } j. \tag{15}$$

This characterizes the country's evolution of capital stocks. Intuition?

Equilibrium Characterization

- Put the static and dynamic equations (7) and (15) together:

$$r_j(t)\, K_j(t) = \mu_j r_j(t)^{1-\varepsilon} \sum_{i=1}^{J} r_i(t)\, K_i(t), \text{ for each } j,$$

$$\frac{\dot{K}_j(t)}{K_j(t)} = r_j(t)^{\tau} - \rho_j, \text{ for each } j.$$

- For each $\{K_j(t)\}_j$, the first equation solves for the prices. Given these prices, the second equation determines how capital evolves into the future. Hence, these two equations completely characterize the equilibrium.
- From these equations, it can be seen that there exists a BGP allocation in which capital in each country grows at the same rate g.

BGP Equilibrium Characterization

- At the BGP, using the dynamic equation, the prices are given by:

$$r_j(t) = r_j^* = (\rho_j + g)^{1/\tau}. \tag{16}$$

- To solve for the BGP growth rate g, sum over the static equation (7) to get:

$$\sum_{j=1}^{J} \mu_j r_j(t)^{1-\varepsilon} = 1.$$

Using Eq. (16), this gives

$$\sum_{j=1}^{J} \mu_j \left(\frac{1}{\rho_j + g}\right)^{(\varepsilon-1)/\tau} = 1,$$

which uniquely solves for g under appropriate parametric conditions.

Transitional Dynamics

- Finally, note that relative capital levels must be at a certain (constant) level on a BGP. To see this, pick a country as reference, i.e., country 1. Divide the condition (7) for country j and country 1 to get:

$$\frac{Y_j(t)}{Y_1(t)} = \frac{r_j(t) K_j(t)}{r_1(t) K_1(t)} = \frac{\mu_j r_j(t)^{1-\varepsilon}}{\mu_1 r_1(t)^{1-\varepsilon}}.$$

Using the BGP expression (16) for the interest rates, this shows that, on a BGP:

$$\frac{K_j(t)}{K_1(t)} = \frac{\mu_j}{\mu_1}\left(\frac{\rho_j + g}{\rho_1 + g}\right)^{-\varepsilon/\tau}.$$

- What happens when initial capital levels do not satisfy this relative condition? What is the equilibrating force?
- What would happen if there was no trade, and each country was a closed economy?
- Also, on a BGP, countries' relative output is given by:

$$\frac{Y_j(t)}{Y_1(t)} = \frac{\mu_j}{\mu_1}\left(\frac{\rho_j + g}{\rho_1 + g}\right)^{-(\varepsilon-1)/\tau}.$$

What determines countries' relative *levels* of output?

- Goal: Develop a framework to think about the direction of technological change. Use the framework to address a number of interesting phenomena. E.g., over the past 60 years, the supply of high skilled workers and their relative wages have increased together. Labor economists' standard explanation: demand for high skill also increased, because technological change has been skill biased. But why has technological change been skill biased?
- Basic concepts: *skill augmenting technological change* and *skill biased technological change*.
- Main results: *weak equilibrium bias* of technological change and *strong equilibrium bias* of technological change.
  Definitions on the next two slides.

- Lecture notes provide a definition of these concepts for more general production functions. For simplicity, consider a CES production function:

$$\left[\gamma_L (A_L L)^{\frac{\sigma-1}{\sigma}} + \gamma_H (A_H H)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

- **H-augmenting** (resp. **L-augmenting**) **technological change** is an increase in A_H (resp. an increase in A_L).
- To define **H-biased technological change**, consider the relative demand function:

$$\frac{w_H}{w_L} = \frac{MP_H}{MP_L} = \left(\frac{A_H}{A_L}\right)^{(\sigma-1)/\sigma} \left(\frac{H}{L}\right)^{-1/\sigma}.$$

  H-biased technological change is a technological change that *increases* the relative wages for skilled workers for any given level of supply. That is, it shifts the demand curve up. In the CES case, this is equivalent to say that $\left(\frac{A_H}{A_L}\right)^{(\sigma-1)/\sigma}$ is increasing in $\frac{H}{L}$ (when it is viewed as an endogenous function of $\frac{H}{L}$).
- Note that *H*-augmenting technological change is *not necessarily* H-biased technological change. In particular:
  - When $\sigma > 1$, H-augmenting technological change is H-biased.
  - When $\sigma < 1$, H-augmenting technological change is L-biased.

  Why?

- We say that technological change features **weak equilibrium bias** if an increase in the relative abundance (supply) of a factor creates technological change biased towards that factor. (Put differently, a positive shock to supply creates a shift of demand that tends to alleviate some of the fall in the price of the factor).
- In the above CES production function, weak equilibrium bias if $\frac{d\left(\frac{A_H}{A_L}\right)^{(\sigma-1)/\sigma}}{d(H/L)} \geq 0$.
- We say that technological change features **strong equilibrium bias** if an increase in the relative abundance of a factor creates sufficiently strong biased technological change that the technology adjusted (long run) demand curve is upwards sloping.
- In the above CES production function, strong equilibrium bias if $\frac{d\left(\frac{MP_H}{MP_L}\right)}{d(H/L)} \geq 0$.
- **Preview of results:** Under fairly general conditions equilibrium features weak equilibrium bias. Under slightly stronger conditions, it features strong equilibrium bias.

- Builds upon the baseline endogenous growth model with lab equipment.
- Consumer side is the same.
- On the production side, consider two factors supplied inelastically, H and L. Final good produced competitively according to:

$$Y(t) = \left[\gamma_L Y_L(t)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_H Y_H(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

  where $Y_H(t)$ and $Y_L(t)$ are H and L-intensive intermediate goods, produced competitively according to:

$$Y_L(t) = \frac{1}{1-\beta}\int_0^{N_L(t)} x_L(\nu,t)^{1-\beta}\, d\nu L^{\beta},$$

$$Y_H(t) = \frac{1}{1-\beta}\int_0^{N_H(t)} x_H(\nu,t)^{1-\beta}\, d\nu H^{\beta}.$$

- Why this functional form? Retains the simplicity of the earlier model, while allowing for studying the direction of technological change.
- Intermediate machines $x_H(\nu,t), x_L(\nu,t)$ produced by monopolists with perpetual patent.
- Normalize the price of the final good to one, and let $p_H(t), p_L(t)$ denote the price of the H and L-intensive intermediates.
- Definition of equilibrium in lecture notes. I will focus on characterization of equilibrium.

The usual approach: First solve the static allocations and prices. In particular, for a

Static Allocation: Intermediate Good Sectors and Monopolists

- Consider the H-intensive sector, which solves:

$$\max_{[x_H(\nu,t)]_t, H} p_H(t) \frac{1}{1-\beta} \int_0^{N_H(t)} x_H(\nu,t)^{1-\beta} d\nu H^\beta - \int_0^{N_H(t)} p(\nu,t) x_H(\nu,t) d\nu - w_H(t) H.$$

- The demand for machine ν in sector H is given by:

$$p_H(t) x_H(\nu,t)^{-\beta} H^\beta = p(\nu,t). \tag{17}$$

This is isoelastic. Leads to the monopoly price:

$$p(\nu,t) = \frac{\psi}{1-\beta} = 1$$

(where last line uses the usual normalization).

- Moreover, using Eq. (17), the output of the monopolist is given by:

$$x_H(\nu,t) = p_H(t)^{1/\beta} H. \tag{18}$$

- The total profits of monopolists are then given by:

$$\pi_L(\nu,t) = (p(\nu,t) - (1-\beta)) x_H(\nu,t) = \beta p_H(t)^{1/\beta} H. \tag{19}$$

- Note the presence of prices. A similar analysis for the L-intensive sector, shows symmetric expressions

Static Allocation: Final Good Sector

- Use Eq. (18) in the expression for $Y_H(t)$ to get:

$$p_H(t) Y_H(t) = p_H(t)^{1/\beta} \frac{1}{1-\beta} H N_H(t). \tag{20}$$

This also leads to

$$X_H(t) = p_H(t)^{1/\beta} (1-\beta) H N_H(t)$$

and

$$w_H(t) = p_H(t)^{1/\beta} \frac{\beta}{1-\beta} N_H(t). \tag{21}$$

- The final good firms also choose $Y_H(t)$ and $Y_L(t)$ optimally, which is equivalent to the following:
  - Relative demand for intermediate goods is isoelastic:

$$\frac{Y_H(t)}{Y_L(t)} = \frac{\gamma_H}{\gamma_L} \left(\frac{p_H(t)}{p_L(t)} \right)^{-\varepsilon} \tag{22}$$

  - The ideal price index over $p_H(t)$ and $p_L(t)$ is equal to the price of the final good (which is equal to one):

$$\left(\gamma_H^{\varepsilon} p_H(t)^{1-\varepsilon} + \gamma_L^{\varepsilon} p_L(t)^{1-\varepsilon} \right)^{1/(1-\varepsilon)} = 1. \tag{23}$$

Static Allocation: Determination of Prices

- Use Eq. (20) in Eq. (22) to get:

$$\left(\frac{p_H(t)}{p_L(t)}\right)^{(1-\beta)/\beta} \frac{HN_H(t)}{LN_L(t)} = \frac{\gamma_H}{\gamma_L}\left(\frac{p_H(t)}{p_L(t)}\right)^{-\varepsilon}.$$

  Note that prices appear on both sides. This is why the derived elasticity business comes in.

  - Relative price of intermediate goods tends to decrease in the relative effective labor (because more effective labor produces more intermediates: the usual supply effect). This is captured by the price term on the right.
  - But the intermediate good production is not only determined by the effective supply of labors in each sector. It also uses machine inputs. The level of these inputs (Eq. (18)) responds to the price of intermediates. This is captured by the price term on the left.

- Rearranging the previous equation, we have

$$\frac{p_H(t)}{p_L(t)} = \left(\frac{\gamma_H}{\gamma_L}\right)^{\varepsilon\beta/\sigma}\left(\frac{HN_H(t)}{LN_L(t)}\right)^{-\beta/\sigma}, \quad (24)$$

  where $\sigma = 1 + (\varepsilon - 1)\beta$. Note that $\sigma > 1$ iff $\varepsilon > 1$.

- That is, the direct (supply) effect prevails and the relative prices for intermediates are decreasing in the relative effective supply of factors.

- Note that Eq. (24) characterizes relative prices $p_H(t), p_L(t)$. Combining this expression with the ideal price index in Eq. (23) solves for $p_H(t)$ and $p_L(t)$. Using these prices, the above analysis characterizes all static equilibrium allocations in terms of $N_H(t)$ and $N_L(t)$.
- Using Eq. (21) for, Eq. (24) also implies

$$\frac{w_H(t)}{w_L(t)} = \left(\frac{\gamma_H}{\gamma_L}\right)^{\varepsilon/\sigma} \left(\frac{N_H(t)}{N_L(t)}\right)^{(\sigma-1)/\sigma} \left(\frac{H}{L}\right)^{-1/\sigma}. \tag{25}$$

- Note that this is the same expression we had earlier (in reduced form). This is why σ is called the derived elasticity.
- However, note that the technology levels in this model are endogenous. This is a major advance in terms of the economics of the model.

Dynamic Allocation: Trade-off

- Next consider the dynamic evolution of $N_H(t), N_L(t)$. Innovation frontier:

$$\dot{N}_H(t) = \eta_H Z_H(t), \text{ and } \dot{N}_L(t) = \eta_L Z_L(t)$$

- The dynamic trade-off is (as usual) about how to allocate output between consumption $C(t)$ and investment $Z_H(t), Z_L(t)$. The dynamic trade-off (loosely speaking) can be broken into two sub-problems:
  - Given a level of investment, how to divide it across the two sectors, i.e., determination of $Z_H(t)/Z_L(t)$.
  - How much to invest and how much to consume, i.e., determination of levels of $Z_H(t)$ and $Z_L(t)$.
- The resolution of the second trade-off will be very similar to the baseline case (with some additional algebra). Not our focus here, so we postpone it to the end. Our focus in this model is on the first trade-off, which we turn to next.

Dynamic Allocation: Free Entry Conditions

- The free entry conditions into the H and L sectors is given by:

$$\eta_H V_H(t) \leq 1 \text{ with inequality only if } Z_H(t) = 0 \qquad (26)$$
$$\eta_H V_H(t) \leq 1 \text{ with inequality only if } Z_L(t) = 0.$$

- The HJB equations for $V_H(t)$ and $V_L(t)$ are given by:

$$r(t) V_H(t) = \pi_H(t) + \dot{V}_H(t), \qquad (27)$$
$$r(t) V_L(t) = \pi_L(t) + \dot{V}_L(t).$$

- Consider a BGP in which the the interest rate $r(t)$ and the prices $p_H(t), p_L(t)$ remain constant. By Eq. (24), this means that $N_H(t)/N_L(t)$ remains constant. This can only be constant if there is positive entry into both sectors. Thus, both conditions in (26) are satisfied with equality. Using this in (27), and substituting for profits from Eq. (19) shows

$$V_H = \frac{\beta p_H^{1/\beta} H}{r^*} \text{ and } V_L = \frac{\beta p_L^{1/\beta} L}{r^*}. \qquad (28)$$

- Dividing the value functions in (28) gives an expression for the relative values that determine the **direction** of the technological change:

$$\frac{V_H}{V_L} = \frac{H}{L}\frac{p_H^{1/\beta}}{p_L^{1/\beta}} = \left(\frac{\gamma_H}{\gamma_L}\right)^{\varepsilon\beta/\sigma}\frac{H}{L}\left(\frac{N_H H}{N_L L}\right)^{-1/\sigma}.$$

Here, the second equality substitutes relative prices from Eq. (24).

- There is a **market size effect** (as before) and a **price effect** (new). Intuition?

- Using Eq. (26) to set V_H and V_L equal to marginal cost of innovation, BGP level of relative technologies is determined by:

$$\left(\frac{N_H}{N_L}\right)^* = \left(\frac{\eta_H}{\eta_L}\right)^\sigma \left(\frac{\gamma_H}{\gamma_L}\right)^{\varepsilon\beta} \left(\frac{H}{L}\right)^{\sigma-1}. \tag{29}$$

- Recall also the equation for relative wages in equilibrium (from Eq. (25)):

$$\left(\frac{w_H}{w_L}\right)^* = \left(\frac{\gamma_H}{\gamma_L}\right)^{\varepsilon/\sigma} \left(\left(\frac{N_H}{N_L}\right)^*\right)^{(\sigma-1)/\sigma} \left(\frac{H}{L}\right)^{-1/\sigma}. \tag{30}$$

- These two expressions prove the weak and the strong equilibrium bias results.
- **Weak bias result:** An increase in H/L always leads to an H-biased technological change.
  For proof, use Eqs. (29) and (30), and consider cases $\sigma > 1$ and $\sigma < 1$ separately.
- **Strong bias result:** When $\sigma > 2$, the weak bias force is so strong that an increase in H/L increases the BGP relative prices $(w_H/w_L)^*$.

Weak and Strong Equilibrium Bias Results: Intuition

- The deeper intuition for the biased technological change is that equilibrium features an optimality property. If a factor becomes more abundant, then it is more efficient to do R&D that will increase the relative marginal productivity of that factor (controlling for the costs of R&D). This is what the equilibrium does.
- This intuition also suggests that the results do not depend on the CES form. Acemoglu (2007, EMA) shows that this is indeed the case: these results naturally generalize to the case in which production function is given by $F(N_H H, N_L L)$.

- Let us next complete the characterization of the BGP equilibrium, by determining the *level* of investment and the growth rate.
- Note that both Eqs. in (26) are satisfied with equality. Thus,

$$\eta_H \beta p_H^{1/\beta} H = r^* \text{ and } \eta_L \beta p_L^{1/\beta} L = r^*.$$

- Solve for p_H and p_L and substitute these expressions into the ideal price index equation (23) to get:

$$\gamma_H^{\varepsilon} \left(\frac{r^*}{\eta_H \beta H} \right)^{\beta(1-\varepsilon)} + \gamma_L^{\varepsilon} \left(\frac{r^*}{\eta_L \beta L} \right)^{\beta(1-\varepsilon)} = 1.$$

From here solve for r^* as:

$$r^* = \beta \left(\gamma_H^{\varepsilon} (\eta_H H)^{(\sigma-1)} + (1-\gamma) (\eta_L L)^{\beta(\varepsilon-1)} \right)^{1/(\sigma-1)} \quad (31)$$

(where we have also used $\sigma - 1 = \beta(\varepsilon - 1)$).

- Note that Eq. (31) is similar to the baseline version: The interest rate must be at a particular level to ensure free-entry into research. In view of Euler equation, this interest rate will dictate consumers' consumption/saving decision (and thus the investment level, and the growth rate of consumption).
- In particular (under appropriate parametric assumptions that ensure positive growth and the transversality condition) the BGP growth rate is given by:

$$g^* = \frac{1}{\theta}(r^* - \rho) = \frac{1}{\theta}\left(\beta\left(\gamma_H^{\varepsilon}(\eta_H H)^{(\sigma-1)} + (1-\gamma)(\eta_L L)^{\beta(\varepsilon-1)}\right)^{1/(\sigma-1)} - \rho\right). \quad (32)$$

- What is the level of investment? On a BGP, given the innovation equation $\dot{N}_H(t) = \eta_H Z_H(t)$, we need to invest $Z_H(t) = g^* \frac{N_H(t)}{\eta_H}$ (to ensure that $N_H(t)$ grows at rate g^*). Similarly, $Z_L(t) = g^* \frac{N_L(t)}{\eta_L}$. Hence, aggregate investment on a BGP is:

$$Z_H(t) + Z_L(t) = g^* \left(\frac{N_H(t)}{\eta_H} + \frac{N_L(t)}{\eta_L}\right).$$

Given this aggregate investment, initial level of consumption can be calculated as the residual of net output. Check that all equilibrium objects are characterized.

Transitional Dynamics

- The above described BGP allocation is an equilibrium if initial level of relative technology $N_H(t)/N_L(t)$ satisfies Eq. (29).
- If not, there are *transitional dynamics*. Recall that, on a BGP:

$$\frac{V_H}{V_L} = \left(\frac{\gamma_H}{\gamma_L}\right)^{\varepsilon\beta/\sigma} \frac{H}{L} \left(\frac{N_H H}{N_L L}\right)^{-1/\sigma}.$$

- If $N_H(0)/N_L(0) < (N_H/N_L)^*$, then innovation incentives are stronger for H sector. Hence,for a while, there is only innovation on H-intensive intermediate goods. As $N_H(t)$ increases and $N_L(t)$ remains constant, eventually $N_H(t)/N_L(t) = (N_H/N_L)^*$. From this point onwards, the economy is on the BGP and there is innovation on both sectors.
(This analysis is loose, because we are no longer on a BGP, and thus previous displayed equation does not apply exactly. But the intuition carries to a rigorous analysis.)

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