14.452 Recitation Notes:
1. Neoclassical Model in Discrete Time
2. Inefficiency in the OLG Model
3. Rebelo's Two Sector $AK$ Model

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## Neoclassical Model in Discrete Time

- Consider the neoclassical model in discrete time with no population growth and no technological progress. Periods denoted by $t \in \{0, 1, .., \infty\}$.
- Preferences of representative household:

$$\sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right)$$

- Technology:

$$Y\left(t\right) = F\left(K\left(t\right), L\left(t\right)\right) \text{ and} \tag{1}$$
$$K\left(t+1\right) = \left(1 - \delta\right) K\left(t\right) + I\left(t\right).$$

- Aggregate resource constraint:

$$C\left(t\right) + K\left(t+1\right) \leq F\left(K\left(t\right), L\left(t\right)\right) + \left(1 - \delta\right) K\left(t\right).$$

In per-capita terms:

$$c\left(t\right) + k\left(t+1\right) \leq f\left(k\left(t\right)\right) + \left(1 - \delta\right) k\left(t\right). \tag{2}$$

- Equilibrium is a path of per-capita allocations $\{a(t), c(t), k(t)\}_{t=0}^{\infty}$ and prices $\{R(t), w(t), r(t)\}_{t=0}^{\infty}$ such that:

  ① Household solves

  $$\max_{\{c(t),a(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t)) \tag{3}$$

  s.t. $a(t+1) = (1+r(t))\,a(t) + w(t) - c(t)$ for each $t$,

  $$\lim_{t\to\infty} a(t) \prod_{s=0}^{t-1} \frac{1}{1+r(s)} \geq 0. \text{ (No Ponzi condition)}$$

  ② Competitive firms maximize. Using factor market clearing, this is equivalent to:

  $$R(t) = f'(k(t)),\ w(t) = f(k(t)) - k(t)\,f'(k(t)),$$

  and $1 + r(t) = R(t) + 1 - \delta$, for each $t$.

  ③ Factor, good and asset markets clear, i.e., the resource constraint in (2) satisfied with equality and

  $$k(t) = a(t) \text{ for each } t.$$

- Eliminate $c(t)$ from Problem (3) to get:

$$\max_{\{a(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left((1+r(t))a(t) + w(t) - a(t+1)\right) \tag{4}$$

$$\text{s.t.} \quad \lim_{t \to \infty} a(t) \prod_{s=0}^{t-1} \frac{1}{1+r(s)} \geq 0.$$

- This fits into the structure of the sequence problem in Chapter 6. Can write it in recursive form and invoke dynamic programming results. For our purposes, there is a simpler approach:

- Theorem 6.10 (Sufficiency Theorem for discrete time): Under standard assumptions (most importantly convexity of choice set and concavity of the objective function) an interior allocation that satisfies the Euler equation and the transversality condition is optimal.

- The Euler equation (first order condition) for Problem (3) is given by:

$$u'(c(t)) = \beta (1 + r(t+1)) u'(c(t+1)) \text{ for each } t. \tag{5}$$

(Note: the previous can be written as an equation only in terms of $\{a(t)\}_{t=0}^{\infty}$, i.e., we can always substitute $c(t) = (1 + r(t)) a(t) + w(t) - a(t+1)$. I use $c(t)$ notation to give a better interpretation to the equation).

- The transversality condition (TC) adapted to Problem (4) is given by (cf. Eq. (6.31) of the book):

$$\lim_{t \to \infty} \beta^t u'(c(t)) (1 + r(t)) a(t) = 0.$$

- We can simplify this further to get an equation with better interpretation. Using the Euler equation

$$u'(c(s+1)) = \frac{u'(c(s))}{\beta(1 + r(s+1))} \text{ for each } s \in \{0, 1.., t-1\}.$$

Multiply these equations to get

$$u'(c(t)) = u'(c(0)) \frac{1}{\beta^t} \prod_{s=0}^{t-1} \frac{1}{1 + r(s+1)}.$$

- Plug in the last equation into TC and get rid of the constant $u'(c(0))$ to get

$$\lim_{t \to \infty} a(t) \prod_{s=0}^{t-1} \frac{1}{1+r(s)} = 0. \qquad (6)$$

- This is the No-Ponzi condition satisfied with equality. Intuition?
- The optimal sequence $\{a(t)\}_{t=0}^{\infty}$ is characterized by the Euler equation

$$u' \left( \begin{array}{c} (1+r(t))\,a(t) + w(t) \\ -a(t+1) \end{array} \right) \qquad (7)$$

$$= \beta(1+r(t+1))\,u' \left( \begin{array}{c} (1+r(t+1))\,a(t+1) + w(t) \\ -a(t+2) \end{array} \right) \text{ for each } t,$$

along with the initial condition $a(0)$ and the transversality condition (6).

- Combine household maximization with market clearing *to eliminate prices*. The Euler equation leads to:

$$u'\left(c\left(t\right)\right) = \beta\left(f'\left(k\left(t+1\right)\right)+1-\delta\right)u'\left(c\left(t+1\right)\right) \text{ for each } t, \qquad (8)$$

Consumer's budget constraint leads to the resource constraint

$$c\left(t\right)+k\left(t+1\right) = f\left(k\left(t\right)\right)+\left(1-\delta\right)k\left(t\right) \text{ for each } t.$$

Transversality condition leads to:

$$u'\left(c\left(0\right)\right)\lim_{t\to\infty}k\left(t\right)\prod_{s=0}^{t-1}\frac{1}{f'\left(k\left(s\right)\right)+1-\delta} = 0. \qquad (9)$$

- After eliminating $c\left(t\right)$ (using the budget constraint), the sequence $\{k\left(t\right)\}_{t=1}^{\infty}$ is characterized by Euler equations (8) for each $t \in \{0,1,..\}$, along with the initial condition $k\left(0\right)$ and end-value condition (9).

## Social Planner's Problem

- Consider next the social planner's problem (also known as the optimal growth problem in this context, see Section 6.9)

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right) \qquad (10)$$

$$\text{s.t. } k\left(t+1\right) = f\left(k\left(t\right)\right) + \left(1-\delta\right)k\left(t\right) - c\left(t\right) \text{ and } k\left(t\right) \geq 0.$$

- Similar steps to above show that the solution to this problem is also *characterized* by Eqs. (8) for each $t$, along with the initial condition $k\left(0\right)$ and the end-value condition (9).

- This establishes the Pareto optimality of equilibrium. In addition, we can use use the social planner's problem to *characterize the properties* of the solution to the above system (and thus the properties of equilibrium).

- The problem in (10) has a recursive form. Using dynamic programming results (of Chapter 6), one can show that the solution $\{c\left(t\right), k\left(t\right)\}_{t=0}^{\infty}$ converges to the steady state $\left(c^*, k^*\right)$ characterized by

$$\beta\left(f'\left(k^*\right) + 1 - \delta\right) = 1$$
$$c^* = f\left(k^*\right) - \delta k^*.$$

- One can also establish that $\left(c\left(t\right), k\left(t\right)\right)$ monotonically converges to $\left(c^*, k^*\right)$ (see Propositions 6.3 and 6.4).

- Consider the OLG model with log utilty, no technological progress, Cobb-Douglas technology, full depreciation. Capital accumulation characterized by

$$k\left(t+1\right) = \frac{1}{1+n}\frac{\beta}{1+\beta}\left(1-\alpha\right)\left(k\left(t\right)\right)^{\alpha}$$

and the steady state capital labor ratio is given by

$$k^{*} = \left[\frac{\beta\left(1-\alpha\right)}{\left(1+n\right)\left(1+\beta\right)}\right]^{1/(1-\alpha)}. \tag{11}$$

The steady state consumption of each generation is given by

$$c_{1}^{*} = \frac{1}{1+\beta}w^{*} \text{ and } c_{2}^{*} = \frac{\beta}{1+\beta}w^{*}R^{*},$$

$$\text{where } w^{*} = \left(1-\alpha\right)\left(k^{*}\right)^{\alpha} \text{ and } R^{*} = 1 + r^{*} = \alpha\left(k^{*}\right)^{\alpha-1}.$$

## Dynamic Inefficiency

- Consider a steady state allocation (not necessarily equilibrium) in which capital labor ratio is constant at $\tilde{k}$ and all generations consume $(\tilde{c}_1, \tilde{c}_2)$. The resource constraints in per-capita terms imply

$$\tilde{c}_1 + \frac{\tilde{c}_2}{1+n} = f\left(\tilde{k}\right) - (1+n)\tilde{k}. \tag{12}$$

- Define $k^{gold}$ as the level that maximizes net resources on a steady state allocation

$$k^{gold} = \arg\max_{\tilde{k}} f\left(\tilde{k}\right) - (1+n)\tilde{k},$$

i.e. $k^{gold} = f'^{-1}(1+n)$.

- Suppose $k^*$ in Eq. (11) is such that $k^* > k^{gold}$ (note: $k^* = (f')^{-1}(1+n) > k^{gold} = f'^{-1}(1+r^*)$ is equivalent to $r^* < n$).

- Suppose $k(0) = k^*$ (we are on a steady state). As a social planner, consider the deviation: drop capital stock to $k^{gold}$ and keep it at $k^{gold}$ starting for all $t \geq 1$. This yields net resources at period 0

$$f(k^*) - k^{gold}(1+n) > f(k^*) - k^*(1+n)$$

and net resources in (12) (evaluated for $\tilde{k} = k^{gold}$) at all periods. Thus, this deviation increases net resources at all periods!

- It follows that Pareto improvements are possible and the OLG equilibrium when $r^* < n$.

## The Technical Explanation for Dynamic Inefficiency

- Why the inefficiency is surprising: The OLG economy can be mapped into a standard Arrow-Debreu economy with complete markets, no externalities in preferences or technology. Hence, intuitively, we would expect from the First Welfare Theorem that it should be efficient. But...
- Technical explanation: FWT reasoning fails for the OLG economy. In the proof of the FWT, there is one step where the sum of the value of all equilibrium endowments (evaluated at equilibrium prices) must be compared to the sum of the value of an alternative allocation. This comparison yields a contradiction and proves the FWT. The comparison cannot be done when sums are infinite.
- The endowments in the OLG economy are labor for each generation. In the steady state equilibrium, the Arrow-Debreu prices for period $t$ good is $w^*/(1+r^*)^t$, hence the sum of all endowments is

$$\sum_{t=0}^{\infty} \frac{w^*(1+n)^t}{(1+r^*)^t}.$$

  This is finite if $n < r^*$, which proves that the OLG equilibria are Pareto efficient when there is no dynamic inefficiency. But the sum is infinite if $n \geq r^*$.
- This doesn't prove that the OLG economy with $n > r^*$ is Pareto inefficient, but shows that the standard proof doesn't apply. In fact, we have shown above, by construction, that the equilibrium is inefficient when $r^* < n$. So it is not just a technical curiosity.

## The Semi-Technical Explanation for Dynamic Inefficiency

- Source: pecuniary (price) externalities. Illustration: Suppose I am a bread producer. When I produce less bread, I have small (positive) effect on the price of bread. This has a negative effect on bread consumers and positive effect on other bread producers (or people endowed with bread). But these effects are present in any Walrasian economy. They typically do not cause inefficiencies: with complete markets and no non-pecuniary (technological or preference) externalities, the welfare effect from the change in prices *can be written* as (see Greenwald-Stiglitz (1986))

$$\sum_{\text{good}} (\text{Total production(good) - Total consumption(good)}) \times \text{Marginal price change}$$

From market clearing, we have production=consumption. So when there are finitely many goods with finite production, welfare effect is 0. This is the sense in which pecuniary externalities typically "net out". This is the content of the first welfare theorem, from a different perspective.

- But this reasoning is fragile when there are infinitely many goods. In this case, these pecuniary externalities don't necessarily net out. Consider (loosely)

$$\sum_{t=0}^{\infty} (2-2) = 2 + \sum_{t=1}^{\infty} 2 - \sum_{t=0}^{\infty} 2 = 2 + \sum_{t=1}^{\infty} 2 - \sum_{t=1}^{\infty} 2 = 2 + \sum_{t=1}^{\infty} (2-2)$$

$$0 = 2$$

what is wrong with this algebra?!

# The Economic Explanation

- The pecuniary externalities in the OLG economy: Savings by current young increases wages and decreases rental rate of capital tomorrow (which is also the gross interest rate received on savings). Thus, savings by the current young generation has a negative externality on the same generation and a positive externality on the next period's young generation.

- As explained above, in an infinite economy, negative externalities may dominate the positive externalities, and thus saving by the young people may make all generations worse off... and it does, when the economy is dynamically inefficient. There is overaccumulation of capital.

- This reasoning also points to potential solutions.

## How to Restore Efficiency: Social Security

- Funded social security: government invests savings on behalf of you and gives you your fair returns when you are old. This has no effect in our framework. You choose your personal savings exactly to offset government and save the same amount in total.

- Unfunded social security: take from current young, give to current old. Tomorrow's old receives receipts from tomorrow's young etc. This scheme reduces total savings. To see this, suppose current young receives wages $w^*$ and faces interest rate $1 + r^* < 1 + n$. Suppose the government introduces social security scheme that makes every young generation contribute $d(t) = d^*$. Per-capita savings in this economy change from $s^* = w^* - c_1^*$ to $s(0) = w^* - d^* - c_1(0)$. So:

  1. There is a direct effect. Government takes from savers (young people), gives it to consumers (old people), which reduces savings. Unless young people cut back on current consumption by a lot (i.e. $c_1(0)$ much lower than $c_1^*$), this will decrease and capital accumulation.

  2. There is also an indirect wealth effect that works in the same direction. Since young receive return $n > r^*$ from their contribution to social security, they have a greater overall budget, i.e. there is a wealth effect. Since utility is separable, consumption in both periods is a normal good hence (at the margin, before interest rate adjusts) consumption today must increase in response to a wealth effect. So the direct effect in 1 is reinforced (rather than overturned).

- National debt (Diamond, AER 1965): Suppose the government borrows from current young $d(t) = d^*$ at market interest rate and pays back next period by borrowing also $d(t) = d^*$ from the next generation. Whenever $r(t+1) < n^*$, this scheme will generate some extra revenues at time $t+1$ which the government redistributes to the old generation at time $t+1$ through lump sum transfers. This economy is similar to the economy with unfunded social security. Intuitively, government takes away from the wages of the current young (and does not invest it) which directly reduces savings.

- Fiat money (Samuelson, JPE 1958) or bubbles in general. Normalize $L(0) = 1$ so $L(t) = (1+n)^t$. Consider an intrinsically worthless asset, which we call bubbly, that has a fixed (one unit) supply and consider the steady state equilibrium of the OLG economy with bubbly. We conjecture a steady state equilibrium on which the price of bubbly evolves according to

$$p(t) = p(0)(1+n)^t$$

for some $p(0)$, and an individual of generation $t$ holds $1/(1+n)^t$ units of bubbly. Note that the market for bubbly clears (demand at each period=supply=1). Note that the individuals save both in capital and bubbly, hence (at an interior solution) must receive same return from both. This implies that the steady state capital-labor ratio is $k^{gold} = f'^{-1}(1+n)$. Note also that, the total savings of individuals in bubbly is $p(t) \times 1/(1+n)^t = p(0)$, hence by adjusting $p(0)$ we can adjust the residual savings of individuals that go to capital sector. There is a level of $p(0)$ that makes each individual invest exactly $k^{gold}(1+n)$ in capital, proving that there is an equilibrium for this level of $p(0)$.

Note that due to dynamic inefficiency, bubbly returns more than capital at the old steady state ($r^* < n$) and thus will be in positive demand in the new steady state (i.e. $p(0) > 0$ and there is an interior solution).

A generation holds the asset to sell it to another generation. Yes, this is speculative trade but of the good kind!

- Based on: Sergio Rebelo, "Long Run Policy Analysis and Long Run Growth", JPE 1991. Also covered in Section 11.2 of "Introduction to Modern Economic Growth".
- Goal: Build an $AK$ model in which share of capital and labor remains constant as economy grows.
- Method: assume capital and consumption sectors have different production technologies. What is essential for sustained growth is linear production in capital producing sector. Consumption good producing sector could use scarce factors (such as labor or land), hence share of labor in output can remain constant despite growth driven by capital accumulation.

- Capital producing sector

$$I(t) = AK_I(t) \tag{13}$$

$$\dot{K}(t) = I(t) - \delta K(t). \tag{14}$$

- Consumption sector ($\alpha = 1$ is like baseline $AK$ model, here we allow for $\alpha < 1$)

$$C(t) = BK_C(t)^\alpha L(t)^{1-\alpha} \tag{15}$$

- Market clearing in capital

$$K_I(t) + K_C(t) \leq K(t)$$

- We need prices for investment and consumption goods $p_I(t), p_C(t)$. We will take $p_C(t) = 1$ as normalization. Wage rate given by $w(t)$. Loans (assets), are denominated in consumption good, so we will denote rate of return on assets by $r_C(t)$.

- Let $K_C(t) = (1 - \kappa(t)) K(t)$ and $K_I(t) = \kappa(t) K(t)$, where $\kappa(t)$ is share of capital used in investment sector.

- Is a path of $\{K(t), \kappa(t), C(t), p_I(t), r_C(t), w(t)\}_{t=0}^{\infty}$ such that
  1. Household solves

  $$\max_{\{C(t), A(t)\}_{t=0}^{\infty}} \int_0^{\infty} \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt$$

  s.t. $\dot{A}(t) = r_C(t) A(t) + w(t) - C(t)$ and $\lim_{t \to \infty} \exp\left(\int_0^t -r(s) ds\right) A(t) \geq 0$.

  2. Competitive firms maximize, which by production technologies in (13) and (15) gives:

  $$r_C(t) = A - \delta + \frac{\dot{p}_I(t)}{p_I(t)} \tag{16}$$

  [Consumption can be converted to investment good, which can then be converted to $A - \delta$ units of next period capital by the linear technology (a portion of capital depreciates), which can then be converted to next period consumption. During the conversion from consumption to investment good and then back to consumption good next period, the asset loses value if investment good is getting cheaper.]

  $$p_I(t) A = \alpha B \left(\frac{L}{(1 - \kappa(t)) K(t)}\right)^{1-\alpha} \tag{17}$$

  (by equating marginal product of capital in two sectors)

  3. Asset and good markets clear.
     $K_I(t) + K_C(t) = K(t)$
     $K(t) p_I(t) = A(t)$ (note that assets are denominated by the consumption good)

## Equilibrium Characterization

- We conjecture a steady-state allocation and prove that the allocation is indeed an equilibrium.
- Conjecture: $\kappa(t) = \kappa^*$ is constant. In any such equilibrium, growth of $K(t)$ is also constant by capital accumulation equation. Denote this by $g_K$.
- Differentiating Eq. (17) gives

$$\frac{\dot{p}_I}{p_I} = -(1-\alpha)\, g_K.$$

Intuition?

- Plug this in Eq. (16) which gives

$$r_C(t) = r_C^* \equiv A - \delta - (1-\alpha)\, g_K.$$

By Euler equation

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left(A - \delta - (1-\alpha)\, g_K - \rho\right). \tag{18}$$

- Next, differentiate the production of the consumption good Eq. (15), to get:

$$\frac{\dot{C}}{C} = \alpha \frac{\dot{K}_C}{K_C} = \alpha g_K. \tag{19}$$

Intuition?

- Combining Eqs. (18) and (19), we uniquely solve for $g_K$ as:

$$g_K = \frac{A - \delta - \rho}{1 - \alpha + \alpha\theta} \text{ and } g_C = \alpha g_K. \tag{20}$$

- From Eq. (14), we can solve for $\kappa^*$ uniquely as

$$\kappa^* = \frac{g_K + \delta}{A}.$$

- We also need $g_K < r_I^*$ to satisfy the transversality condition. In terms of parameters

$$\frac{A - \delta - \rho}{1 - \alpha + \alpha\theta} < A - \delta,$$

(Note that this also ensures $\kappa^* \in [0, 1]$).

- Then, the path in which $\kappa(t) = \kappa^*$ for all $t$ (and the rest of the variables determined uniquely as above) is an equilibrium under the above parametric restriction. (Check that this path satisfies all of the equilibrium conditions).

- Constant growth at all times. No transitional dynamics.
- We can also define the total "output" as the value of the consumption and investment production"

$$Y(t) = C(t) + p_I(t) I(t).$$

Check that output grows at the same rate as consumption, i.e., $g_Y = g_C$ (which is equal to $\alpha g_K$).

- Growth rate in (20) responds to policy as in the standard $AK$ model.
- What happens to wages: Note $w(t) = \beta (K_C(t)/L)^\alpha$, hence

$$\frac{\dot{w}(t)}{w(t)} = \alpha g_K = g_C.$$

Thus, wages grow at the same rate as consumption (and output). Share of labor is constant, which is different than the standard $AK$ model.

- There is, however, capital deepening. $K(t)$ grows faster than output $(Y(t) = C(t))$. Not consistent with Kaldor facts strictly read. But is this a problem?

14.452 Economic Growth
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