February 1984

LIDS-P-1362

THE SCHUR ALGORITHM AND ITS APPLICATIONS

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*The work of this author was supported by the Exxon Education Foundation.

**The work of this author was supported by the Air Force Office of Scientific Research under Grant AFOSR-82-0135A.

ABSTRACT

The Schur algorithm and its times-domain counterpart, the fast Cholseky recursions, are some efficient signal processing algorithms which are well adapted to the study of inverse scattering problems. These algorithms use a layer stripping approach to reconstruct a lossless scattering medium described by symmetric two-component wave equations which model the interaction of right and left propagating waves. In this paper, the Schur and fast Cholesky recursions are presented and are used to study several inverse problems such as the reconstruction of nonuniform lossless transmission lines, the inverse problem for a layered acoustic medium, and the linear least-squares estimation of stationary stochastic processes. The inverse scattering problem for asymmetric two-component wave equations corresponding to lossy media is also examined and solved by using two coupled sets of Schur recursions. This procedure is then applied to the inverse problem for lossy transmission lines.

I. Introduction

The Schur algorithm [1] - [2] is a fast algorithm well-suited to high-speed data processing. This algorithm is obtained by using a layer stripping procedure to reconstruct a lossless scattering medium described by symmetric two-component wave equations. Two-component wave equations describe many real-world physical phenomena, and the purpose of this paper is to examine several inverse problems which can be solved efficiently by using the Schur algorithm. The problems that we will consider include the reconstruction of nonuniform lossless transmission lines, the inverse problem for an acoustic layered medium and the linear least-squares estimation of stationary stochastic processes.

In addition, we will consider the case when the scattering medium that we want to reconstruct is lossy and is described by asymmetric two-component wave equations. It will be shown that in this case the medium can be reconstructed by using coupled Schur recursions, and this procedure will be illustrated for the case of nonuniform lossy transmission lines.

The Schur algorithm has a breadth of applications which is nothing short of astonishing. It is used for example in the Schur-Cohn test [3] for checking the stability of discrete-time polynomials, and in the Darlington procedure [3], [4] for synthesizing an impedance function by a cascade of elementary lossless sections of degree one terminated by a resistor. The fast Cholesky recursions, which constitute the time-domain version of the Schur algorithm, may be used to obtain a lower triangular-diagonal-upper triangular (LDU) factorization of a Toeplitz matrix [5] - [7]. The continuous parameter version of this algorithm, which will be employed throughout this paper, similarly performs a causal-anticausal factorization of a Toeplitz operator [8].

This paper is organized as follows. In Section II, we consider the problem of reconstructing a lossless scattering medium described by symmetric two-component wave equations and we show that the medium can be reconstructed layer by layer by using the fast Cholesky recursions or the Schur algorithm. This procedure is then used in Section III to solve the inverse problem for nonuniform lossless transmission lines, and scattering concepts such as those of impedance, reflection coefficient, right and left propagating waves are illustrated. In Section IV the inverse problem for an acoustic layered medium is examined, and the Schur algorithm (which is known in this context as the dynamic deconvolution algorithm [9]) is used to reconstruct the medium parameters from scattering data. The data are obtained by probing the medium with plane waves, first at normal incidence and then at non-normal incidence. In Section V, the problem of linear least-squares estimation of a stationary stochastic process in white noise is discussed. It is shown that the forwards and backwards estimation residuals satisfy two component wave equations and that the lattice estimation filter obtained by discretizing these equations can be constructed from the spectral function of the observed process by using the Schur algorithm. In Section VI, the inverse problem for a lossy scattering medium is considered, and is solved by using two coupled Schur recursions. As an example of this result, the inverse problem for lossy nonuniform transmission lines is examined. Section VII contains some conclusions and suggests topics for future research.

Notation

Unless specifically identified otherwise, all variables are scalar. Fourier transforms will be designated by a carat, e.g. $\hat{x}$, and partial derivatives by subscripts: $\frac{d^2 f}{dxdt} = f_{xt}$. Dependent variables will generally be omitted for brevity.

II. The Continuous-Time Schur and Fast Cholesky Algorithms

In this section, we quickly review the continuous-time Schur and fast Cholesky algorithms. No derivations will be attempted since all results can be found in the given references. We consider a lossless scattering medium described by the symmetric two-component wave equations

$$p_x + p_t = -r(x)q(x,t) \tag{1a}$$

$$q_x - q_t = -r(x)p(x,t) \tag{1b}$$

which constitute a special case of the equations discussed in [10] - [11]. Here $r(x)$ is the reflectivity function and $p(x,t)$ and $q(x,t)$ are the rightward and leftward propagating waves in the medium at point x and time t. Note that if $r(x) \equiv 0$ over some interval, then

$$p = p(x-t) \qquad q = q(x+t) \tag{2}$$

over the interval, so that p and q correspond effectively to waves propagating rightward and leftward with unit velocity.

In the following, it will be assumed that $r(x) \equiv 0$ for $x<0$ and that $r \in L_1[0,\infty)$, so that for $x<0$ and as $x\to\infty$, $p(x,t)$ and $q(x,t)$ have the form (2).

The Scattering Matrix

By taking the Fourier transform of (1), we obtain

$$\frac{d}{dx}\begin{bmatrix}\hat{p}\\ \hat{q}\end{bmatrix} = \begin{bmatrix}-j\omega & -r(x)\\ -r(x) & j\omega\end{bmatrix}\begin{bmatrix}\hat{p}\\ \hat{q}\end{bmatrix} \tag{3}$$

and a simple discretization of x in (3) gives the elementary scattering section described in Figure 1. This figure shows that $r(x)\Delta$ is the fraction of the rightgoing wave $\hat{p}$ which is reflected by a section of thickness Δ at depth x inside the medium. The discrete ladder structure displayed by Figure 1 has been used to design signal processing architectures for speech processing [12], digital wave-filter synthesis [13], spectral estimation [14] and linear estimation [4], [15] - [16].

The elementary scattering layers of Figure 1 can be composed by using the rules of composition for scattering layers described in Redheffer [17]. The resulting aggregate medium is described by the scattering matrix

$$S(\omega) = \begin{bmatrix} \hat{T}_L(\omega) & \hat{R}_R(\omega) \\ \hat{R}_L(\omega) & \hat{T}_R(\omega) \end{bmatrix} \tag{4}$$

which relates the incoming and outgoing waves appearing in Figures 2a and 2b. In Figure 2a, the medium is probed from the left by a rightward propagating wave $e^{-j\omega x}$, and $\hat{R}_L(\omega)e^{j\omega x}$ and $\hat{T}_L(\omega)e^{-j\omega x}$ are respectively the reflected and transmitted waves. Figure 2b corresponds to the case when the medium is probed from the right. More generally, for arbitrary waves $\hat{p}(x,\omega)$ and $\hat{q}(x,\omega)$

$$\hat{p}(x,\omega) = \hat{p}_L(\omega)e^{-j\omega x} \qquad \hat{q}(x,\omega) = \hat{q}_L(\omega)e^{j\omega x} \tag{5a}$$

for x<0, and

$$\hat{p}(x,\omega) = \hat{p}_R(\omega)e^{-j\omega x} \qquad \hat{q}(x,\omega) = \hat{q}_R(\omega)e^{j\omega x} \tag{5b}$$

and $x\to\infty$, and

$$\begin{bmatrix} \hat{p}_R(\omega) \\ \hat{q}_L(\omega) \end{bmatrix} = S(\omega) \begin{bmatrix} \hat{p}_L(\omega) \\ \hat{q}_R(\omega) \end{bmatrix} \tag{6}$$

expresses the outgoing waves $(\hat{p}_R, \hat{q}_L)$ in function of the incoming waves $(\hat{p}_L, \hat{q}_R)$.

If

$$a_i(x,\omega) = \begin{bmatrix} \hat{p}_i(x,\omega) \\ \hat{q}_i(x,\omega) \end{bmatrix} \quad i = 1,2 \tag{7}$$

are two arbitrary solutions of (3), and if $\Sigma \triangleq \mathrm{diag}(1,-1)$, the system (3) has the property that

$$\frac{d}{dx}\left(a_1^H(x,\omega)\ \Sigma\ a_2(x,\omega)\right) = 0 \tag{8}$$

and

$$\frac{d}{dx} W(a_1(x,\omega),\ a_2(x,\omega)) = 0, \tag{9}$$

where H denotes the Hermitian transpose, and where

$$W(a_1,a_2) \triangleq \hat{p}_1\hat{q}_2 - \hat{q}_1\hat{p}_2 \tag{10}$$

is the Wronskian of a_1 and a_2. The relation (8) expresses the fact that the medium is lossless and it can be used to show that the matrix $S(\omega)$ is unitary, i.e.

$$S^H(\omega)S(\omega) = I \quad . \tag{11}$$

This property is valid only when the two component system (3) is symmetric, whereas the identity (9) holds even for asymmetric two-component systems of the type that will be considered in Section VI.

A consequence of (9) is that

$$\hat{T}_L(\omega) = \hat{T}_R(\omega). \tag{12}$$

Physically, this means that the transmission loss through the system is the same going in either direction. It can also be shown (see [10]) that the assumption that $r \in L_1[0,\infty)$ implies that $\hat{T}_L(\omega)$ has no poles in the lower half-plane, so that the system (3) has no bound states. By combining this observation with (11) and (12), we can conclude [18], [19] that the entries of $S(\omega)$ can all be computed from either $\hat{R}_L(\omega)$ or $\hat{R}_R(\omega)$. This property is very important, since in some inverse scattering applications such as the inverse seismic problem, we have access to only one side of the scattering medium. Note that our sign convention for the Fourier transform is the opposite of that of [18] - [19], which explains why we use the lower half-plane to study the properties of $S(\omega)$, instead of the upper half-plane in [18] - [19].

Fast Cholesky Recursions

To obtain the fast Cholesky recursions, we assume that the medium is quiescent at t=0, and that is is probed from the left by a known rightward propagating wave

$$p(0,t) = \delta(t) + \tilde{p}(0,t)u(t) \tag{13}$$

which is incident on the medium at t=0. Here $\delta(\cdot)$ denotes the Dirac delta function and

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{14}$$

is the unit step function. Note that the main feature of p(0,t) is that it contains a leading impulse which is used as a tag indicating the wavefront of the probing wave. The measured data is the reflected wave

$$q(0,t) = \tilde{q}(0,t)u(t) \tag{15}$$

recorded at x=0. In the special case when $\tilde{p}(0,t) \equiv 0$, $\tilde{q}(0,t) = R_L(t)$

is the impulse response of the scattering medium and its Fourier transform $\hat{R}_L(\omega)$ is the left reflection coefficient. Note that $\hat{R}_L(\omega)$ can also be measured by sending into the medium sinusoidal waveforms at various frequencies and measuring the magnitude and phase shift of the reflected sinusoidal wave. In the following, for convenience we will omit the subscript L of $R_L(t)$ and $\hat{R}_L(\omega)$.

Since the medium is causal and originally at rest, the waves $p(x,t)$ and $q(x,t)$ inside the medium must have the form

$$\begin{aligned} p(x,t) &= \delta(t-x) + \tilde{p}(x,t)u(t-x) \\ q(x,t) &= \tilde{q}(x,t)u(t-x) \end{aligned} \tag{16}$$

where $\tilde{p}(x,t)$ and $\tilde{q}(x,t)$ are smooth functions. By substituting (16) inside (1), and identifying coefficients of the impulse $\delta(t-x)$ on both sides of (1b), we find that

$$r(x) = 2\tilde{q}(x,x) \tag{17}$$

and

$$\tilde{p}_x + \tilde{p}_t = -r(x)\tilde{q}(x,t) \tag{18a}$$

$$\tilde{q}_x - \tilde{q}_t = -r(x)\tilde{p}(x,t) \ . \tag{18b}$$

The recursions (17) - (18) constitute the <u>fast Cholesky recursions</u> [20], and have also been called the downward continuation recursions by Bube and Burridge [21].

The initial data for these recursions are the measured waves $\tilde{p}(0,t)$ and $\tilde{q}(0,t)$. The algorithm (17)-(18) can be viewed as using a layer stripping principle to identify the parameters of the scattering medium. Thus, assume that the waves $\tilde{p}(x,t)$ and $\tilde{q}(x,t)$ at depth x have been computed. The reflectivity function $r(x)$ is obtained from (17) and is used in (18) to compute the waves $\tilde{p}(x+\Delta,t)$ and $\tilde{q}(x+\Delta,t)$ at depth $x+\Delta$. The effect of the recursions (17) - (18) is therefore to identify and then strip away the layer $[x,x+\Delta)$.

The Schur Recursions

The medium can also be reconstructed by using the recursions (3) for the transformed waves $\hat{p}(x,\omega)$ and $\hat{q}(x,\omega)$ with the expression

$$\begin{aligned} r(x) &= 2\tilde{q}(x,x) \\ &= \lim_{\omega\to\infty} 2j\omega \exp j\omega x \; \hat{q}(x,\omega) \end{aligned} \tag{19}$$

where we have assumed that the waves p and q have the form (16). The recursions (3), (19) constitute the frequency domain counterpart of the time-domain recursions (17)-(18).

An alternate method is to consider the left reflection coefficient

$$\hat{R}(x,\omega) = \hat{q}(x,\omega)/\hat{p}(x,\omega) \tag{20}$$

which is associated to the section of the scattering medium extending over $[x,\infty)$. The expression (20) assumes that the medium is probed from the left and that no wave is incident from the right. By using the recursions (3) for $\hat{p}$ and $\hat{q}$, we find that $\hat{R}(x,\omega)$ satisfies the Riccati equation

$$\hat{R}_x = 2j\omega\hat{R} + r(x)\;(\hat{R}^2-1) \tag{21}$$

and the initial value theorem can be used (see [20]) to show that

$$r(x) = \lim_{\omega\to\infty} 2j\omega\hat{R}(x,\omega). \tag{22}$$

When (22) is substituted in (21), the Riccati equation (21) can be propagated autonomously and provides a way of reconstructing the reflectivity function r. The initial condition for these recursions is

$$\hat{R}(0,\omega) = \hat{R}(\omega), \tag{23}$$

and is obtained from the measured waves $\hat{p}(0,\omega)$ and $\hat{q}(0,\omega)$, or from direct frequency-domain measurements of the reflection coefficient $\hat{R}(\omega)$.

It should be emphasized that (17)-(18), (3) and (19), and (21)-(22) are all just different versions of the same algorithm and are thus interchangeable. In this paper, we will refer to (17)-(18) as the fast Cholesky recursions, and to (3) and (19) or to (21)-(22) as the Schur algorithm, but all of these are just different forms of the Schur algorithm. Note that the Riccati equation (21) for the reflection coefficient $\hat{R}$ is well-known in scattering theory [22] and is direct consequence of the rules of composition of scattering layers [17]. This equation was in fact used by Gjevick et al. [23] to develop an iterative method for reconstructing the reflectivity function r(x). What distinquishes the Schur algorithm from these results is the observation that the relation (22) can be used to compute $\hat{R}(x,\omega)$ recursively for increasing values of x.

The recursions (21)-(22) are the continuous version of an algorithm obtained by Schur [1]-[2] for testing the boundedness of a function R(z) which is analytic outside the unit disk. Given R(z), Schur showed that $|R(z)| \leq 1$ outside the unit disk if and only if the reflection coefficients r_n obtained from the recursions

$$R_{n+1}(z) = \frac{R_n(z)-r_n}{z(1-r_n R_n(z))}, \quad R_0(z) = R(z) \tag{24a}$$

$$r_n = \lim_{z\to\infty} R_n(z) \tag{24b}$$

are such that $|r_n| \leq 1$. Some recursions similar to (24) can in fact be obtained by performing a backwards-difference discretization of the Riccati equation (21). Similarly the fast Cholesky recursions (17)-(18) were used in [8] to perform the causal-anticausal factorization of a Toeplitz operator and constitute the continuous counterpart of a discrete algorithm which was obtained in [5]-[7] to construct the LDU factorization of a Toeplitz matrix.

Finally, it is worth noting that the layer stripping principle which was used here to solve the inverse scattering problem for the two-component wave system (1) can also be used for other physical models of scattering media. Some inverse scattering techniques based on a layer stripping principle were in fact developed in [24], [25] for the telegrapher's equation, and in [26] for the Schrodinger equation (see also [20]). However, instead of considering separately the inverse scattering problem for a Schrodinger equation, we can use the results developed above for two-component wave equations.

The Schrodinger Equation

Consider the equation

$$y_{xx}-y_{tt} = V(x)y(x,t) \tag{25}$$

which is associated to an elastically braced string [27], where $y(x,t)$

denotes the displacement of the string at point x and time t, and V(x) is the elasticity constant at x. We assume that V(x) is localized, i.e. $V(x) = 0$ for $x<0$ and

$$\int_0^\infty (1+x)\,|V(x)|\,dx < \infty ,$$

so that as $x<0$ and $x\to\infty$

$$y(x,t) = y_1(x-t) + y_2(x+t) \tag{26}$$

is the superposition of rightward and leftward propagating waves. Taking the Fourier transform of (25) yields the <u>Schrödinger equation</u>

$$\hat{y}_{xx} + (\omega^2 - V(x))\hat{y}(x,\omega) = 0 . \tag{27}$$

Then, the inverse scattering problem for this equation is expressed in terms of the solution $\hat{y}_L(x,\omega)$ and $\hat{y}_R(x,\omega)$ such that

$$\hat{y}_L(x,\omega) = \begin{cases} e^{-j\omega x} + \hat{R}_L(\omega)e^{j\omega x} & \text{for } x<0 \\ \hat{T}_L(\omega)e^{-j\omega x} & \text{as } x\to\infty \end{cases} \tag{28}$$

and

$$\hat{y}_R(x,\omega) = \begin{cases} \hat{T}_R(\omega)e^{j\omega x} & \text{for } x<0 \\ e^{j\omega x} + \hat{R}_R(\omega)e^{-j\omega x} & \text{as } x\to\infty \end{cases} \tag{29}$$

which define the scattering matrix $S(\omega)$ associated to V(x). These solutions correspond to the case when the string is probed from the left or from the right by an impulsive wave. The problem is to determine V(x) given the reflection coefficient function $\hat{R}_L(\omega)$ or $\hat{R}_R(\omega)$.

Several solutions of this problem based on the Gelfand-Levitan procedure [18]-[19], [28] or on trace formulas [29] have been proposed. We will now show that the Schur algorithm provides a solution which is computationally quicker. To see this, note that by taking the derivative of the two-component system (3) with respect to x, we obtain the matrix Schrödinger equation

$$\left(\left(\frac{d^2}{dx^2} + \omega^2 \quad I_2\right) - \begin{bmatrix} r^2 & -r_x \\ -r_x & r^2 \end{bmatrix}\right) \begin{bmatrix} \hat{p}(x,\omega) \\ \hat{q}(x,\omega) \end{bmatrix} = 0 \tag{30}$$

where I_2 denotes the 2x2 identity matrix. By making the change of variable

$$\hat{y}_1(x,\omega) = \hat{p}(x,\omega) + \hat{q}(x,\omega) \tag{31a}$$

$$\hat{y}_2(x,\omega) = \hat{p}(x,\omega) - \hat{q}(x,\omega) \tag{31b}$$

this equation can be decoupled into two scalar Schrödinger equation

$$\hat{y}_{1xx} + (\omega^2 - V_1(x))\hat{y}_1(x,\omega) = 0 \tag{32a}$$

$$\hat{y}_{2xx} + (\omega^2 - V_2(x))\hat{y}_2(x,\omega) = 0 \tag{32b}$$

where

$$V_1(x) = r^2(x) - r_x(x) \tag{33a}$$

$$V_2(x) = r^2(x) + r_x(x) \tag{33b}$$

In addition, we observe from (31) and from the definition of the scattering matrix $S(\omega)$ of the two-component system (3) that the scattering matrix associated to $V_1(x)$ is <u>identical</u> to that of (3), and that the scattering matrix $S_2(\omega)$ associated to $V_2(x)$ is given by

$$S_2(\omega) = \begin{bmatrix} \hat{T}_L(\omega) & -\hat{R}_R(\omega) \\ -\hat{R}_L(\omega) & \hat{T}_R(\omega) \end{bmatrix} = \Sigma S(\omega)\Sigma \ , \tag{34}$$

i.e. it is obtained by changing the sign of the reflection coefficients $\hat{R}_L$ and $\hat{R}_R$ of (3).

Consequently, given a potential $V(x)$, we can always view its left reflection coefficient $\hat{R}_L(\omega)$ as arising from a two-component system such as (3). Then, given $\hat{R}_L(\omega)$ or the impulse response $R_L(t)$, we can use the Schur or fast Cholesky recursions to reconstruct the reflectivity function $r(x)$, which in turn can be used to recover $V(x)$ from the relation (33a). The relation (33a) is known in soliton theory as the <u>Miura transformation</u> [10], [27], and it maps solutions of the modified Korteweg-de Vries equation into solutions of the Korteweg-de Vries equation.

If the potential $V(\cdot)$ extends only over a finite interval $[0,L]$, the interval $[0,L]$ may be divided into N subintervals of length $\Delta = L/N$, and the Schur and fast Cholesky recursions may be discretized accordingly. It is shown in [20], [21] that the resulting procedure requires only $O(N^2)$ operations to recover $r(\cdot)$ and $V(\cdot)$, instead of $O(N^3)$ if we discretize the Gelfand-Levitan equation and solve the resulting system of linear equations. The Schur and fast Cholesky algorithms are therefore quite efficient. Note however that if we exploit the structure of the Gelfand-Levitan equation and use the Levinson recursions to solve this equation [20], we obtain a reconstruction procedure which is as efficient as the Schur algorithm.

III. The Lossless Non-uniform Transmission Line

In this section we study the inverse problem for the lossless non-uniform transmission line, and show that its solution is given by the Schur or fast Cholesky algorithms (see [30] for an earlier solution of this problem). In the process, we give a scattering interpretation of transmission line phenomena such as waves, reflections, impedances.

Consider an infinitesimal section of length Δ of a lossless non-uniform transmission line. Such a section is illustrated in Fig. 3. Note that $L(x)$ and $C(x)$ represent inductance and capacitance per unit length, i.e. they are distributed quantities. Writing equations for Fig. 3, we have

$$\begin{aligned} v(x,t) &= Li_t\Delta + v(x+\Delta,\ t) \\ i(x,t) &= Cv_t\Delta + i(x+\Delta,\ t) \end{aligned} \tag{35}$$

Dividing by Δ and letting $\Delta \to 0$, we obtain the telegrapher's equations

$$\begin{aligned} v_x + L(x)i_t &= 0 \\ i_x + C(x)v_t &= 0 \end{aligned} \tag{36}$$

which also arise in acoustics [24] and in studies of the human vocal tract [25], [31] under the assumption of losslessness.

For a uniform line, it is well known (see [32]) that (36) admits wave solutions, and that for such waves the ratio of the amplitudes of the voltage and current is the characteristic impedance $Z_0 = (L/C)^{1/2}$. Since the quantities p and q appearing in the two-component wave equations must be dimensionally equivalent, this suggests defining for the non-uniform line the dimensionally equivalent variables

$$\begin{aligned} V(x,t) &= Z_0^{-1/2}\ v(x,t) \\ I(x,t) &= Z_0^{1/2}\ i(x,t) \end{aligned} \tag{37}$$

where $Z_0(x) = (L(x)/C(x))^{1/2}$. Substituting (37) in (36) yields

$$
\begin{aligned}
V_x + (LC)^{1/2}\, I_t &= -\frac{1}{2}\frac{d}{dx}\,\ell n\, Z_0\, V(x,t) \\
I_x + (LC)^{1/2}\, V_t &= \frac{1}{2}\frac{d}{dx}\,\ell n\, Z_0\, I(x,t) \quad .
\end{aligned}
\tag{38}
$$

In order to make the dependent variables x and t dimensionally equivalent, we replace x with the <u>travel time</u> z defined by

$$
z(x) = \int_0^x (L(u)\ C(u))^{1/2}\, du \tag{39}
$$

Since $(L(x)\ C(x))^{-1/2}$ is the local wave speed at x, z(x) is the time required for a wave, starting at x = 0, to reach position x. Making the additional change of variables

$$
p(z,t) = \frac{1}{2}\,(V(z,t) + I(z,t)) \tag{40a}
$$

$$
q(z,t) = \frac{1}{2}\,(V(z,t) - I(z,t)) \quad , \tag{40b}
$$

and defining the reflectivity function

$$
r(z) = \frac{1}{2}\frac{d}{dz}\,\ell n\, Z_0(z) \tag{41}
$$

we obtain the two-component wave equations (1). The relations (40) provide an interpretation of the right and left propagating waves in terms of the normalized voltage and current.

<u>Interpretation of the Reflection Coefficient</u>

Suppose a uniform transmission line is terminated with a load Z_L. Then a wave travelling down the line will be reflected back by the load. Define $\hat{R}(\omega)$, the reflection coefficient for the load, to be the ratio of the Fourier transforms of the primary and reflected voltage waves, at the frequency ω. It is easy to show (see [32]) that

$$\hat{R}(\omega) = \frac{\hat{v}_{REFL}(\omega)}{\hat{v}_{PRIM}(\omega)} = \frac{Z_L(\omega) - Z_0}{Z_L(\omega) + Z_0} \quad . \tag{42}$$

For the non-uniform transmission line considered here, since there is a one-to-one correspondence between position x and travel time z, we will use x instead of z in the qualitative analysis to follow. Then, at point x on the line, the load perceived due to all of the line to the right of x is (see Fig. 4)

$$Z_L(x,\omega) = \hat{v}(x,\omega)/\hat{i}(x,\omega) \quad . \tag{43}$$

By substituting this expression in (42), we find that for the non-uniform transmission line, the reflection coefficient at point x is

$$\begin{aligned} \hat{R}(x,\omega) &= \frac{\hat{v}/\hat{i} - Z_0(x)}{\hat{v}/\hat{i} + Z_0(x)} = \frac{\hat{V}/\hat{I} - 1}{\hat{V}/\hat{I} + 1} \\ &= \hat{q}(x,\omega)/\hat{p}(x,\omega) \end{aligned} \tag{44}$$

This is precisely the expression (20) for the left reflection coefficient of the section of the two-component system (1) extending over $[x,\infty)$.

We see therefore the meaning of $\hat{R}(z,\omega)$. For a given point x on the line, and any given frequency ω, it is the ratio of the reflected and primary voltage waves, with the reflection due to the inhomogeneity of the line at x. From Section II, we know that $\hat{R}(z,\omega)$ satisfies the Riccati equation (21), and that $r(x)$ may be found from $\hat{R}(x,\omega)$ by using (22). Also note that if the line is locally uniform at point x_0, $\frac{dZ_0}{dx}(x_0) = 0$ hence $r(x_0) = 0$ and no reflection occurs. Reflections occur only where the line is inhomogeneous.

Inverse Problem

Suppose now that the line characteristics $L(x)$ and $C(x)$ are unknown and that we want to determine them from the measured impedance $Z(\omega) = Z_L(0,\omega)$. This problem arises not only when we want to find the characteristics of an existing transmission line, but also if we want to synthesize a transmission line with prescribed impedance $Z(\omega)$. It is assumed here that we have access to only one end of the line. The line characteristics can be partially reconstructed as follows. First, set $Z_0(0) = 1$ and consider the reflection coefficient

$$\hat{R}(\omega) = \frac{Z(\omega) - 1}{Z(\omega) + 1} \quad . \tag{45}$$

Then, run the Schur recursions (21) - (22), using $\hat{R}(\omega)$ as initial condition, to obtain $r(z)$. Alternately, we may compute the inverse Fourier transform $R(t)$ of $\hat{R}(\omega)$, and use the fast Cholesky recursions (17) - (18) to obtain $r(z)$. Given $r(z)$, the expression

$$Z_0(z) = \exp 2 \int_0^z r(u)\, du \tag{46}$$

enables us to recover the characteristic impedance $Z_0(z) = (L(z)/C(z))^{1/2}$ as a function of the travel time z. However, we cannot reconstruct $L(x)$ and $C(x)$ separately as functions of the position x.

The same difficulty will appear in Section IV for the inverse seismic problem of geophysics, except that in this case we will be able to use an additional degree of freedom, the angle of incidence of the probing waves, in order to reconstruct the medium completely.

IV. The Inverse Seismic Problem

In this section we examine and solve, using the Schur algorithm, the inverse problem for a one-dimensional acoustic medium probed with plane waves. We will consider first the case of plane waves at normal incidence, and then the offset problem in which the probing waves come in at an angle, as shown in Fig. 5.

For the normal incidence case, the equations we will obtain are almost identical to those of last section, and even though the physical situations are quite different, the inverse problem is the same as for the lossless non-uniform transmission line.

A simple transformation will allow the offset problem to be solved by using the same procedure as for the normal incidence problem. By probing the medium at two different angles, it will be shown that the medium density and velocity profiles can be reconstructed separately as functions of depth. The use of the Schur algorithm to solve the offset problem has not to our knowledge appeared in the literature.

The Normal Incidence Problem

The problem to be considered in this section corresponds to the case when the angle of incidence $\theta = 0$ in Fig. 5. The acoustic medium that we want to reconstruct is constituted of a homogeneous half-space with known density ρ_0 and sound speed c_0 extending over $x < 0$, and of an inhomogeneous half-space with unknown density $\rho(x)$ and unknown local sound speed $c(x)$ extending over $x \geq 0$. For convenience, we assume that inhomogeneities are localized, so that for $x > L$ the density ρ_1 and sound speed c_1 are constant. Physically, the region $x < 0$

corresponds to the air or ocean located above the medium to be probed, and the region $x > L$ corresponds to the substrate or bedrock located below it. An impulsive plane pressure wave, propagating downwards, is incident upon the inhomogeneous region at $t = 0$, and the reflections or reverberations making their way back to the surface of the inhomogeneous medium are measured either at the surface of this medium (land case) or above the surface (marine case). Our goal is to botain profiles of $\rho(x)$ and $c(x)$ as functions of depth. The presentation will follow that of [33] - [36].

The two basic equations we start with are the acoustic equation [33], [35]

$$\rho w_{tt} = - P_x \tag{47}$$

and the stress-strain equation

$$P = - \rho c^2 w_x \ , \tag{48}$$

where $w(x,t)$ and $P(x,t)$ denote respectively the displacement and pressure (negative stress) at depth x.

The first step is to change variables from depth x to travel-time $z(x)$, which is the time it takes for a wave starting at the surface of the inhomogeneous medium to reach depth x. (Recall a similar definition in the last section). Thus we have

$$z(x) = \int_0^x du/c(u) \quad . \tag{49}$$

By substituting (49) inside (47) - (48), and defining

$$Z(z) = \rho(z)c(z) = \text{characteristic impedance} \tag{50a}$$

$$v(z,t) = w_t(z,t) = \text{particle velocity} \tag{50b}$$

we obtain the system

$$P_z = -Z(z)v_t$$
$$v_z = - Z^{-1}(z)P_t \tag{51}$$

which in the Fourier domain takes the form

$$\hat{P}_z = - j\omega Z(z)\hat{v}$$
$$\hat{v}_z = - j\, Z^{-1}(z)\hat{P} \quad . \tag{52}$$

Then, if

$$\Psi(z,t) = Z^{-1/2}(z)\; P(z,t) = \text{normalized pressure} \tag{53a}$$

$$\phi(z,t) = Z^{1/2}(z)\; v(z,t) = \text{normalized velocity} \tag{53b}$$

and if we make the change of variables

$$p(z,t) = \frac{1}{2}\,(\Psi + \phi)$$
$$q(z,t) = \frac{1}{2}\,(\Psi - \phi) \quad , \tag{54}$$

the system (51) can be transformed into the two-component wave system

$$p_z + p_t = -r(z)q(z,t)$$
$$q_z - q_t = - r(z)p(z,t) \tag{55}$$

where the reflectivity function $r(z)$ is given by

$$r(z) = \frac{1}{2}\,\frac{d}{dz}\,\ell n\; Z(z) \quad . \tag{56}$$

The definition of the normalized variables $\Psi(z,t)$ and $\phi(z,t)$ and of the waves $p(z,t)$ and $q(z,t)$ is identical to the one we used in last section for the normalized voltage and current $V(z,t)$ and $I(z,t)$ and the associated right and left going waves. The inversion problem for the 1-D acoustic medium probed by plane waves at normal incidence is therefore the same as that for the nonuniform lossless transmission line.

The data which is used in the inversion is obtained by sending a downward impulsive plane pressure wave which is normally incident upon the inhomogeneous half-space at $t = 0$. Then, for the land case, we measure the particle velocity $v(0,t)$ at the earth's surface as a function of time. Since the difference in density between the air and the earth is very large, the earth's surface acts like a free surface, i.e. the pressure on it is zero for positive times. We can therefore express the pressure and velocity on the surface as

$$P(0,t) = P_0\delta(t) \tag{57a}$$

$$v(0,t) = v_0(\delta(t) + 2\tilde{h}(t)u(t)) \tag{57b}$$

where $P_0/v_0 = Z(0)$. After normalization, the downgoing and upgoing waves $p(0,t)$ and $q(0,t)$ take the form

$$\begin{aligned} p(0,t) &= \delta(t) + \tilde{h}(t)u(t) \\ q(0,t) &= -\tilde{h}(t)u(t) \quad , \end{aligned} \tag{58}$$

and the fast Cholesky recursions (17) - (18) or the Schur recursions (3) and (22) can be applied to these waves to reconstruct the reflectivity function $r(z)$.

For the case of a marine seismogram, the reflected pressure wave $R(\cdot)$ is measured at some point inside the homogeneous half-space $x < 0$. The pressure and velocity in this half-space are

$$\begin{aligned} P(x,t) &= P_0(\delta(t-x/c_0) + R(t + x/c_0)\ u(t + x/c_0)) \\ v(x,t) &= v_0(\delta(t-x/c_0) - R(t + x/c_0)\ u(t + x/c_0)) \end{aligned} \tag{59}$$

so that at the surface of the inhomogeneous medium (the ocean floor), the downgoing and upgoing waves are given by

$$p(0,t) = \delta(t), \quad q(0,t) = R(t)u(t) \quad . \tag{60}$$

These can then be used to reconstruct the reflectivity function $r(z)$.

Given $r(z)$, the impedance $Z(z)$ can be obtained by using (46). Thus, as in the case of the nonuniform lossless transmission line, the best we can do is to reconstruct $Z(z) = \rho c$ as a function of the travel time z. We cannot recover $\rho(x)$ and $c(x)$ separately as functions of depth. The procedure consisting of using the Schur recursions to reconstruct the impedance $Z(z)$ is commonly called dynamic deconvolution. It was developed first, using the discrete Schur algorithm, for the case of a layered medium divided up into homogeneous layers of equal travel-time (the so-called Goupillaud model) and it is described in [9].

Another feature of this reconstruction method is that the quantities $p(x,t)$ and $q(x,t)$ represent respectively downgoing and upgoing waves. So when we run the fast Cholesky or Schur recursions on the experimental data, we are decomposing the pressure and particle velocity at each depth into a superposition of upgoing and downgoing waves. Thus, we gain not only information about the medium parameters, but also information about what is happening to the medium. This could prove useful in evaluating how realistic the model is.

The Offset Problem

We now consider the problem in which an impulsive plane pressure wave is obliquely incident on the medium at an angle θ to the vertical, as shown in Fig. 5. In this case, the impulse response $R(t, y; \theta)$ is a function of the horizontal coordinate y (in the normal incidence case there is of course no horizontal variation). We will once again obtain a dynamic deconvolution procedure that uses the Schur algorithm to recover an impedance as a function of travel time, although this impedance differs slightly from (50a). However, running the offset experiment twice for two different angles of incidence $\theta = \theta_1, \theta_2$ will allow us to recover $\rho(x)$ and $c(x)$ separately, as functions of depth, a significant improvement over the normal incidence experiment. The problem is set-up as in [37] - [39].

An impulsive plane pressure wave $P_0\ \delta(t-(x \cos\theta + y \sin\theta)/c_0)$ is incident at an angle θ from the vertical, where x and y are as in Fig. 5. The Fourier transform of this wave is $P_0 e^{-j(k_x x + k_y y)}$, where $k_x = \omega \cos\theta/c_0$ and $k_y = \omega \sin\theta/c_0$ are the vertical and lateral wavenumbers in the upper homogeneous half-space. For the case of a marine seismogram, the pressure field for $x < 0$ is therefore

$$\hat{P}(x,y,\omega;\theta) = P_0 e^{-jk_y y} (e^{-jk_x x} + \hat{R}(\omega;\theta) e^{jk_x x} \tag{61}$$

(compare this to the Schrodinger equation boundary condition (28)). This shows that in the time domain the impulse response $R(t,y;\theta)$ has the form

$$R(t,y;\theta) = R(t-y \sin\theta/c_0;\theta) \tag{62}$$

where $R(t;\theta)$ is the inverse Fourier transform of the reflection coefficient $\hat{R}(\omega;\theta)$. This form is also valid for the case of a land seismogram. Thus, in theory it should only be necessary to measure $R(t,y;\theta)$ at a single surface point (e.g. $y = 0$). However, in practice we need to take data for a range of y and filter or stack it to the form (62). This is because any real-world impulsive wave can only be locally planar, while the form (61) of the pressure field assumes an incident plane wave of infinite extent.

With y dependence added, the acoustic and stress-strain equations (47), (48) become

$$\rho \underline{w}_{tt} = - \nabla P(x,y,t) \tag{63a}$$

$$P(x,y,t) = - \rho c^2 \nabla \cdot \underline{w}(x,y,t) \tag{63b}$$

where the displacement $\underline{w}(x,y,t)$ is now a vector. Then, if $\underline{v} = \underline{w}_t$ is the particle velocity, and if v^x and v^y are its vertical and lateral components, by taking Fourier transforms we obtain

$$\hat{P}_x = - j\omega\rho(x)\hat{v}^x \tag{64a}$$

$$\hat{P}_y = - j\omega\rho(x)\hat{v}^y \tag{64b}$$

$$\rho(x)c^2(x)(\hat{v}^x_x + \hat{v}^y_y) = - j\omega\hat{P} \quad . \tag{64c}$$

Since the medium properties vary only with depth x, the horizontal wavenumber k_y is preserved, and we may write, as in [37]

$$\hat{P}(x,y,\omega) = \hat{\pi}(x,\omega)e^{-jk_y y} \quad . \tag{65}$$

Substituting this expression in (64b) yields

$$\hat{v}^y = (\sin\theta/\rho(x)c_0)\hat{P} \quad , \tag{66}$$

and by inserting (66) in (64c) and using Snell's law

$$\sin\theta(x)/c(x) = \sin\theta/c_0 = \text{ray parameter (constant)} \tag{67}$$

where $\theta(x)$ is the local angle that a ray path makes with the vertical, we get

$$\rho(x)c^2(x)\hat{v}^x_x = - j\omega\cos^2\theta(x)\hat{P} \ . \tag{68}$$

Now, define

$$c'(x) = c(x)/\cos\,\theta(x) = \text{local vertical wave speed} \tag{69}$$

$$z(x) = \int_0^x du/c'(u) = \text{vertical travel time to depth } x \tag{70}$$

$$Z(z) = \rho(z)c'(z) = \text{effective impedance .} \tag{71}$$

Using (69) = (71) in (64a) and (68) gives

$$\hat{P}_z = -\,j\omega\, Z(z)\hat{v}^x \tag{72a}$$

$$\hat{v}^x_z = -\,j\omega\, Z^{-1}(z)\hat{P} \quad , \tag{72b}$$

and once again defining the downgoing and upgoing waves as

$$\hat{p}(z,y,\omega) = \frac{1}{2}(Z^{-1/2}\hat{P}(z,y,\omega) + Z^{1/2}\hat{v}^x(z,y,\omega)) \tag{73a}$$

$$\hat{q}(z,y,\omega) = \frac{1}{2}(Z^{-1/2}\hat{P}(z,y,\omega) - Z^{1/2}\hat{v}^x(z,y,\omega)), \tag{73b}$$

we obtain the two-component wave system

$$\hat{p}_z = -\,j\omega\hat{p} - r(z)\hat{q} \tag{74a}$$

$$\hat{q}_z = -\,r(z)\hat{p} + j\omega\hat{q} \tag{74b}$$

where the reflectivity function $r(z)$ is given by (56). Here y is fixed at whatever value of y we measure $R(t,y;\theta)$ (e.g. $y = 0$), and θ is a parameter on which all quantities depend.

Note that once again the quantities in the wave system (74) are the Fourier transform of the downgoing and upgoing waves, so that the vertical motion of the medium is again decomposed into upgoing and downgoing waves.

Furthermore, for both the case of a land and marine seismogram, these waves have the form (16), so that we can use the fast Cholesky or Schur recursions to reconstruct the effective impedance $Z(z)$ given by (71).

Now, suppose that the offset experiment is run twice, for two angles of incidence $\theta = \theta_1, \theta_2$. Two different imepdances $Z_1(z_1)$ and $Z_2(z_2)$ are obtained, which are functions of two different travel times z_1 and z_2. From (69) - (71), we get

$$\frac{dz_1}{dz_2} = \cos\,\theta_1(x)/\cos\,\theta_2(x) = Z_2(z_2)/Z_1(z_1) \tag{75}$$

which can be solved or integrated numerically to obtain the monotone increasing function $z_1 = z_1(z_2)$. This unables us to express Z_1 and Z_2 as functions of the same travel time z_2. Then, using (67) and (69) - (71), we can reconstruct $\rho(z_2)$ and $c(z_2)$ separately as functions of the known impedances Z_1 and Z_2. Finally, inverting (70) and using (69) gives the effective travel time $z_2(x)$, yielding $\rho(x)$ and $c(x)$ <u>separately</u>, as functions of <u>depth</u>.

This reconstruction procedure has only been sketched, since it has nothing to do with the Schur algorithm; it is presented in more detail in [38]. Note however that the dynamic deconvolution method described above to compute the impedances $Z_i(z_i)$ $i = 1, 2$ is new. An alternate reconstruction procedure was described in [39] which recovers $\rho(x)$ and $c(x)$ recursively by operating directly on the need to compute the impedances $Z_i(z_i)$ $i = 1, 2$ as a preliminary step.

The reason that the profiles $\rho(x)$ and $c(x)$ recursively by operating directly on the waves associated to θ_1 and θ_2, thereby obviating the need to compute the impedances $Z_i(z_i)$ $i = 1, 2$ as a preliminary step.

The reason that the profiles $\rho(x)$ and $c(x)$ can be recovered separately for the oblique incidence problem, but not for the normal incidence problem, is that by running the oblique experiment twice information has been gained along two <u>different</u> ray paths. This option is not available for the normal incidence problem.

Note that along any given ray path the ray parameter (67) is constant, so that unless the angle of incidence θ is less than the <u>critical angle</u> $\sin^{-1}(c_0/\max c(x))$, the angle $\theta(x)$ will become imaginary at some depth. Physically, this situation results in <u>evanescent waves</u>, in which the pressure field decays exponentially with depth. This causes no difficulty in the Schur algorithm until the ray path becomes horizontal, prior to turning back up. When this turning point is reached, $|r(x)| \to \infty$. However, since no new information can be gained beyond the turning point, the exceeds a pre-set value. For the reconstruction procedure described above, this means that we can recover $\rho(x)$ and $c(x)$ only until a turning point occurs in either of the two oblique incidence experiments.

V. Linear Estimation of a Stationary Stochastic Process

In this section the problem of finding the linear least-squares estimate of a stationary stochastic process given some observations of this process over a finite interval is posed as an inverse scattering problem, and solved using the Schur algorithm. This formulation of the estimation problem for a stationary stochastic process is due to Dewilde and his coworkers [4], [15], [16], [40].

The basic problem to be considered is as follows. Let

$$y(t) = z(t) + v(t) \tag{76}$$

be some observations of a zero-mean stationary stochastic process $z(\cdot)$ with covariance

$$E[z(t)z(s)] = k(|t-s|), \tag{77a}$$

where $v(\cdot)$ is a white noise process with unit intensity, i.e.

$$E[v(t)v(s)] = \delta(t-s) \ . \tag{77b}$$

We assume that $z(\cdot)$ and $v(\cdot)$ are uncorrelated and that $k(\cdot) \in L_1[0,\infty)$, so that its Fourier transform

$$\hat{k}(\omega) = \int_0^\infty k(t) \exp{-j\omega t}\, dt \tag{78}$$

exists. In this case, the spectral density of $y(\cdot)$ is $\hat{w}(\omega) = 1 + \hat{k}(\omega) + \hat{k}(-\omega)$.

Given the Hilbert space

$$Y(t;\ x) = H(y(t+s),\ -x \leq s \leq x) \tag{79}$$

spanned by the observations over the interval $[t-x,\ t+x]$, our objective is to compute the forwards and backwards linear least-squares estimates of z at the endpoints of this interval. These estimates can be denoted as

$$\hat{z}(t+x \ Y(t;\ x)) = \int_{-x}^{x} A(x;\ u)\ y(t+u)du \tag{80a}$$

$$\hat{z}(t-x|Y(t;\ x)) = \int_{-x}^{x} B(x;\ u)\ y(t+u)du\ , \tag{80b}$$

where $A(x;\ \cdot)$ and $B(x;\ \cdot)$ are the optimal forwards and backwards prediction filters, respectively. Note that since the process $z(\cdot)$ is stationary the filters $A(x;\ \cdot)$ and $B(x;\ \cdot)$ do not depend on t, the center of the interval $[t-x,\ t+x]$. Then, if the forwards and backwards residuals are defined as

$$e(t,x) = y(t+x) - \hat{z}(t+x|Y(t;\ x)) \tag{81a}$$

$$b(t,x) = y(t-x) - \hat{z}(t-x|Y(t;\ x)), \tag{81b}$$

by using the orthogonality property

$$e(t,x),\ b(t,x) \perp Y(t;\ x) \tag{82}$$

of linear least-squares estimates, we find that the filters $A(x;\ \cdot)$ and $B(x;\ \cdot)$ satisfy the Wiener-Hopf equations

$$A(x;\ s) + \int_{-x}^{x} A(x;\ u)\ k(|u-s|)\ du = k(x-s) \tag{83a}$$

$$B(x;\ s) + \int_{-x}^{x} B(x;\ u)\ k(|u-s|)\ du = k(x+s) \tag{83b}$$

with $-x \leq s \leq x$.

Applying the operators $\frac{\partial}{\partial x} + \frac{\partial}{\partial s}$ and $\frac{\partial}{\partial x} - \frac{\partial}{\partial s}$ to (83a) and (83b) respectively, and using the linearity of the resulting equations yields the <u>Krein-Levinson recursions</u> [15], [41]

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial s}\right) A(x;\ s) = -\ r(x)\ B(x;\ s) \tag{84a}$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial s}\right) B(x;\ s) = -\ r(s)\ A(x;\ s) \tag{84b}$$

with $-\ x \leq s \leq x$, and where

$$r(x) = 2A(x;\ -x) = 2B(x;\ x) \tag{85}$$

is the reflectivity function. The last identity in (85) is obtained by noting from a time-reversal argument that $B(x;\ s) \equiv A(x;\ -s)$. The Krein-Levinson recursions (84) have the same form as the fast Cholesky recursions. However, as noted in [20], these two sets of recursions differ by the fact that the

Krein-Levinson recursions correspond to a boundary value problem where r(x) is computed at every step from

$$r(x) = 2(k(2x) - \int_{-x}^{x} A(x;\ u)\ k(x+u)du), \tag{86}$$

whereas the fast Cholesky recursions give rise to an initial value problem.

The recursions (84), (86) can be used to compute efficiently the filters $A(x;\ \cdot)$ and $B(x;\ \cdot)$. Furthermore, if we apply the operators $\frac{\partial}{\partial x} \mp \frac{\partial}{\partial t}$ to the definition (81) of the forwards and backwards residuals e(t,x) and b(t,x) and use the Krein-Levinson recursions (84), we obtain

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})\ e(t,x) = -\ r(x)\ b(t,x) \tag{87a}$$

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial t})\ b(t,x) = -\ r(x)\ e(t,x)\quad . \tag{87b}$$

This shows that the residuals satisfy a two-component wave system, where e(t,x) and b(t,x) propagate respectively leftward and rightward, and where the waves at $x = 0$ are given by

$$e(t,\ 0) = b(t,\ 0) = y(t)\quad . \tag{88}$$

As a consequence of this observation, the process y(t) can be viewed as the output of a modeling filter driven by e(t,x) as shown in Fig. 6a. This modeling filter is obtained by aggregating infinitesimal ladder sections of the type described in Fig. 6.b.

The scattering matrix associated to the two-component wave system (87) can be identified by noting that as $x \to \infty$

$$e(t,x) = \nu_F(t+x),\ b(t,x) = \nu_B(t-x) \tag{89}$$

where $\nu_F(\cdot)$ and $\nu_B(\cdot)$ denote respectively the forwards and backwards innovations processes associated to $y(\cdot)$ [41]. The processes $\nu_F(\cdot)$ and $\nu_B(\cdot)$ are white noise processes and are related to the observations $y(\cdot)$ through the identities

$$\hat{y}(\omega) = F(\omega)\ \hat{\nu}_F(\omega) \tag{90a}$$

$$\hat{y}(\omega) = F(-\omega)\ \hat{\nu}_B(\omega) \tag{90b}$$

where $\hat{y}(\omega)$, $\hat{\nu}_F(\omega)$ and $\hat{\nu}_B(\omega)$ denote formally the Fourier transforms of $y(\cdot)$, $\nu_F(\cdot)$ and $\nu_B(\cdot)$, and where the shaping filter $F(\omega)$ is the outer spectral factor of $\hat{w}(\omega)$, i.e.

$$\hat{w}(\omega) = |F(\omega)|^2 \tag{91}$$

on the real axis, and $F(\omega)$ and $F^{-1}(\omega)$ are analytic in the lower half-plane.

The relations (88) and (89) imply that the scattering matrix $S(\omega)$ satisfies

$$\begin{bmatrix} \hat{\nu}_B(\omega) \\ \hat{y}(\omega) \end{bmatrix} = S(\omega) \begin{bmatrix} \hat{y}(\omega) \\ \hat{\nu}_F(\omega) \end{bmatrix}, \tag{92}$$

and by substituting (90) inside this relation, we obtain the identity

$$\begin{bmatrix} F^{-1}(-\omega) \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{T}_L & \hat{R}_R \\ \hat{R}_L & \hat{T}_R \end{bmatrix} \begin{bmatrix} 1 \\ F^{-1}(\omega) \end{bmatrix} \tag{93}$$

for the entries of $S(\omega)$. By using the properties (11), (12) of the scattering matrix, this gives after some algebra

$$\hat{R}_L(\omega) = \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)} \tag{94a}$$

$$\hat{T}_L(\omega) = \hat{T}_R(\omega) = \frac{F(\omega)}{1 + \hat{k}(\omega)} \tag{94b}$$

$$\hat{R}_R(\omega) = \frac{-\hat{k}(-\omega)}{1 + \hat{k}(\omega)} \; \frac{F(\omega)}{F(-\omega)} \quad , \tag{94c}$$

where the left reflection coefficient $\hat{R}_L(\omega)$ depends only on the covariance data given by $\hat{k}(\omega)$.

Then, we observe that the Krein-Levinson recursions (84) and the system (87) for the residuals $e(t,x)$ and $b(t,x)$ are parametrized entirely by the reflectivity function $r(x)$. Consequently, the linear least-squares estimation problem over an arbitrary finite interval will be solved completely once we reconstruct $r(\cdot)$. This problem can be formulated as an inverse scattering problem where $\hat{R}_L(\omega)$ is given in the form (94a), and where we want to recover $r(x)$.

To do so, one method is to apply the Schur or fast Cholesky recursions directly to $\hat{R}_L(\omega)$ or its inverse Fourier transform $R_L(t)$. However, the special form of (94a) can be exploited by selecting

$$\begin{aligned} p(0,t) &= \delta(t) + k(t)u(t) \\ q(0, t) &= k(t)u(t) \end{aligned} \tag{95}$$

as probing waves (see [8], [40]), to which we can then apply the fast Cholesky recursions. In this case, as a byproduct of the fast Cholesky algorithm, we obtain a factorization of the covariance operator $w(t-s) = \delta(t-s) + k(|t-s|)$ in terms of causal times anticausal Volterra operators [8]. Furthermore by noting that

$$\begin{bmatrix} F(\omega) \\ \hat{k}(\omega) \end{bmatrix} = S(\omega) \begin{bmatrix} 1 + \hat{k}(\omega) \\ 0 \end{bmatrix} \tag{96}$$

we see that as $x \to \infty$, the rightward propagating wave $\hat{p}(x,\omega)$ corresponding to the probing waves (95) takes the form

$$\hat{p}(x,\omega) = F(\omega)\ \exp - j\omega x\ , \tag{97}$$

so that the Cholesky recursions provide an approximation of the spectral factor $F(\omega)$ of $\hat{w}(\omega)$.

The inverse scattering formulation of the linear least-squares estimation problem that we have described above can also be used to study the properties of orthogonal ladder filters of the type described in Fig. 6b. For example, the stability and lack of sensitivity of these filters to roundoff errors are a direct consequence of the losslessness property of the scattering medium [4], [42]. These properties, as well as the modularity and pipelinability of ladder filters have motivated their widespread use for adaptive equalization [43], speech processing [12], [44], and spectral estimation [14], [45].

VI. Inverse Scattering for Asymmetric Two-Component Wave Systems

In this section, the inverse scattering problem for asymmetric two-component wave equations is examined, and solved by using two coupled sets of Schur recursions. The systems which are described by asymmetric two-component wave equations are not necessarily lossless, and we can therefore use these equations to describe a larger class of physical phenomena than those that we have studied in the previous sections. Our results will be illustrated by considering the inverse problem for a nonuniform transmission line with losses. It is worth noting that a solution of the inverse scattering problem for asymmetric two-component wave equations was presented in [10] and was used by Jaulent [46] to solve the inverse problem for lossy transmission lines. However, this method relied on the solution of two coupled Marchenko equations, whereas the solution that we present here is differential, and uses the layer stripping principle that we have been advocating throughout this paper.

The system that we consider is described by the asymmetric two-component wave equations

$$\frac{d}{dx}\begin{bmatrix}\hat{p}\\ \hat{q}\end{bmatrix} = \begin{bmatrix}-j\omega & -s(x)\\ -r(x) & j\omega\end{bmatrix}\begin{bmatrix}\hat{p}\\ \hat{q}\end{bmatrix} \tag{98}$$

which, in the time-domain correspond to

$$p_x + p_t = -s(x)\ q(x,t) \tag{99a}$$

$$q_x - q_t = -r(x)\ p(x,t) \tag{99b}$$

It is assumed that $r(x) = s(x) = 0$ for $x < 0$, and that $r, s \in L_1[0,\infty)$, so that $r(x)$ and $s(x)$ are localized, i.e. they go to zero as $x \to \infty$.

Then, the scattering matrix $S(\omega)$ can be defined as in Section II by relating the outgoing and incoming waves appearing in Fig. 2. In addition, the property (9) for the Wronskian of two independent solutions $a_i^T(x,\omega) = (\hat{p}_i(x,\omega),\ \hat{q}_i(x,\omega))$ i=1, 2 of (98) remains valid, and by applying it to the waves $a_1(x,\omega)$ and $a_2(x,\omega)$ appearing in Figs. 2a and 2b respectively, we obtain the reciprocity relation

$$\hat{T}_L(\omega) = \hat{T}_R(\omega) . \tag{100}$$

However, if $a^T(x,\omega) = (\hat{p}(x,\omega), \hat{q}(x,\omega))$ is an arbitrary solution of (98), we have

$$\frac{d}{dx}(|\hat{p}|^2 - |\hat{q}|^2) = 2(r(x) - s(x)) \operatorname{Re}(\hat{p}(x,\omega)\hat{q}^*(x,\omega)) \tag{101}$$

so that the scattering medium associated to (98) is <u>not lossless</u> unless $r(x) = s(x)$ which corresponds to the case when the two-component wave equations are symmetric. This implies that $S(\omega)$ is not a unitary matrix, and consequently we cannot recover $S(\omega)$ from the knowledge of the left reflection coefficient $\hat{R}_L(\omega)$ only.

<u>Inverse Scattering Procedure</u>

The inverse scattering method that we develop here relies on the observation that if time is reversed (i.e. t is changed to -t in (99), or ω is changed to $-\omega$ in (98)), and if the waves p and q are interchanged, we obtain an asymmetric two-component wave system

$$p_x^A + p_t^A = -r(x)\ q^A(x,t) \tag{102a}$$

$$q_x^A - q_t^A = -s(x)\ p^A(x,t) \tag{102b}$$

where $r(x)$ replaces $s(x)$ and vice-versa. The scattering matrix associated to this system is

$$S^A(\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S^{-1}(-\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= (S^H(\omega))^{-1}\ , \tag{103}$$

where to obtain (103) we have used the reciprocity relation (100). The system (102) is a fake system, which does not exist really, but its scattering matrix is entirely specified by the knowledge of $S(\omega)$.

Then, in order to reconstruct $r(x)$ and $s(x)$, we assume that the true system (98) and the fake system (102) are probed simultaneously by some waves which have the form

$$\begin{aligned} p(x,t) &= \delta(t-x) + \tilde{p}(x,t)\, u(t-x) \\ q(x,t) &= \tilde{q}(x,t)\, u(t-x) \end{aligned} \tag{104}$$

and

$$\begin{aligned} p^A(x,t) &= \delta(t-x) + \tilde{p}^A(x,t)u(t-x) \\ q^A(x,t) &= \tilde{q}^A(x,t)u(t-x) \quad . \end{aligned} \tag{105}$$

By substituting these waves in (98) and (102), we obtain the system of coupled fast Cholesky recursions

$$\begin{aligned} \tilde{p}_x + \tilde{p}_t &= -s(x)\ \tilde{q}(x,t) \\ \tilde{q}_x - \tilde{q}_t &= -r(x)\ \tilde{p}(x,t) \end{aligned} \tag{106a}$$

and

$$\begin{aligned} \tilde{p}^A_x + \tilde{p}^A_t &= -r(x)\ \tilde{q}^A(x,t) \\ \tilde{q}^A_x - \tilde{q}^A_t &= -s(x)\ \tilde{p}^A(x,t) \end{aligned} \tag{106b}$$

with

$$r(x) = 2\tilde{q}(x,x), \quad s(x) = 2\tilde{q}^A(x,x) \tag{106c}$$

which can be propagated recursively for increasing values of x, starting from x = 0. The specification of the initial conditions for these recursions is very important, since as noted above, the system (102) does not exist really and cannot be relied upon to provide some experimental waves $\tilde{p}^A(0,t)$ and $\tilde{q}^A(0,t)$.

The initial conditions that we select are

$$\tilde{p}(0,t) = \tilde{p}^A(0,t) = 0 \tag{107a}$$

$$\tilde{q}(0,t) = R_L(t) \ , \ \tilde{q}^A(0,t) = R_L^A(t) \tag{107b}$$

where $R_L(t)$ and $R_L^A(t)$ denote the inverse Fourier transforms of the left reflection coefficients $\hat{R}_L(\omega)$ and $\hat{R}_L^A(\omega)\cdot\hat{R}_L(\omega)$ can be measured directly, and from (103)

$$\hat{R}_L^A(\omega) = (S^{-H}(\omega))_{21} \tag{108}$$

i.e. $\hat{R}_L^A(\omega)$ is the (2, 1) entry of the inverse of $S^H(\omega)$. Thus, $\hat{R}_L^A(\omega)$ can be expressed as a function of the whole scattering matrix $S(\omega)$, and it will be specified provided that we can measure all the entries of $S(\omega)$. This implies that we must have access to both ends of the scattering medium. In some cases, such as for the inverse problem of geophysics, this is impossible; but for some other problems, such as for the reconstruction of nonuniform transmission lines, the medium can be probed from both sides, and all the entries of $S(\omega)$ can be measured.

Instead of expressing our reconstruction procedure in terms of the coupled fast Cholesky recursions described above, we can use a set of coupled

Schur recursions. Let

$$\hat{R}(x,\omega) = \frac{\hat{q}(x,\omega)}{\hat{p}(x,\omega)} \quad \text{and} \quad \hat{R}^A(x,\omega) = \frac{\hat{q}^A(x,\omega)}{\hat{p}^A(x,\omega)} \tag{109}$$

be the left reflection coefficients for the true and fake systems over the interval $[x,\infty)$, where the waves $\hat{p}$, $\hat{q}$, p^A, q^A in the definition (109) are assumed to have the form (104) - (105). Then, $\hat{R}(x,\omega)$ and $\hat{R}^A(x,\omega)$ satisfy the Riccati equations

$$\hat{R}_x = 2j\omega\hat{R} + s(x)\,\hat{R}^2 - r(x) \tag{110a}$$

$$\hat{R}^A_x = 2j\omega\hat{R}^A + r(x)\,\hat{R}^{A2} - s(x) \tag{110b}$$

with intial conditions

$$\hat{R}(0,\omega) = \hat{R}_L(\omega),\ \hat{R}^A(0,\omega) = \hat{R}^A_L(\omega) \quad . \tag{111}$$

By using the initial value theorem for the reflection coefficient (109), and taking into account the form of the waves (104) - (105), we get

$$\lim_{\omega\to\infty} 2j\omega\hat{R}(x,\omega) = r(x) \tag{112a}$$

$$\lim_{\omega\to\infty} 2j\omega\hat{R}^A(x,\omega) = s(x) \tag{112b}$$

which can be combined with (110a) and (110b) to propagate $\hat{R}(x,\omega)$ and $\hat{R}^A(x,\omega)$ recursively, and to reconstruct $r(x)$ and $s(x)$ for all x. This algorithm constitutes the generalization of the Schur algorithm (21) - (22).

Reconstruction of Non-uniform Transmission Lines with Losses

In Section III, the reconstruction problem for a nonuniform lossless transmission line was solved using the Schur algorithm. We now consider the more general case where some losses, in the form of series and shunt resistances per unit length have been addded to the transmission line. This reconstruction problem is then solved as an asymmetric two-component inverse scattering problem, using the method obtained at the beginning of this section. The problem is set up as in [46].

An infinitesimal section of the line is shown in Fig. 7. $R(x)$ is the non-uniform series resistance per unit length, representing the finite resistance of the wires, and $G(x)$ is the shunt conductance per unit length, representing leakage current between the wires. The circuit equations are

$$\begin{aligned} v(x,t) &= (Li_t + Ri)\Delta + v(x+\Delta,\ t) \\ i(x,t) &= (Cv_t + Gv)\Delta + i(x+\Delta,\ t) \quad . \end{aligned} \tag{113}$$

Dividing by Δ, and letting $\Delta \to 0$ yields the transmission line equations

$$\begin{aligned} v_x + Li_t + Ri &= 0 \\ i_x + Cv_t + Gv &= 0 \quad . \end{aligned} \tag{114}$$

As in Section III, we replace the position x by the travel time $z(x)$ given by (39), and we introduce the dimensionally equivalent variables

$$V(z,t) = Z^{-1/2}v(z,t),\ I(z,t) = Z^{1/2}i(x,t) \tag{115}$$

where $Z(z) = (L(z)/C(z))^{1/2}$ is the characteristic impedance. Then, the equations (114) take the form

$$\begin{aligned} V_z + I_t &= -\frac{R}{L}\ I - m(z)V \\ I_z + V_t &= m(z)\ I - \frac{G}{C}\ V \end{aligned} \tag{116}$$

where

$$m(z) = \frac{1}{2}\frac{d}{dz}\ \ell n\ Z(z) \quad . \tag{117}$$

Making the change of variables

$$p(z,t) = \frac{1}{2}(V+I),\ q(z,t) = \frac{1}{2}(V-I) \tag{118}$$

gives

$$p_z + p_t = -a(z)p(z,t) - (m(z) + b(z))q(z,t) \tag{119a}$$

$$q_z - q_t = -(m(z) - b(z))p(z,t) + a(z)q(z,t) \tag{119b}$$

which is almost in the desired form, and where

$$a(z) = \frac{1}{2}\left(\frac{G}{C} + \frac{R}{L}\right),\ b(z) = \frac{1}{2}\left(\frac{G}{C} - \frac{R}{L}\right). \tag{120}$$

Considering the scaled variables

$$p^1(z,t) = p(z,t)\exp\int_0^z a(u)\,du \tag{121a}$$

$$q^1(z,t) = q(z,t)\exp -\int_0^z a(u)\,du, \tag{121b}$$

and taking Fourier transforms yields the asymmetric two-component wave equations

$$\begin{aligned} \hat{p}^1_z &= -j\omega\hat{p}^1(z,\omega) - s(z)\hat{q}^1(z,\omega) \\ \hat{q}^1_z &= -r(z)\hat{p}^1(z,\omega) + j\omega\,\hat{q}^1(z,\omega) \end{aligned} \tag{122}$$

where

$$r(z) = (m-b)\exp -2\int_0^z a(u)du = \left(\frac{1}{4}\frac{d}{dz}\left(\ell n\frac{L}{C}\right) - \frac{1}{2}\left(\frac{G}{C} - \frac{R}{L}\right)\right)\exp -\int_0^z \left(\frac{R}{L} + \frac{G}{C}\right)du \tag{123a}$$

$$s(z) = (m+b)\exp 2\int_0^z a(u)du = \left(\frac{1}{4}\frac{d}{dz}\left(\ell n\frac{L}{C}\right) + \frac{1}{2}\left(\frac{G}{C} - \frac{R}{L}\right)\right)\exp\int_0^z \left(\frac{R}{L} + \frac{G}{C}\right)du. \tag{123b}$$

Thus, if we are given the scattering matrix $S(\omega)$ associated to the system (122), the coupled fast Cholesky or Schur recursions (106) and (110)-(112) may be used to reconstruct the rather bizarre quantities $r(z)$ and $s(z)$. Further, these two quantities are the most information about the line that can be obtained from this data. Although $r(z)$ and $s(z)$ may seem to be peculiar quantities, this result is in agreement with [46].

Note that in the event

$$R(z)/L(z) = G(z)/C(z) \tag{124}$$

we may recover $Z(z)$ and $R(z)/L(z)$ by multiplying and dividing $r(z)$ and $s(z)$, and then solving two differential equations. Thus, in this case it is possible to recover $R(z)$, $L(z)$, $C(z)$, and $G(z)$ in various ratios quite easily. This case is referred to as the <u>Heaviside condition</u> for a distortionless line [32], since if (124) holds then the true characteristic impedance $((R + j\omega L)/(G + j\omega C))^{1/2}$ which relates the current and voltage for a wave travelling down the line, is real. Thus, the current and voltage for such a wave are in phase, just as in the lossless line, and it is not surprising that ratios of various line parameters can be recovered, as in the lossless case.

VII. Conclusion

In this paper, the widespread applicability of the fast Cholesky and Schur recursions for the study of inverse scattering problems has been demonstrated. These algorithms were derived by using a layer stripping principle to reconstruct a scattering medium described by symmetric two-component wave equations, for the case when the medium is probed by impulsive waves. The applicability of these algorithms to the reconstruction of a nonuniform lossless transmission line, and to the inverse problem for a one-dimensional layered acoustic medium was demonstrated. In addition, it was shown that the linear least-squares estimation problem for a stationary process could be posed as an inverse scattering problem, and solved by the Schur algorithm.

Next, an asymmetric two-component inverse scattering problem was considered and solved by using a coupled set of fast Cholesky or Schur recursions. This was then applied to the inverse problem for non-uniform transmission lines with losses.

There are several topics which have not been discussed in this paper and which deserve further investigation. One of them is the study of the numerical properties of the Schur algorithm in the presence of noise or modelling uncertainties. The discrete-parameter Schur algorithm was recently shown to be numerically stable by Bultheel [47]. However, a numerically stable algorithm can perform poorly if it operates on ill conditioned data, which could happen for several of the physical problems that we have examined in this paper. This issue deserves therefore to be addressed. An additional feature of the layer stripping principle that we have used here to derive the fast Cholesky and Schur recursions is that it is quite general, and it is applicable to more general physical systems than those described by second order differential equations. For example, in [48], [49], it is shown that this principle can be applied to the reconstruction of a one-dimensional elastic medium described by four coupled first-order differential equations. A natural extension of this result would be to the study of general Hamiltonian systems.

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FIGURE CAPTIONS

Fig. 1: Elementary scattering sections obtained by discretizing the two-component wave equations.

Fig. 2: (a) Scattering for an impulsive wave incident from the left, and (b) from the right.

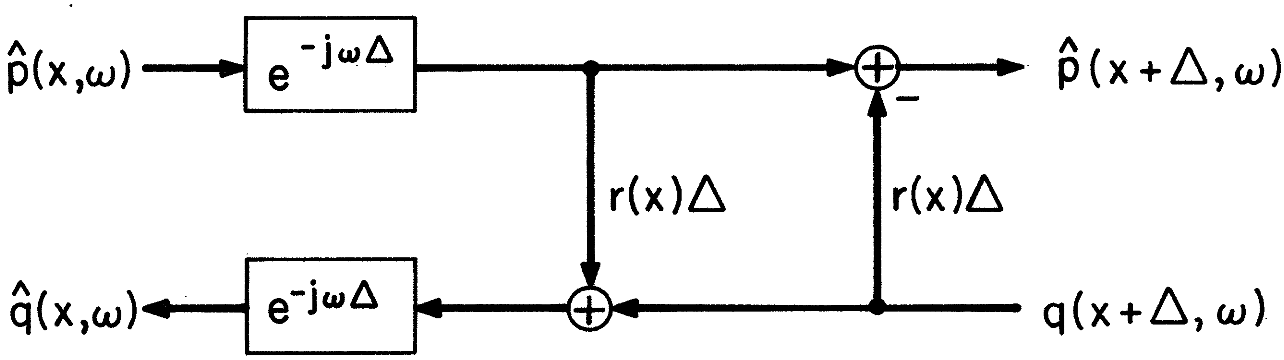
Fig. 3: Infinitesimal section of a lossless non-uniform transmission line.

Fig. 4: The perceived load to the right of x.

Fig. 5: Inverse Problem for a layered acoustic medium.

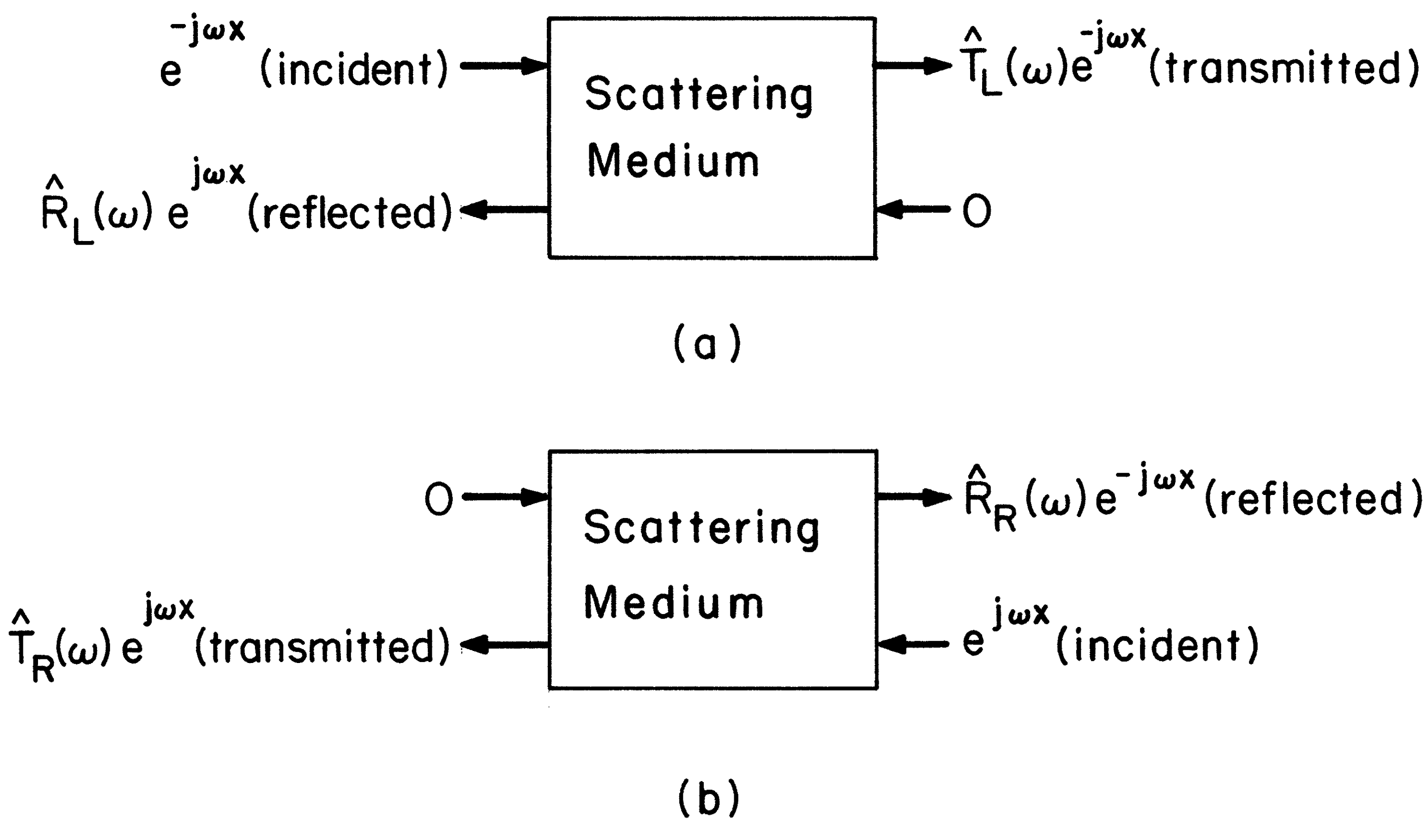
Fig. 6: (a) Aggregate modeling filter for $y(\cdot)$, and (b) infinitesimal ladder sections associated to the Krein-Levinson recursions.

Fig. 7: Infinitesimal section of a lossy non-uniform transmission line.



p̂(x,ω)
e^{-jωΔ}
r(x)Δ
r(x)Δ
p̂(x+Δ,ω)
q̂(x,ω)
e^{-jωΔ}
q(x+Δ,ω)

Fig. 1



e^{-jωx} (incident)
R̂_L(ω) e^{jωx} (reflected)
Scattering Medium
T̂_L(ω)e^{-jωx} (transmitted)
0
(a)
0
T̂_R(ω) e^{jωx} (transmitted)
Scattering Medium
R̂_R(ω) e^{-jωx} (reflected)
e^{jωx} (incident)
(b)

Fig. 2

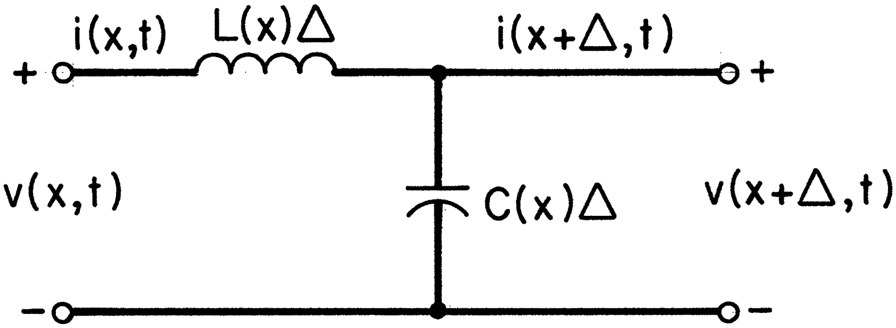

i(x,t)
L(x)Δ
i(x+Δ,t)
+
+
v(x,t)
C(x)Δ
v(x+Δ,t)
−
−

Fig. 3

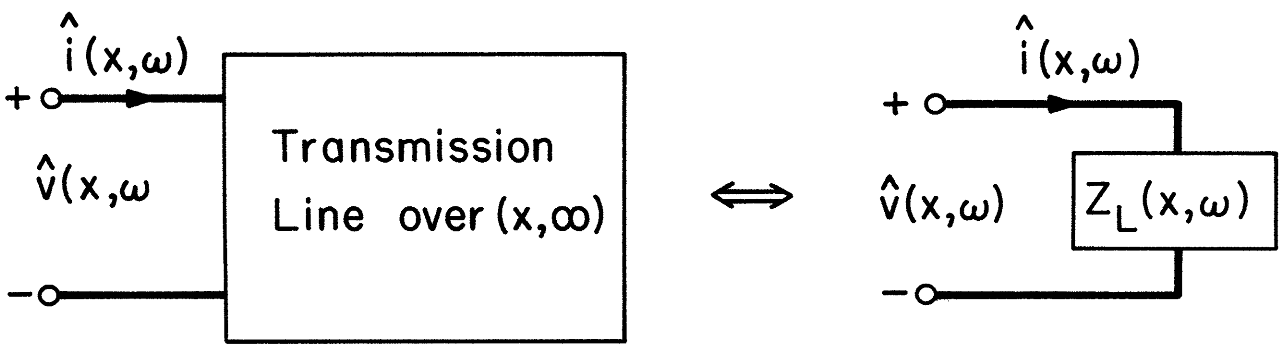

î(x,ω)
+
v̂(x,ω
−
Transmission
Line over (x,∞)
⟺
î(x,ω)
+
v̂(x,ω)
Z_L(x,ω)
−

Fig. 4

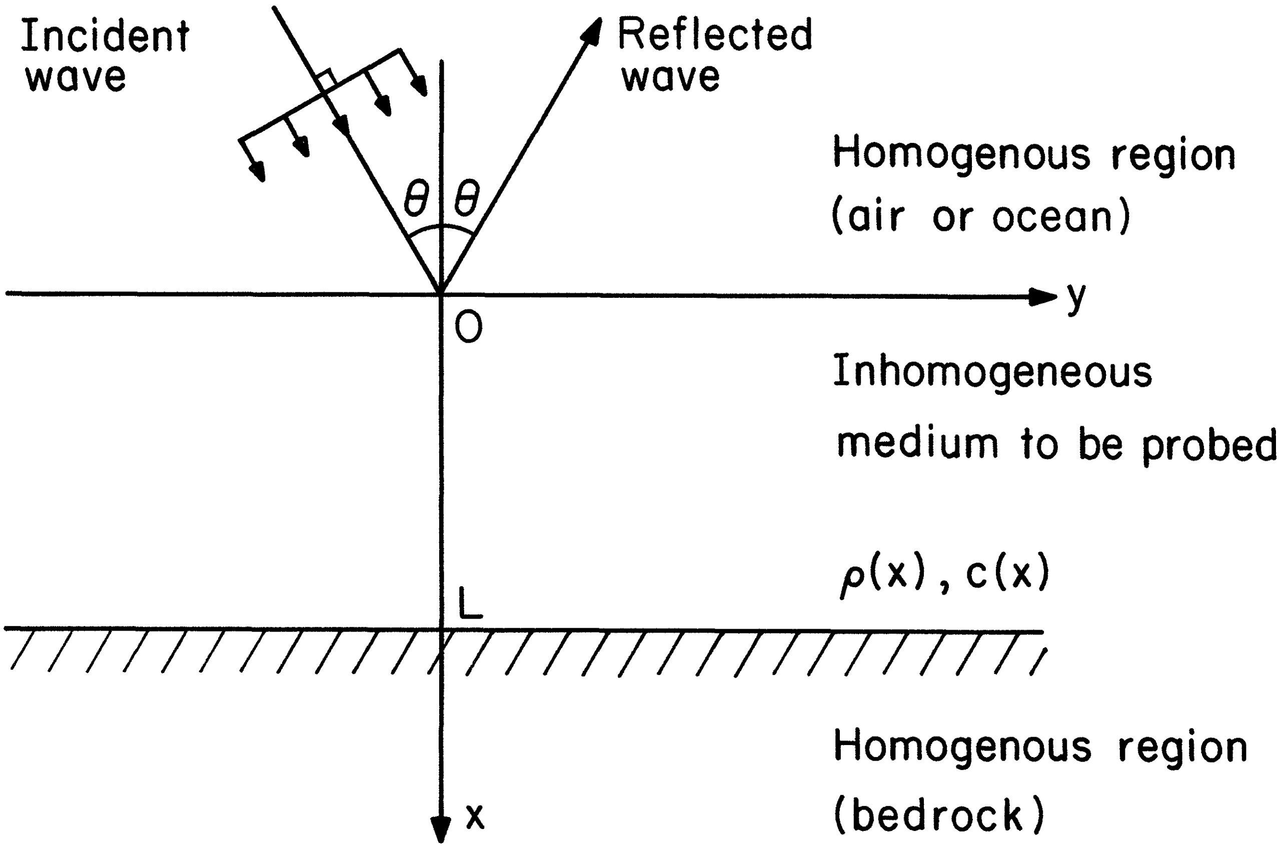

Incident
wave
Reflected
wave
θ
θ
Homogenous region
(air or ocean)
y
O
Inhomogeneous
medium to be probed
ρ(x), c(x)
L
Homogenous region
(bedrock)
x

Fig. 5

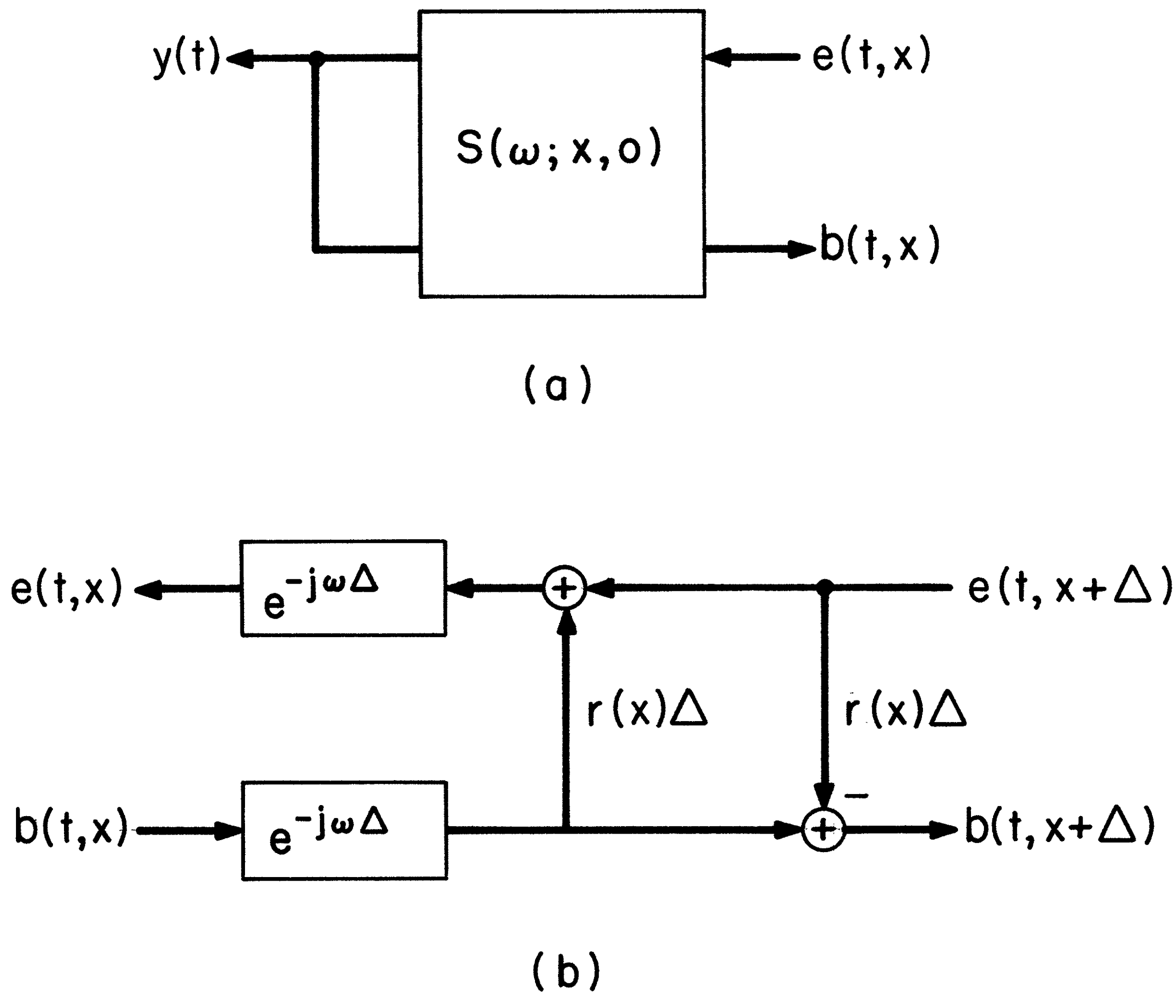

y(t)
e(t,x)
S(ω;x,o)
b(t,x)
(a)
e(t,x)
$e^{-j\omega\Delta}$
e(t,x+Δ)
r(x)Δ
r(x)Δ
b(t,x)
$e^{-j\omega\Delta}$
−
b(t,x+Δ)
(b)

Fig. 6

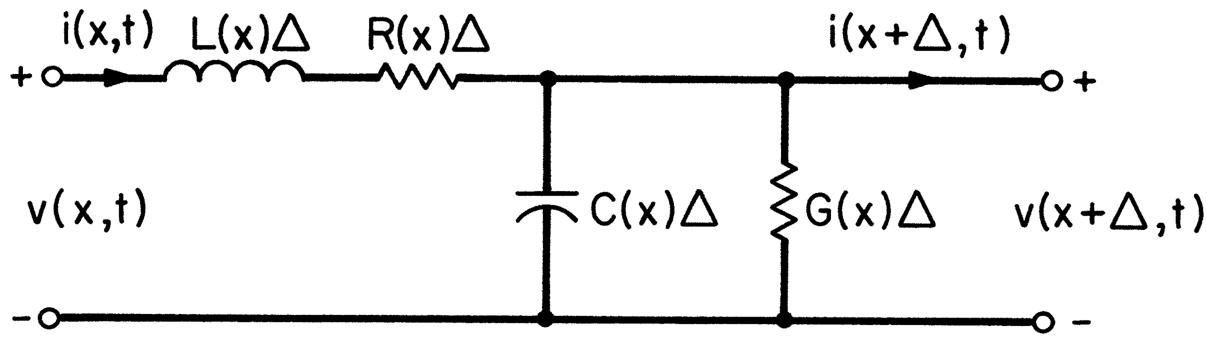


Fig. 7