# Delay-Aware Control Designs of Wide-Area Power Networks: Proofs of the Results ⋆


**Seyed Mehran Dibaji * Yildiray Yildiz ** Anuradha Annaswamy * Aranya Chakrabortty *** Damoon Soudbakhsh ****** 

* *Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139 USA, {dibaji,aannu}@mit.edu.*
** *Department of Mechanical Engineering, Bilkent University, Ankara, Turkey, yyildiz@bilkent.edu.tr*
*** *Department of Electrical and Computer Engineering, North Carolina State University, Raleigh, NC 27695 USA, achakra2@ncsu.edu*
**** *Department of Mechanical Engineering, George Mason University, Fairfax, VA 22030 USA, dsoudbak@gmu.edu*



**Abstract:** The purpose of this document is to provide proofs of the results in Dibaji et al. (2017).



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## 1. A DELAY-AWARE CONTROL DESIGN

### 1.1 Main Equations

The physical network follows the ordinary continuous-time LTI model,

$$\dot{x} = A_c x(t) + B_c u(t), \tag{1}$$

where $B_c = [B_c^1, \ldots, B_c^n]$. However, if there is no delay present in the network, the following sampled-data model is applied for the purpose of control designs.

$$x[k+1] = Ax[k] + Bu[k], \tag{2}$$

where

$$A = e^{A_c h} \quad \text{and} \quad B = \int_0^h e^{A_c s} B_c ds. \tag{3}$$

The delay-aware ANCS co-design model leads to (4),

$$x[k+1] = Ax[k] + B_1 U[k] + B_2 U[k-1], \tag{4}$$

where

$$B_{j1}^i = \begin{cases} \displaystyle\int_0^{h-\tau'_{ij}} e^{A_c s} ds B_c^i & \text{if } \ j = g(i), \quad \text{(5a)} \\ \displaystyle\int_{h-\tau'_{i(j+1)}}^{h-\tau'_{ij}} e^{A_c s} ds B_c^i, & \text{if } \ j \neq g(i), \quad \text{(5b)} \end{cases}$$

$$B_{i2}^i = \int_{h-\tau'_{i1}}^{h} e^{A_c s} ds B_c^i, \tag{6}$$

and $B_{j1}^i \in \mathbb{R}^{N\times 1}$ is the coefficient of $u_{ij}[k]$ and $B_{i2}^i \in \mathbb{R}^{N\times 1}$ is the coefficient of $u_{ig(i)}[k-1]$ in (7).

$$U_i(t) = \begin{cases} u_{ij}[k] & \text{if } t-kh \in [\tau'_{ij}, \tau'_{i(j+1)}), \ j < g(i), \\ u_{ig(i)}[k] & \text{if } t-kh \in [\tau'_{ig(i)}, h), \ j = g(i), \\ u_{ig(i)}[k-1] & \text{if } t-kh \in [0, \tau'_{i1}), \end{cases} \tag{7}$$

The extended form of (4) is given by

$$W_{4h}[k+1] = A_{4h} W_{4h}[k] + B_{4h} U[k], \tag{8}$$

where

$$W_{4h}[k] = \begin{bmatrix} x[k] \\ U[k-1] \\ x[k-1] \\ U[k-2] \\ \vdots \\ x[k-5] \\ U[k-6] \end{bmatrix}, \quad B_{4h} = \begin{bmatrix} B_1 \\ I_{\mathcal{G}} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

and $A_{4h} \in \mathbb{R}^{6(N+\mathcal{G})\times 6(N+\mathcal{G})}$ is

$$A_{4h} = \begin{bmatrix} A & B_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ I_N & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I_{\mathcal{G}} & 0 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & I_{\mathcal{G}} & 0 & 0 \end{bmatrix}. \tag{9}$$

### 1.2 Stability Analysis

*Lemma 1.* (Yuz and Goodwin, 2014) The matrix $A$ in (3) is non-singular and all of its eigenvalues, provided that $A_c$ in (1) is Hurwitz, are inside the unit circle.

**Proof.** Denoting $\mu_i$ as the $i$-th eigenvalue of $A$, it follows that $\mu_i = e^{h\lambda_i}$, where $\lambda_i$ is the $i$-th eigenvalue of $A_c$. If $\alpha_i =$


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$\mathrm{Re}(\lambda_i)$ and $\beta_i = \mathrm{Im}(\lambda_i)$, it follows that $\mu_i = e^{h\alpha_i}e^{jh\beta_i}$. If $A_c$ is Hurwitz, $\alpha_i$ is negative which in turn implies that $0 < |\mu_i| = e^{h\alpha_i} < 1$.

*Proposition 2.* If the system (1) is stable, $A_{4h}$ is Schur-stable with $N$ eigenvalues coinciding with all eigenvalues of $A$ and the remaining ones at zero.

**Proof.** The structure of $A_{4h}$ in (9) implies that $A_{4h}$ can be written in a block diagonal form as $A_{4h} = \begin{bmatrix} \Delta_{1,1} & \Delta_{2,1} \\ \Delta_{3,1} & \Delta_{4,1} \end{bmatrix}$, where $\Delta_{4,1} \in \mathbb{R}^{\mathcal{G}\times\mathcal{G}}$ is a zero matrix in the right lowest corner. It can also be shown that $\Delta_{2,1} = 0$. Hence, the eigenvalues of $A_{4h}$ include $\mathcal{G}$ zeros (Demmel, 1997). We now further partition $\Delta_{1,1} = \begin{bmatrix} \Delta_{1,2} & \Delta_{2,2} \\ \Delta_{3,2} & \Delta_{4,2} \end{bmatrix}$, where $\Delta_{4,2} \in \mathbb{R}^{N\times N}$ is also a zero matrix. Therefore it follows that $A_{4h}$ has an additional $N$ zero eigenvalues. By repeating the same process, we obtain the sub-partition $\Delta_{1,9} = \begin{bmatrix} A_h & 0 \\ [I_N \; 0] & 0 \end{bmatrix}$. Consequently, $\Delta_{1,10} = A_h$ whose eigenvalues include the $N$ eigenvalues of $A$ and additional $\mathcal{G}$ zeros. The complete process yields a total of $N$ eigenvalues which correspond to those of $A$ and all remaining $5N + 6\mathcal{G}$ eigenvalues at zero.

### 1.3 Controllability Analysis

*Lemma 3.* (Liu and Fong, 2012) The time-delay system

$$x[k+1] = \sum_{i=0}^{d_x} A_i x[k-i] + \sum_{i=0}^{d_u} B_i u[k-i],$$

is completely controllable if and only if

$$Y = [\sum_{i=0}^{d_x} \lambda^{d_x-i}A_i - \lambda^{d_x+1}I \quad \sum_{i=0}^{d_u} \lambda^{d_u-i}B_i]$$

has full rank at all roots of

$$|\sum_{i=0}^{d_x} \lambda^{d_x-i}A_i - \lambda^{d_x+1}I| = 0.$$

*Proposition 4.* If $(A_h, B_h)$ is controllable, $(A_{4h}, B_{4h})$ is stabilizable.

**Proof.** First, we restate the sufficient condition in terms of the criterion given in Lemma 3. For (4), $d_u = 1$ and $d_x = 0$. Hence $Y_1 = \big[A - \lambda I \;\; \lambda B_1 + B_2\big]$, where $\lambda$ is the solution to the equation $|A - \lambda I| = 0$. This means that $\lambda$ is an eigenvalue of $A$. But from Lemma 1, we know that $\lambda \neq 0$. Then, in (8), we have $d_x = 5$ and $d_u = 6$. So $Y_2 = \big[\nu^5 A - \nu^6 I \;\; \nu^6 B_1 + \nu^5 B_2\big]$, where $\nu$ is the solution to the equation $|\nu^5 A - \nu^6 I| = \nu^5|A - \nu I| = 0$. This means that either $\nu = 0$ or $\nu$ is an eigenvalue of $A$ which is non-zero. If $\nu$ is zero, $Y_2$ has not full rank. From the sufficient condition, we know that for all $\nu$ as eigenvalues of $A$, $\big[A - \nu I \;\; \nu B_1 + B_2\big]$ has full rank. Hence, rank of $Y_2 = \nu^5\big[A - \nu I \;\; \nu B_1 + B_2\big]$ for eigenvalues of $A$ is equal to the rank of $\big[A - \nu I \;\; \nu B_1 + B_2\big]$ which is full. Thus, the extended system (8) is not completely controllable; however, its non-zero eigenvalues, corresponding to $A_h$, can be altered using a proper control signal $U[k]$ and this makes $(A_{4h}, B_{4h})$ stabilizable.

## 2. EMPLOYING OUTPUT FEEDBACK

### 2.1 Main Equations

When the states are not accessible, the control input is modified as

$$U[k] = K_0^{(1)} x_o[k] + K_0^{(2)} \hat{x}_o[k] + G_0 U[k-1], \tag{10}$$

where the *estimated observer state vector* $\hat{x}_o$ is calculated using (11) as

$$\begin{aligned} \hat{x}_o[k] &= A x_o[k-1] + B_2 U[k-2] + B_1 U[k-1] \\ &\quad + L(y[k-1] - C x_o[k-1]). \end{aligned} \tag{11}$$

By using an appropriate output matrix $C \in \mathbb{R}^{s\times N}$, we determine the output vector $y \in \mathbb{R}^s$ as

$$y[k] = Cx[k]. \tag{12}$$

The observer error dynamics is obtained as

$$\tilde{x}[k+1] = (A - LC)\tilde{x}[k], \tag{13}$$

where $\tilde{x} = x - x_o$ is the observer error vector.

### 2.2 Stability Analysis

*Theorem 5.* The closed-loop output feedback control system consisting of (4), (10) and (11) is stable.

**Proof.** To obtain the closed-loop dynamics of the overall system, we first rewrite (4), and then by using (10) and (11):

$$\begin{aligned} x[k+1] &= Ax[k] + B_2 U[k-1] \\ &\quad + B_1\big(K_0^{(1)} x_o[k] + K_0^{(2)} \hat{x}_o[k] + G_0 U[k-1]\big) \\ &= Ax[k] + (B_2 + B_1 G_0)U[k-1] \\ &\quad + B_1 K_0^{(1)} x_o[k] + B_1 K_0^{(2)} \hat{x}_o[k] \\ &= Ax[k] + (B_2 + B_1 G_0)U[k-1] + \\ &\quad B_1 K_0^{(1)} x_o[k] + B_1 K_0^{(2)}\big(A x_o[k-1] + B_2 U[k-2] \\ &\quad + B_1 U[k-1] + L(y[k-1] - C x_o[k-1])\big) \\ &= Ax[k] + (B_2 + B_1 G_0 + B_1 K_0^{(2)} B_1)U[k-1] \\ &\quad + B_1 K_0^{(1)} x_o[k] + B_1 K_0^{(2)}\big(A - LC\big)x_0[k-1] \\ &\quad + B_1 K_0^{(2)} B_2 U[k-2] + B_1 K_0^{(2)} L y[k-1]. \end{aligned} \tag{14}$$

Using (12) and the fact that $x_0 = x - \tilde{x}$, (14) can be rewritten as

$$\begin{aligned} x[k+1] &= Ax[k] + (B_2 + B_1 G_0 + B_1 K_0^{(2)} B_1)U[k-1] \\ &\quad + B_1 K_0^{(1)}(x[k] - \tilde{x}[k]) \\ &\quad + B_1 K_0^{(2)}\big(A - LC\big)(x[k-1] - \tilde{x}[k-1]) \\ &\quad + B_1 K_0^{(2)} B_2 U[k-2] + B_1 K_0^{(2)} L C x[k-1] \\ &= \big(A + B_1 K_0^{(1)}\big)x[k] + B_1 K_0^{(2)} A x[k-1] \\ &\quad + (B_2 + B_1 G_0 + B_1 K_0^{(2)} B_1)U[k-1] \\ &\quad + B_1 K_0^{(2)} B_2 U[k-2] - B_1 K_0^{(1)} \tilde{x}[k] \\ &\quad - B_1 K_0^{(1)} \tilde{x}[k-1]. \end{aligned} \tag{15}$$

Similarly (10) can be rewritten as

$$
\begin{aligned}
U[k] &= K_0^{(1)}\big(x[k] - \tilde{x}[k]\big) \\
&\quad + K_0^{(2)}\big(Ax_o[k-1] + B_2U[k-2] + B_1U[k-1] \\
&\quad + L(y[k-1] - Cx_o[k-1])\big) + G_0U[k-1] \\
&= K_0^{(1)}x[k] - K_0^{(1)}\tilde{x}[k] + K_0^{(2)}Ax[k-1] \\
&\quad - K_0^{(2)}A\tilde{x}[k-1] + \big(K_0^{(2)}B_1 + G_0\big)U[k-1] \\
&\quad + K_0^{(2)}B_2U[k-2] + K_0^{(2)}LC\tilde{x}[k-1] \\
&= K_0^{(1)}x[k] - K_0^{(1)}\tilde{x}[k] + K_0^{(2)}Ax[k-1] \\
&\quad - K_0^{(2)}(A - LC)\tilde{x}[k-1] + \big(K_0^{(2)}B_1 + G_0\big)U[k-1] \\
&\quad + K_0^{(2)}B_2U[k-2]. \qquad (16)
\end{aligned}
$$

Defining $X[k] \equiv \big[x[k]^T\ U[k-1]^T\ \tilde{x}[k]^T\big]^T$ and using (13), (15) and (16), and by using the fact that $K_0^{(1)} + K_0^{(1)} = K_0$, we obtain the closed-loop dynamics as

$$
\begin{bmatrix} X[k+1] \\ X[k] \end{bmatrix} = H \begin{bmatrix} X[k] \\ X[k-1] \end{bmatrix}, \qquad (17)
$$

where

$$
H = \begin{bmatrix}
A + B_1K_0 & B_2 + B_1G_0 & -B_1K_0 & 0 & 0 & 0 \\
K_0 & G_0 & -K_0 & 0 & 0 & 0 \\
0 & 0 & A - LC & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0
\end{bmatrix}
$$

The characteristic equation for the system (17) can be calculated as

$$
\begin{aligned}
&\begin{vmatrix} \lambda I - (A + B_1K_0) & B_2 + B_1G_0 \\ K_0 & \lambda I - G_0 \end{vmatrix} \times |\lambda I - (A - LC)| \\
&\qquad \times \begin{vmatrix} \lambda I & 0 & 0 \\ 0 & \lambda I & 0 \\ 0 & 0 & \lambda I \end{vmatrix} = 0 \qquad (18)
\end{aligned}
$$

As seen from (18), the closed-loop system poles of the delayed power network with output feedback are the same as the state accessible case with the additional poles coming from the observer dynamics and additional poles at the origin. It was shown in Section 1 that the state accessible case is stable. We also showed that observer dynamics are stable. Therefore, the closed-loop system with output feedback is stable.

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