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AGGREGATE FEASIBILITY SETS FOR LARGE POWER NETWORKS*

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ABSTRACT

An aggregate feasibility set for large electric power systems is derived. The feasibility set consists of the set of substation loads that can be served in steady-state with the available generation without overloading the transmission lines or transformers. The derivation is based on an aggregate continuum model of large power systems that is obtained from the DC load flow model.

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Keywords. Power Systems, Large Scale Systems, Load Flow, Feasibility Set

INTRODUCTION

An integrated methodology that takes into account generation, transmission, and distribution systems simultaneously would be a useful planning and evaluation tool for electric utilities. Such a methodology, based on the evaluation of a hierarchy of attributes that describe the supply of and the demand for electricity independently from each other, has been presented in a series of documents, Dersin (1980), Dersin and Levis (1982a), and Dersin and Levis (1982b). A key issue in this work has been the characterization of electric energy service through the introduction of the feasibility set. This was defined as the set of steady state substation loads that can be served with the available generation resources without overloading the transmission lines or transformers. The feasibility set can also be viewed as a projection of a general security region in the space of loads and generation output levels (Hnyilicza et al, 1975; Fischl et al, 1976; Galiana, 1977; Banakar, 1980) on the space of loads.

An explicit description of the feasibility set, based on the DC load flow model, has been presented in Dersin and Levis (1982a) in which the requirements of load feasibility reduced to the requirement that a system of linear inequalities have a solution. However, the total number of conditions is $2(n+N)!/(n!)^2$, where N and n are the number of nodes and branches, respectively. The number of conditions can be reduced substantially, but not enough, by taking into account that at many nodes there is neither generation nor load. Therefore, an aggregate description of the feasibility set was sought which would be a good approximation when the systems considered are large. To that effect, a continuum model that corresponds to the DC load flow model is introduced. While the use of continuum models for describing large scale systems is not new —there is a long history, —this particular formulation has been developed for the explicit purpose of deriving an aggregate, approximate description of the feasibility set.

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AGGREGATE DC MODEL OF THE POWER SYSTEM

On the Continuum Viewpoint

The aggregation method involves substituting a geographical description of the power system for the topological (i.e., network-based) one. Let the geographical domain, D, over which the power system extends, be covered by a square grid. The spacings of the grid are of length Δ , and the elementary meshes of the grid are referred to as Δ-cells. On each Δ-cell, cell variables are defined in terms of the network variables that refer to branches and nodes contained in the cell. The network description is thereby replaced by an aggregate description in terms of cells variables. The more nodes and branches each Δ-cell contains, the less detailed the description. Therefore, the aggregation level is defined as the average number nodes in a Δ-cell. The scale, defined as the ratio of h, the greatest distance between any two points in D, to the cell side Δ, is an indicator of the number of Δ-cells needed to cover the power system and, therefore, of the number of aggregate variables. The proposed approach is applicable to very large systems where, simultaneously, the scale is large and the aggregation is reasonably high; the ratio Δ/h will be small and linear expansions will be allowed.

To simplify the notation, it will be convenient from now on to take the length h as the unit of distance. This is equivalent to representing the power system on a map in such a way that a physical length h corresponds to a unit length on the map. Then for a large system, modeled with a large scale, the map representation of the side of a Δ-cell is much smaller than 1. Consequently, from now on, Δ will be used instead of Δ/h.

The cell variables, obtained by aggregating the network variables over the respective cells, can be viewed as resulting from the discretization of continuous point functions. Namely, instead of dealing with a tableau of numbers — one number for each Δ-cell — one can deal with a continuous function of the location (x,y) whose average over the k^{th} Δ-cell is the k^{th} entry of the tableau. Dealing with continuous functions rather than network data makes it possible to circumvent combinatorial steps.

Defining aggregate load and generation variables is rather straightforward; an aggregate description of the transmission is more subtle. The load aggregate variable $\lambda(\Delta)$ associated with a given Δ-cell is defined as the ratio of the total load contributed by the nodes in that cell over the cell area:

$$\lambda(\Delta) = (\sum_i L_i)/\Delta^2 \quad (1)$$

Likewise, the cell variable $g(\Delta)$, which refers to the generated output, is defined as

$$g(\Delta) = (\sum_i G_i)/\Delta^2 \quad (2)$$

The continuum viewpoint implies that $\lambda(\Delta)$ and $g(\Delta)$ should be regarded as discretizations of density functions $\lambda(x,y)$ and $g(x,y)$.

Transmission Modeling from the Continuum Viewpoint

By relying on the analogy between the DC load flow equations and the relations between curents and voltages in a purely resistive circuit (Elgerd, 1971) it is possible to define quantities which are continuum equivalents of the flows f_m and the voltage angles δ_i.

In each Δ-cell, a value for the current density $\underline{i}(\Delta)$ is defined in terms of the branch power flows f_m corresponding to that cell, as follows: If a branch m is totally or partially contained in a Δ-cell (Fig. 1), its contribution to the $\underline{i}(\Delta)$ vector for that cell is defined by:

$$i_x = \frac{RS}{R_1 S_1} \varepsilon_m(x)\ f_m/\Delta \quad (3)$$

$$i_y = \frac{RS}{R_2 S_2} \varepsilon_m(y)\ f_m/\Delta \quad (4)$$

where f_m is the power (in megawatts) flowing through branch m; R, S, R_1, S_2 are defined in Figure 1, and $\varepsilon_m(x)$ is equal to +1 or −1, depending on whether the arbitrary orientation of branch m (used to define f_m) coincides with the direction of increasing x or decreasing x; $\varepsilon_m(y)$ has the corresponding meaning with respect to the y direction.

The total density vector $\underline{i}(\Delta)$ for a Δ-cell is obtained by adding up the vectors produced by all branches m in the cell.

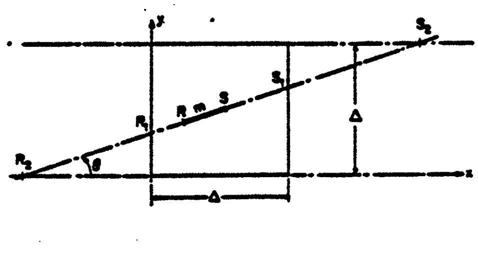

Figure 1. Aggregation Procedure within a Δ-cell

The node voltage angles δ are now approximated by assuming that they vary linearly within the Δ-cell as functions of the position coordinates, or, equivalently, by setting the electrical field $\underline{E}$

$$\underline{E} = -\ \underline{\text{grad}}\ \delta \quad (5)$$

and considering $\underline{E}$ to be constant within each Δ-cell. Denoting by (r_1,r_2) and (s_1,s_2) the coordinates of R and S, respectively, and by (E_x,E_y) the components of $\underline{E}$, it follows:

$$\delta_R - \delta_S = [E_x(s_1 - r_1) + E_y(s_2 - r_2)]\Delta + o(\Delta) \quad (6)$$

The constitutive relation of the DC model, which relates the real power flow f_m in a branch m and the voltage angles δ_j, δ_k at its extremity nodes j,k, is:

$$f_m = b_m\ (\delta_j - \delta_k) \quad (7)$$

where b_m is the susceptance (Sullivan,1977) of branch m (in megawatts). The above definitions of $\underline{i}$ and $\underline{E}$, and eq. (7) imply a linear relation between $\underline{i}$ and $\underline{E}$, to within $o(\Delta)$ terms. This relation is formally analogous to Ohm's law at a point (Kraus and Carver, 1975), and involves a matrix $\underline{\sigma}$, defined on each Δ-cell, as the following theorem indicates.

Theorem 1: The constitutive relation (7) of the DC flow model, which relates branch power flows and node voltage angles, can be expressed in a form analogous to Ohm's law at a point, in terms of cell aggregate variables:

$$\underline{i} = \underline{\sigma}\ \underline{E} \quad (8)$$

where terms that vanish with Δ are neglected. The conductance matrix $\underline{\sigma}$ takes a value on each Δ-cell, which is defined in terms of the discrete transmission parameters in that cell by:

$$\underline{\sigma} = \sum_m \begin{bmatrix} b_m\beta_m^2 \cos^2\theta_m & b_m\beta_m^2 \sin\theta \cos\theta_m \\ b_m\beta_m^2 \sin_m\theta_m \cos\theta_m & b_m\beta_m^2 \sin^2\theta_m \end{bmatrix} \quad (9)$$

where b_m is the susceptance of the portion of branch m in the cell; β_m is the length of branch m relative to Δ (thus, a dimensionless number) $\beta_m = RS/\Delta$; and θ_m is the angle that defines the orientation of branch m, i.e.,

$$\theta_m = \tan^{-1} \frac{s_2 - r_2}{s_1 - r_1}$$

if branch m connects nodes R and S.

The proof is given in Dersin, (1980) ch. 5.

The DC model includes the flow conservation law, a continuum expression of which can be given by

$$\text{div}(\underline{i}) = g(x,y) - \lambda(x,y) \quad (10)$$

It can be shown (Dersin 1980) that in the limit of an infinitely large system $(\Delta\to 0)$, both formulations, the discrete and the continuous, coincide. A complete continuum formulation of the DC load flow problem can now be given:

Theorem 2: In the limit of an infinitely large system $(\Delta\to 0)$, the DC load flow problem approaches the following boundary value problem in the function $\delta(x,y)$:

$$\text{div}\,(\underline{\sigma}\ \underline{\text{grad}}\ \delta) = \lambda(x,y) - g(x,y) \qquad (x,y) \in D^o \qquad (11)$$

$$\underline{\text{grad}}\ \delta^T\ \underline{\sigma}\ \underline{n} = 0 \qquad (x,y) \in \partial D \qquad (12)$$

The following compatibility condition, which expresses the equality of load and generation, must hold:

$$\iint_D [\lambda(x,y) - g(x,y)]\,dxdy = 0 \qquad (13)$$

The matrix $\underline{\sigma}$ is the location-dependent conductance, as defined by (9) from the discrete data; the quantities $\lambda(x,y)$ and $g(x,y)$ are the load and generation densities, respectively; the geographical domain which contains the power system is denoted by D (with interior D^o and boundary ∂D); the outward normal to the boundary ∂D is denoted by $\underline{n}$.

Proof: Dersin (1980),ch. 5.

Isotropy and Homogeneity

If the $\underline{\sigma}(x,y)$ matrix correponds to an isotropic medium, i.e., if

$$\underline{\sigma}(x,y) = \sigma(x,y)\ I$$

where I is the identity matrix, then (11) reduces to:

$$\sigma\left(\frac{\partial^2\delta}{\partial x^2} + \frac{\partial^2\delta}{\partial y^2}\right) + \frac{\partial\sigma}{\partial x}\frac{\partial\delta}{\partial x} + \frac{\partial\sigma}{\partial y}\frac{\partial\delta}{\partial y} = \lambda(x,y) - g(x,y) \qquad (14)$$

If furthermore, homogeneity prevails, i.e., in (14), $\sigma(x,y)$ is independent of the location, the partial differential equation becomes Poisson's:

$$\frac{\partial^2\delta}{\partial x^2} + \frac{\partial^2\delta}{\partial y^2} = (1/\sigma)\ [\lambda(x,y) - g(x,y)] \qquad (15)$$

with the boundary condition

$$\frac{\partial\delta}{\partial n} = 0 \qquad \text{for } (x,y) \in \partial D$$

so that the classical Neumann problem is obtained. A straightforward example of a network that leads to an isotropic medium is provided by a square grid.

More general networks can also give rise to isotropic — and sometimes homogeneous— continua, when certain statistical properties occur which refer to the density of the transmission (in miles per square mile) and the orientation of transmission lines. These properties involve (1) variation of the transmission density (in miles per square mile) around a constant as the aggregation level increases; (2) uniform distribution of branch orientations. The homogeneous-isotropic case is important because it leads to explicit results on the feasibility set.

Comparison Between Discrete and Continuous Solution

The aggregate variables have been defined so as to be close to the network variables they approximate when the level of aggregation is at its lowest (one node per cell) and the scale is large (small Δ). This means that i_x and i_y are close to f_m, $\lambda(x,y)\Delta^2$ close to L_i, and $g(x,y)\Delta^2$ close to G_i. The analogous property for the voltage angles δ_i and their continuous approximation $\delta(x,y)$ can be proved directly, by comparing the partial differential equation (11) with the discrete load flow equations in the case of the lowest aggregation level and for a special class of networks that consists of rectangular grids. The physical network can then be used to discretize the partial differential equation, i.e., to replace it by a system of finite-difference equations, which is found to be precisely the system of the DC load flow equations.

The low aggregation level is considered first with each Δ-cell containing only one node, (Fig. 2). Using the notation N,W, S, E (North, West, South, East) to refer to the nodes neighboring a given node 0, the result for the homogeneous isotropic case can be stated as follows.

Theorem 3: For the network of Fig. 2, where all branches have the same susceptance b, the equations satisfied by the node variables δ_i are:

$$\frac{\delta_W - 2\delta_0 + \delta_E}{\Delta^2} + \frac{\delta_N - 2\delta_0 + \delta_S}{\Delta^2} = \frac{1}{b}\ [\lambda(0) - g(0)] \qquad (16)$$

where $[g(0) - \lambda(0)]\Delta^2$ is the injection at node 0.

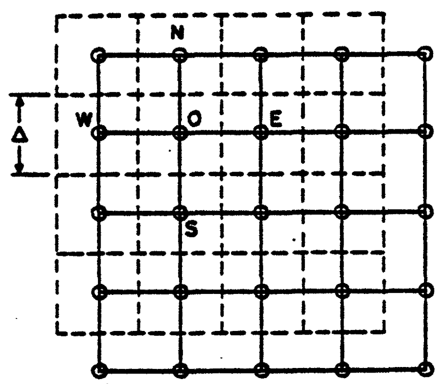

Figure 2. Lowest Aggregation Level

Equation (16) is the finite-difference approximation, with step Δ, of the Poisson equation (15). Accordingly, the following bound holds:

$$|\delta(x_o,y_o) - \delta_o| < C\ \Delta^2|\log\Delta| \qquad (17)$$

where C is a constant independent of Δ.

Proof: See Dersin (1980). The bound (17) is specific of the Poisson equation (Wedland, 1979).

Consider now a higher aggregation level, when each Δ-cell would contain k^2 nodes instead of one. The result of theorem 3 is then weakened, as indicated in the following corrollary.

Corrollary 3: For a rectangular grid network where the aggregation level is chosen such that each Δ-

cell contains k^2 nodes, the following bound holds for the error between the value of the continuous point function δ at any point (x,y) of the Δ-cell and the arithmetic average of the node variables in the cell.

$$\left|\delta(x,y) - \sum_{i=1}^{k^2} \frac{\delta_i}{k^2}\right| \le O(\Delta) + C\frac{\Delta^2}{k^2}\left|\log\frac{\Delta}{k}\right| \qquad (18)$$

The constant C is independent of Δ and $O(\Delta)$ denotes a quantity which vanishes with Δ.

The proof relies on theorem 3 and the triangle inequality. These approximation results can be extended to the general (inhomogeneous, anisotropic) case (Dersin, 1980).

THE AGGREGATE FEASIBILITY SET

The Aggregate Coefficients

The feasibility set was introduced in Dersin and Levis (1982a) to denote the set of substation loads than can be served in steady state with the available generation resources without overloading the transmission lines or transformers. When the distribution of power flow through the network is represented by the DC load flow (Sullivan, 1977) then the feasibility set can be described by a system of inequalities which are linear in the real bus loads. Therefore, the set is a convex polyhedron in the space of substation load vectors. The defining inequalities are:

$$\sum_{i\epsilon A} (tP_i^*)L_i - \sum_{i\epsilon A} (tP_i^*)^+ S_i$$

$$\le \sum_{m\epsilon B} \varphi_m |t|\ \left|b_m(P_j^* - P_k^*) + \alpha^*(z_r) - \alpha^*(z_h)\right|$$

$$\text{for } t > 0 \text{ and } t < 0 \qquad (19)$$

where (1) L_i is the real bus load at bus i (in megawatts), S_i is the maximum generating capacity at bus i (in megawatts), and φ_m is the maximum phase angle bound across branch m; (2) A is a subset of the set of node indices and B is a subset of the set of branch indices; (3) For each compatible choice of A and B, the coefficients P_i^* and $\alpha^*(z_r)$ are determined as the solution of a linear algebraic system; furthermore, the P_i^* coefficients constitute the solution of a DC load flow in the network with the branches of B removed; (4) The factor t is arbitrary and can be positive or negative; (5) the notation $(\ )^+$ refers to the positive part of that quantity.

The aggregate description of large power systems will be used to derive an aggregate feasibility set. First, one obtains cell variables P_k which are approximations of the coefficients P_i^* in (19). A continuous function $P(x,y)$, the solution of the continuum DC load flow corresponding to the discrete DC load flow of which P_i^* is the solution, satisfies the following equation:

$$\text{div}\ (\underline{\sigma}\ \underline{\text{grad}}\ P) = \sum_{m=1}^{\nu} F_m[\delta(x-x_j, y-y_j)$$

$$-\delta(x-x_k, y-y_k)] \qquad (20)$$

with boundary conditions as stated in Theorem 2. In (20), δ refers to the Dirac impulse function; (x_j,y_j) and (x_k,y_k) are respectively the coordinates of j and k, the extremities of branch m. Thus, the P function satisfies the homogeneous equation

$$\text{div}\ (\underline{\sigma}\ \underline{\text{grad}}\ \underline{P}) = 0 \qquad (21)$$

An explicit assumption is now made about the scale and the aggregation level: the <u>scale</u> is <u>large</u>, so that Δ is small as compared to the extent of the domain D and the <u>aggregation level</u> is <u>high</u> enough so that the length β of a branch is smaller than Δ.

To translate those assumptions into simplifications in the analytical expression for the solution $P(x,y)$, the two branch extremities (x_j,y_j) and (x_k,y_k) are modeled as though they were infinitely close to each other. Then the distance β is made to go to zero while the intensity F of the source goes to infinity in such a way that the product $F\beta$ remains constant, say M. Then, the function $P(x,y)$ reaches a limit, which is the potential created by a <u>dipole</u> (Webster, 1957) of <u>moment</u> M located ar the limit location (ξ,η) and directed along the straight line from source to sink. As the process just described is carried out, r_1 and r_2 tend to a common value r (Fig. 3), and P reaches a limit. For an arbitrary number ν of singularities,

$$P(x,y) = \frac{1}{2\pi\sigma} \sum_m (M_m \cos\theta_m / r_m) + C_1 \qquad (22)$$

The parameters $M_1, M_2, \ldots, M\nu$, and C_1 in the dipole approximation are determined to within an arbitrary multiplier by requiring $P(x,y)$ to vanish on the ν nodes of the subset A.

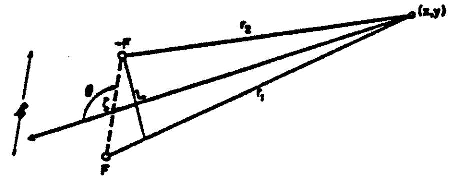

Figure 3. Dipole model of a branch

In the homogeneous-isotropic case, $P(x,y)$ is a harmonic function. Therefore, there exists an analytic function of the complex variable $z=x+iy$ of which the imaginary part is $\sigma P(x,y)$ and $\alpha(x,y)$ its real part. Let the branch set B consist of one single branch, modeled as a dipole and let the boundary ∂D of the domain be far enough from the dipole location ζ. If the dipole has moment M and orientation γ, the corresponding functions P and α are obtained as follows:

$$\alpha(x,y) + i\sigma\, P(x,y) = \frac{M}{2\pi}\ \frac{e^{i\gamma}}{z-\zeta} + C$$

$$= \frac{1}{2\pi}\ \frac{m}{z-\zeta} + C \qquad (23)$$

where C is an arbitrary complex constant. For the derivation, see Dersin (1980), ch 6.

Once the $P(x,y)$ function is known, the $\alpha(x,y)$ function can be obtained from it by integration, using the Cauchy-Riemann equations. From this property, it can be shown that the $\alpha(x,y)$ function approximates the $\alpha(z_k)$ coefficients in (19) in the same way as $P(x,y)$ approximates the P_i^* coefficients.

When the assumptions of homogeneity and isotropy are removed, a function $\alpha(x,y)$ can still be defined, which is related to $P(x,y)$ by a system of two linear first-order partial differential equations. In the case of several dipoles ($\nu>1$), the analytic function is obtained as a summation over all the dipoles, analogously to (22).

The Feasibility Conditions (homogeneous-isotropic case)

Theorem 4: In accordance with the continuous-space modeling of load, generation and the node and region variables, the feasibility conditions are modeled by the following integral inequalities.

$$\iint \{\lambda(x,y)[t\,P(x,y)]-s(x,y)[tP(x,y)]^{+}\}dxdy \le \sum_{k=1}^{n} \varphi(\zeta_k)|M_k| \quad ; \quad t>0,\ t<0 \qquad (24)$$

One such pair of inequalities arises for each choice of an integer ν where $0 \le \nu \le q$; a node subset $\bar{A}\subset\{1,\dots,q\}$ with ν nodes; a distribution of ν dipoles located at $\zeta_1,\dots,\zeta_\nu$, with given orientations $\gamma_1,\dots,\gamma_\nu$ and moments $M_1,\dots,M_\nu$ to be determined.

Given $\bar{A}$ and the dipole distribution, the function $P(x,y)$ is expressed by:

$$P(x,y) = \frac{1}{2\pi\sigma} \sum_{k=1}^{\nu} M_k \quad \mathrm{Im}\left[\frac{e^{i\gamma_k\zeta}}{z-k}\right] + C_1 \qquad (25)$$

The ν dipole moments M_k and the constant C_1 are determined to within an arbitrary multipler t by requiring that P vanish on the ν points $z = a_i$ of $\bar{A}$. Accordingly, $P(x,y)$ is determined to within an arbitrary multiplier t.

Equation (25) is valid only if the dipoles are far enough from the boundary ∂D so that boundary condition ($\partial p/\partial n = 0$) is automatically satisfied; if this is not the case, a harmonic function $u(x,y)$ must be added to fulfill the boundary condition.

The variable φ_m is a bound on the magnitude of a difference of voltage phase angles. If branch m connects nodes j and k, the constraint is:

$$|\delta_j - \delta_k| \le \varphi_m \qquad (26)$$

The bound φ_m is modeled by a scalar positive function $\varphi(x,y)$, which acts as a bound on the magnitude of the electrical field:

$$\max_{\text{over } \underline{n} \text{ s.t. } \|\underline{n}\| = 1} |\underline{E}.\underline{n}| \le \max_{\text{over } \underline{n}} \varphi(x,y,\underline{n}) = \varphi(x,y) \qquad (27)$$

As pointed out earlier, the aggregate feasibility set is defined in R^K, where K is the number of cells that contain loads. The expression (24) for the feasibility conditions is a continuous-space approximation of the aggregate conditions; these can be written as follows:

$$\sum_{k=1}^{K} \{[t\,\bar{P}^k]\,\lambda_k(\Delta) - [t\,\bar{P}^k]^{+}\,s_k(\Delta)\}\,\Delta_2 \le \sum^{n} \varphi(\zeta_j)\,|M_j|\,|t| \qquad t>0 \text{ and } t<0 \qquad (28)$$

where $\bar{P}^k$ is the average of the function $P(x,y)$ given by eq. (25) over all k.

Since each dipole distribution produces a pair of conditions (28) there are an infinity of linear inequalities such as (28) in the lumped loads $\lambda(\Delta_k)$. This implies that the feasibility set is now a smooth convex set instead of a convex polyhedron. However, not all constraints involve all loads necessarily: it may occur that $\bar{P}^k$ is zero for some cells k. Therefore, there may be points of the boundary where the tangent hyperplane is not defined; the set boundary is then only piecewise continuously differentiable.

CONCLUSIONS

The formulation of the DC load flow problem as a boundary problem for a partial differential equation, the coefficients of which are derived from the discrete transmission data, is used to derive directly an aggregate feasibility set in the space of lumped load vectors. This feasibility set is an aggregate in that no distinction can be made between load patterns which differ only by the internal distribution of the load among buses of the same Δ-cell. The method is direct in that all the necessary calculations are performed at the aggregate level from aggregate data: the method does not involve deriving the exact feasibility conditions and the somehow aggregating them.

The method applies to large power networks that extend over large enough geographical domain so that a square grid can be superimposed on that domain with most Δ-cells not empty and with Δ small with respect to the dimensions of the domain. These conditions permit the application of the continuum viewpoint, namely, the substitution of analytical formulations for combinatorial (i.e., graph-theoretic) ones.

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