14.126 GAME THEORY

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1. Existence and Continuity of Nash Equilibria

Follow Muhamet's slides. We need the following result for future reference.

Theorem 1. *Suppose that each S_i is a convex and compact subset of an Euclidean space and that each u_i is continuous in s and quasi-concave in s_i. Then there exists a* ***pure strategy*** *Nash equilibrium.*

2. Bayesian Games

When some players are uncertain about the characteristics or **types** of others, the game is said to have **incomplete information**. Most often a player's type is simply defined by his payoff function. More generally, types may embody any private information that is relevant to players' decision making. This may include, in addition to the player's payoff function, his beliefs about other players' payoff functions, his beliefs about what other players believe his beliefs are, and so on. The idea that a situation in which players are unsure about each other's payoffs and beliefs can be modeled as a Bayesian game, in which a player's type encapsulates all his uncertainty, is due to Harsanyi (1967, 1968) and has been formalized by Mertens and Zamir (1985). For simplicity, we assume that a player's type is his own payoff and the type captures all the private information.

A **Bayesian game** is a list $\mathcal{B} = (N, S, \Theta, u, p)$ with

- $N = \{1, 2, \ldots, n\}$ is a finite set of **players**
- S_i is the set of **pure strategies** of player i; $S = S_1 \times \ldots \times S_n$
- Θ_i is the set of **types** of player i; $\Theta = \Theta_1 \times \ldots \times \Theta_n$

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- $u_i : \Theta \times S \to \mathbb{R}$ is the **payoff function** of player i; $u = (u_1, \ldots, u_n)$
- $p \in \Delta(\Theta)$ is a common prior (we can relax this assumption).

We often assume that Θ is finite and the marginal $p_i(\theta_i)$ is positive for each type θ_i.

Example 1 (First Price Auction with I.I.D. Private Values). *One object is up for sale. Suppose that the value θ_i of player $i \in N$ for the object is uniformly distributed in $\Theta_i = [0, 1]$ and that the values are independent across players. This means that if $\tilde{\theta}_i \in [0, 1], \forall i$ then $p(\theta_i \leq \tilde{\theta}_i, \forall i) = \prod_i \tilde{\theta}_i$. Each player i submits a bid $s_i \in S_i = [0, \infty)$. The player with the highest bid wins the object and pays his bid. Ties are broken randomly. Hence the payoffs are given by*

$$u_i(\theta, s) = \begin{cases} \frac{\theta_i - s_i}{|\{j \in N | s_i = s_j\}|} & \text{if } s_i \geq s_j, \forall j \in N \\ 0 & \text{otherwise.} \end{cases}$$

Example 2 (An exchange game). *Each player $i = 1, 2$ receives a ticket on which there is a number in some finite set $\Theta_i \subset [0, 1]$. The number on a player's ticket represents the size of a prize he may receive. The two prizes are independently distributed, with the value on i's ticket distributed according to F_i. Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player's prize, hence $S_i = \{agree, \ disagree\}$. If both players agree then the prizes are exchanged; otherwise each player receives his own prize. Thus the payoff of player i is*

$$u_i(\theta, s) = \begin{cases} \theta_{3-i} & \text{if } s_1 = s_2 = agree \\ \theta_i & \text{otherwise.} \end{cases}$$

In the **normal form representation** $G(\mathcal{B})$ **of the Bayesian game** $\mathcal{B}$ player i has strategies $(s_i(\theta_i))_{\theta_i \in \Theta_i} \in S_i^{\Theta_i}$ and utility function U_i given by

$$U_i((s_i(\theta_i))_{\theta_i \in \Theta_i, i \in N}) = \mathbb{E}_p(u_i(\theta, s_1(\theta_1), \ldots, s_n(\theta_n))).$$

The **agent-normal form representation** $AG(\mathcal{B})$ **of the Bayesian game** $\mathcal{B}$ has player set $\cup_i \Theta_i$. The strategy space of each player θ_i is S_i. A strategy profile $(s_{\theta_i})_{\theta_i \in \Theta_i, i \in N}$ yields utility

$$U_{\theta_i}((s_{\theta_i})_{\theta_i \in \Theta_i, i \in N}) = \mathbb{E}_p(u_i(\theta, s_{\theta_1}, \ldots, s_{\theta_n}) | \theta_i)$$

for player θ_i. For the conditional expectation to be well-defined we need $p_i(\theta_i) > 0$.

Definition 1. *A Bayesian Nash equilibrium of $\mathcal{B}$ is a Nash equilibrium of $G(\mathcal{B})$.*

Proposition 1. *If $p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i, i \in N$, a strategy profile is a Nash equilibrium of $G(\mathcal{B})$ iff it is a Nash equilibrium of $AG(\mathcal{B})$ (strategies are mapped across the two games by $s_i(\theta_i) \to s_{\theta_i}$).*

Theorem 2. *Suppose that*

- *N and Θ are finite*
- *each S_i is a compact and convex subset of an Euclidean space*
- *each u_i is continuous in s and concave in s_i.*

Then $\mathcal{B}$ has a pure strategy Bayesian Nash equilibrium.

Proof. By Proposition 1, it is sufficient to show that $AG(\mathcal{B})$ has a pure Nash equilibrium. The latter follows from Theorem 1. We use the concavity of u_i in s_i to show that the corresponding U_{θ_i} is quasi-concave in s_{θ_i}. Quasi-concavity of u_i in s_i does not typically imply quasi-concavity of U_{θ_i} in s_{θ_i} because U_{θ_i} is an integral of u_i over variables other than s_{θ_i}.[1] □

We can show that the set of Bayesian Nash equilibria of $\mathcal{B}^x$ is upper-hemicontinuous with respect to x when payoffs are given by u^x, assumed continuous in x in a compact set X, if S, Θ are finite. Indeed, $BNE(\mathcal{B}^x) = NE(AG(\mathcal{B}^x))$. Furthermore, we have upper-hemicontinuity with respect to beliefs.

Theorem 3. *Suppose that N, S, Θ are finite. Let $P \subset \Delta(\Theta)$ be such that for every $p \in P$ $p_i(\theta_i) > 0, \forall \theta_i \in \Theta_i, i \in N$. Then $BNE(\mathcal{B}^p)$ is upper-hemicontinuous in p over P.*

Proof. Since $BNE(\mathcal{B}^p) = NE(G(\mathcal{B}^p))$, it is sufficient to note that

$$U_i((s_i(\theta_i))_{\theta_i \in \Theta_i, i \in N}) = \mathbb{E}_p(u_i(\theta, s_1(\theta_1), \ldots, s_n(\theta_n)))$$

(as defined for $G(\mathcal{B}^p)$) is continuous in p. □

[1]Sums of quasi-concave functions are not necessarily quasi-concave.

3. Extensive Form Games

An **extensive form game** consists of

- a finite set of **players** $N = \{1, 2, \ldots, n\}$; nature is denoted as "player 0"
- the order of moves specified by a **tree**
- each player's payoffs at the terminal nodes in the tree
- information partition
- the set of actions available at every information set and a description of how actions lead to progress in the tree
- moves by nature.

A **tree** is a **directed graph** $(X, >)$—there is a link from node x to node y if $x > y$, which we interpret as "x precedes y." We assume that X is **finite**, there is an initial node $\phi \in X$, $>$ is transitive $(x > y, y > z \Rightarrow x > z)$ and asymmetric $(x > y \Rightarrow y \not> x)$. Hence the tree has no cycles. We also require that each node $x \neq \phi$ has exactly one immediate predecessor, i.e., $\exists x' > x$ such that $x'' > x, x'' \neq x'$ implies $x'' > x'$. A node is **terminal** if it does not precede any other node; this means that the set of terminal nodes is $Z = \{z | \not\exists x, z > x\}$. Each $z \in Z$ completely determines a path of moves though the tree (recall the finiteness assumption), with associated payoff $u_i(z)$ for player i.

An **information partition** is a partition of $X \setminus Z$. Node x belongs to the information set $h(x)$. The same player, denoted $i(h) \in N \cup \{0\}$, moves at each node $x \in h$ (otherwise players might disagree on whose turn to move is). The interpretation is that $i(h)$ is uncertain whether he is at x or some other $x' \in h(x)$. We abuse notation writing $i(x) = i(h(x))$.

The set of available **actions** at $x \in X \setminus Z$ for player $i(x)$ is $A(x)$. We assume that $A(x) = A(x') =: A(h), \forall x' \in h(x)$ (otherwise $i(h)$ might play an infeasible action). A function l labels each node $x \neq \phi$ with the last action taken to reach it. We require that the restriction of l to the immediate successors of x be bijective on $A(x)$. Finally, a move by nature at some node x corresponds to a probability distribution over $A(x)$.

Let $H_i = \{h | i(h) = i\}$. The set of **pure strategies** for player i is $S_i = \prod_{h \in H_i} A(h)$. As usual, $S = \prod_{i \in N} S_i$. A strategy is a complete contingent plan specifying an action to be taken at each information set (if reached). We can define **mixed strategies** as probability distributions over pure strategies, $\sigma_i \in \Delta(S_i)$. Any mixed strategy profile $\sigma \in \prod_{i \in N} \Delta(S_i)$, along with the distribution of the moves by nature and the labeling of nodes with actions, leads to a probability distribution $O(\sigma) \in \Delta(Z)$. We denote by $u_i(\sigma) = \mathbb{E}_{O(\sigma)}(u_i(z))$. The associated **normal form** game is (N, S, u).

Two strategies $s_i, s_i' \in S_i$ are equivalent if $O(s_i, s_{-i}) = O(s_i', s_{-i}), \forall s_{-i}$, that is, they lead to the same distribution over outcomes regardless of how the opponents play. See figure 3.9 in FT p. 86. S_i^R is a subset of S_i that contains exactly one strategy from each equivalence class. The **reduced normal form** game is given by (N, S^R, u).

A **behavior strategy** specifies a distribution over actions for each information set. Formally, a behavior strategy $b_i(h)$ for player $i(h)$ at information set h is an element of $\Delta(A(h))$. We use the notation $b_i(a|h)$ for the probability of action a at information set h. A behavior strategy b_i for i is an element of $\prod_{h \in H_i} \Delta(A(h))$. A profile b of behavior strategies determines a distribution over Z in the obvious way. Clearly, b_i is equivalent to σ_i with

$$\sigma_i(s_i) = \prod_{h \in H_i} b_i(s_i(h)|h),$$

where $s_i(h)$ denotes the projection of s_i on $A(h)$.

To guarantee that every mixed strategy is equivalent to a behavior strategy we need to impose the additional requirement of **perfect recall**. Basically, prefect recall means that no player ever forgets any information he once had and all players know the actions they have chosen previously. See figure 3.5 in FT, p. 81. Formally, perfect recall stipulates that if $x'' \in h(x')$, x is a predecessor of x' and the same player i moves at both x and x' (and thus at x'') then there is a node $\hat{x}$ in the same information set as x (possibly x itself) such that $\hat{x}$ is a predecessor of x'' and the action taken at x along the path to x' is the same as the action taken at $\hat{x}$ along the path to x''. Intuitively, the nodes x' and x'' are distinguished by information i does not have, so he cannot have had it at $h(x)$; x' and x'' must be consistent with the same action at $h(x)$ since i must remember his action there.

Let $R_i(h)$ be the set of pure strategies for player i that do not preclude reaching the information set $h \in H_i$, i.e., $R_i(h) = \{s_i | h \text{ is on the path of some } (s_i, s_{-i})\}$. If the game has prefect recall, a mixed strategy σ_i is equivalent to a behavior strategy b_i defined by

$$b_i(a|h) = \frac{\sum_{\{s_i \in R_i(h) | s_i(h) = a\}} \sigma_i(s_i)}{\sum_{s_i \in R_i(h)} \sigma_i(s_i)},$$

when the denominator is positive and any distribution when it is zero.

Many different mixed strategies can generate the same behavior strategy. Consider the example from FT p. 88, figure 3.12. Player 2 has four pure strategies, $s_2 = (A, C), s_2' = (A, D), s_2'' = (B, C), s_2''' = (B, D)$. Now consider two mixed strategies, $\sigma_2 = (1/4, 1/4, 1/4, 1/4)$, which assigns probability $1/4$ to each pure strategy, and $\sigma_2 = (1/2, 0, 0, 1/2)$, which assigns probability $1/2$ to each of s_2 and s_2'''. Both of these mixed strategies generate the behavior strategy b_2 with $b_2(A|h) = b_2(B|h) = 1/2$ and $b_2(C|h') = b_2(D|h') = 1/2$. Moreover, for any strategy σ_1 of player 1, all of σ_2, σ_2', b_2 lead to the same probability distribution over terminal nodes. For example, the probability of reaching node z_1 equals the probability of player 1 playing U times $1/2$.

The relationship between mixed and behavior strategies is different in the game illustrated in FT p. 89, figure 3.13, which is not a game of perfect recall (player 1 forgets what he did at the initial node; formally, there are two nodes in his second information set which obviously succeed the initial node, but are not reached by the same action at the initial node). Player 1 has four strategies in the strategic form, $s_1 = (A, C), s_1' = (A, D), s_1'' = (B, C), s_1''' = (B, D)$. Now consider the mixed strategy $\sigma_1 = (1/2, 0, 0, 1/2)$. As in the last example, this generates the behavior strategy $b_1 = \{(1/2, 1/2), (1/2, 1/2)\}$, where player 1 mixes $50 - 50$ at each information set. But b_1 is *not* equivalent to the σ_1 that generated it. Indeed (σ_1, L) generates a probability $1/2$ for the terminal node corresponding to (A, L, C) and a $1/2$ probability for (B, L, D). However, since behavior strategies describe independent randomizations at each information set, (b_1, L) assigns probability $1/4$ to each of the four paths $(A, L, C), (A, L, D), (B, L, C), (B, L, D)$. Since both A vs. B and C vs. D are choices made by player 1, the strategy σ_1 under which player 1 makes all his decisions at once allows choices at different information sets to be correlated. Behavior strategies cannot produce this correlation in the example, because when it comes time to choose between C and D, player 1 has forgotten whether he chose A or B.

Theorem 4 (Kuhn 1953)**.** *Under perfect recall, mixed and behavioral strategies are equivalent.*

Hereafter we restrict attention to games with perfect recall, and use mixed and behavior strategies interchangeably. Behavior strategies prove more convenient in many arguments and constructions. We drop the notation b for behavior strategies and instead use $\sigma_i(a_i|h)$ to denote player i's probability of playing action a_i at information set h...

4. Backward Induction and Subgame Perfection

An extensive form game has **perfect information** if all information sets are singletons. Backward induction can be applied to any extensive form game of perfect information with finite X (which means that the number of "stages" and the number of actions feasible at any stage are finite). The idea of backward induction is formalized by Zermelo's algorithm. Since the game is finite, it has a set of penultimate nodes, i.e., nodes whose (all) immediate successors are terminal nodes. Specify that the player who moves at each such node chooses the strategy leading to the terminal node with the highest payoff for him. In case of a tie, make an arbitrary selection. Next each player at nodes whose immediate successors are penultimate (or terminal) nodes chooses the action maximizing his payoff over the feasible successors, given that players at the penultimate nodes play as assumed. We can now roll back through the tree, specifying actions at each node. At the end of the process we have a pure strategy for each player. It is easy to check that the resulting strategies form a Nash equilibrium.

Theorem 5 (Zermelo 1913; Kuhn 1953)**.** *A finite game of perfect information has a pure-strategy Nash equilibrium.*

Moreover, the backward induction solution has the nice property that each player's actions are optimal at every possible history if the play of the opponents is held fixed, which we call subgame perfection. More generally, subgame perfection extends the logic of backward induction to games with imperfect information. The idea is to replace the "smallest" proper subgame with one of its Nash equilibria and iterate this procedure on the reduced tree. In stages where the "smallest" subgame has multiple Nash equilibria, the procedure implicitly assumes that all players believe the same equilibrium will be played. To define subgame perfection formally we first need the definition of a proper subgame. Informally, a proper subgame is a portion of a game that can be analyzed as a game in its own right.

Definition 2. *A* ***proper subgame*** *G of an extensive form game T consists of a single node x and all its successors in T, with the property that if $x' \in G$ and $x'' \in h(x')$ then*

$x'' \in G$. The information sets and payoffs of the subgame are inherited from the original game. That is, two nodes are in the same information set in G if and only if they are in the same information set in T, and the payoff function on the subgame is just the restriction of the original payoff function to the terminal nodes of G.

The requirements that all the successors of x be in the subgame and that the subgame does not "chop up" any information set ensure that the subgame corresponds to a situation that could arise in the original game. In figure 3.16, p. 95 of FT, the game on the right is not a subgame of the game on the left, because on the right player 2 knows that player 1 has not played L, which he did not know in the original game.

Together, the requirements that the subgame begin with a single node x and respect information sets imply that in the original game x must be a singleton information set, i.e. $h(x) = \{x\}$. This ensures that the payoffs in the subgame, conditional on the subgame being reached, are well defined. In figure 3.17, p. 95 of FT, the "game" on the right has the problem that player 2's optimal choice depends on the relative probabilities of nodes x and x', but the specification of the game does not provide these probabilities. In other words, the diagram on the right cannot be analyzed as a separate game; it makes sense only as a component of the game on the left, which provides the missing probabilities.

Since payoffs conditional on reaching a proper subgame are well-defined, we can test whether strategies yield a Nash equilibrium when restricted to the subgame.

Definition 3. *A behavior strategy profile σ of an extensive form game is a* ***subgame perfect equilibrium*** *if the restriction of σ to G is a Nash equilibrium of G for every proper subgame G.*

Because any game is a proper subgame of itself, a subgame perfect equilibrium profile is necessarily a Nash equilibrium. If the only proper subgame is the whole game, the sets of Nash and subgame perfect equilibria coincide. If there are other proper subgames, some Nash equilibria may fail to be subgame perfect.

It is easy to see that subgame perfection coincides with backward induction in finite games of perfect information. Consider the penultimate nodes of the tree, where the last choices are made. Each of these nodes begins a trivial one-player proper subgame, and Nash equilibrium in these subgames requires that the player make a choice that maximizes his payoff. Thus any subgame perfect equilibrium must coincide with a backward induction solution at every penultimate node, and we can continue up the tree by induction.

5. Important Examples of Extensive Form Games

5.1. Repeated games with observable actions.

- **time** $t = 0, 1, 2, \ldots$ (usually infinite)
- **stage game** is a normal-form game G
- G is played every period t
- players **observe** the realized actions at the end of each period
- **payoffs** are the sum of discounted payoffs in the stage game.

Repeated games are a widely studied class of dynamic games. There is a lot of research dealing with various restrictions on the information about past play.

5.2. Multistage games with observable actions.

- **stages** $k = 0, 1, 2, \ldots$
- at stage k, after having **observed** a "non-terminal" history of play $h = (a^0, \ldots, a^{k-1})$, each player i simultaneously chooses an **action** $a_i^k \in A_i(h)$
- **payoffs** given by $u(h)$ for each terminal history h.

Often it is natural to identify the "stages" of the game with time periods, but this is not always the case. A game of perfect information can be viewed as a multistage game in which every stage corresponds to some node. At every stage all but one player (the one moving at the node corresponding to that stage) have singleton action sets ("do nothing"; can refer to these players as "inactive"). Repeated versions of perfect information extensive form games also lead to multistage games, e.g., the Rubinstein (1982) alternating bargaining game, which we discuss later.

6. Single (or One-Shot) Deviation Principle

Consider a multi-stage game with observed actions. We show that in order to verify that a strategy profile σ is subgame perfect, it suffices to check whether there are any histories h_t where some player i can gain by deviating from the actions prescribed by σ_i at h_t and conforming to σ_i elsewhere.

Definition 4. *A strategy σ_i is* ***unimprovable*** *given σ_{-i} if $u_i(\sigma_i, \sigma_{-i}| h_t) \geq u_i(\sigma_i', \sigma_{-i}| h_t)$ for every $t \geq 0, h_t \in H_i$ and $\sigma_i' \in \Delta(S_i)$ with $\sigma_i'(h_{t'}') = \sigma_i(h_{t'}')$ for all $h_{t'}' \in H_i \setminus \{h_t\}$.*

Hence a strategy σ_i is unimprovable if after every history, no strategy that differs from it at only one information set can increase utility. Obviously, if σ is a subgame perfect equilibrium then σ_i is unimprovable given σ_{-i}. To establish the converse, we need an additional condition.

Definition 5. *A game is* ***continuous at infinity*** *if for each player i the utility function u_i satisfies*

$$\lim_{t\to\infty} \sup_{\{(h,\tilde{h})|h_t=\tilde{h}_t\}} |u_i(h) - u_i(\tilde{h})| = 0.$$

This condition requires that events in the distant future are relatively unimportant. It is satisfied if the overall payoffs are a discounted sum of per-period payoffs and the stage payoffs are uniformly bounded.

Theorem 6. *Consider a (finite or infinite horizon) multi-stage game with observed actions*[2] *that is continuos at infinity. If σ_i is unimprovable given σ_{-i} then σ_i is a best response to σ_{-i} conditional on any history h_t.*

Proof. Suppose that σ_i is unimprovable given σ_{-i}, but σ_i is not a best response to σ_{-i} following some history h_t. Let σ_i^1 be a strictly better response and define

$$\varepsilon = u_i(\sigma_i^1, \sigma_{-i}|h_t) - u_i(\sigma_i, \sigma_{-i}|h_t) > 0. \tag{6.1}$$

Since the game is *continuos at infinity*, there exists $t' > t$ and σ_i^2 such that σ_i^2 is identical to σ_i^1 at all information sets up to (and including) stage t', σ_i^2 coincides with σ_i across all longer histories and

$$|u_i(\sigma_i^2, \sigma_{-i}|h_t) - u_i(\sigma_i^1, \sigma_{-i}|h_t)| < \varepsilon/2. \tag{6.2}$$

In particular, 6.1 and 6.2 imply that

$$u_i(\sigma_i^2, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t).$$

[2] We allow for the possibility that the action set be infinite at some stages.

Denote by σ_i^3 the strategy obtained from σ_i^2 by replacing the stage t' actions following any history $h_{t'}$ with the corresponding actions under σ_i. Conditional on any history $h_{t'}$, the strategies σ_i and σ_i^3 coincide, hence

$$u_i(\sigma_i^3, \sigma_{-i}|h_{t'}) = u_i(\sigma_i, \sigma_{-i}|h_{t'}). \tag{6.3}$$

As σ_i is *unimprovable* given σ_{-i}, and σ_i and σ_i^2 only differ at stage t' conditional on $h_{t'}$, we need

$$u_i(\sigma_i, \sigma_{-i}|h_{t'}) \geq u_i(\sigma_i^2, \sigma_{-i}|h_{t'}). \tag{6.4}$$

Then 6.3 and 6.4 lead to

$$u_i(\sigma_i^3, \sigma_{-i}|h_{t'}) \geq u_i(\sigma_i^2, \sigma_{-i}|h_{t'})$$

for all histories $h_{t'}$ (consistent with h_t). Since σ_i^2 and σ_i^3 coincide before reaching stage t', we obtain

$$u_i(\sigma_i^3, \sigma_{-i}|h_t) \geq u_i(\sigma_i^2, \sigma_{-i}|h_t).$$

Similarly, we can construct $\sigma_i^4, \ldots, \sigma_i^{t'-t+3}$. The strategy $\sigma_i^{t'-t+3}$ is identical to σ_i conditional on h_t and

$$u_i(\sigma_i, \sigma_{-i}|h_t) = u_i(\sigma_i^{t'-t+3}, \sigma_{-i}|h_t) \geq \ldots \geq u_i(\sigma_i^3, \sigma_{-i}|h_t) \geq u_i(\sigma_i^2, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t),$$

a contradiction. □

7. Iterated Conditional Dominance

Definition 6. *In a multistage game with observable actions, an action a_i is conditionally dominated at stage t given history h_t if in the subgame starting at h_t every strategy for player i that assigns positive probability to a_i is strictly dominated.*

Proposition 2. *In any perfect information game, every subgame perfect equilibrium survives iterated elimination of conditionally dominated strategies.*

8. Bargaining with Alternating Offers

The set of players is $N = \{1, 2\}$. For $i = 1, 2$ we write $j = 3 - i$. The set of feasible utility pairs is $U \subset \mathbb{R}^2$, assumed to be compact and convex with $(0, 0) \in U$.[3] Time is discrete and infinite, $t = 0, 1, \ldots$ Each player i discounts payoffs by δ_i, so receiving u_i at time t is worth $\delta_i^t u_i$.

Rubinstein (1982) analyzes the following perfect information game. At every time $t = 0, 1, \ldots$, player $i(t)$ proposes an alternative $u = (u_1, u_2) \in U$ to player $j = 3 - i(t)$; the bargaining protocol specifies that $i(t) = 1$ for t even and $i(t) = 2$ for t odd. If j accepts the offer, then the game ends yielding a payoff vector $(\delta_1^t u_1, \delta_2^t u_2)$. Otherwise, the game proceeds to period $t + 1$. If agreement is never reached, each player receives a 0 payoff.

For each player i, it is useful to define the function g_i by setting

$$g_i(u_j) = \max\{u_i | (u_1, u_2) \in U\}.$$

Notice that the graphs of g_1 and g_2 coincide with the Pareto-frontier of U.

8.1. Stationary subgame perfect equilibrium.

Let (m_1, m_2) be the unique solution to the following system of equations

$$\begin{aligned} m_1 &= \delta_1 g_1(m_2) \\ m_2 &= \delta_2 g_2(m_1). \end{aligned}$$

Note that (m_1, m_2) is the intersection of the graphs of the functions $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$.

We are going to argue that the following "stationary" strategies constitute the unique subgame perfect equilibrium. In any period where player i has to make an offer to j, he offers u with $u_j = m_j$ and j accepts only offers u with $u_j \geq m_j$. We can use the *single-deviation principle* to check that this is a subgame perfect equilibrium.

8.2. Equilibrium uniqueness.

We prove that the subgame perfect equilibrium is unique by arguing that it is essentially the only strategy profile that survives *iterated conditional dominance*.

[3] The set of feasible utility outcomes U can be generated from a set of contracts or decisions X in a natural way. Define $U = \{(v_1(x), v_2(x)) | x \in X\}$ for a pair of utility functions v_1 and v_2 over X. With additional assumptions on X, v_1, v_2 we can ensure that the resulting U is compact and convex.

Theorem 7. *If a strategy profile survives iterative elimination of conditionally dominated strategies, then it is identical to the stationary subgame perfect equilibrium except for the nodes at which a player is indifferent between accepting and rejecting an offer in the subgame perfect equilibrium.*

Proof. Since player i can get 0 by never reaching an agreement, offering an alternative that gives him less than

$$m_i^0 = 0$$

or accepting such an offer at any history is conditionally dominated. All such offers are eliminated at the first stage of the iteration. Then i should never expect to receive more than

$$M_i^0 = \delta_i g_i (0)$$

in any future period following a disagreement. Hence rejecting an offer u with $u_i > M_i^0$ is conditionally dominated by accepting such an offer for i. Once we eliminate the latter strategies, i always accepts offers u with $u_i > M_i^0$ from j. Then making offers u with $u_i > M_i^0$ is dominated for j by offers $\bar{u} = \lambda u + (1 - \lambda) (M_i^0, g_j (M_i^0))$ for $\lambda \in (0, 1)$. We remove all the strategies involving such offers.

Under the surviving strategies, j can reject an offer from i and make an offer next period that leaves him with slightly less than $g_j (M_i^0)$, which i accepts. Hence accepting any offer that gives him less than

$$m_j^1 = \delta_j g_j \left(M_i^0\right)$$

is dominated for j. Moreover, making such offers is dominated for j because we argued above that offers with $u_i > M_i^0$ are dominated. After we eliminate such moves, i cannot expect more than

$$M_i^1 = \delta_i g_i \left(m_j^1\right) = \delta_i g_i \left(\delta_j g_j \left(M_i^0\right)\right)$$

in any future period following a disagreement.

We can recursively define the sequences

$$\begin{aligned} m_j^{k+1} &= \delta_j g_j \left(M_i^k\right) \\ M_i^{k+1} &= \delta_i g_i \left(m_j^{k+1}\right) \end{aligned}$$

for $i = 1, 2$ and $k \geq 1$. Since both g_1 and g_2 are decreasing functions, we can easily show that the sequence (m_i^k) is increasing and (M_i^k) is decreasing. By arguments similar to those above, we can prove by induction on k that, at some stage in the iteration, player $i = 1, 2$

- never accepts or makes an offer with $u_i < m_i^k$
- always accepts offers with $u_i > M_i^k$, but making such offers is dominated for j.

The sequences (m_i^k) and (M_i^k) are monotonic and bounded, so they need to converge. The limits satisfy

$$\begin{aligned} m_j^\infty &= \delta_j g_j \left(\delta_i g_i \left(m_j^\infty\right)\right) \\ M_i^\infty &= \delta_i g_i \left(m_j^\infty\right). \end{aligned}$$

It follows that (m_1^∞, m_2^∞) is the (unique) intersection point of the graphs of the functions $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$. Moreover, $M_i^\infty = \delta_i g_i \left(m_j^\infty\right) = m_i^\infty$. Therefore, no strategy for i that rejects u with $u_i > M_i^\infty = m_i^\infty$ or accepts u with $u_i < m_i^\infty = M_i^\infty$ survives iterated elimination of conditionally dominated strategies. Also, no strategy for i to offer u with $u_i \neq M_i^\infty = m_i^\infty$ survives. □

9. Nash Bargaining

Assume that U is such that g_2 is decreasing, strictly concave and continuously differentiable (derivative exists and is continuous). The **Nash (1950) bargaining solution** u^* is defined by $\{u^*\} = \arg\max_{u \in U} u_1 u_2 = \arg\max_{u \in U} u_1 g_2(u_1)$. It is the outcome $(u_1^*, g_2(u_1^*))$ uniquely pinned down by the first order condition $g_2(u_1^*) + u_1^* g_2'(u_1^*) = 0$. Indeed, since g_2 is decreasing and strictly concave, the function f, given by $f(x) = g_2(x) + x g_2'(x)$, is strictly decreasing and continuous and changes sign on the relevant range.

Theorem 8. *Suppose that $\delta_1 = \delta_2 =: \delta$ in the alternating bargaining model. Then the unique subgame perfect equilibrium payoffs converge to the Nash bargaining solution as $\delta \to 1$.*

Proof. Recall that the subgame perfect equilibrium payoffs are given by $(g_1(m_2), m_2)$ where (m_1, m_2) satisfies

$$\begin{aligned} m_1 &= \delta g_1 \left(m_2\right) \\ m_2 &= \delta g_2 \left(m_1\right). \end{aligned}$$

It follows that $g_1(m_2) = m_1/\delta$, hence $m_2 = g_2(g_1(m_2)) = g_2(m_1/\delta)$. We rewrite the equations as follows

$$\begin{aligned} g_2(m_1/\delta) &= m_2 \\ g_2(m_1) &= m_2/\delta. \end{aligned}$$

By the mean value theorem, there exists $\xi \in (m_1, m_1/\delta)$ such that $g_2(m_1/\delta) - g_2(m_1) = (m_1/\delta - m_1)g_2'(\xi)$, hence $(m_2 - m_2/\delta) = (m_1/\delta - m_1)g_2'(\xi)$ or, equivalently, $m_2 + m_1 g_2'(\xi) = 0$. Substituting $m_2 = \delta g_2(m_1)$ we obtain $\delta g_2(m_1) + m_1 g_2'(\xi) = 0$.

Note that $(g_1(m_2), m_2)$ converges to u^* as $\delta \to 1$ if and only if (m_1, m_2) does. In order to show that (m_1, m_2) converges to u^* as $\delta \to 1$, it is sufficient to show that any limit point of (m_1, m_2) as $\delta \to 1$ is u^*. Let (m_1^*, m_2^*) be such a limit point corresponding to a sequence $(\delta_k)_{k \geq 0} \to 1$. Recognizing that m_1, m_2, ξ are functions of δ, we have

$$\delta_k g_2(m_1(\delta_k)) + m_1(\delta_k) g_2'(\xi(\delta_k)) = 0. \tag{9.1}$$

Since $\xi(\delta_k) \in (m_1(\delta_k), m_1(\delta_k)/\delta_k)$ with $m_1(\delta_k), m_1(\delta_k)/\delta_k \to m_1^*$ as $k \to \infty$ and g_2' is continuos by assumption, in the limit 9.1 becomes $g_2(m_1^*) + m_1^* g_2'(m_1^*) = 0$. Therefore, $m_1^* = u_1^*$. $\square$