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An Algebraic Approach to Analysis and Control of Time-Scales†

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Abstract

The structure of time-scales in systems of the form $\dot{x} = A(\varepsilon)x$ is related to the invariant factors of $A(\varepsilon)$ when this matrix is over the ring of functions analytic at 0. This relationship motivates the study of invariant factor assignment in the matrix $A(\varepsilon) + B(\varepsilon) K(\varepsilon)$ by choice of $K(\varepsilon)$. Results on this problem have implications for assignment of time-scales by state feedback in systems of the form $\dot{x} = A(\varepsilon)x + B(\varepsilon)u$. Work in this direction is presented.

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1. Introduction

Perturbed, linear, time-invariant systems of the form

$$\dot{x}(t) = A(\varepsilon)x(t) + B(\varepsilon)u(t) \tag{1.1}$$

are the focus of this paper. Here x and u are n- and m-dimensional state and control vectors respectively; ε is a small positive perturbation parameter; and $A(\varepsilon)$, $B(\varepsilon)$ are analytic at $\varepsilon = 0$.

We shall show that an algebraic approach to the study of the above system, with $A(\varepsilon)$, $B(\varepsilon)$ considered as matrices over the (local) ring W of functions of ε that are analytic at $\varepsilon = 0$, leads to new perspectives and results on multiple-time-scale behavior in this system. Proofs are omitted here, but relevant ones will appear in the final version.

We begin with the undriven situation, where

$$\dot{x} = A(\varepsilon)x \ . \tag{1.2}$$

We assume, with no essential loss of generality, that $A(\varepsilon)$ is nonsingular for ε in $(0, \varepsilon_0)$, for some $\varepsilon_0 > 0$; the situation of interest is where $A(0)$ is singular. Section 2 reviews a familiar special case of this, namely the class of two-time-scale systems extensively studied by Kokotovic and co-workers, [1], [2], in the so-called explicit form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ . \tag{1.3}$$

(The forms actually used in [1], [2] can be brought to (1.3) by a simple change of time-scale.) The section then outlines the extension of these results, to multiple-time-scale decompositions of systems of the form (1.2), that have recently been obtained by Coderch _et_ _al_., [3].

The procedure suggested in [3] for extracting and displaying the multiple-time-scale structure of (1.2) is described in terms of operations such as projection and pseudo-inversion. We complement that viewpoint in this paper by presenting a slightly more concrete version of it that turns out to be very fruitful. Our procedure is the natural generalization of the one used for (1.3), and makes clear the role of the _invariant factors_ of $A(\varepsilon)$ (in the ring W), a role that is suggested (but not developed) in [3]. This algorithm is presented in Section 3; the basis for it is a transformation of $A(\varepsilon)$ to Smith form, which is outlined in that section.

Section 4 turns to questions of _feedback control of time-scales_ in the system (1.1), assuming state feedback of the form

$$u = K(\varepsilon)x \quad , \tag{1.4}$$

with $K(\varepsilon)$ again a matrix over W. Noting the above interpretation of invariant factors of the system matrix, the question of _invariant-factor assignment_ in $A(\varepsilon) + B(\varepsilon)K(\varepsilon)$ is raised and, for the case of left-coprime $A(\varepsilon)$ and $B(\varepsilon)$, answered rather completely. To actually assign time-scales by such feedback requires that certain stability conditions be also satisfied, but our invariant factor results show what the limits are.

2. Background

A special case of (1.2) that has received a great deal of attention in the control literature, see [1], [2], is the system (1.3). It is known that, if A_{11} and $A_{22} - A_{21} A_{11}^{-1} A_{12}$ are nonsingular, the eigenvalues of (1.3) occur in two groups, one being of order 1 and lying "close" to the eigenvalues of A_{11}, and the other being of order ε and close to the eigenvalues of $\varepsilon(A_{22} - A_{21} A_{11}^{-1} A_{12})$. If both the latter matrices are Hurwitz, then the system exhibits well-behaved two-time-scale structure, in the following sense:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_{1f}(t) + x_{1s}(\varepsilon t) + 0(\varepsilon) \\ x_{2s}(\varepsilon t) + 0(\varepsilon) \end{bmatrix} \quad , \ t \geq 0, \tag{2.1}$$

where

$$\dot{x}_{1f} = A_{11} x_{1f} , \quad x_{1f} = x_1(0)$$

$$x_{1s} = -A_{11}^{-1} A_{12} x_{2s} , \text{ and}$$

$$\dot{x}_{2s} = (A_{22} - A_{21} A_{11}^{-1} A_{12}) x_{2s}, \quad x_{2s}(0) = x_2(0) \quad .$$

The subscripts s and f denote <u>slow</u> and <u>fast</u> subsystems.

For some appropriate constant, nonsingular matrix T, it can be shown that

$$\lim_{\varepsilon \downarrow 0} \sup_{t \geq 0} \| Te^{A(\varepsilon)t} T^{-1} - e^{A_d t} \| = 0, \tag{2.2}$$

where the block diagonal matrix A_d is given by

$$A_d = \begin{bmatrix} A_{11} & 0 \\ 0 & \varepsilon\tilde{A}_{22} \end{bmatrix} , \quad \tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12} \quad . \tag{2.3}$$

(2.2) and (2.3) provide an alternative definition of what it means to have well-behaved two-time-scale structure.

The above decomposition has found significant applications. Two criticisms that may, however, be noted are: firstly, that the system is assumed <u>given</u> in the explicit form (1.3); and secondly, that the nonsingularity assumptions on A_{11} and $\tilde{A}_{22}$ restrict the system behavior to <u>two</u> time scales. Recent work of Coderch <u>et al</u>., [3], has attempted to address both objections. It starts with the more general form (1.2), assuming however that a Taylor series for $A(\varepsilon)$ is given. It is then shown that an expression of the form (2.2) holds, with a different T and with A_d of the form

$$A_d = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon^m \tilde{A}_m \end{bmatrix} . \tag{2.4}$$

The $\tilde{A}_i$ above govern behavior at the various time-scales. They are obtained through a rather elaborate, though systematic sequence of operations on the Taylor series coefficients of $A(\varepsilon)$, involving cascaded projections onto progressively "slower" subspaces. The convergence in (2.2) is proved under a so-called semi-stability condition on matrices derived from the Taylor series; this condition implies that the $\tilde{A}_i$ are Hurwitz.

With this as background, the role of Section 3 may be stated more clearly. We show in that section that the Smith decomposition of $A(\varepsilon)$ over the ring W makes possible a change of variables in (1.2) that brings it to what can be termed the explicit form of (1.2). This form is the natural extension of that in (1.3) to the case of multiple-time-scales. It is then shown that a simple nested iteration of the familiar procedure of [1] used for (1.3) will result in (2.2), and directly give the $\tilde{A}_i$ of (2.3); it is assumed, for (2.2), that the $\tilde{A}_i$ are Hurwitz. We also show that the same result is obtained by application of the procedure in [3] to the explicit form of (1.2).

While the above approach provides some valuable additional perspectives on known results, the real pay-off appears in the results on feedback control, described in Section 4.

3. An Algebraic Transformation for Multiple-Time-Scale Decomposition

A transformation of (1.2) that more explicitly displays its amenability to multiple-time-scale decomposition is obtained by employing the Smith decomposition of $A(\varepsilon)$ over the ring W of functions of ε that are analytic at 0; see [4] and [5] for example.

It is easily seen that W is a Euclidean ring, with the degree of a scalar being defined as the order of the first nonzero term in its Taylor expansion (e.g. $\varepsilon^2 + \varepsilon^3 + \cdots$ has degree 2). $A(\varepsilon)$ therefore has the Smith decomposition

$$A(\varepsilon) = P(\varepsilon)D(\varepsilon)Q(\varepsilon) \quad , \tag{3.1}$$

where P, D, Q are all nxn matrices over W; P, Q are unimodular, i.e. $|P(0)| \neq 0$ and $|Q(0)| \neq 0$; and*

$$D(\varepsilon) = \begin{bmatrix} \varepsilon^{i_1} I & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{i_m} I \end{bmatrix} \quad , \tag{3.2}$$

where $0 \leq i_1 < \cdots < i_m$. (We have used the assumption that $A(\varepsilon)$ is nonsingular in the neighborhood of 0 in writing (3.2); in the more general case, some of the diagonal terms would be 0). Actual computation of such decompositions is discussed in [4] and [5]. (In the terminology of [5], what is required is to transform $A(\varepsilon)$ to the matrix $D(\varepsilon)\ Q(\varepsilon)$, which is "row-reduced at 0", through row operations embodied in $P^{-1}(\varepsilon)$.)

* The identity matrices I may have different dimensions.

Using (3.1), (1.2) becomes

$$\dot{x} = P(\varepsilon)D(\varepsilon)Q(\varepsilon)x \quad . \tag{3.3}$$

Because $P(\varepsilon)$ is unimodular, $P^{-1}(\varepsilon)$ exists in the neighborhood of 0. Let

$$y = P^{-1}(\varepsilon)x \tag{3.4}$$

to obtain what we shall term the <u>explicit form</u> of (1.2):

$$\dot{y} = D(\varepsilon)Q(\varepsilon)P(\varepsilon)y \quad . \tag{3.5}$$

Now, noting that $Q(\varepsilon)P(\varepsilon)$ is unimodular, we denote the <u>nonsingular</u> matrix $Q(0)P(0)$ by $\bar{A}$, and study the related system

$$\dot{z} = D(\varepsilon)\bar{A}z \quad , \quad \bar{A} = Q(0)P(0) \quad . \tag{3.6}$$

This, by (3.2), is of the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_m \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \cdots A_{1m} \\ \hline \varepsilon^{i_2} A_{21} & \cdots\cdots \varepsilon^{i_2} A_{2m} \\ \vdots & \vdots \qquad \vdots \\ \varepsilon^{i_m} A_{m1} & \cdots\cdots \varepsilon^{i_m} A_{mm} \end{array}\right] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \tag{3.7}$$

We term (3.7) the <u>reduced explicit form</u> of (1.2), since it is obtained by simplifying the explicit form (3.5). (We have assumed, with no loss of generality, that $i_1 = 0$; this can always be obtained by a change of time scale in (1.2).) The partitioning indicated in (3.7) will be explained shortly.

The rest of this section is devoted to establishing the following:

1. The system (3.7) can, under a natural set of stability conditions,

be decomposed to exhibit well-behaved multiple-time-scale structure. The decomposition procedure is a natural extension of the familiar one of [1] for (1.3). It is also shown to yield an equivalent decomposition to that obtained by the procedure in [3].

2. When (3.7) has a well-behaved multiple-time-scale structure, (1.2) has this same structure as well, and

$$\lim_{\varepsilon \downarrow 0} \sup_{t \geq 0} \|x - P(0)\, z\| = 0 \quad . \tag{3.8}$$

To establish these results, note first that the indicated partitioning of (3.7) puts it in the form (1.3), so that the familiar decomposition procedure of [1], [2] can be used. We then get, as in (2.1), the following decomposition into slow and fast subsystems:

$$\begin{bmatrix} z_1(t) \\ z_{2,m}(t) \end{bmatrix} = \begin{bmatrix} z_{1f}(t) + z_{1s}(\varepsilon t) + 0(\varepsilon) \\ z_{2s}(\varepsilon t) + 0(\varepsilon) \end{bmatrix} , \tag{3.9a}$$

where

$$z_{2,m} = \begin{bmatrix} z_2 \\ \vdots \\ z_m \end{bmatrix} , \tag{3.9b}$$

$$\dot{z}_{1f} = A_{11} z_{1f} \quad , \tag{3.9c}$$

$$z_{1s} = - A_{11}^{-1} [A_{12} \dots A_{1m}] z_{2s} \quad , \tag{3.9d}$$

$$\dot{z}_{2s} = \begin{bmatrix} \tilde{A}_{22} & & \tilde{A}_{2m} \\ & & \vdots \\ \varepsilon^{m-1}\tilde{A}_{m2} & \cdots & \varepsilon^{m-1}\tilde{A}_{mm} \end{bmatrix} z_{2s} \quad , \tag{3.9e}$$

and

$$\tilde{A}_{ij} = A_{ij} - A_{i1} A_{11}^{-1} A_{1j} \quad . \tag{3.9f}$$

The decomposition holds under the condition that the subsystems in (3.9c) and (3.9e) are stable.

Observe now that the system in (3.9e) is itself in reduced explicit form, and may be subjected to the same procedure. Iteration of this (Schur complementation) procedure leads to a decomposition into m subsystems of the form

$$\dot{z}^{(i)} = \tilde{A}_i z^{(i)} \quad , \quad i = 1 \text{ to } m, \tag{3.10}$$

where the $\tilde{A}_i$ are given by

$$\tilde{A}_i = A_{ii}^{(i)} \quad ,$$

with

$$\tilde{A}_{ij}^{(k)} = \tilde{A}_{ij}^{(k-1)} - \tilde{A}_{ik}^{(k-1)} \tilde{A}_{kk}^{(k-1)^{-1}} \tilde{A}_{kj}^{(k-1)} \quad ,$$

and

$$\tilde{A}_{ij}^{(0)} = A_{ij} \quad .$$

The $\tilde{A}_{ij}$ in (3.9e) are actually $\tilde{A}_{ij}^{(1)}$ in the present notation, and z_{1f} of (3.9c) is $z^{(1)}$ of (3.10). Also, the $\tilde{A}_i$ in (3.10) are precisely those referred to in (2.2), (2.4). Under the assumption that the $\tilde{A}_i$ are Hurwitz, we find that (3.7) has well-behaved time-scales. The <u>number</u> of variables of (3.7) at each time-scale is precisely given by the degrees i_k of the invariant factors of $A(\varepsilon)$ (which appear on the diagonal of (3.2)).

With this result in hand, the rest of the results listed above are fairly directly obtained. The detailed development is deferred to the final paper.

4. Assignment of Time-Scales by State Feedback

The results of Section 3 have established the role of the invariant factors (i.e. entries of $D(\varepsilon)$ in (3.2)) of the matrix $A(\varepsilon)$ in determining the time-scales of the undriven system (1.2). For the driven system (1.1), it is now natural to ask what freedom there is in (re-) assigning time-scales by application of the state feedback of (1.4). This feedback yields the closed-loop system

$$\dot{x} = F(\varepsilon)x, \quad F(\varepsilon) = A(\varepsilon) + B(\varepsilon)\, K(\varepsilon) \quad . \tag{4.1}$$

A key question, then, is the question of invariant factor assignment: what freedom is there in assigning the invariant factors of $F(\varepsilon)$ by choice of $K(\varepsilon)$? The following theorem provides a result in this direction.

Theorem: Assume that $A(\varepsilon)$, $B(\varepsilon)$ are left coprime, i.e. that $[A(0)\ B(0)]$ has full row rank. (Recall that A and B have dimensions nxn and nxm respectively.) Let b denote the rank of $B(0)$.

1. $F(\varepsilon)$ can have no more than b non-unit invariant factors.
2. There exists a $K(\varepsilon)$ such that $F(\varepsilon)$ has $\varepsilon^{j_1},\ldots,\varepsilon^{j_b}$ as its invariant factors, for arbitrary non-negative integers $j_1,\ldots,j_b$ (with the convention that $\varepsilon^{\infty} = 0$).

The existence of well-behaved multiple-time-scale structure in (4.1) that corresponds to the above invariant-factor structure can be guaranteed if the various $\tilde{F}_i$ (defined as the $\tilde{A}_i$ were) turn out to be Hurwitz.

Some results are also available for the case of non-coprime $A(\varepsilon)$, $B(\varepsilon)$. In this case, $F(\varepsilon)$ is of the form

$$F(\varepsilon) = W(\varepsilon)\bar{F}(\varepsilon), \quad \bar{F}(\varepsilon) = \bar{A}(\varepsilon) + \bar{B}(\varepsilon)K(\varepsilon) \quad , \tag{4.2}$$

where $W(\varepsilon)$ is a greatest common left divisor of $A(\varepsilon)$, $B(\varepsilon)$, and $\bar{A}(\varepsilon)$, $\bar{B}(\varepsilon)$ are left coprime. If the invariant factors of $F(\varepsilon)$, $W(\varepsilon)$ and $\bar{F}(\varepsilon)$ are denoted by $f_i(\varepsilon)$, $w_i(\varepsilon)$ and $\bar{f}_i(\varepsilon)$, and ordered such that the i-th one divides the (i + 1)-th one, we will have

$$w_i(\varepsilon)\,|\,f_i(\varepsilon) \text{ and } \bar{f}_i(\varepsilon)\,|\,f_i(\varepsilon) \quad . \tag{4.3}$$

The first divisibility condition in (4.3) shows that every invariant factor of $F(\varepsilon)$ must contain the corresponding invariant factor of $W(\varepsilon)$. The $\bar{f}_i(\varepsilon)$ are governed by the above Theorem, and conclusions about the $f_i(\varepsilon)$ can then be drawn from the second divisibility condition in (4.3).

Conclusion

A promising basis for an algebraic treatment of time-scale structure and assignment in linear, time-invariant systems has been presented. A wide range of research questions has thereby been exposed, and preliminary results have been outlined.

References

1. P.V. Kokotovic, "Subsystems, time scales, and multimodeling," Automatica, Vol. 17, pp. 789-795, 1981.

2. J.H. Chow (Ed.), Time-Scale Modeling of Dynamic Networks, with Applications to Power Systems, Berlin: Springer-Verlag, 1982.

3. M. Coderch, A.S. Willsky, S.S. Sastry and D.A. Castanon, Hierarchical Aggregation of Linear Systems with Multiple Time Scales," Proc. First IEEE Symposium on Large Scale Systems, Virginia Beach, October 1982 (also to appear in IEEE Trans. Automatic Control).

4. P. Van Dooren, P. Dewilde and J. Vandewalle, "On the determination of the Smith-McMillan form of a rational matrix from its Laurent expansion," IEEE Trans. Circuits and Systems, March 1979.

5. G. Verghese and T. Kailath, "Rational matrix structure," IEEE Trans. Automatic Control, Vol. AC-26, No. 2, pp. 434-439, April 1981.