Generating Functions

Generating functions are one of the most surprising, useful, and clever inventions in discrete math. Roughly speaking, generating functions transform problems about *sequences* into problems about *functions*. This is great because we've got piles of mathematical machinery for manipulating functions. Thanks to generating functions, we can apply all that machinery to problems about sequences. In this way, we can use generating functions to solve all sorts of counting problems. There is a huge chunk of mathematics concerning generating functions, so we will only get a taste of the subject.

In this lecture, we'll put sequences in angle brackets to more clearly distinguish them from the many other mathemtical expressions floating around.

1 Generating Functions

The ***ordinary generating function*** for the infinite sequence $\langle g_0, g_1, g_2, g_3 \ldots \rangle$ is the formal power series:

$$G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots$$

A generating function is a "formal" power series in the sense that we usually regard x as a placeholder rather than a number. Only in rare cases will we let x be a real number and actually evaluate a generating function, so we can largely forget about questions of convergence. Not all generating functions are ordinary, but those are the only kind we'll consider here.

Throughout the lecture, we'll indicate the correspondence between a sequence and its generating function with a double-sided arrow as follows:

$$\langle g_0, g_1, g_2, g_3, \ldots \rangle \;\longleftrightarrow\; g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots$$

For example, here are some sequences and their generating functions:

$$\begin{aligned}
\langle 0, 0, 0, 0, \ldots \rangle &\;\longleftrightarrow\; 0 + 0x + 0x^2 + 0x^3 + \cdots = 0 \\
\langle 1, 0, 0, 0, \ldots \rangle &\;\longleftrightarrow\; 1 + 0x + 0x^2 + 0x^3 + \cdots = 1 \\
\langle 3, 2, 1, 0, \ldots \rangle &\;\longleftrightarrow\; 3 + 2x + 1x^2 + 0x^3 + \cdots = 3 + 2x + x^2
\end{aligned}$$

The pattern here is simple: the i-th term in the sequence (indexing from 0) is the coefficient of x^i in the generating function.

Recall that the sum of an infinite geometric series is:

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1 - z}$$

This equation does not hold when $|z| \geq 1$, but once again we won't worry about convergence issues. This formula gives closed-form generating functions for a whole range of sequences. For example:

$$\begin{aligned}
\langle 1, 1, 1, 1, \dots \rangle &\longleftrightarrow 1 + x + x^2 + x^3 + \cdots &&= \frac{1}{1-x} \\
\langle 1, -1, 1, -1, \dots \rangle &\longleftrightarrow 1 - x + x^2 - x^3 + x^4 - \cdots &&= \frac{1}{1+x} \\
\langle 1, a, a^2, a^3, \dots \rangle &\longleftrightarrow 1 + ax + a^2x^2 + a^3x^3 + \cdots &&= \frac{1}{1-ax} \\
\langle 1, 0, 1, 0, 1, 0, \dots \rangle &\longleftrightarrow 1 + x^2 + x^4 + x^6 + \cdots &&= \frac{1}{1-x^2}
\end{aligned}$$

2 Operations on Generating Functions

The magic of generating functions is that we can carry out all sorts of manipulations on sequences by performing mathematical operations on their associated generating functions. Let's experiment with various operations and characterize their effects in terms of sequences.

2.1 Scaling

Multiplying a generating function by a constant scales every term in the associated sequence by the same constant. For example, we noted above that:

$$\langle 1, 0, 1, 0, 1, 0, \dots \rangle \longleftrightarrow 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

Multiplying the generating function by 2 gives

$$\frac{2}{1-x^2} = 2 + 2x^2 + 2x^4 + 2x^6 + \cdots$$

which generates the sequence:

$$\langle 2, 0, 2, 0, 2, 0, \dots \rangle$$

Rule 1 (Scaling Rule). *If*

$$\langle f_0, f_1, f_2, \dots \rangle \longleftrightarrow F(x),$$

then

$$\langle cf_0,\ cf_1,\ cf_2,\ \dots \rangle \longleftrightarrow c \cdot F(x).$$

Proof.

$$\begin{aligned}\langle cf_0, cf_1, cf_2, \ldots \rangle &\longleftrightarrow cf_0 + cf_1x + cf_2x^2 + \cdots \\ &= c \cdot (f_0 + f_1x + f_2x^2 + \cdots) \\ &= cF(x)\end{aligned}$$

□

2.2 Addition

Adding generating functions corresponds to adding the two sequences term by term. For example, adding two of our earlier examples gives:

$$\begin{array}{rcccccccl} & \langle\ 1, & 1, & 1, & 1, & 1, & 1, & \ldots\ \rangle & \longleftrightarrow \dfrac{1}{1-x} \\ + & \langle\ 1, & -1, & 1, & -1, & 1, & -1, & \ldots\ \rangle & \longleftrightarrow \dfrac{1}{1+x} \\ \hline & \langle\ 2, & 0, & 2, & 0, & 2, & 0, & \ldots\ \rangle & \longleftrightarrow \dfrac{1}{1-x} + \dfrac{1}{1+x}\end{array}$$

We've now derived two different expressions that both generate the sequence $\langle 2, 0, 2, 0, \ldots \rangle$. Not surprisingly, they turn out to be equal:

$$\frac{1}{1-x} + \frac{1}{1+x} = \frac{(1+x)+(1-x)}{(1-x)(1+x)} = \frac{2}{1-x^2}$$

Rule 2 (Addition Rule). *If*

$$\begin{aligned}\langle f_0, f_1, f_2, \ldots \rangle &\longleftrightarrow F(x), && \textit{and} \\ \langle g_0, g_1, g_2, \ldots \rangle &\longleftrightarrow G(x),\end{aligned}$$

then

$$\langle f_0 + g_0,\ f_1 + g_1,\ f_2 + g_2,\ \ldots \rangle \longleftrightarrow F(x) + G(x).$$

Proof.

$$\begin{aligned}\langle f_0 + g_0,\ f_1 + g_1,\ f_2 + g_2,\ \ldots \rangle &\longleftrightarrow \sum_{n=0}^{\infty} (f_n + g_n)x^n \\ &= \left(\sum_{n=0}^{\infty} f_n x^n\right) + \left(\sum_{n=0}^{\infty} g_n x^n\right) \\ &= F(x) + G(x)\end{aligned}$$

□

2.3 Right Shifting

Let's start over again with a simple sequence and its generating function:

$$\langle 1, 1, 1, 1, \ldots \rangle \longleftrightarrow \frac{1}{1-x}$$

Now let's *right-shift* the sequence by adding k leading zeros:

$$\begin{aligned}
\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}}, 1, 1, 1, \ldots \rangle \quad &\longleftrightarrow \quad x^k + x^{k+1} + x^{k+2} + x^{k+3} + \cdots \\
&= \quad x^k \cdot (1 + x + x^2 + x^3 + \cdots) \\
&= \quad \frac{x^k}{1-x}
\end{aligned}$$

Evidently, adding k leading zeros to the sequence corresponds to multiplying the generating function by x^k. This holds true in general.

Rule 3 (Right-Shift Rule). *If* $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$, *then:*

$$\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}}, f_0, f_1, f_2, \ldots \rangle \longleftrightarrow x^k \cdot F(x)$$

Proof.

$$\begin{aligned}
\langle \overbrace{0, 0, \ldots, 0}^{k \text{ zeroes}}, f_0, f_1, f_2, \ldots \rangle \quad &\longleftrightarrow \quad f_0 x^k + f_1 x^{k+1} + f_2 x^{k+2} + \cdots \\
&= \quad x^k \cdot (f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots) \\
&= \quad x^k \cdot F(x)
\end{aligned}$$

□

2.4 Differentiation

What happens if we take the *derivative* of a generating function? As an example, let's differentiate the now-familiar generating function for an infinite sequence of 1's.

$$\begin{aligned}
\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) \quad &= \quad \frac{d}{dx}\left(\frac{1}{1-x}\right) \\
1 + 2x + 3x^2 + 4x^3 + \cdots \quad &= \quad \frac{1}{(1-x)^2} \\
\langle 1, 2, 3, 4, \ldots \rangle \quad &\longleftrightarrow \quad \frac{1}{(1-x)^2}
\end{aligned}$$

We found a generating function for the sequence $\langle 1, 2, 3, 4, \ldots \rangle$!

In general, differentiating a generating function has two effects on the corresponding sequence: each term is multiplied by its index and the entire sequence is shifted left one place.

Rule 4 (Derivative Rule). *If*

$$\langle f_0, f_1, f_2, f_3, \ldots \rangle \longleftrightarrow F(x),$$

then

$$\langle f_1, 2f_2, 3f_3, \ldots \rangle \longleftrightarrow F'(x).$$

Proof.

$$\begin{aligned} \langle f_1, 2f_2, 3f_3, \ldots \rangle &= f_1 + 2f_2x + 3f_3x^2 + \cdots \\ &= \frac{d}{dx}\left(f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots\right) \\ &= \frac{d}{dx} F(x) \end{aligned}$$

□

The Derivative Rule is very useful. In fact, there is frequent, independent need for each of differentiation's two effects, multiplying terms by their index and left-shifting one place. Typically, we want just one effect and must somehow cancel out the other. For example, let's try to find the generating function for the sequence of squares, $\langle 0, 1, 4, 9, 16, \ldots \rangle$. If we could start with the sequence $\langle 1, 1, 1, 1, \ldots \rangle$ and multiply each term by its index two times, then we'd have the desired result:

$$\langle 0 \cdot 0,\ 1 \cdot 1,\ 2 \cdot 2,\ 3 \cdot 3,\ \ldots \rangle = \langle 0, 1, 4, 9, \ldots \rangle$$

A challenge is that differentiation not only multiplies each term by its index, but also shifts the whole sequence left one place. However, the Right-Shift Rule 3 tells how to cancel out this unwanted left-shift: multiply the generating function by x.

Our procedure, therefore, is to begin with the generating function for $\langle 1, 1, 1, 1, \ldots \rangle$, differentiate, multiply by x, and then differentiate and multiply by x once more.

$$\begin{aligned} \langle 1, 1, 1, 1, \ldots \rangle &\longleftrightarrow \frac{1}{1-x} \\ \langle 1, 2, 3, 4, \ldots \rangle &\longleftrightarrow \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \\ \langle 0, 1, 2, 3, \ldots \rangle &\longleftrightarrow x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2} \\ \langle 1, 4, 9, 16, \ldots \rangle &\longleftrightarrow \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3} \\ \langle 0, 1, 4, 9, \ldots \rangle &\longleftrightarrow x \cdot \frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3} \end{aligned}$$

Thus, the generating function for squares is:

$$\frac{x(1+x)}{(1-x)^3}$$

3 The Fibonacci Sequence

Sometimes we can find nice generating functions for more complicated sequences. For example, here is a generating function for the Fibonacci numbers:

$$\langle 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \rangle \longleftrightarrow \frac{x}{1 - x - x^2}$$

The Fibonacci numbers are a fairly nasty bunch, but the generating function is simple!

We're going to derive this generating function and then use it to find a closed form for the n-Fibonacci number. Of course, we already *have* a closed form for Fibonacci numbers, obtained from the cookbook procedure for solving linear recurrences. But there are a couple reasons to cover the same ground again. First, we'll gain some insight into why the cookbook method for linear recurrences works. And, second, the techniques we'll use are applicable to a large class of recurrence equations, including some that we have no other way to tackle.

3.1 Finding a Generating Function

Let's begin by recalling the definition of the Fibonacci numbers:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \qquad \text{(for } n \geq 2\text{)} \end{aligned}$$

We can expand the final clause into an infinite sequence of equations. Thus, the Fibonacci numbers are defined by:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= f_1 + f_0 \\ f_3 &= f_2 + f_1 \\ f_4 &= f_3 + f_2 \\ &\vdots \end{aligned}$$

Now the overall plan is to *define* a function $F(x)$ that generates the sequence on the left side of the equality symbols, which are the Fibonacci numbers. Then we *derive* a function that generates the sequence on the right side. Finally, we equate the two and solve for $F(x)$. Let's try this. First, we define:

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

Now we need to derive a generating function for the sequence:

$$\langle 0,\ 1,\ f_1+f_0,\ f_2+f_1,\ f_3+f_2,\ \ldots \rangle$$

One approach is to break this into a sum of three sequences for which we know generating functions and then apply the Addition Rule:

$$\begin{array}{rcccccccl}
 & \langle\ 0, & 1, & 0, & 0, & 0, & \ldots\ \rangle & \longleftrightarrow & x \\
 & \langle\ 0, & f_0, & f_1, & f_2, & f_3, & \ldots\ \rangle & \longleftrightarrow & xF(x) \\
+ & \langle\ 0, & 0, & f_0, & f_1, & f_2, & \ldots\ \rangle & \longleftrightarrow & x^2F(x) \\
\hline
 & \langle\ 0, & 1+f_0, & f_1+f_0, & f_2+f_1, & f_3+f_2, & \ldots\ \rangle & \longleftrightarrow & x+xF(x)+x^2F(x)
\end{array}$$

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is $1 + f_0$ instead of simply 1. However, this amounts to nothing, since $f_0 = 0$ anyway.

Now if we equate $F(x)$ with the new function $x+xF(x)+x^2F(x)$, then we're implicitly writing down *all* of the equations that define the Fibonacci numbers in one fell swoop:

$$\begin{array}{ccccccccccccc}
F(x) & = & f_0\ + & f_1 & x\ + & f_2 & x^2\ + & f_3 & x^3\ + & f_4 & x^4 + \ldots \\
\shortparallel & & \shortparallel & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & \\
x+xF(x)+x^2F(x) & = & 0\ + & (1+f_0) & x\ + & (f_1+f_0) & x^2\ + & (f_2+f_1) & x^3\ + & (f_3+f_2) & x^4 + \cdots
\end{array}$$

Solving for $F(x)$ gives the generating function for the Fibonacci sequence:

$$\begin{aligned}
& F(x) = x + xF(x) + x^2F(x) \\
\Rightarrow \qquad & F(x) = \frac{x}{1-x-x^2}
\end{aligned}$$

Sure enough, this is the simple generating function we claimed at the outset!

3.2 Finding a Closed Form

Why should one care about the generating function for a sequence? There are several answers, but here is one: if we can find a generating function for a sequence, then we can often find a closed form for the n-th coefficient— which can be pretty useful! For example, a closed form for the coefficient of x^n in the power series for $x/(1-x-x^2)$ would be an explicit formula for the n-th Fibonacci number.

So our next task is to extract coefficients from a generating function. There are several approaches. For a generating function that is a ratio of polynomials, we can use the method of partial fractions, which you learned in calculus. Just as the terms in a partial fractions expansion are easier to integrate, the coefficients of those terms are easy to compute.

Let's try this approach with the generating function for Fibonacci numbers. First, we factor the denominator:

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1+\sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1-\sqrt{5})$. Next, we find A_1 and A_2 which satisfy:

$$\frac{x}{1-x-x^2} = \frac{A_1}{1-\alpha_1 x} + \frac{A_2}{1-\alpha_2 x}$$

We do this by plugging in various values of x to generate linear equations in A_1 and A_2. We can then find A_1 and A_2 by solving a linear system. This gives:

$$\begin{aligned} A_1 &= \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}} \\ A_2 &= \frac{-1}{\alpha_1 - \alpha_2} = -\frac{1}{\sqrt{5}} \end{aligned}$$

Substituting into the equation above gives the partial fractions expansion of $F(x)$:

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right)$$

Each term in the partial fractions expansion has a simple power series given by the geometric sum formula:

$$\begin{aligned} \frac{1}{1-\alpha_1 x} &= 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots \\ \frac{1}{1-\alpha_2 x} &= 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots \end{aligned}$$

Substituting in these series gives a power series for the generating function:

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right) \\ &= \frac{1}{\sqrt{5}}\left((1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots) - (1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots)\right) \\ \Rightarrow f_n &= \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \end{aligned}$$

This is the same scary formula for the n-th Fibonacci number that we found using the method for solving linear recurrences. And this alternate approach sheds some light on that method. In particular, the strange rules involving repeated roots of the characteristic equation are reflections of the rules for finding a partial fractions expansion!

4 Counting with Generating Functions

Generating functions are particularly useful for solving counting problems. In particular, problems involving choosing items from a set often lead to nice generating functions. When generating functions are used in this way, the coefficient of x^n is the number of ways to choose n items.

4.1 Choosing Distinct Items from a Set

The generating function for binomial coefficients follows directly from the Binomial Theorem:

$$\begin{aligned}\left\langle \binom{k}{0}, \binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k}, 0, 0, 0, \ldots \right\rangle \quad &\longleftrightarrow \quad \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{k}x^k \\ &= \quad (1+x)^k\end{aligned}$$

Thus, the coefficient of x^n in $(1+x)^k$ is the number of ways to choose n distinct items from a k-element set. For example, the coefficient of x^2 is $\binom{k}{2}$, the number of ways to choose 2 items from a k-element set. Similarly, the coefficient of x^{k+1} is the number of ways to choose $k+1$ items from a k-element set, which is zero.

4.2 Building Generating Functions that Count

Often we can translate the description of a counting problem directly into a generating function for the solution. For example, we could figure out that $(1+x)^k$ generates the number of ways to select n distinct items from a k-element subset without resorting to the Binomial Theorem or even fussing with binomial coefficients!

Here is how. First, consider a single-element set $\{a_1\}$. The generating function for the number of ways to choose n elements from this set is simply $1 + x$: we have 1 way to choose zero elements, 1 way to choose one element, and 0 ways to choose more than one element. Similarly, the number of ways to choose n elements from the set $\{a_2\}$ is also given by the generating function $1 + x$. The fact that the elements differ in the two cases is irrelevant.

Now here is the the main trick: *the generating function for choosing elements from a union of disjoint sets is the product of the generating functions for choosing from each set.* We'll justify this in a moment, but let's first look at an example. According to this principle, the generating function for the number of ways to choose n elements from the $\{a_1, a_2\}$ is:

$$\underbrace{(1+x)}_{\substack{\text{OGF for}\\ \{a_1\}}} \cdot \underbrace{(1+x)}_{\substack{\text{OGF for}\\ \{a_2\}}} = \underbrace{(1+x)^2}_{\substack{\text{OGF for}\\ \{a_1, a_2\}}} = 1 + 2x + x^2$$

Sure enough, for the set $\{a_1, a_2\}$, we have 1 way to choose zero elements, 2 ways to choose one element, 1 way to choose two elements, and 0 ways to choose more than two elements.

Repeated application of this rule gives the generating function for choosing n items from a k-element set $\{a_1, a_2, \ldots, a_k\}$:

$$\underbrace{(1+x)}_{\substack{\text{OGF for}\\ \{a_1\}}} \cdot \underbrace{(1+x)}_{\substack{\text{OGF for}\\ \{a_2\}}} \cdots \underbrace{(1+x)}_{\substack{\text{OGF for}\\ \{a_k\}}} = \underbrace{(1+x)^k}_{\substack{\text{OGF for}\\ \{a_1, a_2, \ldots, a_k\}}}$$

This is the same generating function that we obtained by using the Binomial Theorem. But this time around we translated directly from the counting problem to the generating function.

We can extend these ideas to a general principle:

Rule 5 (Convolution Rule). *Let $A(x)$ be the generating function for selecting items from set $\mathcal{A}$, and let $B(x)$ be the generating function for selecting items from set $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then the generating function for selecting items from the union $\mathcal{A} \cup \mathcal{B}$ is the product $A(x) \cdot B(x)$.*

This rule is rather ambiguous: what exactly are the rules governing the selection of items from a set? Remarkably, the Convolution Rule remains valid under *many* interpretations of selection. For example, we could insist that distinct items be selected or we might allow the same item to be picked a limited number of times or any number of times. Informally, the only restrictions are that (1) the order in which items are selected is disregarded and (2) restrictions on the selection of items from sets $\mathcal{A}$ and $\mathcal{B}$ also apply in selecting items from $\mathcal{A} \cup \mathcal{B}$. (Formally, there must be a bijection between n-element selections from $\mathcal{A} \cup \mathcal{B}$ and ordered pairs of selections from $\mathcal{A}$ and $\mathcal{B}$ containing a total of n elements.)

Proof. Define:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n x^n, \qquad C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Let's first evaluate the product $A(x) \cdot B(x)$ and express the coefficient c_n in terms of the a and b coefficients. We can tabulate all of the terms in this product in a table:

	b_0x^0	b_1x^1	b_2x^2	b_3x^3	$\ldots$
a_0x^0	$a_0b_0x^0$	$a_0b_1x^1$	$a_0b_2x^2$	$a_0b_3x^3$	$\ldots$
a_1x^1	$a_1b_0x^1$	$a_1b_1x^2$	$a_1b_2x^3$	$\ldots$	
a_2x^2	$a_2b_0x^2$	$a_2b_1x^3$	$\ldots$		
a_3x^3	$a_3b_0x^3$	$\ldots$			
$\vdots$	$\ldots$				

Notice that all terms involving the same power of x lie on a /-sloped diagonal. Collecting these terms together, we find that the coefficient of x^n in the product is:

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

Now we must show that this is also the number of ways to select n items from $\mathcal{A} \cup \mathcal{B}$. In general, we can select a total of n items from $\mathcal{A} \cup \mathcal{B}$ by choosing j items from $\mathcal{A}$ and $n-j$ items from $\mathcal{B}$, where j is any number from 0 to n. This can be done in $a_j b_{n-j}$ ways. Summing over all the possible values of j gives a total of

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

ways to select n items from $\mathcal{A} \cup \mathcal{B}$. This is precisely the value of c_n computed above. □

The expression $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$ may be familiar from a signal processing course; the sequence $\langle c_0, c_1, c_2, \ldots \rangle$ is the ***convolution*** of sequences $\langle a_0, a_1, a_2, \ldots \rangle$ and $\langle b_0, b_1, b_2, \ldots \rangle$.

4.3 Choosing Items with Repetition

The first counting problem we considered asked for the number of ways to select a dozen doughnuts when there were five varieties available. We can generalize this question as follows: in how many ways can we select k items from an n-element set if we're allowed to pick the same item multiples times? In these terms, the doughnut problem asks in how many ways we can select a dozen doughnuts from the set:

$$\{\text{chocolate}, \text{lemon-filled}, \text{sugar}, \text{glazed}, \text{plain}\}$$

if we're allowed to pick several doughnuts of the same variety. Let's approach this question from a generating functions perspective.

Suppose we choose n items (with repetition allowed) from a set containing a single item. Then there is one way to choose zero items, one way to choose one item, one way to choose two items, etc. Thus, the generating function for choosing n elements with repetition from a 1-element set is:

$$\begin{aligned}
\langle 1, 1, 1, 1, \ldots \rangle \quad &\longleftrightarrow \quad 1 + x + x^2 + x^3 + \cdots \\
&= \quad \frac{1}{1-x}
\end{aligned}$$

The Convolution Rule says that the generating function for selecting items from a union of disjoint sets is the product of the generating functions for selecting items from each set:

$$\underbrace{\frac{1}{1-x}}_{\substack{\text{OGF for} \\ \{a_1\}}} \cdot \underbrace{\frac{1}{1-x}}_{\substack{\text{OGF for} \\ \{a_2\}}} \cdots \underbrace{\frac{1}{1-x}}_{\substack{\text{OGF for} \\ \{a_n\}}} = \underbrace{\frac{1}{(1-x)^n}}_{\substack{\text{OGF for} \\ \{a_1, a_2, \ldots, a_n\}}}$$

Therefore, the generating function for selecting items from a n-element set with repetition allowed is $1/(1-x)^n$.

Now we need to find the coefficients of this generating function. We could try to use partial fractions, but $(1-x)^n$ has a nasty repeated root at 1. An alternative is to use Taylor's Theorem:

Theorem 1 (Taylor's Theorem).

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

This theorem says that the k-th coefficient of $1/(1-x)^n$ is equal to its k-th derivative evaluated at 0 and divided by $k!$. And computing the k-th derivative turns out not to be very difficult. Let

$$g(x) = \frac{1}{(1-x)^n} = (1-x)^{-n}$$

Then we have:

$$\begin{aligned} G'(x) &= n(1-x)^{-n-1} \\ G''(x) &= n(n+1)(1-x)^{-n-2} \\ G'''(x) &= n(n+1)(n+2)(1-x)^{-n-3} \\ G^{(k)}(x) &= n(n+1)\cdots(n+k-1)(1-x)^{-n-k} \end{aligned}$$

Thus, the coefficient of x^k in the generating function is:

$$\begin{aligned} G^{(k)}(0)/k! &= \frac{n(n+1)\cdots(n+k-1)}{k!} \\ &= \frac{(n+k-1)!}{(n-1)!\,k!} \\ &= \binom{n+k-1}{k} \end{aligned}$$

Therefore, the number of ways to select k items from an n-element set with repetition allowed is:

$$\binom{n+k-1}{k}$$

This makes sense, since there is a bijection between such selections and $(n+k-1)$-bit sequences with k zeroes (representing the items) and $n-1$ ones (separating the n different types of item).

5 An "Impossible" Counting Problem

So far everything we've done with generating functions we could have done another way. But here is an absurd counting problem— really over the top! In how many ways can we fill a bag with n fruits subject to the following constraints?

- The number of apples must be even.
- The number of bananas must be a multiple of 5.
- There can be at most four oranges.
- There can be at most one pear.

For example, there are 7 ways to form a bag with 6 fruits:

Apples	6	4	4	2	2	0	0
Bananas	0	0	0	0	0	5	5
Oranges	0	2	1	4	3	1	0
Pears	0	0	1	0	1	0	1

These constraints are so complicated that the problem seems hopeless! But let's see what generating functions reveal.

Let's first construct a generating function for selecting apples. We can select a set of 0 apples in one way, a set of 1 apples in zero ways (since the number of apples must be even), a set of 2 applies in one way, a set of 3 apples in zero ways, and so forth. So we have:

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

Similarly, the generating function for selecting bananas is:

$$B(x) = 1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5}$$

Now, we can select a set of 0 oranges in one way, a set of 1 orange in one ways, and so on. However, we can not select more than four oranges, so we have the generating function:

$$O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

Here we're using the geometric sum formula. Finally, we can select only zero or one pear, so we have:

$$P(x) = 1 + x$$

The Convolution Rule says that the generating function for selecting from among all four kinds of fruit is:

$$\begin{aligned}
A(x)B(x)O(x)P(x) &= \frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1-x^5}{1-x}(1+x) \\
&= \frac{1}{(1-x)^2} \\
&= 1 + 2x + 3x^2 + 4x^3 + \cdots
\end{aligned}$$

Almost everything cancels! We're left with $1/(1-x)^2$, which we found a power series for earlier: the coefficient of x^n is simply $n+1$. Thus, the number of ways to form a bag of n fruits is just $n+1$. This is consistent with the example we worked out, since there were 7 different fruit bags containing 6 fruits. *Amazing!*