Hitting time behavior for the solution of a stochastic differential equation

Narmada Herath and Domitilla Del Vecchio

September 7, 2016

Introduction

In this report, we analyze the behavior of the hitting time of diffusion processes modeled by chemical Langevin equations [1].

System Model and Problem Formulation

We consider chemical Langevin equations, which take the form of stochastic differential equations given by

$$\dot{X} = f(X, k(t)) + \sigma(X, k(t))\Gamma, \tag{1}$$

where $X \in \mathbb{R}^n$ denote the state variables, $k(t) \in \mathbb{R}$ is a bounded input and Γ is a $d-$dimensional white noise process.

Due to the definition of the chemical Langevin equations, we have that the drift function $f(X, u(t))$ is a polynomial functions of the state variables and the diffusion matrix $\sigma(X, u(t))$ consists of square-root functions of the state variables. Thus, the equation (1) will have a unique, well-defined solution until a stopping time defined by the state $X(t)$ reaching zero.

In the following section, we consider the time for the state variables $X(t)$ to reach a lower bound and analyze how this hitting time changes with the initial condition $X(0)$. Specifically, we define the function $r : \mathbb{R}^n \to \mathbb{R}$ such that

$$r(x) = \sum_{i=1}^{n} \frac{1}{x_i^2},$$

and consider the minimum first hit time for the process $X(t)$ to reach a lower bound defined by $r(X(t)) = a$ where $a > 0$, over a range of inputs $k(t)$. We show that the minimum first hit time for the process $X(t)$ starting within a given set of initial conditions $r(X(0)) = p$ increases as p decreases.

Hitting time behavior

In order to analyze the hitting time of a boundary, we first define the sets

$$B_p = \left\{ x \in \mathbb{R}^n_{\geq 0} \middle| r(x) = p \right\},$$
$$U = \{u(t) \in \mathbb{R} | u(t) = k(t+\alpha) \text{ for all } \alpha > 0\}.$$

Figure 1 shows an illustration of the set B_p for $n = 2$.

We define the hitting time for the process $X(t)$ to reach the set A, starting within the set I with the given input $u(t)$ as

$$\tau_I(A, u(t)) = \inf\{t > 0 \text{ such that } X(t) \in A \text{ given } X(0) \in I \text{ and } u(t) \in U\}.$$

Then, the first time for the process $X(t)$ to reach a lower bound defined by $r(X(t)) = a$ where $a > 0$, starting within the set of initial conditions where $r(X(0)) = r_1$ is given by $\tau_{B_{r_1}}(B_a, u(t))$.

We consider $\min_{u(t)\in U} \tau_{B_{r_1}}(B_a, u(t))$ where $0 < r_1 < a$. Since $X(t)$ is a markov diffusion process with continuous sample paths, we have that

$$\tau_{B_{r_1}}(B_a, u(t)) \geq \tau_{B_{r_1}}(B_{r_0}, u(t)) + \tau_{B_{r_0}}(B_a, u(t + \tau_{B_{r_1}}(B_{r_0}, u(t)))),$$

where $r_1 < r_0 < a$. Then, taking the minimum time over the set of inputs U, we have that

$$\begin{aligned} \min_{u(t)\in U} \tau_{B_{r_1}}(B_a, u(t)) &\geq \min_{u(t)\in U} \tau_{B_{r_1}}(B_{r_0}, u(t)) + \min_{u(t)\in U} \tau_{B_{r_0}}(B_a, u(t)), \\ &\geq \min_{u(t)\in U} \tau_{B_{r_0}}(B_a, u(t)) \end{aligned}$$

Thus, we have that the minimum time for the process $X(t)$ to reach a lower bound is higher for the set of initial conditions with lower $r(X(0))$, where $r(X(0))$ decreases as the magnitude of the elements of $X(0)$ increases.

Acknowledgment

We would like to thank Dr. Mohammad Naghnaeian for helpful discussions and suggestions in developing this work.

References

[1] D. T. Gillespie. The chemical langevin equation. *The Journal of Chemical Physics*, 113(1):297–306, 2000.

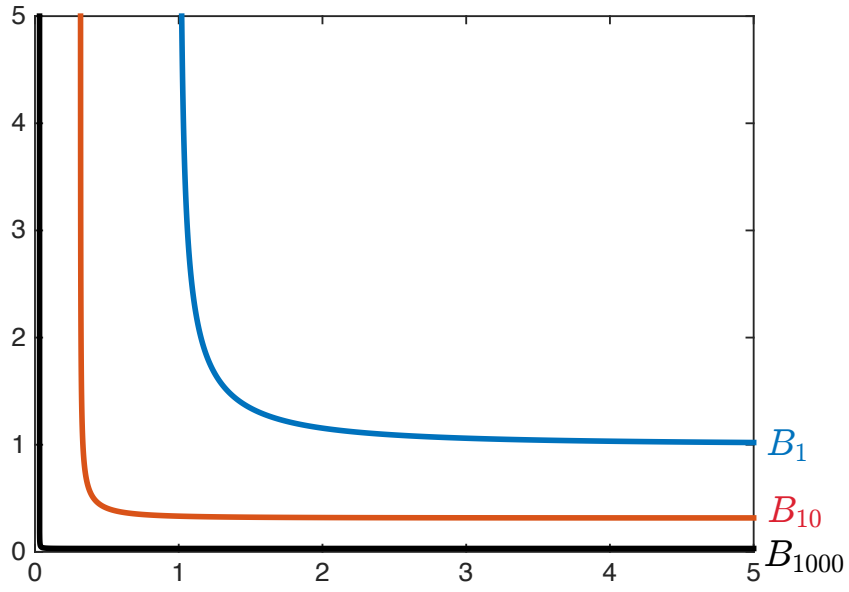

Figure 1: Plot of the sets B_p for $p = 1, 10, 1000$.