SOME IRREDUCIBLE COMPLEX REPRESENTATIONS OF A FINITE GROUP WITH BN PAIR

by

ROBERT WILLIAM KILMOYER, JR.

A.B., Lebanon Valley College

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Signature Redacted

Signature of Author..
Department of Mathematics,
Signature Redacted 7, 1969

Certified by..

Signature Redacted

Accepted by..
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ABSTRACT

Some Irreducible Complex Representations of a Finite Group with BN Pair.

Robert W. Kilmoyer, Jr.

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In this thesis the irreducible constituents of the permutation representation of G on the homogeneous space G/B are studied where G is a finite group with BN pair and B is a Borel subgroup of G.

§1 establishes the correspondence $\tilde{\chi} \leftrightarrow \chi$ between the irreducible constituents $\tilde{\chi}$ of the induced character $(1_P)^G$ and the irreducible characters χ of the Hecke algebra $H=H_C(G,P)$, where G is an arbitrary finite group and P a subgroup of G. A theorem is proved which expresses $\tilde{\chi}(g)$, $g \in G$, purely in terms of the character χ on H.

§2 is a resumé of the known properties of finite groups with BN pair which are needed for this thesis.

In §3 a semisimple algebra H (also called a Hecke algebra) is attached to every finite Coxeter system (W,I). H is a generalization of both $H_C(G,B)$ and $C[W]$, if G is a finite group with BN pair having (W,I) as its associated Coxeter system. The center of H is characterized and the one-dimensional representations of H are classified.

§4 consists of a complete classification of the irreducible representations of the Hecke algebra attached to a dihedral group.

In §5 a distinguished absolutely irreducible representation π of H (the reflection representation), and its compounds are constructed. $\tilde{\pi}$, the corresponding irreducible character of G, is uniquely characterized by its multiplicities in the induced representations from parabolic subgroups of G.

In §6 a theorem is proved about the stabilizers of the orbits of certain permutation representations of a Weyl group W. Information is obtained about the structure of double cosets of W.

In §7 a polynomial $p(x,y)$ in two variables (the Poincaré polynomial) is attached to every finite Coxeter system. The results of §6 are applied inductively to obtain

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a multiplicative formula for $p(x,y)$ and hence for $[G: B]$.

In §8 and §9 the results of §1 are applied to the linear representations and the reflection representation to obtain formulas for the degrees of the corresponding irreducible representations of G.

Thesis Supervisor: Bertram Kostant

Title: Professor of Mathematics

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§1. HECKE ALGEBRAS AND THE IRREDUCIBLE CHARACTERS OF A FINITE GROUP

Let G be a finite group and P a subgroup of G. Denote by $A = C[G]$ the group algebra of G over the complex number field C. Then $e(P) = |P|^{-1} \sum_{x \varepsilon P} x$ is an idempotent in A; and the left A-module $Ae(P)$ affords the character $(1_P)^G$ induced from the trivial character 1_P of P. $Ae(P)$ is equivalent to the permutation representation of G on G/P. We identify $\mathrm{End}_A(Ae(P))$ with the semisimple subalgebra $e(P)Ae(P)$ of A as an algebra of right operators, where $e(P)Ae(P)$ operates on $Ae(P)$ by right multiplication.

$e(P)Ae(P)$ is called the Hecke algebra over C of G relative to P, and is denoted by $H_C(G,P)$. It is clear that for x and y in G one has $e(P)xe(P) = e(P)ye(P)$ if and only if $PxP = PyP$; that is, if and only if x and y lie in the same (P,P) double coset of G. Moreover, if D is a complete set of representatives for the (P,P) double cosets of G, then $\{e(P)ae(P) | a \varepsilon D\}$ is a C-basis for $H_C(G,P)$. For each double coset PaP, $(PaP)^{-1} = Pa^{-1}P$ is also a double coset. Thus we may choose D in such a way that if a is the representative of PaP, then a^{-1} is the representative of $(PaP)^{-1}$. We also make the convention that 1, the identity of G, is the representative of the double coset P.

PROPOSITION 1: In the above notation put $X_a = |P|^{-1} \sum_{x \varepsilon PaP} x$, and put $\zeta(X_a) = [P: P \cap aPa^{-1}]$, $a \varepsilon D$, then

(i) $\zeta(X_a) = |(PaP)/P|$, hence $\zeta(X_a)$ depends only on the double coset PaP.

(ii) $\sum_{a \varepsilon D} \zeta(X_a) = [G: P]$.

(iii) $\zeta(X_a) = \zeta(X_a{}^{-1})$.

(iv) $X_a = \zeta(X_a)e(P)ae(P)$.

(v) $\{X_a | a \varepsilon D\}$ is a C-basis of $H_C(G,P)$. X_1 is the identity of $H_C(G,P)$.

(vi) ζ: $H_C(G,P) \to C$, extended to $H_C(G,P)$ by linearity, is an algebra homomorphism.

(vii) One has $X_a X_b = \sum_{c \varepsilon D} n_{a,b}^{c} X_c$, where $n_{a,b}^{c} = |(PaP \cap cPb^{-1}P)/P|$.

In particular the $n_{a,b}^{c}$ are rational integers, $a,b,c \varepsilon D$.

PROOF: (i) The cosets of G/P which lie in PaP form a P-orbit under the action of P by left multiplication. The stabilizer of this orbit is $P \cap aPa^{-1}$.

(ii) is immediate from (i).

(iii) follows from the fact that $|P \cap aPa^{-1}| = |a^{-1}Pa \cap P|$.

(iv) Note that if $x,y \varepsilon P$, then $xae(P) = yae(P)$ if and only if $y^{-1}x \varepsilon P \cap aPa^{-1}$. Thus $e(P)ae(P) = |P \cap aPa^{-1}||P|^{-2} \sum_{x \varepsilon PaP} x = \zeta(X_a)^{-1}X_a$.

(v) $\{X_a | a \varepsilon D\}$ is a basis because $\{e(P)ae(P) | a \varepsilon D\}$ is a basis. Note that $X_1 = |P|^{-1} \sum_{x \varepsilon P} x = e(P)$.

(vi) follows from the observation that ζ is just the restriction to $H_C(G,P)$ of the trivial character 1_G of G.

(vii) $X_a X_b = |P|^{-2}(\sum_{x \varepsilon PaP} x)(\sum_{y \varepsilon PbP} y) = |P|^{-1} \sum_{c \varepsilon D} (n_{a,b}^{\ c} \sum_{z \varepsilon PcP} z)$.

Comparing the coefficient of c on both sides of this equation we have

$$\begin{aligned} n_{a,b}^{\ c} &= |P|^{-1}|\{(x,y)|x \varepsilon PaP,\ y \varepsilon PbP,\ xy = c\}| \\ &= |P|^{-1}|\{x|x \varepsilon PaP,\ x^{-1}c \varepsilon PbP\}| \\ &= |P|^{-1}|\{x|x \varepsilon PaP,\ x \varepsilon cPb^{-1}P\}| \\ &= |(PaP \cap cPb^{-1}P)/P|. \end{aligned}$$

We call $\{X_a | a \varepsilon D\}$ the natural basis of the Hecke algebra $H_C(G,P)$. As the constants of structure $\{n_{a,b}^{\ c}\}$ relative to this basis are rational integers, the Hecke algebra of G relative to P is really defined over Z; namely, let $H(G,P) = H_Z(G,P)$ be the free Z-module whose basis is $\{X_a | a \varepsilon D\}$, and define multiplication in $H(G,P)$ by $X_a X_b = \sum_{c \varepsilon D} n_{a,b}^{\ c} X_c$. We call $H(G,P)$ the Hecke ring of G relative to P. If k is an arbitrary field, put $H_k(G,P) = k \otimes_Z H(G,P)$; and call $H_k(G,P)$ the Hecke algebra over k of G relative to P. Call $\zeta: H(G,P) \to Z$ the trivial character of $H(G,P)$.

We shall show that information about the representation theory of $H_C(G,P)$ can be used to deduce information on the irreducible complex representations of G which appear as the irreducible constituents of the induced representation $Ae(P)$. Our method is based on an observation that is valid in any finite dimensional semisimple C-algebra.

Let A be a finite dimensional semisimple C-algebra. Let $\pi: A \to \operatorname{End} V$ be a representation of A on the finite dimensional complex vector space V. Thus V is a left A-module where $a \cdot v = \pi(a)v$ for $a \in A$, $v \in V$. Put $\dim \pi = \dim V$. Let χ be the character of π; i.e., $\chi(a) = \operatorname{trace} \pi(a)$ for all $a \in A$. Call χ irreducible if and only if π is irreducible. In any case $\chi(1) = \dim \pi$.

A is the direct sum of simple two-sided ideals each of which is isomorphic to a finite dimensional matrix algebra over C. The identities of these simple subalgebras are the minimal central idempotents of A. Thus there is a natural one-to-one correspondence between the set of all minimal central idempotents of A and the set of all irreducible characters of A; namely, if $\tilde{\chi}$ is an irreducible character of A and $\tilde{e}$ is a minimal central idempotent of A, then $\tilde{\chi}$ corresponds to $\tilde{e} \Leftrightarrow \tilde{\chi}(\tilde{e}) \neq 0$. If $\tilde{\chi}$ corresponds to $\tilde{e}$, then $\tilde{\chi}(1) = \tilde{\chi}(\tilde{e}) = \dim \tilde{\pi}$ where $\tilde{\pi}$ is an irreducible representation of A affording the character $\tilde{\chi}$.

Let A be a finite dimensional semisimple C-algebra, and let e be an arbitrary idempotent in A. Consider $V=Ae$ as a left A-module by left multiplication in A. Then we may identify the commuting algebra, $\operatorname{End}_A V$ with the semisimple subalgebra eAe of A as an algebra of right operators, where eAe operates on V by right multiplication. If $\{V_1, V_2, \ldots, V_m\}$ is a complete set of A-irreducible constituents of V, then

$$V \simeq \bigoplus_{i=1}^{m} V_i \underset{C}{\otimes} U_i$$

where U_i is a C-vector space such that $\dim U_i$ is equal to the multiplicity of V_i in V. A operates on $V_i \otimes U_i$ by $a\cdot(v_i \otimes u_i) = (a\cdot v_i) \otimes u_i$, $a \in A$, $v_i \in V_i$, $u_i \in U_i$. We may identify $\mathrm{End}_A V = eAe$ with $\bigoplus_{i=1}^{m} \mathrm{End}\, U_i$. Let $\tilde{e}_i$ be the minimal central idempotent in A corresponding to V_i $(1 \le i \le m)$. It is clear then that $f_i = \tilde{e}_i e = e\tilde{e}_i e$ is a minimal central idempotent in eAe. In fact, under right multiplication in A, f_i is just the projection of V onto the primary component $V_i \otimes U_i$ of V. Thus $\tilde{e}_i \leftrightarrow f_i = e\tilde{e}_i e$ sets up a one-to-one correspondence between the set of all minimal central idempotents e_i of A which correspond to the distinct irreducible constituents of Ae and the set of all minimal central idempotents of eAe. We summarize the facts pertinent to this situation that we will be needing in the following:

<u>LEMMA 1</u>: Let A be a finite dimensional semisimple C-algebra and e an arbitrary idempotent in A. Identify the (semisimple) subalgebra eAe with $\mathrm{End}_A\, Ae$ as an algebra of right operators. Let $\tilde{\pi}: A \to \mathrm{End}\, M$ be an irreducible representation of A such that M is equivalent to an

irreducible constituent of the left A-module Ae. Let $\tilde{\chi}$ be the character of $\tilde{\pi}$ and $\tilde{e}$ the corresponding minimal central idempotent of A. Then the following conclusions are valid:

(i) $f = e\tilde{e} = e\tilde{e}e$ is a minimal central idempotent in eAe.

(ii) $\tilde{\pi}$ restricted to eAe induces an irreducible representation $\pi\colon eAe \to \mathrm{End}(e\cdot M)$.

(iii) $\tilde{\chi}|eAe = \chi$ is an irreducible character of eAe.

(iv) $A\cdot f$ is a primary component of Ae of type $\tilde{\chi}$.

(v) $\tilde{\chi}(e) = \chi(f) = \dim(e\cdot M) =$ multiplicity of $\tilde{\pi}$ in Ae $=$ multiplicity of $\tilde{\pi}$ in $Af = \dim \pi$.

(vi) $\tilde{e} \leftrightarrow e$ (resp. $\tilde{\chi} \leftrightarrow \chi$) sets up a one-to-one correspondence between the set of all minimal central idempotents (resp. irreducible characters) of A which correspond to the distinct A-irreducible constituents of Ae and the set of all minimal central idempotents (resp. irreducible characters) of eAe.

PROOF: (i) and (iv) are obvious from the discussion of the preceding paragraph. To see (ii), note that $e\cdot M \neq 0$ because M is A-isomorphic to an irreducible constituent of Ae. $e\cdot M$ is eAe-irreducible because if x is any non-zero vector in $e\cdot M$, one has $(eAe)x = (eA)x = e\cdot M$. Thus π is irreducible. Obviously χ is the character of π, hence χ is an irreducible character of eAe, proving (iii). It is clear that $\tilde{\chi}(e) = \tilde{\chi}(\tilde{e}e) = \chi(f)$. Also $\tilde{\chi}(e) = \mathrm{trace}\ \tilde{\pi}(e)$

$= \dim(e\cdot M)$. In the notation of the preceding paragraph we may take $\tilde{e} = \tilde{e}_i$, $f = f_i$, $Af = V_i \otimes U_i$, $\pi = \pi_i$: $eAe \to \mathrm{End}\, U_i$, and χ the character of π_i. Thus $\chi(e) = \chi(f) = \dim U_i$ $= \dim \pi =$ the multiplicity of V_i in Af = the multiplicity of V_i in Ae. This establishes (v). (vi) is also immediate from the preceding paragraph.

Applying lemma 1 to the case where $A = C[G]$, $e = e(P)$ we have the following

<u>PROPOSITION 2</u>: Let $H = H_C(G,P)$ be the Hecke algebra of the finite group G relative to the subgroup P. Put $X_1 = e(P)$ = the identity of H. Let $\tilde{\chi}$ be an irreducible character (complex) of G such that $\tilde{\chi}$ is a constituent of $(1_P)^G$. Let $\tilde{\pi}$: $C[G] \to \mathrm{End}\, M$ be a representation affording $\tilde{\chi}$, and $\tilde{e}$ the minimal central idempotent of $C[G]$ corresponding to $\tilde{\chi}$. Then

(i) $f = \tilde{e}X_1 = X_1\tilde{e}X_1$ is a minimal central idempotent of H.

(ii) $\tilde{\pi}$ restricted to H yields an irreducible representation π of H on $M^P = \{v \in M | x\cdot v = v$ for all $x \in P\}$.

(iii) $\tilde{\chi}|H = \chi$ is an irreducible character of H.

(iv) Af is a primary G-module of type $\tilde{\chi}$.

(v) $\tilde{\chi}(e) = \chi(f) = \dim(M^P) = (\tilde{\chi},(1_P)^G)_G = (\tilde{\chi}|P,1_P)_P$ = the multiplicity of $\tilde{\pi}$ in Af.

(vi) $\tilde{e} \leftrightarrow e$ (resp. $\tilde{\chi} \leftrightarrow \chi$) sets up a one-to-one correspondence between the set of all minimal central idempotents (respectively irreducible characters) of $C[G]$

which appear as irreducible constituents of $(1_P)^G$ and the set of all minimal central idempotents (respectively irreducible characters of H.

<u>COROLLARY</u>: In the above notation, if χ is a character of H of degree 1; that is, χ is an algebra homomorphism of H into C, then $f = f(\chi)$ is a primitive idempotent in $C[G]$, and $\tilde{\chi}$ is irreducible.

This corollary was proved by Janusz in [9].

There is quite a bit more that we can say about the relationship between the minimal central idempotents and the irreducible characters of H. The following theorem was proved independently by C. Curtis, [4].

<u>THEOREM 1</u>: Let $H = H_C(G,P)$ be the Hecke algebra of the finite group G relative to the subgroup P of G. Let $\{X_a | a \in D\}$ be the natural basis of H, where D is a set of double coset representatives for $P\backslash G/P$. For any two characters χ and ψ of H, put $\langle\chi,\psi\rangle_P = \sum_{a\in D} \chi(X_a^{-1})\psi(X_a)\zeta(X_a)^{-1}$. Then the following conclusions are valid:

(i) If χ is an irreducible character of H, then the minimal central idempotent of H corresponding to χ is given by

$$f(\chi) = \langle\chi,\chi\rangle_P^{-1}\chi(X_1) \sum_{a\in D} \chi(X_a^{-1})\zeta(X_a)^{-1}X_a$$

(ii) If χ is an irreducible character of H and $\tilde{\chi}$ the unique irreducible character of G such that $\tilde{\chi}|H = \chi$, then

$$\tilde{\chi}(1) = \chi(X_1)[G: P]\langle\chi,\chi\rangle_P^{-1}.$$

(iii) If χ and ψ are distinct irreducible characters of H, then $\langle\chi,\psi\rangle_P = 0$.

PROOF: By proposition 1 if $\tilde{\chi}$ is the unique irreducible character of G such that $\tilde{\chi}|H = \chi$, then $f(\chi) = e(P)\tilde{e}(\tilde{\chi})e(P)$, where $\tilde{e}(\tilde{\chi})$ is the minimal central idempotent of $C[G]$ corresponding to $\tilde{\chi}$. Now $\tilde{e}(\tilde{\chi}) = |G|^{-1}\tilde{\chi}(1)\sum_{x\in G}\tilde{\chi}(x^{-1})x$.

$$\begin{aligned}\text{Thus}\quad f(\chi) &= |G|^{-1}\tilde{\chi}(1)\sum_{x\in G}\tilde{\chi}(x^{-1})e(P)xe(P)\\ &= |G|^{-1}\tilde{\chi}(1)\sum_{a\in D}\ \sum_{x\in PaP}\tilde{\chi}(x^{-1})e(P)ae(P)\\ &= [G: P]^{-1}\tilde{\chi}(1)\sum_{a\in D}\chi(X_a^{-1})\zeta(X_a)^{-1}X_a \qquad (1)\end{aligned}$$

But then $\chi(X_1) = \chi(f(\chi)) = [G: P]^{-1}\tilde{\chi}(1)\sum_{a\in D}\chi(X_a^{-1})\chi(X_a)\zeta(X_a)^{-1}$ $= [G: P]^{-1}\tilde{\chi}(1)\langle\chi,\chi\rangle_P$. Thus $\tilde{\chi}(1) = \chi(X_1)[G: P]\langle\chi,\chi\rangle_P^{-1}$, proving (ii). (i) follows from equation (1) upon substituting $\langle\chi,\chi\rangle_P^{-1}\chi(X_1)$ for $[G: P]^{-1}\tilde{\chi}(1)$. Finally, if ψ is an irreducible character of H distinct from χ, then $0=\psi(f(\chi))$ $= \langle\chi,\chi\rangle_P^{-1}\chi(X_1)\sum_{a\in D}\chi(X_a^{-1})\psi(X_a)\zeta(X_a)^{-1} = \langle\chi,\chi\rangle_P^{-1}\chi(X_1)\langle\chi,\psi\rangle_P$. Hence $\langle\chi,\psi\rangle_P = 0$, proving (iii).

Note that theorem 1 tells us that if an irreducible character χ of $H_C(G,P)$ is known, in the sense that $\chi(X_a)$ is known for all $a \in D$, then the degree $\tilde{\chi}(1)$ of the corresponding irreducible character of G is known. Actually, the conclusion of theorem 1 can be sharpened so as to give all the values $\tilde{\chi}(g)$ of the irreducible character $\tilde{\chi}$ of G provided sufficient information is known about the conjugacy classes of G, and how they intersect the (P,P) double cosets of G. I wish to thank C. Curtis for pointing out to me the fact that the following proposition appears (without proof) in [10].

PROPOSITION 3 (Littlewood): Let G be a finite group and let $e = \sum_{g \in G} \lambda_g g$ be a primitive idempotent in $C[G]$ affording the irreducible character χ. Let S be a conjugacy class in G and let $g \in S$, then

$$\zeta(g^{-1}) = |Z(g)| \sum_{g \in S} \lambda_g,$$

where $Z(g)$ denotes the centralizer of g in G.

Proposition 3 can be sharpened to deal with the case of a primary idempotent f of $C[G]$. By a primary idempotent f we mean an idempotent f such that $C[G]\cdot f$ is a primary $C[G]$-module of type χ for some irreducible character χ of G. Thus the character of G afforded by $C[G]\cdot f$ is just $\chi(f)\cdot\chi$.

PROPOSITION 4: Let $f = \sum_{g \in G} \lambda_g \cdot g$ be a primary idempotent in $C[G]$ of type χ. Let S be a conjugacy class of G and let $g \in S$, then $\chi(g^{-1}) = \chi(f)^{-1} |Z(g)| \sum_{g \in S} \lambda_g$.

PROOF: Put $z = \chi(f)^{-1}\chi(1)|G|^{-1} \sum_{x \in G} xfx^{-1}$. Then

$\chi(z) = \chi(1)$ while if ψ is any irreducible character of G distinct from χ, then $\psi(z) = 0$. z is obviously central; hence it follows that z is equal to the minimal central idempotent corresponding to χ. That is,

$$\chi(f)^{-1}\chi(1)|G|^{-1} \sum_{x\varepsilon G} xfx^{-1} = \chi(1)|G|^{-1} \sum_{x\varepsilon G} \chi(x^{-1})x. \tag{2}$$

The assertion of proposition 4 now follows from collecting together the conjugacy classes on both sides of equation (2), and comparing the coefficients.

Applying proposition 4 to the idempotent $f = f(\chi)$ obtained in the proof of theorem 1 we thus obtain

<u>THEOREM 2</u>: Assume the hypothesis of theorem 1. Let $g \varepsilon G$ and S be the conjugacy class of g in G, then one has

$$\tilde{\chi}(g^{-1}) = [G: P]|S|^{-1}\langle\chi,\chi\rangle_P^{-1} \sum_{a\varepsilon D} |PaP\cap S|\chi(X_a^{-1})\zeta(X_a)^{-1}.$$

§2. COXETER SYSTEMS AND GROUPS WITH BN PAIR

In this section we recall some known properties of finite groups with BN pair and their associated Coxeter systems. We omit all proofs in this section. Most of these results can be found in [1].

Coxeter Systems: Let W be a group generated by a set $\{w_i \mid i \in I\}$ of distinct nonidentity involutions. Then every element w of W has an expression of the form $w = w_{i_1} w_{i_2} \ldots w_{i_m}$ $(i_j \in I,\ 1 \leq j \leq m)$. This expression is called a reduced expression if it is not possible to write w as a product of less than m of the involutions w_i, $i \in I$. If $w = w_{i_1} w_{i_2} \ldots w_{i_m}$ is a reduced expression for w, put $\ell(w) = m$. $\ell(w)$ is called the length of w.

PROPOSITION 5: Let W be a group generated by a set $\{w_i \mid i \in I\}$ of distinct nonidentity involutions. Then the following are equivalent:

(i) (Axiom of Cancellation): If $w_{i_1} w_{i_2} \ldots w_{i_m}$ is not a reduced expression, then there exist integers p and q between 1 and m such that $w_{i_1} w_{i_2} \ldots w_{i_m}$ $= w_{i_1} w_{i_2} \ldots \hat{w}_{i_p} \ldots \hat{w}_{i_q} \ldots w_{i_m}$ (where $\hat{}$ means omit).

(ii) If $w_{i_1} w_{i_2} \ldots w_{i_m}$ is a reduced expression, but $w_i w_{i_1} w_{i_2} \ldots w_{i_m}$ is not a reduced expression, then there exists an integer p $(1 \leq p \leq m)$ such that $w_i w_{i_1} \ldots w_{i_m} = w_{i_1} \ldots \hat{w}_{i_p} \ldots w_{i_m}$; and this last expression is reduced.

(iii) Let $m_{ij} = |\langle w_i w_j \rangle|$. Then the generators $\{w_i | i \in I\}$ together with the relations $w_i^2 = 1$, $(w_i w_j)^{m_{ij}} = 1$ $(i,j \in I,\ m_{ij} < \infty)$ form a presentation for the group W.

If the conditions (i)-(iii) of proposition 5 are satisfied, then (W,I) is called a Coxeter system. If (W,I) is a Coxeter system, then one has $\ell(w_i w) = \ell(w) \pm 1$ for all $w \in W$, $i \in I$.

For the rest of this section Coxeter system will always mean finite Coxeter system; that is, $|W| < \infty$.

Let (W,I) be a Coxeter system; and let $w \in W$. If $w_{i_1} w_{i_2} \cdots w_{i_m} = w$ is a reduced expression for w, one defines the support of w (supp(w)) to be the subset $\{i_1, i_2, \ldots, i_m\}$ of I. The supp(w) depends only upon w, not upon the choice of the reduced expression for w. For every subset J of I put W_J equal to the group generated by $\{w_i | i \in J\}$. Then (W_J, J) is again a Coxeter system. One has $w \in W_J$ if and only if $\mathrm{supp}(w) \subseteq J$.

The Coxeter system (W,I) is called irreducible if it is impossible to partition I into two disjoint subsets I' and I" such that w_i commutes with w_j for all $i \in I'$, $j \in J'$. It is easy to see that every finite Coxeter system is the direct product of irreducible Coxeter systems in the obvious sense.

Let (W,I) be a Coxeter system. There exists in W a unique element of maximal length. This element will always be denoted by w_0. w_0 is characterized by the property that one has $\ell(ww_0) = \ell(w_0) - \ell(w)$ for all $w \in W$.

The finite irreducible Coxeter systems have been classified as follows:

(a) The Weyl groups of the simple complex Lie algebras (Coxeter systems of Lie type).

(b) The dihedral groups.

(c) H_3 and H_4.

If W is the Weyl group of the simple complex Lie algebra $\mathfrak{g}$ of rank ℓ, we take $I = \{1,2,\ldots,\ell\}$, $\{\alpha_i | 1 \leq i \leq \ell\}$ to be a set of simple roots of $\mathfrak{g}$ relative to a Cartan subalgebra h of $\mathfrak{g}$, and w_i, $i \in I$ to be the reflection with respect to the simple root α_i. Thus the groups W which appear in (a) are also the finite irreducible groups generated by reflections in a finite dimensional Euclidean vector space.

The dihedral group D_m of order 2m has the presentation: $D_m = \langle w_1, w_2 | w_1^2 = w_2^2 = (w_1w_2)^m = 1\rangle$. Here we take $I = \{1,2\}$.

The groups H_3 and H_4 have the presentations:

$H_3 = \langle w_1, w_2, w_3 | w_1^2 = w_2^2 = w_3^2 = (w_1w_2)^5 = (w_2w_3)^3 = 1\rangle$,

$H_4 = \langle w_1, w_2, w_3, w_4 | w_1^2 = w_2^2 = w_3^2 = w_4^2 = (w_1w_2)^5 = (w_2w_3)^3 = (w_3w_4)^3 = 1\rangle$.

Let (W,I) be a finite irreducible Coxeter system. Put m_{ij} equal to the order of w_iw_j for all $i,j \in I$. The elements w_i and w_j are conjugate in W if and only if there exists a sequence $i_1, i_2, \ldots, i_s$ of elements of I such that $i_1 = i$, $i_s = j$, and $m_{i_k i_{k+1}}$ is odd. Thus from the classification of the finite irreducible Coxeter systems it is easily seen that there are at most two conjugacy classes of the elements $\{w_i | i \in I\}$. If W is of Lie type, then w_i and w_j are conjugate if and only if they correspond to the reflections with respect to simple roots α_i and α_j of the same length. Thus if we identify I with the points of the Dynkin diagram D of $\mathfrak{g}$, the conjugacy classes of the elements $\{w_i | i \in I\}$ are determined by the points of D which lie on opposite sides of a multiple bond.

Let (W,I) be a finite Coxeter system. Let J and K be subsets of I. In each coset wW_J of W/W_J there is a unique element of minimal length called the distinguished coset representative (dcr) for that coset. If w' is the dcr for wW_J, then w' is characterized by the property that $\ell(w'u) = \ell(w') + \ell(u)$ for all $u \in W_J$. In each double coset $W_K w W_J$ of $W_K \backslash W / W_J$ there exists a unique element of minimal length $\tilde{w}$ called the distinguished double coset representative (ddcr) for $W_K w W_J$. $\tilde{w}$ is characterized by the property that $\ell(\tilde{w}u) = \ell(\tilde{w}) + \ell(u)$ for all $u \in W_J$ and $\ell(v\tilde{w}) = \ell(v) + \ell(\tilde{w})$ for all $v \in W_K$.

Let (W,I) be a finite irreducible Coxeter system of Lie type so that we may identify W with the Weyl group of a simple complex Lie algebra $\mathfrak{g}$. Let h be a Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_1, \ldots, \alpha_\ell\}$ a set of simple roots of $\mathfrak{g}$ relative to

h, Δ^+ the corresponding set of positive roots, and Δ the set of all roots of $\mathfrak{g}$ relative to h. We take $I = \{1,2,\ldots,\ell\}$, and w_i, $i \in I$ to be the reflection with respect to the simple root α_i. That is, $w_i(\xi)=\xi-\frac{2(\alpha_i,\xi)}{(\alpha_i,\alpha_i)}\alpha_i$ for all $\xi \in h$, where $(\ , \ )$ denotes the Killing form of $\mathfrak{g}$. Thus h forms a natural irreducible module for W. Let σ be a permutation of the set $I = \{1,2,\ldots,\ell\}$. The element $w_{1\sigma}w_{2\sigma}\cdots w_{\ell\sigma}$ is called a Coxeter transformation in W. The Coxeter transformations in W are all conjugate to one another. The order of a Coxeter transformation is called the Coxeter number of W. Let c be a Coxeter transformation in W, and let h be the Coxeter number of W. As the order of c is h, the characteristic polynomial of c in the natural representation of W on the Cartan subalgebra is of the form

$$\prod_{j=1}^{\ell}[T - \exp(\frac{2\pi i m_j}{h})] ,$$

where the m_i are positive integers and we may assume that $0 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell \leq h$. $\{m_1,m_2,\ldots,m_\ell\}$ are called the exponents of $\mathfrak{g}$ or of W; and $\{d_1,d_2,\ldots,d_\ell\}$ are called the degrees of W, where $d_i = m_i + 1$. We list the properties of $\mathfrak{g}$ and W concerning the exponents and the degrees that we will need for future reference in the following

<u>PROPOSITION 6</u>:

(i) $m_1 = 1$, $m_\ell = h - 1$ (h = the Coxeter number of W)

(ii) $\sum_{j=1}^{\ell} m_j = N = |\Delta^+|$, the number of positive roots of $\mathfrak{g}$.

(iii) $N = \frac{1}{2}\ell h$.

(iv) $\ell(w_0) = N$.

(v) If the Dynkin diagram of $\mathfrak{g}$ is not simply laced, so that there are two nonempty conjugacy classes $\{w_i | i \in I_1\}$ and $\{w_i | i \in I_2\}$ of the involutions $\{w_i | i \in I\}$, put $\ell_1 = |I_1|$, $\ell_2 = |I_2|$, and let $w_0 = w_{i_1} w_{i_2} \ldots w_{i_N}$ be any reduced expression for w_0; then exactly $\frac{1}{2}\ell_1 h$ of the i_j lie in I_1 and $\frac{1}{2}\ell_2 h$ of the i_j lie in I_2 $(1 \leq j \leq N)$. [17]

(vi) Put $p(T) = \prod_{i=1}^{\ell} \left(\frac{T^{d_i} - 1}{T - 1}\right)$. $p(T)$ is called the Poincaré polynomial of $\mathfrak{g}$. One has

$$p(T) = \sum_{w \in W} T^{\ell(w)}.$$

(vii) $p(1) = \prod_{i=1}^{\ell} d_i = |W|$.

Following is a list of the exponents for the Weyl groups of the simple complex Lie algebras.

($\mathfrak{g}$)	$m_1, \ldots, m_\ell$
(A_ℓ)	$1,2,3,\ldots,\ell$
(B_ℓ)	$1,3,5,\ldots,2\ell-1$
(C_ℓ)	$1,3,5,\ldots,2\ell-1$
(D_ℓ)	$1,3,5,\ldots,2\ell-3,\ell-1$
(E_6)	1,4,5,7,8,11
(E_7)	1,5,7,9,11,13,17
(E_8)	1,7,11,13,17,19,23,29
(F_4)	1,5,7,11
(G_2)	1,5

<u>Groups with BN pair</u>: A group with BN pair (called a Tits System in [1]) is a group G together with a pair of subgroups B and N such that

(a) G is generated by $B \cup N$.

(b) $T = B \cap N$ is a normal subgroup of N.

(c) $N/T = W$ is a group generated by a set $\{w_i | i \in I\}$ of distinct nonidentity involutions.

(d) $w_i B w_i \neq B$ for all $i \in I$.

(e) $w_i B w \subseteq BwB \cup Bw_i wB$ for all $i \in I$, $w \in W$.

[If $w \in W$, by wB (respectively Bw) we mean nB (respectively Bn) where $n \to w$ under the natural projection $N \to W = N/T$. The coset wB or Bw depends only on w because T is a subgroup of B.] The group W is called the Weyl group of G.

If G is a group with BN pair, then in the above notation (W,I) is a Coxeter system, called Coxeter system associated to G.

<u>PROPOSITION 7</u>: (Bruhat Decomposition) Let G be a group with BN pair, then the (B,B) double cosets of G are indexed by the Weyl group W of G. That is, one has

$$G = \bigcup_{w \in W} BwB$$

is a disjoint union.

If G is a group with BN pair whose associated

Coxeter system is (W,I), then for each subset J of I $G_J = BW_JB$ is a subgroup of G. Moreover, every subgroup of G which contains B is equal to G_J for some $J \subseteq I$. The mapping $J \to G_J$ is a lattice isomorphism from the lattice of subsets of I onto the lattice of subgroups of G which contain B. The subgroups of G conjugate to B are called Borel subgroups, and the subgroups of G conjugate to the G_J, $J \subseteq I$, are called parabolic subgroups of G. The following proposition is valid concerning the parabolic subgroups.

PROPOSITION 8: (i) Two parabolic subgroups containing the same Borel subgroup are never conjugate in G unless they are equal. (ii) Every parabolic subgroup is its own normalizer in G. (iii) If two parabolic subgroups P_1 and P_2 are conjugate in G, and if $P_i \subset P$, $i = 1,2$, where P is a third parabolic subgroup, then P_1 and P_2 are conjugate in P.

In the sequel we shall deal only with finite groups with BN pair. It is a theorem of Feit and Higman [6] that if G is a finite group with BN pair, then the associated Coxeter system (W,I) of G is isomorphic to a direct product of ordinary Weyl groups (the Weyl groups of simple complex Lie algebras) and dihedral groups of order 16. Thus in particular, the finite Coxeter systems of type (H_3) and (H_4) can never appear as the Weyl group of a finite group G

with BN pair.

We call a finite group G with BN pair irreducible if the associated Coxeter system (W,I) of G is irreducible.

We need a proposition on the double cosets in G also for future reference.

<u>PROPOSITION 9</u>: Let G be a finite group with BN pair and (W,I) the associated Coxeter system. Let J,K be subsets of I. Then the mapping $W_J\backslash W/W_K \to G_J\backslash G/G_K$, $W_JwW_K \to G_JwG_K$ is bijective. In particular, if $\{u_1,u_2,\ldots,u_m\}$ is the set of distinguished double coset representatives for $W_J\backslash W/W_K$, then $\{u_1,\ldots,u_m\}$ is also a complete set of representatives for $G_J\backslash G/G_K$.

§3. THE HECKE ALGEBRA ASSOCIATED TO A FINITE COXETER SYSTEM

Let G be a finite group with BN pair, and let (W,I) be the associated Coxeter system. According to the Bruhat decomposition, the (B,B) double cosets of G are indexed by the Weyl group W of G. Thus one has a natural basis $\{X_w | w \in W\}$ for $H(G,B)$, where $X_w = |B|^{-1} \sum_{x \in BwB} x$. This Hecke ring was first studied by N. Iwahori [8] in the case where G is a Chevalley group and B a Borel subgroup. In [2] Iwahori has proved, using a theorem of J. Tits (unpublished), that if G is any finite group with BN pair and k a field such that the characteristic of k does not divide the order of G, and such that k is a splitting field for both G and W, then $H_k(G,G_J)$ and $H_k(W,W_J)$ are isomorphic as k-algebras, in particular $H_C(G,B) \cong H_C(W,\{1\}) = C[W]$.

The following theorem is due to H. Matsumoto [11].

THEOREM 3: Let G be a finite group with BN pair whose associated Coxeter system is (W,I). For each $i \in I$ put $q_i = |Bw_iB/B| = [B: B \cap w_iBw_i^{-1}]$, then one has

(i) $$X_{w_i}X_w = \begin{cases} X_{w_iw} & , \text{ if } \ell(w_iw) = \ell(w)+1 \\ q_iX_{w_iw} + (q_i-1)X_w, & \text{ if } \ell(w_iw) = \ell(w)-1 \end{cases}$$

for all $i \in I$, $w \in W$.

(ii) The generators $\{X_{w_i} | i \in I\}$ together with the relations:

$$X_1 X_{w_i} = X_{w_i} X_1 = X_{w_i}$$

$$X_{w_i}^2 = q_i X_1 + (q_i - 1) X_{w_i} \tag{3}$$

$$\underbrace{X_{w_i} X_{w_j} X_{w_i} \cdots}_{m_{ij}} = \underbrace{X_{w_j} X_{w_i} X_{w_j} \cdots}_{m_{ij}}$$

$$(i,j \in I,\ m_{ij} = |\langle w_i w_j \rangle|)$$

form a presentation for $H(G,B)$.

Let $\zeta\colon H(G,B) \to Z$ be defined as in §1. That is, $\zeta(X_w) = [B\colon B \cap wBw^{-1}]$. Then ζ is an algebra homomorphism and $\zeta(X_{w_i}) = q_i$, $i \in I$.

PROPOSITION 10: One has $q_i = q_j$ if m_{ij} is odd; and hence $q_i = q_j$ whenever w_i and w_j are conjugate in W.

PROOF: By (3) one has $\underbrace{q_i q_j q_i \cdots}_{m_{ij}} = \underbrace{q_j q_i q_j \cdots}_{m_{ij}}$.

Hence $q_i = q_j$ if m_{ij} is odd. The second assertion of the proposition follows then from §2.

REMARK: There exist finite groups with BN pair such that $q_i \neq q_j$ when m_{ij} is even. For example, the twisted Chevalley groups have this property; cf. [12,13,19,20]. However, in all the known examples q_i and q_j are either equal or are both powers of the same prime for all $i,j \in I$. It is not an open question as to the existence of a finite group G with BN pair such that q_i and q_j are not both powers of the same prime for some $i,j \in I$.

We would like to study some representation theory of a C-algebra whose constants of structure satisfy (3), only with the q_i being of a slightly more general nature. The next proposition shows that such an algebra exists.

<u>PROPOSITION 11</u>: Let (W,I) be a finite Coxeter system and M be a vector space over C having basis $\{X_w | w \in W\}$. Let $\{q_i | i \in I\}$ be a set of complex numbers such that $q_i = q_j$ if w_i and w_j are conjugate in W $(i,j \in I)$. Then there exists on M a unique associative C-algebra structure such that

$$X_{w_i} X_w = \begin{cases} X_{w_i w} & , \ \ell(w_i w) = \ell(w) + 1 \\ q_i X_{w_i w} + (q_i - 1) X_w, & \ell(w_i w) = \ell(w) - 1 \end{cases}$$

Moreover, the generators $\{X_{w_i} | i \in I\}$ and the relations

$$\begin{array}{lll} X_{w_i} X_1 = X_1 X_{w_i} & i \in I & \\ X_{w_i} = q_i X + (q_i - 1) X_{w_i} & i \in I & (4) \\ \underbrace{X_{w_i} X_{w_j} X_{w_i} \cdots}_{m_{ij}} = \underbrace{X_{w_j} X_{w_i} X_{w_j} \cdots}_{m_{ij}} & i,j \in I & \end{array}$$

form a presentation of the C-algebra M.

PROOF: This proposition, in much greater generality, is given as an exercise in [1, p.55].

We denote the algebra M obtained in the preceding theorem by $H(q_1,\ldots,q_\ell)$, where $I = \{1,2,\ldots,\ell\}$, and we refer to $H(q_1,\ldots,q_\ell)$ as a Hecke algebra over C associated to (W,I). Thus $H(1,1,\ldots,1)$ is just $C[W]$ the group algebra of W, while if W happens to be the Weyl group of some finite group G with BN pair, then $H(q_1,\ldots,q_\ell)$ becomes $H_C(G,B)$ upon the appropriate choice of q_i as positive integers. Note that the structural constants of $H(q_1,\ldots,q_\ell)$ are certain polynomials in the q_i with rational integer coefficients, so that this algebra is really defined over the subring $Z[q_1,\ldots,q_\ell]$ of C.

<u>REMARK</u>: Using the techniques of Iwahori and Tits mentioned before, it is easy to show that whenever $H(q_1,\ldots,q_\ell)$ is semisimple, then it is C-isomorphic with $C[W]$ as a C-algebra. It does not seem, however, that there is any natural isomorphism as long as the rank is greater than one. Nevertheless, it is reasonable to expect the representation theory of $H(q_1,\ldots,q_\ell)$ to resemble that of $C[W]$ in the sense that given a representation of $H(q_1,\ldots,q_\ell)$, one should be able to obtain an analogous representation of $C[W]$ by setting "$q_i = 1$" everywhere. We shall see later that this is the case for certain representations that we construct.

As an immediate consequence of proposition 11 we have that the map $X_{w_i} \to q_i$ can be uniquely extended to an algebra homomorphism $\zeta\colon H(q_1,\ldots,q_\ell) \to C$ where if $w_{i_1}w_{i_2}\ldots w_{i_m}$ is any reduced expression for $w \in W$, then $\zeta(X_w) = q_{i_1} q_{i_2} \cdots q_{i_m}$. We shall refer to ζ as the trivial representation of $H(q_1,\ldots,q_\ell)$.

Our next result is that $H(q_1,\ldots,q_\ell)$ is semisimple if the q_i are positive real numbers, but we need a few lemmas.

Define the linear functional ε on $H(q_1,\ldots,q_\ell)$ by $\varepsilon(\sum_{w\in W} c_w X_w) = c_1$, $c_w \in C$.

<u>LEMMA 2</u>: $\varepsilon(X_w X_u) = \begin{cases} 0 & \text{if } wu \neq 1 \\ \zeta(X_w) & \text{if } wu = 1 \end{cases}$ $(w,u \in W)$.

PROOF: By induction on $\ell(w)$. If $\ell(w) = 0$, then $w = 1$ and the result is clear. Otherwise we may write $w = w'w_i$ with $w' \in W$, $i \in I$ and $\ell(w') = \ell(w) - 1$. Now we make two cases.

<u>Case 1</u>: $\ell(w_i u) = \ell(u) + 1$. In this case $X_w X_u = X_{w'} X_{w_i} X_u = X_{w'} X_{w_i u}$. Now $wu \neq 1$ and so $w'(w_i u) \neq 1$. Thus by induction $\varepsilon(X_w X_u) = \varepsilon(X_{w'} X_{w_i u}) = 0$.

<u>Case 2</u>: $\ell(w_i u) = \ell(u) - 1$. In this case $X_w X_u$

$$= X_{w'w_i} X_u = X_{w'} X_{w_i} X_u$$
$$= X_{w'}\{\zeta(X_{w_i}) X_{w_i u} + (\zeta(X_{w_i}) - 1) X_u\}.$$

Now clearly $w'u \neq 1$ in this case so that $\varepsilon(X_{w'} X_u) = 0$ by induction. On the other hand, we have $wu = 1$ if and only if $w'(w_i u) = 1$, so again by induction

$$\varepsilon(X_w X_u) = \varepsilon(X_{w'} X_{w_i u}) = \begin{cases} 0 & \text{if } wu \neq 1 \\ \zeta(X_{w_i})\zeta(X_{w'}) = \zeta(X_w) & \text{if } wu=1 \end{cases}.$$

<u>PROPOSITION 12</u>: Let $x = \sum a_w X_w$ and $y = \sum b_w X_w$ be arbitrary elements of $H(q_1,\ldots,q_\ell)$, $a_w, b_w \in C$. Then $\varepsilon(xy) = \sum_{w\in W} a_w b_{w^{-1}} \zeta(X_w) = \varepsilon(yx)$.

PROOF: This is an immediate consequence of lemma 2.

<u>LEMMA 3</u>: For each $x \in H(q_1,\ldots,q_\ell)$, $x = \sum_{w\in W} c_w X_w$, $c_w \in C$, put $x^* = \sum_{w\in W} \bar{c}_w X_{w^{-1}}$, where $\bar{c}_w$ denotes the complex conjugate of c_w, then the following properties of the mapping $x \to x^*$ are valid:

(i) $(x^*)^* = x$

(ii) $(cx)^* = \bar{c}x^*$

(iii) $(xy)^* = y^*x^*$

(iv) If the q_i are positive real numbers, then $xx^* = 0$ implies $x = 0$.

PROOF: (i) and (ii) are obvious from the definition. To prove (iii) it suffices to show that $(X_w X_u)^* = X_u^* X_w^*$, $w,u \in W$, and this can be shown quite easily by induction on the length of w. Now suppose $xx^* = 0$. Let $x = \sum_{w\in W} c_w X_w$, then $0 = xx^* = \varepsilon(xx^*) = \sum_{w\in W} c_w \bar{c}_w \zeta(X_w) = \sum_{w\in W} |c_w|^2 \zeta(X_w)$. But if the $q_i > 0$, $i \in I$, then $\zeta(X_w) > 0 \;\forall\; w \in W$. Thus we must have $|c_w| = 0 \;\forall\; w \in W$ and $x = 0$.

<u>THEOREM 4</u>: If the $q_i > 0$, $i \in I$, then $H(q_1,\ldots,q_\ell)$ is a semisimple C-algebra.

PROOF: Let J be the radical of $H(q_1,\ldots,q_\ell)$, and let x be an element of J. Then $y = xx^*$ is also an element of J and by lemma 3 we have $y^* = y$. Let k be the smallest positive integer such that $y^k = 0$. Suppose $k > 1$. If k is even $k = 2n$ and $0 = y^k = y^{2n} = (y^n)(y^n)^*$ so that by lemma 3 we have $y^n = 0$, a contradiction. If n is odd, $k = 2n + 1$, then $0 = y^k = y^{k+1} = (y^{n+1})(y^{n+1})^*$ so that by lemma 3 we have $y^{n+1} = 0$, again a contradiction. Hence $k = 1$, and $0 = y = xx^*$ which implies $x = 0$ by lemma 3. Thus $J = (0)$ and $H(q_1,\ldots,q_\ell)$ is semisimple.

We now characterize the center of $H(q_1,\ldots,q_\ell)$ in terms of the natural basis $\{X_w | w \in W\}$.

<u>PROPOSITION 13</u>: Let $H = H(q_1,\ldots,q_\ell)$ and $x = \sum_{w\in W} a_w X_w$ be an element of H. Then x is central in H if and only if the following condition is satisfied on the coefficients a_w, $w \in W$: For all $w \in W$ and $i \in I$ such that $\ell(w_i w w_i) = \ell(w) + 2$, one has

$$\begin{aligned} q_i a_{w_i w w_i} &= a_w + (q_i - 1) a_{w_i w} \\ a_{w_i w} &= a_{w w_i} \end{aligned} \tag{5}$$

PROOF: Since H is generated by $\{X_{w_i} | i \in I\}$ it follows that x will be central if and only if $xX_{w_i} = X_{w_i}x$ for all $i \in I$. Let $i \in I$ and let Γ be a set of distinguished coset representatives for $W/\langle w_i\rangle$, that is, $\ell(w_i w) = \ell(w) + 1$ for all $w \in \Gamma$. Then we may express x as follows in two ways:

$$x = \sum_{w\varepsilon\Gamma} a_w X_w + \sum_{w\varepsilon\Gamma} a_{w_i w} X_{w_i w} \tag{6}$$

$$x = \sum_{w^{-1}\varepsilon\Gamma} a_{w^{-1}} X_{w^{-1}} + \sum_{w^{-1}\varepsilon\Gamma} a_{w^{-1}w_i} X_{w^{-1}w_i} \tag{7}$$

Now if one multiplies equations (6) and (7) on the left and right respectively by X_{w_i}, one obtains after making the appropriate substitutions, the following necessary and sufficient condition for X_{w_i} to commute with x (broken into four separate cases).

$$\begin{array}{llll}
\text{(i)} & \begin{array}{l} \ell(w_i w) = \ell(w) + 1 \\ \ell(w w_i) = \ell(w) + 1 \end{array} & a_{w_i w} = a_{w w_i} & \\
\text{(ii)} & \begin{array}{l} \ell(w_i w) = \ell(w) + 1 \\ \ell(w w_i) = \ell(w) - 1 \end{array} & \begin{array}{l} q_i a_{w_i w} = a_{w w_i} + (q_i - 1) a_w \\ a_w = a_{w_i w w_i} \end{array} & \\
 & & & (8) \\
\text{(iii)} & \begin{array}{l} \ell(w_i w) = \ell(w) - 1 \\ \ell(w w_i) = \ell(w) + 1 \end{array} & \begin{array}{l} q_i a_{w w_i} = a_{w_i w} + (q_i - 1) a_w \\ a_w = a_{w_i w w_i} \end{array} & \\
\text{(iv)} & \begin{array}{l} \ell(w_i w) = \ell(w) - 1 \\ \ell(w w_i) = \ell(w) - 1 \end{array} & a_{w_i w} = a_{w w_i} &
\end{array}$$

But for x to be central it is necessary and sufficient for (8) to be satisfied for all $w \varepsilon W$, $i \varepsilon I$. It follows by making the appropriate substitution for w in the four parts of (8) that (8) may be replaced by the single condition (5), proving the proposition.

PROPOSITION 14: Keeping the above notation, let w_0 be the unique element of W having maximal length, then

(i) X_{w_0} is central in H if and only if w_0 is central in W.

(ii) $X_{w_0}^2$ is always central in H.

PROOF: (i) is an immediate consequence of proposition 13. For each $i \in I$ there exists a unique $j \in I$ such that $w_i w_0 = w_0 w_j$. Thus w_i, w_j are conjugate in W so one has $q_i = q_j$. It follows that $X_{w_i} X_{w_0} = q_i X_{w_i w_0} + (q_i - 1) X_{w_0}$ $= q_j X_{w_0 w_j} + (q_j - 1) X_{w_0} = X_{w_0} X_{w_j}$. Similarly $X_{w_j} X_{w_0} = X_{w_0} X_{w_i}$. Thus $X_{w_i} X_{w_0} X_{w_0} = X_{w_0} X_{w_j} X_{w_0} = X_{w_0} X_{w_0} X_{w_i}$, and $X_{w_0}^2$ is central because it commutes with X_{w_i}, $i \in I$.

PROPOSITION 15: Put $c = \sum_{w \in W} \zeta(X_w)$,

(i) If $c \neq 0$, then $c^{-1} \sum_{w \in W} X_w$ is a primitive central idempotent in H affording the trivial representation ζ.

(ii) If $c = 0$, then H is not semisimple.

PROOF: Let $x = \sum_{w \in W} X_w$. By proposition 13 we know that x is central in H. Now let $i \in I$ and let Γ be the set of distinguished coset representatives for $W/\langle w_i \rangle$, as in the proof of proposition 13. We may write $x = \sum_{w \in \Gamma} X_w + \sum_{w \in \Gamma} X_{w_i w}$ $= (X_1 + X_{w_i}) \sum_{w \in \Gamma} X_w$. Then $X_{w_i} x = q_i (X_1 + X_{w_i}) \sum_{w \in \Gamma} X_w = q_i x$. It follows that if $w \in W$, then $X_w x = \zeta(X_w) x$ and hence $x^2 = (\sum_{w \in W} \zeta(X_w)) x = cx$. Thus if $c = 0$, then x is a central nilpotent and H is not semisimple. But if $c \neq 0$, then $c^{-1}x$ is an idempotent as asserted.

REMARK: It seems likely that H will be semisimple if and only if $\sum_{w \in W} \zeta(X_w) \neq 0$, but we have no result in this direction of a general nature.

For the remainder of this section we assume that $q_i > 0$, $i \in I$, so that $H = H(q_1,\ldots,q_\ell)$ is semisimple. Let J be an arbitrary subset of I. Denote by H_J the subalgebra of H generated by X_1 and $\{X_{w_i} \mid i \in J\}$. Thus H_J is just a Hecke algebra over C associated to the Coxeter subsystem (W_J,J). It is clear that H_J will be semisimple also because $q_i > 0$, $i \in I$. The trivial representation of H_J is given by $\zeta_J = \zeta|H_J$, and

$e_J = \left(\sum_{w \in W_J} \zeta_J(X_w)\right)^{-1} \sum_{w \in W_J} X_w$ is the primitive central idempotent of H_J affording ζ_J. Thus if $\pi\colon H_J \to \text{End } V$ is any representation of H_J, then $\pi(e_J)$ is the projection on the H_J-submodule $e_J \cdot V$ consisting of a certain number of copies of the trivial representation of H_J. Applying lemma 1 to this situation we observe the following simple reciprocity theorem for future reference.

PROPOSITION 16: Let $\pi\colon H \to \text{End } V$ be an irreducible representation affording the character χ, then for any subset J of I one has $\chi(e_J)$ = the multiplicity of π in He_J = the dim $e_J \cdot V$ = the multiplicity of ζ_J in $\pi|H_J$.

The linear characters (one-dimensional representations) of an algebra are just the multiplicative linear functionals on that algebra. The linear characters of $H(G,B)$ have been classified by N. Iwahori [8], when G is a Chevalley group and B a Borel subgroup of G. It is not difficult to extend his argument to our algebra $H = H(q_1,\ldots,q_\ell)$.

Recall that as (W,I) is a finite irreducible Coxeter system, there are at most two conjugacy classes $\{w_i \mid i \in I_1\}$ and $\{w_i \mid i \in I_2\}$ of the elements $\{w_i \mid i \in I\}$. If there is only one conjugacy class we make the convention that $I_1 = I$, $I_2 = \emptyset$. Put $q_i = p$ for all $i \in I_1$, $q_i = q$ for all $i \in I_2$.

<u>PROPOSITION 17</u>: Let $H = H(q_1,\ldots,q_\ell)$. If $I_2 = \emptyset$, then there are exactly two linear characters ζ and σ of H, where $\sigma(X_w) = (-1)^{\ell(w)}$, $w \in W$. If $I_2 \neq \emptyset$, then there are two additional linear characters σ_1 and σ_2 of H, where

$$\sigma_1(X_{w_i}) = \begin{cases} p & i \in I_1 \\ -1 & i \in I_2 \end{cases}$$

$$\sigma_2(X_{w_i}) = \begin{cases} -1 & i \in I_1 \\ q & i \in I_2 \end{cases}$$

PROOF: It is clear that σ can be extended uniquely to a multiplicative linear functional by our presentation for H. Similarly, if there are two conjugacy classes, then σ_1 and σ_2 can be extended to multiplicative linear functionals.

It suffices to show that these are all of the linear characters of H. But if $C_1,\ldots,C_m$ are the conjugacy classes of the elements $\{w_i | i \in I\}$; and if ϕ is any multiplicative linear functional on H, then $\phi(X_{w_i})$ must be equal to -1 or q_i because these are the only roots of the quadratic equation $x^2 = q_i + (q_i-1)x$ which is satisfied by X_{w_i}. Furthermore ϕ must be constant on the conjugate classes $C_1,\ldots,C_m$. Hence the number of linear characters is 2^m.

COROLLARY: Assume that there are two conjugacy classes of the involutions $\{w_i | i \in I\}$; i.e., $I_2 \neq \emptyset$. Let $w = w_{i_1} w_{i_2} \cdots w_{i_m}$ be a reduced expression for $w \in W$. Put $\ell_1(w)$ equal to the number of i_j such that $i_j \in I_1$, and $\ell_2(w)$ equal to the number of i_j such that $i_j \in I_2$ $(1 \leq j \leq m)$. Then $\ell_1(w)$ and $\ell_2(w)$ depend only upon w, not upon the choice of reduced expression for $w \in W$. Moreover, one has

$$
\begin{aligned}
\sigma_1(X_w) &= p^{\ell_1(w)}(-1)^{\ell_2(w)} \\
\sigma_2(X_w) &= (-1)^{\ell_1(w)} q^{\ell_2(w)}
\end{aligned}
\tag{9}
$$

PROOF: (9) is obvious from the proposition. Taking $p,q > 1$, it follows that $\ell_1(w),\ell_2(w)$ are uniquely determined, independent of the choice of reduced expression for w.

Iwahori has also shown in [8] the existence of a canonical involution of $H_C(G,B)$. This involution exists for $H = H(q_1,\ldots,q_\ell)$.

LEMMA 4: X_w is invertible for all $w \in W$.

PROOF: It suffices to show that X_{w_i} is invertible, $i \in I$. But it follows from the fact that $X_{w_i}^2 = q \cdot X_1 + (q-1)X_{w_i}$, that in any irreducible representation of H, the eigenvalues of X_{w_i} are q_i and -1. Hence $X_{w_i}^{-1}$ is given by $X_{w_i}^{-1} = q_i^{-1}(X_{w_i} + (1-q_i)X_1)$.

PROPOSITION 18: Let $\hat{X}_w = \zeta(X_w)\sigma(X_w)X_{w^{-1}}^{-1}$, and for $x = \sum_{w \in W} a_w X_w$ in H put $\hat{x} = \sum_{w \in W} a_w \hat{X}_w$. Then the mapping $x \to \hat{x}$ is an algebra automorphism of H, having order 2.

PROOF: It is obvious that $\hat{\hat{x}} = x$, hence it suffices to prove that the $\{\hat{X}_{w_i} | i \in I\}$ satisfy the relations of our presentation (4) of H. Now $\hat{X}_{w_i} = -q_i^{-1}X_{w_i}^{-1}$, and hence in any representation of H must also have only the eigenvalues q_i or -1. Thus $\hat{X}_{w_i}$ satisfies the same quadratic equation as X_{w_i}, namely $\hat{X}_{w_i}^2 = q_i X_1 + (q_i - 1)\hat{X}_{w_i}$. Let $i, j \in I$, $m_{ij} = |\langle w_i w_j \rangle|$, and put $\underbrace{w_i w_j w_i \ldots}_{m_{ij}} = \underbrace{w_j w_i w_j \ldots}_{m_{ij}} = w$. These are the two distinct reduced expressions for w. Thus one has

$$\underbrace{\hat{X}_{w_i}\hat{X}_{w_j}\hat{X}_{w_i}\ldots}_{m_{ij}} = \zeta(X_w)\sigma(X_w)\underbrace{X_{w_i}^{-1}X_{w_j}^{-1}X_{w_i}^{-1}\ldots}_{m_{ij}} = \zeta(X_w)\sigma(X_w)X_w^{-1} = \underbrace{\hat{X}_{w_j}\hat{X}_{w_i}\hat{X}_{w_j}\ldots}_{m_{ij}} .$$

REMARK 1: Note that this canonical involution, $x \to \hat{x}$ induces a natural pairing of the irreducible characters of H; namely, if χ is an irreducible character of H, then $\hat{\chi}(x) = \chi(\hat{x})$ is also such. If $H = C[W]$, then $\hat{\chi}(w)$

$= \operatorname{sgn}(w)\chi(w)$ for all $w \in W$. Note that in the notation of proposition 17 one has $\hat{\zeta} = \sigma$, $\hat{\sigma}_1 = \sigma_2$.

REMARK 2: If $H = H(q_1,\ldots,q_\ell)$, $q_i > 0$, $i \in I$, one can show, using the characterization of the center of H given in proposition 13, that if one puts $\langle\chi,\chi\rangle = \sum_{w\in W} \chi(X_{w^{-1}})\chi(X_w)\zeta(X_w)^{-1}$, then $\langle\chi,\chi\rangle \neq 0$ and $e = \langle\chi,\chi\rangle^{-1} \sum_{w\in W} \chi(X_{w^{-1}})\zeta(X_w)^{-1}X_w$ is the minimal central idempotent in H corresponding to χ. By theorem 1 we know, of course, that this is true when either $H = C[W]$ or H is the Hecke algebra of $H_C(G,B)$ of some finite group with BN pair.

LEMMA 5: The map $X_w \to X_{w^{-1}}$, extended by linearity to H, is an anti-automorphism of H of order two.

PROOF: This is an immediate consequence of proposition 18.

Let $\pi\colon H \to M_n(C)$ be a representation of H by $n \times n$ complex matrices. Then the preceding lemma enables us to define the contragredient representation π^* of π, namely $\pi^*(X_w) = \pi(X_{w^{-1}})^t$. If χ is the character of π, we denote the character of π^* by χ^*. Thus $\chi^*(X_w) = (X_{w^{-1}})$. Using our presentation, (4), of H we can also define $\bar{\pi}$ the complex conjugate of π by $\bar{\pi}(X_w) = \overline{\pi(X_w)}$ for all $w \in W$. Denote the character of $\bar{\pi}$ by $\bar{\chi}$. Thus $\bar{\chi}(X_w) = \overline{\chi(X_w)}$. It is clear that χ is irreducible, so are χ^* and $\bar{\chi}$.

<u>PROPOSITION 19</u>: One has $\chi^* = \overline{\chi}$.

PROOF: It suffices to prove the proposition when χ is irreducible. Let χ be irreducible and let $e = \langle\chi,\chi\rangle^{-1} \sum_{w\in W} \chi(X_{w^{-1}})\zeta(X_w)^{-1}X_w$ be the minimal central idempotent in H corresponding to χ. Then

$$\overline{\chi}^*(e) = \langle\chi,\chi\rangle^{-1} \sum_{w\in W} \chi(X_{w^{-1}})\zeta(X_w)^{-1}\overline{\chi(X_{w^{-1}})}$$

$$= \langle\chi,\chi\rangle^{-1} \sum_{w\in W} |\chi(X_{w^{-1}})|^2\zeta(X_w)^{-1}.$$ Now $\zeta(X_w)^{-1} > 0$

so that $\overline{\chi}^*(e) \neq 0$. But this implies $\overline{\chi}^* = \chi$, hence $\chi^*=\overline{\chi}$.

<u>COROLLARY</u>: $\chi(X_{w^{-1}}) = \overline{\chi(X_w)}$, $\langle\chi,\chi\rangle = \sum_{w\in W} |\chi(X_w)|^2\zeta(X_w)^{-1} > 0$. If $\chi(X_w)$ is real for all $w \in W$, then $\chi(X_{w^{-1}})=\chi(X_w)$.

§4. CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE HECKE ALGEBRA OF A DIHEDRAL GROUP

In this section we assume that the Coxeter system (W,I) is dihedral of order $2m$; that is, $I = \{1,2\}$, and W has the presentation:

$$W = \langle w_1, w_2 \mid w_1^2 = w_2^2 = (w_1w_2)^m = 1 \rangle.$$

Let $H = H(q_1,q_2)$ be a Hecke algebra, over $\mathbb{C}$, associated to (W,I) as in section 3. We assume that q_1 and q_2 are positive real numbers so that H is semi-simple by theorem 4. Recall also that $q_1 = q_2$ if m is odd.

We shall classify, in this section, all the irreducible complex representations of H.

<u>LEMMA 6</u>: Let q_1 and q_2 be positive real numbers such that $q_1 = q_2$ if m is odd. Let s be a positive integer such that $1 \leq s \leq \frac{m-1}{2}$ if m is odd and $1 \leq s \leq \frac{m-2}{2}$ if m is even. Let a and b be complex numbers such that $ab = q_1 + q_2 + 2\sqrt{q_1q_2}\cos\frac{2\pi s}{m}$. Let A_1 and A_2 be the 2×2 complex matrices:

$$A_1 = \begin{pmatrix} -1 & a \\ 0 & q_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} q_2 & 0 \\ b & -1 \end{pmatrix}. \tag{10}$$

Then the following statements are valid:

(i) $A_i^2 = q_i \cdot I + (q_i-1)A_i \quad (i=1,2)$

(ii) A_1 and A_2 do not commute.

(iii) A_1A_2 and A_2A_1 have the same minimal polynomial equal to $x^2 - 2\sqrt{q_1q_2}\cos\frac{2\pi s}{m}x + q_1q_2$.

(iv) The eigenvalues of A_1A_2, (A_2A_1) are $\sqrt{q_1q_2}\exp[\pm 2\pi i s/m]$.

(v) If m is even one has $(A_1A_2)^{m/2} = (A_2A_1)^{m/2}$.

(vi) If m is odd, one has $(A_1A_2)^{\frac{m-1}{2}}A_1 = (A_2A_1)^{\frac{m-1}{2}}A_2$.

PROOF: $$A_1A_2 = \begin{pmatrix} ab-q_2 & -a \\ q_1b & -q_1 \end{pmatrix},$$

$$A_2A_1 = \begin{pmatrix} -q_2 & q_2a \\ -b & ab-q_1 \end{pmatrix}$$

Hence (i), (ii), (iii), and (iv) are immediate. To verify (v), note that when m is even we have by (iv), that the eigenvalues of $(A_1A_2)^{\frac{m}{2}}$ are equal to $(q_1q_2)^{\frac{m}{4}}\exp(\pm\pi i s)$. That is, $(A_1A_2)^{\frac{m}{2}} = +(q_1q_2)^{\frac{m}{4}}$ if s is even, and $(A_1A_2)^{\frac{m}{2}} = -(q_1q_2)^{\frac{m}{4}}$ when s is odd. Since the same is true of $(A_2A_1)^{\frac{m}{2}}$ we have (v). It remains to prove (vi). As m is odd we have $q_1 = q_2 = q$, say; and by (iv) the eigenvalues of A_1A_2 are $q\cdot\exp(\pm 2\pi i s/m)$. Thus $(A_1A_2)^m = q^m\cdot I$, and

$$(A_1A_2)^{m-1} = q^mA_2^{-1}A_1^{-1}.$$

Now the eigenvalues of $(A_1A_2)^{\frac{m-1}{2}}$ are $q^{\frac{m-1}{2}}\exp(\pm i\theta)$ where $\theta = \frac{2\pi s}{m}.\frac{m-1}{2}$. Hence we have the equation:

$$(A_1A_2)^{m-1} - 2q^{\frac{m-1}{2}}\cos\theta(A_1A_2)^{\frac{m-1}{2}} + q^{m-1} = 0$$

that is,

$$q^m A_2^{-1} A_1^{-1} - 2q^{\frac{m-1}{2}} \cos\theta (A_1A_2)^{\frac{m-1}{2}} + q^{m-1} = 0 \ ;$$

hence

$$q^m A_2^{-1} - 2q^{\frac{m-1}{2}} \cos\theta (A_1A_2)^{\frac{m-1}{2}} A_1 + A_1 q^{m-1} = 0 \qquad (11)$$

similarly

$$q^m A_1^{-1} - 2q^{\frac{m-1}{2}} \cos\theta (A_2A_1)^{\frac{m-1}{2}} A_2 + A_2 q^{m-1} = 0 \qquad (12)$$

Now $qA_i^{-1} = A_i + (1-q)\cdot I \quad (i=1,2)$, and hence

$$q^m(A_2^{-1} - A_1^{-1}) = q^{m-1}(A_2 - A_1).$$

Thus subtracting (12) from (11) we obtain

$$-2q^{\frac{m-1}{2}} \cos\theta [(A_1A_2)^{\frac{m-1}{2}} A_1 - (A_2A_1)^{\frac{m-1}{2}} A_2] = 0 \qquad (13)$$

But our hypothesis implies that $\cos\theta \neq 0$. Hence (vi) follows from equation (13).

<u>THEOREM 5</u>: Let $H = H(q_1, q_2)$ be a Hecke algebra over C associated to the dihedral group of order $2m$. Let s be a positive integer such that $1 \leq s \leq \frac{m-1}{2}$ if m is odd and $1 \leq s \leq \frac{m-2}{2}$ if m is even. Let the 2×2 complex matrices A_1 and A_2 be defined as in (10) where $ab = q_1 + q_2 + 2\sqrt{q_1q_2} \cdot \cos\frac{2\pi s}{m}$. Then there exists a unique irreducible matrix representation

$$\pi_s : H \to M(2; C)$$

such that $\pi_s(X_{w_1}) = A_1$, $\pi_s(X_{w_2}) = A_2$. Moreover, the $\{\pi_s\}$

together with the one-dimensional representations of H form a complete set of inequivalent irreducible representations of H.

PROOF: It follows from the presentation, (4), of H, together with (i), (v), and (vi) of lemma 6 that the map $X_{w_i} \to A_i$ $(i=1,2)$ can be uniquely extended to an algebra homomorphism $\pi_s\colon H \to M(2; C)$. The representation π_s of H is irreducible because A_1 and A_2 do not commute by (ii) of the lemma. The π_s are inequivalent by (iii). Thus if m is even we have found $\frac{m-2}{2}$ inequivalent two-dimensional representations. By proposition 17 there are precisely 4 distinct one-dimensional representations of H when m is even. As $4(\frac{m-2}{2}) + 4 = 2m$, it follows that $\{\pi_s | 1 \leq s \leq \frac{m-2}{2}\}$ together with the four one-dimensional representations of H form a complete set of inequivalent irreducible representations of H. Similarly, if m is odd, then $4(\frac{m-1}{2}) + 2 = 2m$, and it follows that $\{\pi_s | 1 \leq s \leq \frac{m-1}{2}\}$ together with the two one-dimensional representations of H form a complete set of inequivalent irreducible representations of H.

§5. THE REFLECTION REPRESENTATION OF $H_C(G,B)$ AND ITS COMPOUNDS

In this section $H = H(q_1,q_2,\ldots,q_\ell)$ denotes a Hecke algebra attached to a finite irreducible Coxeter system (W,I) of Lie type (cf. §3). Thus W is the Weyl group of a simple complex Lie algebra $\mathfrak{g}$. Let $\{\alpha_1,\alpha_2,\ldots,\alpha_\ell\}$ be a set of simple roots of $\mathfrak{g}$ relative to a Cartan subalgebra of $\mathfrak{g}$ and h the real vector space spanned by $\{\alpha_1,\ldots,\alpha_\ell\}$. Denote by $(\ ,\ )$ the Killing form of $\mathfrak{g}$. We take $I = \{1,2,\ldots,\ell\}$, and w_i, $i \in I$, to be the reflection with respect to the root α_i. We assume the q_i, $i \in I$, are positive real numbers so that H is semisimple by theorem 4.

We shall define a representation of H on h that coincides with the natural action of W on h when $q_i = 1$ for all $i \in I$.

Let $m_{ij} = |\langle w_i w_j\rangle|$, and put

$$u_{ij} = \frac{[q_i + q_j + 2\sqrt{q_i q_j}\cos 2\pi/m_{ij}]^{1/2}}{2\cos \pi/m_{ij}}, \quad m_{ij}\neq 2$$

$$u_{ij} = 0 \qquad\qquad , m_{ij}=2. \tag{14}$$

We define the symmetric bilinear form B on h as follows:

$$B(\alpha_i,\alpha_i) = \frac{1}{2}(q_i + 1)(\alpha_i,\alpha_i)$$

$$B(\alpha_i,\alpha_j) = u_{ij}(\alpha_i,\alpha_j),\ i \neq j.$$

Put
$$C_{ij} = \frac{(q_i + 1)B(\alpha_i,\alpha_j)}{B(\alpha_i,\alpha_i)}$$

and call $C = (C_{ij})$ the <u>Cartan</u> <u>matrix</u> of the Hecke algebra H. Following is a list of all the Cartan matrices of H, along with their determinants.

<u>(A_ℓ)</u>:

$$\begin{array}{ccccccc} \circ & \text{—} & \circ & - \cdots - & \circ & \text{—} & \circ \\ 1 & & 2 & & \ell-1 & & \ell \end{array}$$

Let $q_i = p$, $1 \le i \le \ell$

$$C = \begin{bmatrix} p+1 & -\sqrt{p} & & & \\ -\sqrt{p} & p+1 & -\sqrt{p} & & \\ & -\sqrt{p} & p+1 & & \\ & & & \ddots & -\sqrt{p} \\ & & & -\sqrt{p} & p+1 \end{bmatrix}_{\ell\times\ell}$$

$$\det C = \frac{p^{\ell+1} - 1}{p - 1} = 1 + p + p^2 + \cdots + p^\ell.$$

<u>(B_ℓ)</u>:

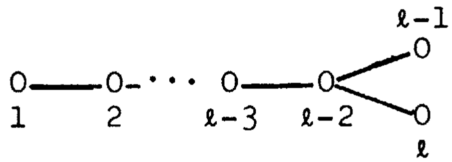
$$\begin{array}{ccccccccccc} \circ & \text{—} & \circ & \text{—} & \circ & - \cdots - & \circ & \text{—} & \circ & \Rightarrow & \circ \\ 1 & & 2 & & 3 & & \ell-2 & & \ell-1 & & \ell \end{array}$$

Let $q_i = p$, $1 \le i \le \ell-1$, $q_\ell = q$.

$\det C$
$= p^{\ell-1}q+1$

$$\begin{bmatrix} p+1 & -\sqrt{p} & & & & \\ -\sqrt{p} & p+1 & -\sqrt{p} & & & \\ & -\sqrt{p} & p+1 & & & \\ & & & \ddots & -\sqrt{p} & \\ & & & -\sqrt{p} & p+1 & -2\sqrt{\frac{p+q}{2}} \\ & & & & -\sqrt{\frac{p+q}{2}} & q+1 \end{bmatrix}_{\ell\times\ell}$$

(D_ℓ)

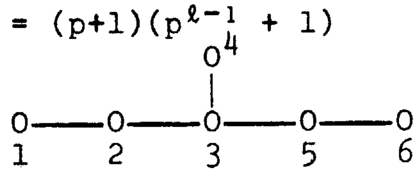


Let $q_i = p, \ 1 \leq i \leq \ell.$

$$C = \begin{bmatrix} p+1 & -\sqrt{p} & & & & & & \\ -\sqrt{p} & p+1 & -\sqrt{p} & & & & & \\ & -\sqrt{p} & p+1 & & & & & \\ & & & \ddots & & & & \\ & & & & p+1 & -\sqrt{p} & & \\ & & & & -\sqrt{p} & p+1 & -\sqrt{p} & -\sqrt{p} \\ & & & & & -\sqrt{p} & p+1 & \\ & & & & & -\sqrt{p} & & p+1 \end{bmatrix}_{\ell\times\ell}$$

$$\det C = (p+1)(p^{\ell-1}+1)$$

$(\underline{E_6})$:

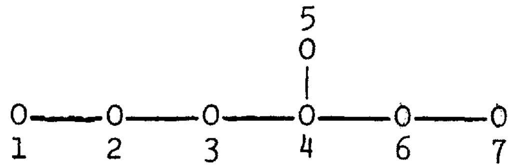


Let $q_i = p, \ 1 \leq i \leq 6.$

$$C = \begin{bmatrix} p+1 & -\sqrt{p} & & & & \\ -\sqrt{p} & p+1 & -\sqrt{p} & & & \\ & -\sqrt{p} & p+1 & -\sqrt{p} & -\sqrt{p} & \\ & & -\sqrt{p} & p+1 & & \\ & & -\sqrt{p} & & p+1 & -\sqrt{p} \\ & & & & -\sqrt{p} & p+1 \end{bmatrix}$$

$$\det C = \frac{p^6+1}{p^2+1}\cdot(p^2+p+1) = p^6 + p^5 - p^3 + p + 1$$

$\underline{(E_7)}$:

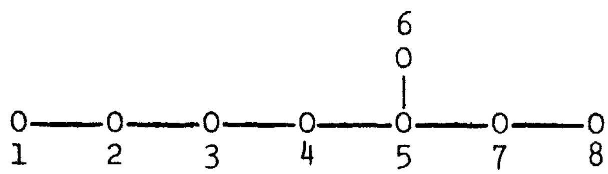


Let $q_i = p$, $1 \leq i \leq 7$.

$$C = \begin{bmatrix} p+1 & -\sqrt{p} & & & & & \\ -\sqrt{p} & p+1 & -\sqrt{p} & & & & \\ & -\sqrt{p} & p+1 & -\sqrt{p} & & & \\ & & -\sqrt{p} & p+1 & -\sqrt{p} & -\sqrt{p} & \\ & & & -\sqrt{p} & p+1 & & \\ & & & -\sqrt{p} & & p+1 & -\sqrt{p} \\ & & & & & -\sqrt{p} & p+1 \end{bmatrix}$$

$$\det C = \frac{(p+1)(p^9+1)}{(p^3+1)} = (p+1)(p^6 - p^3 + 1)$$

$\underline{(E_8)}$:



Let $q_i = p$, $1 \leq i \leq 8$

$$C = \begin{bmatrix} p+1 & -\sqrt{p} & & & & & \\ -\sqrt{p} & p+1 & -\sqrt{p} & & & & \\ & -\sqrt{p} & p+1 & -\sqrt{p} & & & \\ & & -\sqrt{p} & p+1 & -\sqrt{p} & -\sqrt{p} & \\ & & & -\sqrt{p} & p+1 & & \\ & & & -\sqrt{p} & & p+1 & -\sqrt{p} \\ & & & & & -\sqrt{p} & p+1 \end{bmatrix}$$

$$\det C = \frac{(p^{15}+1)(p+1)}{(p^5+1)(p^3+1)} = p^8 + p^7 - p^5 - p^4 - p^3 + p + 1$$

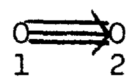
<u>(F_4)</u>:

Let $q_1 = q_2 = p$, $q_3 = q_4 = q$.

$$C = \begin{bmatrix} p+1 & -\sqrt{p} & & \\ -\sqrt{p} & p+1 & -2\sqrt{\frac{p+q}{2}} & \\ & -\sqrt{\frac{p+q}{2}} & q+1 & -\sqrt{q} \\ & & -\sqrt{q} & q+1 \end{bmatrix}$$

$$\det C = p^2q^2 - pq + 1 = \frac{p^3q^3 + 1}{pq + 1}$$

(G_2):



Let $q_1 = p$, $q_2 = q$.

$$\begin{bmatrix} p+1 & -3\sqrt{\frac{p+q+\sqrt{pq}}{3}} \\ -\sqrt{\frac{p+q+\sqrt{pq}}{3}} & q+1 \end{bmatrix}$$

$$\det C = pq - \sqrt{pq} + 1 = \frac{(pq)^{3/2} + 1}{pq + 1}$$

<u>PROPOSITION 20</u>: The bilinear form B is positive definite.

PROOF: It suffices to prove that the principal minors of the matrix $(B(\alpha_i, \alpha_j))$ are all positive, and for this it suffices to show that the principal minors of the Cartan matrix are all positive. Now the preceding list shows that the determinants of all the Cartan matrices are positive, and any principal minor of a Cartan matrix can be construed as the determinant of the Cartan matrix of the Hecke algebra associated to a Weyl group of some semisimple Lie algebra of smaller rank. Hence B is positive definite.

Now for $i \in I$ define $\tilde{X}_i \in \operatorname{End} h$ by

$$\tilde{X}_i(\xi) = q_i\xi - (q_i+1)\frac{B(\alpha_i,\xi)}{B(\alpha_i,\alpha_i)}\alpha_i \quad . \tag{15}$$

As B is an inner product on h it is clear that $R\alpha_i$ is the -1 eigenspace for $\tilde{X}_i$, while, $(R\alpha_i)^{\perp}$, the B-orthogonal complement of $R\alpha_i$, is the q_i-eigenspace for $\tilde{X}_i$. In particular -1 and q_i are the only eigenvalues of $\tilde{X}_i$, and hence we have:

<u>LEMMA 7</u>: $(\tilde{X}_i)^2 = q_i \cdot I + (q_i-1)\tilde{X}_i$.

<u>LEMMA 8</u>: $$\underbrace{\tilde{X}_i\tilde{X}_j\tilde{X}_i\cdots}_{m_{ij}} = \underbrace{\tilde{X}_j\tilde{X}_i\tilde{X}_j\cdots}_{m_{ij}} \tag{16}$$

PROOF: Let $V_{ij} = R\alpha_i + R\alpha_j$ and $U_{ij} = V_{ij}^{\perp}$. It is clear from the definition of $\tilde{X}_i$, that V_{ij} is stable under the action of $\tilde{X}_i$ and $\tilde{X}_j$. Moreover, $\tilde{X}_i$ and $\tilde{X}_j$ operate as the scalars q_i and q_j respectively on U_{ij} so that U_{ij} is also stable under the action of $\tilde{X}_i$ and $\tilde{X}_j$. Now it is clear that (16) holds on U_{ij} (keeping in mind that $q_i=q_j$

if m_{ij} is odd); and on V_{ij} the matrices of $\tilde{X}_i$ and $\tilde{X}_j$ with respect to the basis $\{\alpha_i, \alpha_j\}$ are

$$\begin{bmatrix} -1 & \dfrac{-2(\alpha_i,\alpha_j)u_{ij}}{(\alpha_i,\alpha_i)} \\ 0 & q_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} q_j & 0 \\ \dfrac{-2(\alpha_j,\alpha_i)}{(\alpha_j,\alpha_j)}u_{ij} & -1 \end{bmatrix}$$

respectively.

Now,

$$\frac{4(\alpha_i,\alpha_j)^2}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}u_{ij}^2 = \begin{cases} q_i + q_j + 2\sqrt{q_i q_j}\cos\dfrac{2\pi}{m_{ij}} & m_{ij} \neq 2 \\ 0 & m_{ij} = 2 \end{cases}$$

It follows therefore from the representation theory of the Hecke algebra of the dihedral groups (lemma 6) that on V_{ij} we have

$$\underbrace{\tilde{X}_i\tilde{X}_j\tilde{X}_i\cdots}_{m_{ij}} = \underbrace{\tilde{X}_j\tilde{X}_i\tilde{X}_j\cdots}_{m_{ij}}$$

and this proves the lemma.

It follows from the presentation (4) of H together with the above lemmas that the map $X_{w_i} \to \tilde{X}_i$ can be uniquely extended to an algebra homomorphism $\pi\colon H \to \operatorname{End} h$; namely, if $w_{i_1}w_{i_2}\cdots w_{i_m}$ is any reduced expression for $w \in W$, then $\pi(X_w) = \tilde{X}_{i_1}\tilde{X}_{i_2}\cdots\tilde{X}_{i_m}$. We call $\pi\colon H \to \operatorname{End} h$ the reflection representation of H because it reduces to the usual representation of W on h as a group generated by reflections when all the q_i are set equal to 1. We use the notation

$x \cdot \alpha = \pi(x) \cdot \alpha$ when $x \in H$, $\alpha \in h$.

PROPOSITION 21: Relative to the inner product B, $\pi(X_{w_i})$ is self adjoint. The adjoint of $\pi(X_w)$ is $\pi(X_{w^{-1}})$.

PROOF: The first statement is easily checked from the definition of $\pi(X_{w_i}) = \tilde{X}_i$. The second assertion follows from the fact that if $w_{i_1} w_{i_2} \dots w_{i_m}$ is a reduced expression for $w \in W$, then $X_w = X_{w_{i_1}} X_{w_{i_2}} \dots X_{w_{i_m}}$, and $X_{w^{-1}}$ $= X_{w_{i_m}} X_{w_{i_{m-1}}} \dots X_{w_{i_1}}$.

Note that if we put $a_{ij} = \dfrac{-(q_i+1)B(\alpha_i,\alpha_j)}{B(\alpha_i,\alpha_i)}$, then we have

$$X_{w_i} \cdot (\alpha_j) = q_i \alpha_j + a_{ij} \alpha_i, \quad i,j \in I.$$

we have the relations

$$\left.\begin{aligned}
a_{ii} &= -(q_i + 1) \\
a_{ij} &= a_{ji} = 0 && \text{if } m_{ij} = 2 \\
a_{ij}a_{ji} &= q_i = q_j && \text{if } m_{ij} = 3 \\
a_{ij}a_{ji} &= q_i + q_j && \text{if } m_{ij} = 4 \\
a_{ij}a_{ji} &= q_i + q_j + \sqrt{q_i q_j} && \text{if } m_{ij} = 6
\end{aligned}\right\} \qquad (17)$$

REMARK: It is not difficult to show, using the fact that the Dynkin graph of W is a tree, that given any set $\{b_{ij}\}$ of ℓ^2 complex numbers which satisfy equations (17), there exist complex numbers $\{c_1, c_2, \dots, c_\ell\}$ such that if $\alpha_i' = c_i \alpha_i$, $(1 \leq i \leq \ell)$, then one has $X_{w_i} \cdot (\alpha_j') = q_i \alpha_j' + b_{ij} \alpha_i'$. Thus the reflection representation is determined up to complex equivalence by equations (17).

Let $\nu: H \to \mathrm{End}\, V$ be a representation of H on the complex vector space V. We say that ν has an integral form or simply that ν is defined over Z if there exists a basis of V such that the matrices of $\nu(X_w)$, relative to that basis, have rational integral coefficients for all $w \in W$.

PROPOSITION 22: If W is not of type (G_2), and if the q_i $(1 \leq i \leq \ell)$ are positive integers, then the reflection representation of H is defined over Z. If W is of type (G_2), then H is defined over Z if and only if q_1, q_2 and $\sqrt{q_1 q_2}$ are positive integers.

PROOF: If W is not of type (G_2) and the q_i $(1 \leq i \leq \ell)$ are positive integers, or if W is of type (G_2) such that q_1, q_2 and $\sqrt{q_1 q_2}$ are positive integers, then the fact that the reflection representation π is defined over Z is immediate from the preceding remark. Suppose conversely that W is of type (G_2) and that π is defined over Z. If we denote by χ the character of π, then $\chi(X_{w_1}) = q_1 - 1$, $\chi(X_{w_2}) = q_2 - 1$, $\chi(X_{w_1 w_2}) = \sqrt{q_1 q_2}$. Hence q_1, q_2 and $\sqrt{q_1 q_2}$ must be integers.

Let $\overset{k}{\Lambda} h$ denote the k-fold exterior product of h; we consider $\overset{k}{\Lambda} h$ as a subspace of the exterior algebra of h. Define the operator $\tilde{X}_i^{(k)} \in \mathrm{End}\, \overset{k}{\Lambda} h$ by

$$\tilde{X}_i^{(k)}(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k) = q_i^{-(k-1)} \tilde{X}_i(\xi_1) \wedge \tilde{X}_i(\xi_2) \wedge \cdots \wedge \tilde{X}_i(\xi_k)$$

$$(\xi_1, \xi_2, \ldots, \xi_k \in h).$$

<u>LEMMA 9</u>: $(\tilde{X}_i^{(k)})^2 = q_i \cdot I + (q_i - 1)\tilde{X}_i^{(k)}$ (18)

$$\underbrace{\tilde{X}_i^{(k)}\tilde{X}_j^{(k)}\cdots}_{m_{ij}} = \underbrace{\tilde{X}_j^{(k)}\tilde{X}_i^{(k)}\cdots}_{m_{ij}} \tag{19}$$

PROOF: It is obvious that equation (19) holds from the definition because the relation is satisfied in End h. Now let $U = R\alpha_i$ and $V = U^{\perp}$ = the orthogonal complement of U relative to the inner product B. Then

$$\overset{k}{\Lambda} h = \overset{k}{\Lambda} V \oplus (\overset{k-1}{\Lambda} V)\wedge U$$

is a linear direct sum. As U is the (-1)-eigenspace for $\tilde{X}_i$ and V is the q_i-eigenspace for $\tilde{X}_i$, it follows from the definition that every element of $\overset{k}{\Lambda} V$ is an eigenvector for $\tilde{X}_i^{(k)}$ with eigenvalue q_i, and every element of $(\overset{k-1}{\Lambda} V)\wedge U$ is an eigenvector for $\tilde{X}_i^{(k)}$ with eigenvalue -1. Hence the eigenvalues of $\tilde{X}_i^{(k)}$ are q_i and -1; and equation (18) is verified.

It is immediate from the lemma that the mapping $X_{w_i} \to \tilde{X}_i^{(k)}$ can be extended uniquely to an algebra homomorphism $\pi^{(k)}: H \to \mathrm{End}\,(\overset{k}{\Lambda} h)$. We call the representation $\pi^{(k)}: H \to \mathrm{End}\,\overset{k}{\Lambda} h$ the k^{th} compound of π, and denote the character of $\pi^{(k)}$ by $\chi^{(k)}$. We identify $\overset{0}{\Lambda} h$ with the trivial representation $\pi^0(X_w) = \zeta(X_w)$ (cf §3).

<u>PROPOSITION 23</u>: Assume the conditions of proposition 22 are satisfied. Then the $\pi^{(k)}$ are defined over Z $(0 \le k \le \ell)$.

PROOF: One has $X_{w_i}(\alpha_j) = q_i\alpha_j + a_{ij}\alpha_i$ where the a_{ij} are integers $1 \leq i,j \leq \ell$. We choose as a canonical basis for $\overset{k}{\Lambda}h$ $\{\alpha_{i_1}\wedge\alpha_{i_2}\wedge\cdots\wedge\alpha_{i_k}\}$ where $(i_1,i_2,\ldots,i_k)$ runs over all sequences of positive integers such that $i_1 < i_2 < \ldots < i_k$. We apply X_{w_j} to $\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_k}$:

$$X_{w_j}\cdot(\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_k}) = q_j^{-(k-1)}X_{w_j}\alpha_{i_1}\wedge\cdots\wedge X_{w_j}\alpha_{i_k}$$

$$= q_j^{-(k-1)}[(q_j\alpha_{i_1} + a_{ji_1}\alpha_j)\wedge\cdots\wedge(q_j\alpha_{i_k} + a_{ji_k}\alpha_j)] =$$

$$q_j\alpha_{i_1}\wedge\alpha_{i_2}\wedge\cdots\wedge\alpha_{i_k} + \sum_{m=1}^{k}(-1)^{m+1}a_{ji_m}\alpha_j\wedge\alpha_{i_1}\wedge\cdots\wedge\hat{\alpha}_{i_m}\wedge\cdots\wedge\alpha_{i_k}, \tag{20}$$

where the symbol $\hat{\alpha}_{i_m}$ means that the factor α_{i_m} is omitted. Now rearranging (20) so as to get everything expressed in terms of the canonical basis of $\overset{k}{\Lambda}h$ only involves changing the signs of certain coefficients. Hence relative to this basis, the matrix of $\pi^{(k)}(X_{w_i})$ has integral coefficients and consequently $\pi^{(k)}$ is defined over Z.

<u>THEOREM 6</u>: If the Coxeter system (W,I) is irreducible, then the representations $\pi^{(k)}: H \to \mathrm{End}\,\overset{k}{\Lambda}h$ are distinct and absolutely irreducible $(0 \leq k \leq \ell)$.

PROOF: To simplify the notation let $h = C \otimes_R h$, $H = C \otimes_R H$, and extend B to a symmetric nondegenerate bilinear form on h in the obvious way. We argue by induction on the rank of W. If the rank is one, then $\overset{0}{\Lambda}h$ and $\overset{1}{\Lambda}h$ are both 1-dimensional, hence irreducible. Suppose then that rank $(W) = \ell > 1$, and let J be a subset of I such that $|J| = \ell-1$ and the Dynkin diagram of W_J is connected, where $W_J = \langle w_i \mid i \in J\rangle$. Then $h_J = \sum_{i \in J} C\alpha_i$ can be identified with a Cartan subalgebra of a simple complex Lie algebra of rank $\ell-1$, and H_J = the subalgebra of H generated by $\{X_{w_i} \mid i \in J\}$ is a Hecke algebra over C associated to W_J. Let $V = h_J^{\perp}$ be the orthogonal complement of h_J relative to B. Then

$$\overset{k}{\Lambda} h = \overset{k}{\Lambda} h_J \oplus \overset{k-1}{\Lambda} h_J \wedge V \tag{21}$$

Now considering $\overset{k}{\Lambda} h$ as an H_J-module by restriction, V is 1-dimensional affording the representation $\pi^{(0)}(X_{w_i}) = q_i$, $i \in J$. Thus it follows from the definition of the action of H on h that $\overset{k-1}{\Lambda} h_J \wedge V \cong \overset{k-1}{\Lambda} h_J$. But by induction $\overset{k}{\Lambda} h_J$ and $\overset{k-1}{\Lambda} h_J$ are distinct and irreducible as H_J-modules. Thus as an H-module, either $\overset{k}{\Lambda} h$ is irreducible or (21) is the decomposition of $\overset{k}{\Lambda} h$ into distinct irreducible constituents. But it is easily seen that $\overset{k}{\Lambda} h_J$ is not stable under the action of H. Hence $\overset{k}{\Lambda} h$ is irreducible. It remains to show that $\overset{k}{\Lambda} h$ is not H-isomorphic with $\overset{k'}{\Lambda} h$ if $k \neq k'$ $(0 \leq k,k' \leq \ell)$. In the proof of lemma 9 we have seen that the dimension of the q_i-eigenspace for $\pi^{(k)}(X_{w_i})$ is $\binom{\ell-1}{k}$. Hence if $\overset{k}{\Lambda} h$ and $\overset{k-1}{\Lambda} h$ are H-equivalent we must have $\binom{\ell}{k} = \binom{\ell}{k'}$ and $\binom{\ell-1}{k} = \binom{\ell-1}{k'}$ which implies that $k = k'$. This completes the induction argument.

Recall that the involution $x \to \hat{x}$ of proposition 18 sets up a natural pairing among the irreducible characters of

H. The compound characters $\chi^{(k)}$ of the reflection character are naturally paired as we shall see below, but first we need a lemma.

LEMMA 10: $\overset{\ell}{\Lambda} h$ affords the linear character σ, where $\sigma(X_w) = (-1)^{\ell(w)}$, $w \in W$.

PROOF: Let $i \in I$, and choose a basis $\{\xi_1, \ldots, \xi_\ell\}$ of h such that $X_{w_i} \cdot \xi_j = q_i \cdot \xi_j$, $(1 \leq j \leq \ell-1)$ and $X_{w_i} \cdot \xi_\ell = -\xi_\ell$. Then by the definition of the action of H on $\overset{\ell}{\Lambda} h$ it is obvious that if $\xi = \xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_\ell$, one has $X_{w_i} \cdot \xi = -\xi$. Thus $\chi^{(\ell)}(X_{w_i}) = -1$, and as $\chi^{(\ell)}$ is linear we must have $\chi^{(\ell)}(X_w) = (-1)^{\ell(w)} = \sigma(X_w)$ for all $w \in W$.

PROPOSITION 24: One has $\hat{\chi}^{(k)} = \chi^{(\ell-k)}$.

PROOF: Let λ be a nonzero vector in $\overset{\ell}{\Lambda} h$, and let $\xi \in \overset{k}{\Lambda} h$, $\eta \in \overset{\ell-k}{\Lambda} h$. Then $\xi \wedge \eta = c \cdot \lambda$ for some unique scalar c and we have a nonsingular pairing $<\ ,\ >$ between $\overset{k}{\Lambda} h$ and $\overset{\ell-k}{\Lambda} h$, namely $<\xi, \eta> = c$. Let $i \in I$ and let $\xi = \xi_1 \wedge \ldots \wedge \xi_k$, $\eta = \eta_1 \wedge \ldots \wedge \eta_{\ell-k}$, $\xi_j \in h$, $\eta_j \in h$. Then $(X_{w_i} \cdot \xi) \wedge (X_{w_i} \cdot \eta)$ $= q_i^{-(\ell-2)} (X_{w_i} \cdot \xi_1) \wedge \ldots \wedge (X_{w_i} \cdot \xi_k) \wedge (X_{w_i} \cdot \eta_1) \wedge \ldots \wedge (X_{w_i} \cdot \eta_{\ell-k})$ $= q_i X_{w_i} \cdot (\xi \wedge \eta) = -q_i (\xi \wedge \eta)$. In other words $<X_{w_i} \cdot \xi, X_{w_i} \cdot \eta>$ $= -q_i <\xi, \eta>$. Now we can rewrite this equation in the form $<X_{w_i} \cdot \xi, \eta> = <\xi, -q_i^{-1} X_{w_i}^{-1} \cdot \eta> = <\xi, \hat{X}_{w_i} \eta>$. Hence for any $w \in W$ we have the equation $<X_w \cdot \xi, \eta> = <\xi, \hat{X}_w^{-1} \cdot \eta>$. This implies, using the natural identification of $\overset{k}{\Lambda} h$ with $(\overset{\ell-k}{\Lambda} h)^*$, that the contragredient H-module $({}^{\ell}\Lambda^{k} h)^*$ (cf. proposition 19) is equivalent to $\overset{\hat{k}}{\Lambda} h$. But as the $\overset{k}{\Lambda} h$ are all defined over R it follows from proposition 19 that $(\overset{\ell-k}{\Lambda} h)^*$ is equivalent to $\overset{\ell-k}{\Lambda} h$. Hence we have $\hat{\chi}^{(k)} = \chi^{(\ell-k)}$ as asserted.

<u>THEOREM 7</u>: Let J be any subset of I, $e_J = \left(\sum_{w \in W_J} \zeta(X_w)\right)^{-1} \sum_{w \in W_J} X_w$, then $\chi^{(k)}(e_J) = \binom{|I-J|}{k}$.

PROOF: Let h_J be the subspace of h spanned by $\{\alpha_i | i \in J\}$, and $h_J^{\perp}$ = the orthogonal complement of h_J with respect to the bilinear form B. Then $h_J^{\perp}$ obviously affords $|I-J|$ copies of the trivial representation of H_J. One has $h = h_J \oplus h_J^{\perp}$ and hence $\overset{k}{\Lambda} h = \bigoplus_{i=0}^{k} (\overset{i}{\Lambda} h_J) \wedge (\overset{k-i}{\Lambda} h_J^{\perp})$. Now it is easily seen from the definition of the action of H on $\overset{k}{\Lambda} h$, that as an H_J-module one has

$$(\overset{i}{\Lambda} h_J) \wedge (\overset{k-i}{\Lambda} h_J^{\perp}) \simeq (\overset{i}{\Lambda} h_J)^{\binom{|I-J|}{k-i}}$$

We assert that $\overset{i}{\Lambda} h_J$ does not contain the trivial representation of H_J if $i > 0$. Indeed let $J_1, J_2, \ldots, J_m$ be the decomposition of J into connected subsets considering the elements of I as points of the Dynkin graph of W. Then $h_J = h_{J_1} \oplus h_{J_2} \oplus \cdots \oplus h_{J_m}$ is the decomposition of h_J into distinct irreducible H_J-submodules. Thus $\overset{i}{\Lambda} h_J = \oplus (\overset{i_1}{\Lambda} h_{J_1}) \wedge (\overset{i_2}{\Lambda} h_{J_2}) \wedge \cdots \wedge (\overset{i_m}{\Lambda} h_{J_m})$, where the summation is extended over all sequences $(i_1, i_2, \ldots, i_m)$ of positive integers such that $\sum_{j=1}^{m} i_j = i$. Moreover, each direct summand of $\overset{i}{\Lambda} h_J$ is an H_J-submodule. Suppose $\overset{i}{\Lambda} h_J$ contains a vector ξ which affords the trivial representation of H_J. Then $\xi = \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_m}$, where $\xi_{i_j} \in \overset{i_j}{\Lambda} h_{J_j}$. But then the pairwise orthogonality of the h_{J_i} implies that each ξ_{i_j} affords the trivial representation of H_J in h_{J_j}. By theorem 6 we must have $i_j = 0$, $1 \leq j \leq m$, thus $i = 0$, proving the assertion. Now by proposition 16 $\pi^{(k)}(e_J)$ is the projection of $\overset{k}{\Lambda} h$ onto the primary component of $\overset{k}{\Lambda} h$ corresponding to the trivial representation of H_J. Hence $e_J \cdot \overset{k}{\Lambda} h \cong (\overset{0}{\Lambda} h_J)^{\binom{|I-J|}{k}}$ and in particular $\dim e_J \cdot \overset{k}{\Lambda} h = \binom{|I-J|}{k}$. The assertion of the theorem now follows from proposition 16.

In the case of $\pi = \pi^{(1)}$, the reflection representation, these multiplicities are enough to distinguish it when the rank is greater than 1, as we shall see below.

<u>THEOREM 8</u>: Let (W,I) be an irreducible Coxeter system of Lie type and $H = H(q_1,\ldots,q_\ell)$ a Hecke algebra associated to W over C. Suppose $\pi: H \to \mathrm{End}\, V$ is an irreducible complex representation of H, affording the character χ of H. Assume that $\chi(e_J) = |I-J|$ for every subset J of I. If W is not of type (G_2), then π is equivalent to the reflection representation of H.

<u>PROOF</u>: Taking J equal to the empty set $\emptyset$, we have $e_\emptyset = X_1$, the identity of H. Thus $\dim V = \chi(e_\emptyset) = |I| = \ell$. Let $i \in I$, and take $J = \{i\}$. Then $\pi(e_J)$ is the projection on the q_i-eigenspace for the operator $\pi(X_{w_i})$. As $\chi(e_J) = |I-J| = \ell-1$, the dimension of the q_i-eigenspace for $\pi(X_{w_i})$ is $\ell-1$. Thus the (-1)-eigenspace for $\pi(X_{w_i})$ is one-dimensional for all $i \in I$. Let u_i be a nonzero vector

in the (-1)-eigenspace of $\pi(X_{w_i})$, $i \in I$. We break the rest of the proof into a number of assertions.

<u>ASSERTION (a)</u>: If $\xi \in V$, then $X_{w_i} \cdot \xi = q_i \xi + c u_i$ for some $c \in C$.

PROOF: Note that $\pi(X_{w_i} - q_i X_1)$ must be a scalar multiple of the orthogonal projection on the (-1)-eigenspace for $\pi(X_{w_i})$. Assertion (a) follows from the fact that this eigenspace is one -dimensional spanned by u_i.

<u>ASSERTION (b)</u>: $\{u_1, u_2, \ldots, u_\ell\}$ form a basis for V.

PROOF: It is clear from assertion (a) that the subspace spanned by $\{u_1, \ldots, u_\ell\}$ is H-stable. By the irreducibility of V it must coincide with V; and since $\dim V = \ell$, the u_i are linearly independent.

Let $X_{w_i} \cdot u_j = q_i u_j + b_{ij} u_i$ $(1 \leq i,j \leq \ell)$. It is clear that $b_{ii} = -(q_i + 1)$.

<u>ASSERTION (c)</u>: $\pi(X_{w_i})$ and $\pi(X_{w_j})$ commute if and only if $b_{ij} = b_{ji} = 0$, in which case one has $X_{w_i}(u_j) = q_i u_j$, $X_{w_j}(u_i) = q_j u_i$, $(i \neq j)$.

PROOF: This can be checked directly from the definitions.

<u>ASSERTION (d)</u>: Let $i \neq j$, then $\pi(X_{w_i})$ and $\pi(X_{w_j})$ commute if and only if $m_{ij} = 2$ $(m_{ij} = |\langle w_i w_j \rangle|)$.

PROOF: If $m_{ij} = 2$, then X_{w_i} and X_{w_j} commute, hence so do $\pi(X_{w_i})$ and $\pi(X_{w_j})$. Suppose that $\pi(X_{w_i})$ and

$\pi(X_{w_j})$ commute with $m_{ij} > 2$. Then as the Dynkin graph of W is a tree, it follows that there exists a partition of I into two disjoint nonempty subsets I' and I'' such that $\pi(X_{w_i})$ commutes with $\pi(X_{w_j})$ for all $i \in I'$, $j \in I''$. But then if $V' = \sum_{i\in I'} Cu_i$, $V'' = \sum_{i\in I''} Cu_i$, one has by assertion (c) that V' and V'' are H-stable, contradicting the irreducibility of V.

<u>ASSERTION (e)</u>: If $m_{ij} > 2$, then

$$b_{ij}b_{ji} = q_i + q_j + 2\sqrt{q_iq_j}\cos\frac{2\pi}{m_{ij}}.$$

PROOF: Let H_{ij} be the subalgebra of H generated by X_{w_i}, X_{w_j} and the identity. Let $V_{ij} = Cu_i + Cu_j$. Then H_{ij} is a Hecke algebra of the dihedral group of order $2m_{ij}$, and the restriction of π to H_{ij} induces a representation of H_{ij} on the subspace V_{ij} of V. As $m_{ij} > 2$, $\pi(X_{w_i})$ and $\pi(X_{w_j})$ do not commute on V_{ij} by assertion (c). Consequently V_{ij} is an irreducible H_{ij}-module. But W is of Lie type so that $m_{ij} = 3$ or 4. It follows from theorem 5 that in either case H_{ij} has precisely one irreducible two-dimensional representation, and that the trace of $X_{w_i}X_{w_j}$ in this representation is $2\sqrt{q_iq_j}\cos\frac{2\pi}{m_{ij}}$. Assertion (e) now follows from the fact that the trace of $\pi(X_{w_i}X_{w_j})$ on V_{ij} is equal to $b_{ij}b_{ji} - (q_i + q_j)$.

Thus we have by the above assertions:

$$X_{w_i}(u_j) = q_i u_j + b_{ij} u_i \qquad i,j \in I$$

$$b_{ii} = -(q_i + 1) \qquad i \in I$$

$$b_{ij} = b_{ji} = 0 \qquad m_{ij}=2,\ i,j \in I$$

$$b_{ij} b_{ji} = q_i + q_j + 2\sqrt{q_i q_j}\cos\frac{2\pi}{m_{ij}},\quad m_{ij}>2,\ i,j \in I$$

The fact that π is equivalent to the reflection representation of H is now an immediate consequence of remark following proposition 21.

REMARK: Note that for the proof of theorem 8 we only had to assume that $\chi(e_J) = |I-J|$ when $|J| = 0$ or 1.

Applying these results to the case where H is the Hecke algebra of some finite irreducible group with BN pair we have the following theorem:

THEOREM 9: Let G be a finite irreducible group with BN pair, and assume that the Coxeter system (W,I) of G is of Lie type. Then there exist irreducible complex characters $\{\tilde{\chi}^{(k)} | 0 \leq k \leq \ell\}$ of G such that if $(1_{G_J})^G$ denotes the induced character from the trivial character of the parabolic subgroup G_J, then

$$\left(\tilde{\chi}^{(k)}, (1_{G_J})^G\right)_G = \binom{|I-J|}{k} \qquad (22)$$

Moreover, $\tilde{\chi}^{(1)}$ is uniquely determined by (22) if G is not of type (G_2). The representations affording the characters $\tilde{\chi}^{(k)}$ are all defined over Q.

PROOF: By proposition 2 we know that for each k $(0 \leq k \leq \ell)$ there exists a unique complex irreducible character $\tilde{\chi}^{(k)}$ whose restriction to $H_C(G,B)$ is $\chi^{(k)}$, where $\{\chi^{(i)} | 0 \leq i \leq \ell\}$ are the compounds of the reflection character of $H_C(G,B)$. Now

$$e_J = \Big(\sum_{w \in W_J} \zeta(X_w)\Big)^{-1} \sum_{w \in W_J} X_w = |B|^{-1}\Big(\sum_{w \in W_J} \zeta(X_w)\Big)^{-1} \sum_{x \in G_J} x = e(G_J),$$

is the idempotent of $C[G]$ affording the character $(1_{G_J})^G$. Thus by the Frobenius reciprocity theorem $(\tilde{\chi}^{(k)},(1_{G_J})^G)$ $= (\tilde{\chi}^{(k)}|G_J, 1_{G_J}) = \tilde{\chi}^{(k)}(e(G_J)) = \chi^{(k)}(e_J)$. But by theorem 7, $\chi^{(k)}(e_J) = \binom{|I-J|}{k}$. $\tilde{\chi}^{(1)}$ is uniquely determined by (22) because $\chi^{(1)}$ is uniquely determined by the fact that $\chi^{(1)}(e_J) = |I-J|$. It remains to prove that the representations affording the $\tilde{\chi}^{(k)}$ are defined over Q. By proposition 23, the $\pi^{(k)}$ are defined over Z (except possibly when G is of type G_2 and $\sqrt{q_1 q_2}$ is not rational.) (We exclude this case from the present theorem; it will follow later, from the calculation of the degrees of the characters, that this case can never occur.) Let $\tilde{e}^{(k)}$ be the minimal central idempotent in $C[G]$ corresponding to $\tilde{\chi}^{(k)}$. Then by proposition 2 $\tilde{e}^{(k)}e(B) = e^{(k)}$ is the minimal central idempotent in $H_C(G,B)$ corresponding to the irreducible character $\chi^{(k)}$ of $H_C(G,B)$. Thus by the formula (i) of theorem 1 $e^{(k)} = \sum_{w \in W} a_w X_w$ where $a_w \in Q$ for all $w \in W$. It follows that $\tilde{e}^{(k)}e(B)$ is an element of $Q[G]$. Now let J be any subset of I having cardinality $\ell-k$ where $\ell = |I|$.

Then by theorem 7 we have $\tilde{\chi}^{(k)}(e_J) = \binom{k}{k} = 1$; and it follows from the corollary to proposition 2 that $\tilde{e}^{(k)}e_J$ is a primitive idempotent in $C[G]$ affording the character $\tilde{\chi}^{(k)}$. But $\tilde{e}^{(k)}e_J = \tilde{e}^{(k)}e(B)e_J = e^{(k)}e_J$. Hence $\tilde{e}^{(k)}e_J \in Q[G]$, so that $\tilde{\chi}^{(k)}$ is afforded by the rational irreducible G-module $Q[G]\tilde{e}^{(k)}e_J$.

<u>REMARK</u>: The character $\tilde{\chi}^{(\ell)}$, called the Steinberg character, was first constructed by R. Steinberg for any finite group of Lie type [18]. C. Curtis, [5], has shown that $\tilde{\chi}^{(\ell)}$ exists for any finite group with BN pair, and using methods different from ours has shown that $\tilde{\chi}^{(\ell)}$ is uniquely determined by the fact that $(\tilde{\chi}^{(\ell)},(1_B)^G) = 1$ and $(\tilde{\chi}^{(\ell)},(1_P)^G) = 0$ for any parabolic subgroup P of G having rank exceeding 1. It seems quite likely that the $\tilde{\chi}^{(k)}$ are also uniquely determined by the multiplicities (22), but we have no proof of this as yet.

For future reference we now define the weights of H when H is a Hecke algebra associated to an irreducible Coxeter system of Lie type. It will be seen that upon setting $q_i = 1$ $(1 \leq i \leq \ell)$ one recovers the usual definition of the weights.

Let $\pi: H \to \mathrm{End}\, h$ be the reflection representation of H. For each $i \in I$ let $J_i = I-\{i\}$. Then as an H_{J_i}-module we have $h = h_{J_i} \oplus h_{J_i}^{\perp}$ where h_{J_i} is the subspace of h spanned by the α_j, $j \in J_i$. Thus $h_{J_i}^{\perp}$ is one-dimensional affording the trivial representation of H_{J_i}. It is clear that $\pi(e_{J_i})$ is the orthogonal projection on $h_{J_i}^{\perp}$. Thus $e_{J_i} \cdot \alpha_j = 0$ for $i \neq j$, $e_{J_i} \cdot \alpha_i \neq 0$.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ be the dual basis of $\{\alpha_1', \alpha_2', \ldots, \alpha_\ell'\}$ relative to the inner product B, where $\alpha_i' = \dfrac{(q_i + 1)}{B(\alpha_i, \alpha_i)} \alpha_i$. It follows that one has $\alpha_j = \sum_{i=1} C_{ij} \lambda_i$ where $(C_{ij}) = C$ is the Cartan matrix of H. We call λ_i the weight of H associated to the root α_i.

<u>PROPOSITION 25</u>: Let $\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ be the weights of H. Then one has

(i) $X_{w_i} \cdot \lambda_i = q_i \lambda_i - \alpha_i$, $X_{w_i} \lambda_j = q_i \lambda_j$ for $i \neq j$.

(ii) $e_{J_i} \cdot \lambda_i = \lambda_i$.

(iii) $e_{J_i} \cdot h = C \cdot \lambda_i$.

PROOF: Immediate from the definition.

It is clear that the weights can be computed in each case by finding the inverse of the Cartan matrix of H.

According to proposition 14, $X_{w_0}^2$ is central in H. As the reflection representation $\pi: H \to \text{End } h$ is absolutely simple it follows that $\pi(X_{w_0}^2)$ is a scalar multiple of the identity operator. Let $\{w_i | i \in I_1\}$, $\{w_i | i \in I_2\}$ be the two conjugacy classes of the elements $\{w_i | i \in I\}$. If there is only one conjugacy class we put $I_1 = I$, $I_2 = \emptyset$. Let $\ell_1 = |I_1|$, $\ell_2 = |I_2|$ and put $q_i = p$, $i \in I_1$; $q_i = q$, $i \in I_2$.

<u>PROPOSITION 26</u>: One has $\pi(X_{w_0}^2) = (p^{\ell_1} q^{\ell_2})^{\frac{h(\ell-1)}{\ell}}$,

where h is the Coxeter number of W. (That is, h is the order of a Coxeter transformation of W.)

PROOF: If $w_0 = w_{i_1} w_{i_2} \cdots w_{i_N}$ is a reduced expression for w_0, then $N = \frac{h\ell}{2}$. Exactly $\frac{h\ell_1}{2}$ of the i_j lie in I_1 and exactly $\frac{h\ell_2}{2}$ of the i_j lie in I_2, $(1 \leq j \leq N)$. Now $\det \pi(X_{w_i}) = -q_i^{\ell-1}$ for all $i \in I$. Hence $\det \pi(X_{w_0}^2)$ $= [(-p^{\ell-1})^{\frac{\ell_1 h}{2}}(-q^{\ell-1})^{\frac{\ell_2 h}{2}}]^2 = \eta(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{\ell}}$, where η is some root of unity. But if we replace p and q by 1 we obtain the action of w_0 on h; and $w_0^2 = 1$. Hence we must have $\eta = +1$.

PROPOSITION 27: The eigenvalues of $\pi(X_{w_0})$ are $\pm(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}}$.

PROOF: Immediate from the preceding proposition.

It is known that if $i \in I$, then $w_0 \cdot \alpha_i = -\alpha_j$ for some $j \in I$. Thus w_0 induces a permutation of the set I, and as w_0 preserves the Killing form, w_0 induces a graph automorphism of the Dynkin diagram of W.

PROPOSITION 28: Let $i \in I$ and let $w_0(\alpha_i) = -\alpha_j$ $(j \in I)$. Then $X_{w_0} \cdot \alpha_i = -(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{2}} \alpha_j$.

PROOF: We have $w_i w_0 = w_0 w_j$. From the proof of proposition 14 it follows that $X_{w_i} X_{w_0} = X_{w_0} X_{w_j}$. Thus $X_{w_i} X_{w_0} \cdot \alpha_j = -X_{w_0} \cdot \alpha_j$. As the (-1)-eigenspace of $\pi(X_{w_i})$ is one-dimensional spanned by α_i we have that $X_{w_0} \cdot \alpha_j = c\alpha_i$

for some $c \in R$. Now $w_0^2 = 1$ so that by proposition 21 $\pi(X_{w_0})$ is self adjoint relative to B. Hence $B(X_{w_0} \cdot \alpha_j, X_{w_0} \cdot \alpha_j) = B(\alpha_j, X_{w_0}^2 \alpha_j)$, $c^2 B(\alpha_i, \alpha_i) = (p^{\ell_1} q^{\ell_2})^{\frac{h(\ell-1)}{\ell}} B(\alpha_j, \alpha_j)$. But w_i and w_j are conjugate by w_0 so that $B(\alpha_i, \alpha_i) = B(\alpha_j, \alpha_j)$. Thus $c = \varepsilon(p^{\ell_1} q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}}$ where ε is a root of unity. Setting $p = q = 1$ we obtain the action of w_0 on h, hence $\varepsilon = -1$, proving the proposition.

§6. DOUBLE COSETS IN WEYL GROUPS

Let (W,I) be a finite Coxeter system. If J is subset of I, then each coset wW_J of W/W_J contains a unique element of minimal length, called the distinguished coset representative (dcr) of wW_J. The dcr $\tilde{w}$ of wW_J is distinguished by the fact that $\ell(\tilde{w}w_j) = \ell(\tilde{w}) + 1$ for all $j \in J$. If J_1 and J_2 are subsets of I, then each double coset $W_{J_1}wW_{J_2}$ of $W_{J_1}\backslash W/W_{J_2}$ contains a unique element of minimal length called the distinguished double coset representative (ddcr) of $W_{J_1}wW_{J_2}$. The ddcr w' of $W_{J_1}wW_{J_2}$ is distinguished by the fact that one has $\ell(w_jw') = \ell(w')+1$ for all $j \in J_1$ and $\ell(w'w_j) = \ell(w') + 1$ for all $j \in J_2$ (cf. section 2).

In this section we prove a theorem, based on a theorem of B. Kostant, about the structure of double coset decompositions. We use the following notations and conventions throughout this section: W is the Weyl group of a semisimple complex Lie algebra $\mathfrak{g}$; h is a Cartan subalgebra of $\mathfrak{g}$; Δ is the set of roots of $\mathfrak{g}$ relative to h; Δ^+ is the set of positive roots relative to some ordering of h; $I = \{1,2,\ldots,\ell\}$, and $\{\alpha_i | i \in I\}$ is the set of simple roots. If $\beta = \sum_{i=1}^{\ell} c_i\alpha_i \in \Delta$, we put $\operatorname{supp}(\beta) = \{ i \in I | c_i \neq 0\}$, $\operatorname{ht}(\beta) = \sum_{i=1}^{\ell} c_i$. R_β is the reflection corresponding to the root β, i.e. $R_\beta(\xi) = \xi - \frac{2(\beta,\xi)}{(\beta,\beta)}\beta$, where $(\ ,\ )$ denotes the Killing form. Put $R\alpha_i = w_i$, $i \in I$. If $J \subseteq I$, put $W_J = \langle w_i | i \in J\rangle$, and put

h_J equal to the subspace of h spanned by $\{\alpha_i | i \in J\}$. $\Delta_J = \Delta \cap h_J$, $\Delta_J^+ = \Delta^+ \cap h_J$. $\perp$ denotes orthogonal complement relative to the Killing form.

The following theorem is due to B. Kostant [13]:

THEOREM 10 (Kostant):

(i) Let $w \in W$ be arbitrary. Then $(w-1)h$ has a basis of roots $\{\beta_1, \ldots, \beta_t\}$ such that $w = R_{\beta_1} R_{\beta_2} \ldots R_{\beta_t}$;

(ii) If $w \in W$ and $w = R_{\gamma_1} \ldots R_{\gamma_s}$ where $\{\gamma_1, \ldots, \gamma_s\}$ is a set of linearly independent roots, then $\{\gamma_1, \ldots, \gamma_s\}$ form a basis for $(w-1)h$.

COROLLARY: Let $J \subset I$, $w \in W$, and $w = R_{\beta_1} \ldots R_{\beta_t}$ where $\{\beta_1, \ldots, \beta_t\}$ is a set of linearly independent roots, then $w \in W_J$ if and only if $\beta_i \in \Delta_J$ $(1 \leq i \leq t)$.

PROOF: Assume $\beta_i \in \Delta_J$ $(1 \leq i \leq t)$ then clearly $R_{\beta_i} \in W_J$ and hence so is w. Conversely if $w \in W_J$, then $h_J^\perp \subseteq h^w$ which implies that $(w-1)h \subseteq h_J$. But by theorem 10, $\{\beta_1, \ldots, \beta_t\}$ is a basis of $(w-1)h$. Hence $\beta_i \in h_J \cap \Delta = \Delta_J$.

THEOREM 11: Let J_1, J_2 be subsets of I and let w_* be the distinguished double coset representative for the double coset $W_{J_1} w_* W_{J_2}$. Then the stabilizer of $w_* W_{J_2}$ in W_{J_1} is equal to W_K where

$$K = \{j \in J_1 | w_*^{-1} w_j w_* \in J_2\}.$$

PROOF: It is clear that W_K is contained in the stabilizer. Suppose that w is an element of W_{J_1} which stabilizes the coset $w_* W_{J_2}$. Then $w_*^{-1} w w_* \in W_{J_2}$. By theorem

10 and the corollary we can write $w = R_{\beta_1}\cdots R_{\beta_t}$ where $\{\beta_1,\ldots,\beta_t\}$ is a set of linearly independent roots in $\Delta^+_{J_1}$. As w_* is distinguished we have that $w_*^{-1}(\beta) \in \Delta^+$ for any root β in $\Delta^+_{J_1}$. Now $w_*^{-1}ww_* = R_{w_*^{-1}(\beta_1)}\cdots R_{w_*^{-1}(\beta_t)} \in W_{J_2}$ and $\{w_*^{-1}(\beta_1),\ldots,w_*^{-1}(\beta_t)\}$ is a set of linearly independent roots. Thus by the corollary to theorem 10 we have that $w_*^{-1}(\beta_i) \in \Delta^+_{J_2}$ $(1 \leq i \leq t)$. Hence it suffices to prove that if $\beta \in \Delta^+_{J_1}$ and $w_*^{-1}(\beta) \in \Delta^+_{J_2}$, then $R_\beta \in W_K$; and to prove this it suffices to prove that $\operatorname{supp}(\beta)$ is contained in K. We prove this by induction on $\operatorname{ht}(\beta)$. If $\operatorname{ht}(\beta) = 1$ there is nothing to prove. Suppose that $\operatorname{ht}(\beta) > 1$. Then we can write $\beta = \beta' + \alpha_j$ where $\beta' \in \Delta^+_{J_1}$, $j \in J_1$, and $\operatorname{ht}(\beta') = \operatorname{ht}(\beta) - 1$. Let $w_*^{-1}(\beta') = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_\ell\alpha_\ell$, $w_*^{-1}(\alpha_j) = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_\ell\alpha_\ell$. Then $w_*^{-1}(\beta) = (c_1+d_1)\alpha_1 + \cdots + (c_\ell + d_\ell)\alpha_\ell$. By hypothesis on β, $c_i + d_i = 0$ if $i \notin J_2$. But since w_* is distinguished, $c_i \geqq 0$, $d_i \geqq 0$. Hence $c_i = d_i = 0$ if $i \notin J_2$. This means that β' and α_j satisfy the same hypothesis as β. By the induction assumption we must have $\operatorname{supp}(\beta')$, $\operatorname{supp}(\alpha_j) \subseteq K$. But $\operatorname{supp}(\beta) = \operatorname{supp}(\beta') \cup \operatorname{supp}(\alpha_j)$. Hence $\operatorname{supp}(\beta) \subseteq K$.

As a corollary to theorem 11 we have the following:

<u>PROPOSITION 29</u>: Let the notation be as in theorem 11, and let Γ be the set of distinguished coset representatives for W_{J_1}/W_K, then $\{\gamma w_* \mid \gamma \in \Gamma\}$ is the set of distinguished coset representatives for $(W_{J_1}w_*W_{J_2})/W_{J_2}$. Each element of

$W_{J_1}w_*W_{J_2}$ has a unique expression of the form $\gamma w_* u$ where $\gamma \in \Gamma$, $u \in W_{J_2}$. One has $\ell(\gamma w_* u) = \ell(\gamma) + \ell(w_*) + \ell(u)$.

PROOF: By theorem 11 we know that $\{\gamma w_* | \gamma \in \Gamma\}$ is a set of distinct representatives for the $W_{J_1}w_*W_{J_2}/W_{J_2}$. So it suffices to show that γw_* is distinguished when $\gamma \in \Gamma$. For this it suffices to show that $\ell(\gamma w_* w_j) = \ell(\gamma w_*) + 1$ for all $j \in J_2$. Suppose that $\ell(\gamma w_* w_j) = \ell(\gamma w_*) - 1$ for some $j \in J_2$. Then since w_* is distinguished, by the axiom of cancellation we must have $\gamma w_* w_j = \gamma' w_*$ for some $\gamma' \in W_{J_1}$, and $\ell(\gamma') = \ell(\gamma) - 1$. But then $w_*^{-1}\gamma^{-1}\gamma' w_* = w_j \in W_{J_2}$ implies that $\gamma^{-1}\gamma' \in$ the stabilizer W_K of $w_*W_{J_2}$ in W_{J_1}. Then $\gamma' \in \gamma W_K$. But since γ is distinguished coset representative for W_{J_1}/W_K it follows that $\ell(\gamma') \geqq \ell(\gamma)$, and this is a contradiction. Hence γw_* is a distinguished coset representative for $W_{J_1}w_*W_{J_2}/W_{J_2}$, and consequently each element of $\gamma w_* W_{J_2}$ has a unique expression of the form $\gamma w_* u$ for some $u \in W_{J_2}$.

Theorem 11 does not show how one can find the ddcr's. The remainder of this section is devoted to showing how some ddcr's can be found in special cases.

<u>LEMMA 11</u>: Let J be a subset of I and w_J the unique element of maximal length in W_J, then an element u of W is a dcr of W/W_J if and only if $\ell(uw_J) = \ell(u) + \ell(w_J)$.

PROOF: It is clear that the condition is necessary (cf. section 2). Suppose that $\ell(uw_J) = \ell(u) + \ell(w_J)$. If $j \in J$ we can write $w_J = w_j w'$ where $w' \in W_J$ and $\ell(w_J) = \ell(w_j) + \ell(w')$. Suppose $\ell(uw_j) < \ell(u)$. Then $\ell(uw_J) = \ell(uw_j w') \leq \ell(uw_j) + \ell(w') < \ell(u) + \ell(w') < \ell(u) + \ell(w_J) = \ell(uw_J)$, a contradiction. Hence $\ell(uw_j) = \ell(u) + 1$ for all $j \in J$, and u is a dcr of W/W_J.

<u>LEMMA 12</u>: Let $J \subseteq I$, then there is a unique dcr w_* of W/W_J of maximal length.

PROOF: If w_J is the unique element of maximal length in W_J, and w_* is a dcr of W/W_J of maximal length, then $\ell(w_* w_J) = \ell(w_*) + \ell(w_J)$. It follows that $w_* w_J$ must be the unique element w_0 of maximal length in W. Hence $w_* = w_0 w_J$ is uniquely determined.

<u>LEMMA 13</u>: Suppose that u is a dcr of W/W_J, and that u does not have maximal length among the dcr of W/W_J, then there exists $i \in I$ such that $\ell(w_i u) = \ell(u) + 1$ and $w_i u$ is again a dcr of W/W_J.

PROOF: Let w_J be the unique element of maximal length in W_J. The assumption on u implies that uw_J is not equal to w_0, the unique element of maximal length in W. Hence there exists $i \in I$ such that $\ell(w_i uw_J) = \ell(uw_J)+1 = \ell(u) + \ell(w_J) + 1 = \ell(w_i u) + \ell(w_J)$. Thus $\ell(w_i u) = \ell(u) + 1$ and $w_i u$ is a dcr of W/W_J by lemma 11.

<u>LEMMA 14</u>: If u is a dcr of W/W_J and u is an involution, then u is a ddcr of $W_J \backslash W/W_J$.

PROOF: Let $j \in J$, then $\ell(uw_j) = \ell(w_j u) = \ell(u) + 1$.

LEMMA 15: Let J be a subset of I, and assume that w_*, the unique dcr of W/W_J of maximal length is an involution. Then w_* is also a ddcr of $W_J\backslash W/W_J$ and the stabilizer of w_*W_J in W_J is equal to W_J.

PROOF: The fact that w_* is a ddcr of $W_J\backslash W/W_J$ follows from lemma 14, as w_* is an involution. If the stabilizer of w_*W_J in W_J is not equal to W_J, then by theorem 11 this stabilizer is equal to W_K where K is some proper subset of J. Thus there exists a dcr γ of W_J/W_K such that $\ell(\gamma) \geq 1$. But then by proposition 29 γw_* is a dcr of W/W_J and $\ell(\gamma w_*) = \ell(\gamma) + \ell(w_*) > \ell(w_*)$. This contradicts our assumption about w_*.

For the rest of this section we restrict ourselves to the following situation: The Dynkin graph D of (W,I) is a tree. Thus there exists $i_0 \in I$ such that the point corresponding to i_0 in D is joined to at most one other point j_0 of D. Evidently i_0 is a terminal point of D. After relabeling the set $I = \{1,2,\ldots,\ell\}$, we may assume that $i_0 = 1$, $j_0 = 2$. Put $J = I-\{1\}$, and $K = I-\{1,2\}$. We then have the following propositions about the ddcr's of $W_J\backslash W/W_J$:

PROPOSITION 30: I = the identity of W is a ddcr of $W_J\backslash W/W_J$. The stabilizer of $1 \cdot W_J$ in W_J is W_J.

PROPOSITION 31: w_1 is a ddcr of $W_J\backslash W/W_J$. The stabilizer of w_1W_J in W_J is W_K.

PROOF: The fact that w_1 is a ddcr of $W_J\backslash W/W_J$ is obvious. By theorem 11, the stabilizer of w_1W_J in W_J is $W_{K'}$ where $K' = \{j \in J | w_1w_jw_1 \in J\}$. In other words, $K' = \{j \in J | w_1w_j = w_jw_1\}$. By our assumption on the subsets J and K we have $K' = K$.

<u>PROPOSITION 32</u>: Let w_* be the unique dcr of W/W_J of maximal length, and assume that w_0 is central. Then w_* is an involution, w_* is a ddcr for $W_J\backslash W/W_J$, and the stabilizer of w_*W_J in W_J is W_J.

PROOF: We have $w_0 = w_*w_J$ where w_J is the unique element of maximal length in W_J. Thus $w_* = w_0w_J$ is an involution, being the product of two commuting involutions. The additional assertions of proposition 32 follow now from lemma 15.

<u>PROPOSITION 33</u>: Assume that w_1 is not the ddcr of $W_J\backslash W/W_J$ of maximal length. Let γ_* be the unique dcr of W_J/W_K of maximal length; and suppose that γ_* is an involution. Then $u = w_1\gamma_*w_1$ is a ddcr for $W_J\backslash W/W_J$. Moreover, the stabilizer of uW_J in W_J is either W_K or W_J depending upon whether u is or is not the unique ddcr of $W_J\backslash W/W_J$ of maximal length.

PROOF: The stabilizer of w_1W_J in W_J is W_K by proposition 31. Hence by proposition 29 γ_*w_1 is a dcr for W/W_J, and $\ell(\gamma_*w_1) = \ell(\gamma_*) + \ell(w_1)$. Thus γ_*w_1 is the unique dcr of W/W_J of maximal length contained in the double coset $W_J w_1 W_J$. By lemma 13 there exists $i \varepsilon I$ such that $\ell(w_i \gamma_* w_1) = \ell(\gamma_* w_1) + 1$, and $w_i \gamma_* w_1$ is again a dcr of W/W_J. From our choice of γ_* it follows that i must be equal to 1. Thus $w_1 \gamma_* w_1$ is a dcr of W/W_J. $w_1 \gamma_* w_1$ is an involution because γ_* is an involution. Hence by lemma 14 $w_1 \gamma_* w_1$ is a ddcr of $W_J \backslash W/W_J$. Now the assumptions on γ_* together with lemma 15 imply that the stabilizer of $\gamma_* W_K$ in W_K is W_K. Thus by theorem 11, conjugation by γ_* induces a permutation of the set $\{w_k | k \varepsilon K\}$. As w_1 commutes with w_k, $k \varepsilon K$ it follows that $w_1 \gamma_* w_1 = u$ also permutes the set $\{w_k | k \varepsilon K\}$ under conjugation, and hence by theorem 11 W_K is contained in the stabilizer of uW_J in W_J. Thus the stabilizer of uW_J in W_J is either W_J or W_K, as K is a maximal subset of J. If this stabilizer is W_J, then by lemma 13 u must be the unique ddcr of $W_J \backslash W/W_J$ of maximal length. Conversely if u is the unique ddcr of $W_J \backslash W/W_J$ of maximal length, then as u is an involution we have by lemma 15 that the stabilizer of uW_J in W_J is W_J.

<u>PROPOSITION 34</u>: Let $L \subseteq J$ and assume that u is a ddcr for $W_L \backslash W_{(I-J) \cup L}/W_L$. Then u is also a ddcr for $W_J \backslash W/W_J$.

PROOF: By definition $\ell(w_i u) = \ell(uw_i) = \ell(u) + 1$ for all $i \varepsilon L$; but if $j \varepsilon J-L$, then $j \not\varepsilon (I-J) \cup L$ and consequently w_j is not in the support of u. Thus $\ell(w_j u) = \ell(uw_j)$

$= \ell(u) + 1$.

It turns out that if I is connected (that is, the corresponding Lie algebra $\mathfrak{g}$ is simple) then the preceding propositions are sufficient to determine the ddcr of $W_J \backslash W / W_J$ along with their stabilizers, as we shall see in the next section.

§7. THE POINCARÉ POLYNOMIAL OF A FINITE COXETER SYSTEM

Let (W,I) be a finite irreducible Coxeter system. Let $\{w_i \mid i \in I_1\}$ and $\{w_i \mid i \in I_2\}$ be the conjugacy classes of the elements $\{w_i \mid i \in I\}$. If there is only one conjugacy class we put $I_1 = I$, $I_2 = \emptyset$. Let $w \in W$ and $w_{i_1} w_{i_2} \cdots w_{i_m}$ be a reduced expression for w. By the corollary to proposition 17, $\ell_1(w) = |\{i_j \mid i_j \in I_1,\ 1 \leq j \leq m\}|$ and $\ell_2(w) = |\{i_j \mid i_j \in I_2,\ 1 \leq j \leq m\}|$ are positive integer valued functions on W, independent of the choice of reduced expression for w. We have $\ell(w) = \ell_1(w) + \ell_2(w)$. If $w,u \in W$, then $\ell_i(wu) = \ell_i(w) + \ell_i(u)$, $(i = 1,2)$.

Let $Z[x,y]$ be the polynomial ring in two variables over Z. If S is any subset of W define

$$p(S) = \sum_{w \in S} x^{\ell_1(w)} y^{\ell_2(w)} .$$

We call $p(W)$ the Poincaré polynomial of W. If J is a subset of I, and W_J is of type $(\mathfrak{g})$ we also use the notation $p(\mathfrak{g})$ for $p(W_J)$ provided that there is no confusion about how the variables x and y are arranged, where $\mathfrak{g}$ is a semisimple complex Lie algebra.

In this section we are going to compute $p(\mathfrak{g}) = p(W)$ when (W,I) is of Lie type $(\mathfrak{g})$. We obtain a multiplicative formula for $p(W)$ for each type $\mathfrak{g}$.

If (W,I) is the Coxeter system of a finite group G

with BN pair, then $[B: B \cap w^{-1}Bw] = \zeta(X_w)$ in the notation of §3. Thus $[G: B] = \sum_{w \in W} \zeta(X_w)$ is obtained from $p(W)$ by simply replacing x and y by the positive integers p and q, where $p = \zeta(X_{w_i})$ for all $i \in I_1$ and $q = \zeta(X_{w_i})$ for all $i \in I_2$. Hence we obtain a completely algebraic proof for a multiplicative formula for $[G: B]$. In particular this applies to the groups of Chevalley [3], Steinberg [19], Suzuki [20], and Ree [13, 14].

<u>PROPOSITION 35</u>: Let J be a subset of I, then $p(W_J)$ divides $p(W)$.

PROOF: Let Γ be the set of distinguished coset representatives (dcr) for W/W_J, then clearly $p(W) = p(\Gamma)p(W_J)$.

<u>PROPOSITION 36</u>: Let $J \subseteq I$, and $\{u_1, \ldots, u_m\}$ be the complete set of distinguished double coset representatives (ddcr) for $W_J \backslash W / W_J$. Let W_{K_i} be the stabilizer of $u_i W_J$ in W_J, then one has

$$p(W) = \sum_{i=1}^{m} \frac{p(W_J)}{p(W_{K_i})} p(u_i) p(W_J) \tag{23}$$

PROOF: By theorem 11 the stabilizer of $u_i W_J$ in W_J is of the form W_{K_i} where K_i is a subset of J. By proposition 29 if Γ_i is the set of dcr for W_J/W_{K_i}, then $\Gamma_i u_i$ is the set of dcr for $(W_J u_i W_J)/W_J$, and $\ell(\gamma_i u_i w) = \ell(\gamma_i) + \ell(u_i) + \ell(w)$ for all $\gamma_i \in \Gamma_i$. By proposition 35 $p(\Gamma_i) = p(W_J)/p(W_{K_i})$. Hence (23) is obvious.

It is clear from proposition 36 that one can compute $p(W)$ by induction on the rank provided that one knows sufficient information about the ddcr and the stabilizers of their cosets. We can obtain this information using the results of §6, and we now proceed to calculate $p(W)$ case by case.

(A_ℓ): $\underset{1}{\circ}$ — $\underset{2}{\circ}$ — - - - — $\underset{\ell}{\circ}$

As the diagram is simply laced, there is only one conjugacy class $I_1 = I$. Let $J = \{2,3,\ldots,\ell\}$, then W_J is of type $(A_{\ell-1})$. We prove by induction on ℓ that $p(A_\ell)/p(A_{\ell-1}) = \frac{x^{\ell+1}-1}{x-1}$. The result is clear when $\ell = 1$, so assume $\ell > 1$. By propositions 30 and 31 1 and w_1 are ddcr of $W_J\ W/W_J$. The stabilizer of $1\cdot W_J$ in W_J is W_J; the stabilizer of $w_1\cdot W_J$ in W_J is W_K where $K = \{3,4,\ldots,\ell\}$. W_K is of type $(A_{\ell-2})$. $|(W_J \cup W_J w_1 W_J)/W_J| = 1 + [W_J\colon W_K] = 1 + \ell = [W\colon W_J]$. Hence $\{1,w_1\}$ is the complete set of ddcr of $W_J\ W/W_J$. By proposition 36

$$\frac{p(A_\ell)}{p(A_{\ell-1})} = 1 + x\frac{p(A_{\ell-1})}{p(A_{\ell-2})} = 1 + x\frac{x^\ell - 1}{x-1} = \frac{x^{\ell+1}-1}{x-1}\ .$$

Thus we have $\dfrac{p(A_\ell)}{p(A_{\ell-1})} = \dfrac{x^{\ell+1}-1}{x-1}$ for all ℓ. It follows that $p(A_\ell) = \prod_{i=2}^{\ell+1}\left(\frac{x^i-1}{x-1}\right)$.

(B_ℓ): $\quad \underset{1}{\circ}\!-\!\!-\!\underset{2}{\circ}\!-\!\!-\,-\,-\,-\!\!-\!\underset{\ell-1}{\circ}\!\Rightarrow\!\underset{\ell}{\circ}$

The two conjugacy classes are $I_1 = \{1,2,\ldots,\ell-1\}$, and $I_2 = \{\ell\}$. Let $J = \{2,3,\ldots,\ell-1,\ell\}$. Then W_J is of type $(B_{\ell-1})$. We prove the following by induction on ℓ, $(\ell \geqq 2)$:

(i) The ddcr for $W_J\backslash W/W_J$ are $\{1,w_1,w_*\}$, where $w_* = 123\ldots(\ell-1)\ell(\ell-1)\ldots321$. (The notation $i_1i_2\ldots i_m$ means $w_{i_1}w_{i_2}\ldots w_{i_m}$). $w_*^2 = 1$.

(ii) The stabilizer of w_1W_J in W_J is W_K where $K = \{3,4,\ldots,\ell\}$. W_K is thus of type $(B_{\ell-2})$.

(iii) The stabilizer of w_*W_J in W_J is W_J.

(iv) $\dfrac{p(B_\ell)}{p(B_{\ell-1})} = \dfrac{(x^\ell - 1)(x^{\ell-1}y + 1)}{(x - 1)}$

(v) $p(B_\ell) = \prod_{i=1}^{\ell} \left(\dfrac{(x^i - 1)(x^{i-1}y + 1)}{x - 1}\right)$.

We first consider the case when $\ell = 2$:

(B_2) $\quad \underset{1}{\circ}\!\Rightarrow\!\underset{2}{\circ} \quad$ $J = \{2\}$

It is easy to see by inspection that the ddcr for $W_J\backslash W/W_J$ are $\{1,w_1,w_*\}$, $w_* = 121$, the stabilizer of w_1W_J in W_J is $\{1\}$. The stabilizer of w_*W_J in W_J is W_J. Thus

$$\frac{p(B_2)}{p(A_1)} = 1 + x(1 + y) + x^2y = (1 + x)(1 + xy) = \frac{(x^2-1)(xy+1)}{(x - 1)} .$$

$$p(B_2) = (1+x)(1+xy)(1+y) = \left(\frac{(x-1)(y+1)}{(x-1)}\right)\left(\frac{(x^2-1)(xy+1)}{x - 1}\right) .$$

Now assume $\ell > 2$. 1 and w_1 are ddcr of $W_J\backslash W/W_J$ and the stabilizers in W_J of $1W_J$ and w_1W_J are W_J and W_K respectively where $K = \{3,4,\ldots,\ell\}$ by proposition 31. Now $|(W_J \cup W_Jw_1W_J)/W_J| = 1 + [W_J: W_K] = 1 + 2(\ell-1) = 2\ell-1$; while $[W: W_J] = 2\ell$. It follows that there is precisely one more double coset of $W_J\backslash W/W_J$, and that this double coset consists of precisely one W_J-coset. By the induction hypothesis the unique dcr γ_* of W_J/W_K is given by $\gamma_* = 234\ldots(\ell-1)\ell(\ell-1)\ldots432$. γ_* is an involution. Hence the hypothesis of proposition 33 is satisfied and we have that:

$w_* = w_1\gamma_*w_1 = 1234\ldots(\ell-1)\ell(\ell-1)\ldots4321$ is a ddcr of $W_J\backslash W/W_J$. Thus $\{1,w_1,w_*\}$ is the complete set of ddcr of $W_J\backslash W/W_J$. The stabilizer of w_*W_J in W_J is W_J because $W_Jw_*W_J$ contains only one W_J-coset. w_* is obviously an involution. This proves (i), (ii), and (iii). It remains to establish (iv) and (v). By proposition 36 we have

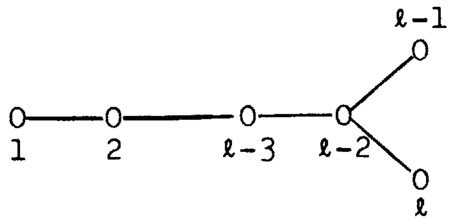
$$\frac{p(B_\ell)}{p(B_{\ell-1})} = 1 + p(w_1)\,\frac{p(B_{\ell-1})}{p(B_{\ell-2})} + p(w_*)$$

$$= 1 + x\,\frac{(x^{\ell-1} - 1)(x^{\ell-2}y + 1)}{x - 1} + x^{2(\ell-1)}y$$

$$= \frac{(x^\ell - 1)(x^{\ell-1}y + 1)}{(x - 1)}.$$

Hence $p(B_\ell) = \prod_{i=1}^{\ell} \frac{(x^i - 1)(x^{i-1}y + 1)}{(x - 1)}$ as asserted.

This completes the induction argument.

(D_ℓ): 

There is only one conjugacy class, $I_1 = \dot{I}$. Let $J = \{2,3,\ldots,\ell\}$, $K = \{3,4,\ldots,\ell\}$. Thus W_J is of type $D_{\ell-1}$ if $\ell \geqq 4$. (We consider (D_3) as being the same as (A_3).) We prove the following by induction on ℓ:

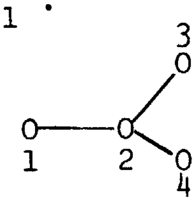
(i) $\{1, w_1, w_*\}$ is the complete set of ddcr for $W_J \backslash W / W_J$, where $w_* = 123\ldots(\ell-2)(\ell-1)\ell(\ell-2)\ldots321$. w_* is an involution.

(ii) The stabilizer of $w_1 W_J$ in W_J is W_K.

(iii) w_* is also the unique maximal dcr for W/W_J. The stabilizer of $w_* W_J$ in W_J is W_J.

(iv) $$\frac{p(D_\ell)}{p(D_{\ell-1})} = \frac{(x^\ell - 1)(x^{\ell-1} + 1)}{x - 1} = \frac{x^\ell - 1}{x - 1}\,\frac{x^{2(\ell-1)} - 1}{x^{\ell-1} - 1}.$$

(v) $$p(D_\ell) = \left(\prod_{i=1}^{\ell-1} \frac{x^{2i} - 1}{x - 1}\right) \cdot \frac{x^\ell - 1}{x - 1}.$$

We first consider (D_4)  , $J = \{2,3,4\}$, $K = \{2,3\}$. 1 and w_1 are ddcr of $W_J \backslash W / W_J$ by proposition 31; and the stabilizer of $1 \cdot W_J$ in W_J is W_J, while the stabilizer of $w_1 W_J$ in W_J is W_K again by proposition 31. Now $|(W_J \cup W_J w_1 W_J)/W_J| = 1 + [W_J : W_K] = 1 + 6 = 7$, while $[W : W_J] = 8$. It follows that there is exactly one additional double coset of $W_J \backslash W / W_J$, and that this double coset contains precisely one W_J-coset. Now it is easy to see that the unique dcr of W_J/W_K is $\gamma_* = 2342$. γ_* is an involution, the hypothesis of proposition 33 is satisfied, hence $w_* = w_1\gamma_* w_1 = 123421$ is a ddcr of $W_J\backslash W/W_J$. w_* is obviously an involution, proving (i) and (ii). As the double coset $W_J w_* W_J$ contains only one W_J-coset we have that the W_J-stabilizer of $w_* W_J$ is W_J, proving (iii). By proposition 36:

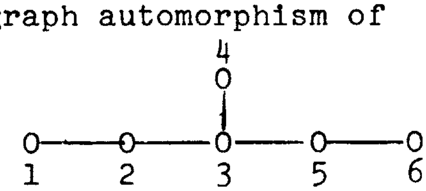
$$\frac{p(D_4)}{p(A_3)} = 1 + x\,\frac{p(A_3)}{p(A_1 \times A_1)} + x^6$$

$$= 1 + x\,\frac{(x^3-1)(x^2+1)}{(x-1)} + x^6 = \frac{(x^4-1)(x^3+1)}{x-1}$$

Thus $p(D_4) = \frac{x^2-1}{x-1} \cdot \frac{x^4-1}{x-1} \cdot \frac{x^6-1}{x-1} \cdot \frac{x^4-1}{x-1}$.

Assuming that $\ell > 4$, the induction argument for D_ℓ is quite similar to the one just given for D_4 and will be omitted.

REMARK: Since the stabilizer of $w_* W_J$ in W_J is W_J, it follows from theorem 11 that conjugation by w_* induces a permutation of the set $\{2,3,\ldots,\ell\}$. As w_* preserves the Killing form, w_* induces a graph automorphism of $(D_{\ell-1})$. It is not difficult to show that w_* induces the nontrivial graph automorphism of $(D_{\ell-1})$.

(E_6): 

There is only one conjugacy class, $I_1 = I$. Let $J = \{2,3,4,5,6\}$. Then W_J is of type (D_5). 1 is a ddcr of $W_J\backslash W/W_J$. The stabilizer of $1 \cdot W_J$ in W_J is W_J. By proposition 31 w_1 is a ddcr of $W_J\backslash W/W_J$, the stabilizer of w_1W_J in W_J is W_K, where $K = \{3,4,5,6\}$. W_K is of type (A_4). By proposition 34 applied to the subset $L = \{2,3,4,5\}$ we have that $u = 12345321$ is a ddcr of $W_J\backslash W/W_J$. From the discussion of type (D_ℓ) we know that u induces a non-trivial graph automorphism of $\{2,3,4,5\}$ under conjugation (it interchanges 4 and 5). It follows from theorem 11 that the stabilizer of uW_J in W_J is W_J because this graph automorphism cannot be extended to the graph of $\{2,3,4,5,6\}$. Note that W_L is of type (D_4).

Now $|(W_J \cup W_Jw_1W_J \cup W_JuW_J)/W_J|$

$= 1 + [W_J\colon W_K] + [W_J\colon W_L] = 1 + 16 + 10 = 27$

$= [W\colon W_J]$.

Hence $\{1,w\ ,u\}$ is the complete list of ddcr for W/W_J.

By proposition 36 we have

$$\frac{p(E_6)}{p(D_5)} = 1 + p(w_1)\frac{p(D_5)}{p(A_4)} + p(u)\frac{p(D_5)}{p(D_4)}$$

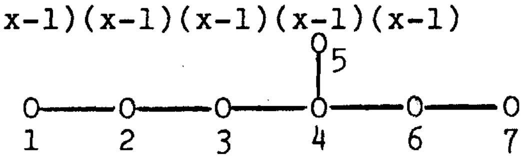
$$= 1 + x\frac{(x^8-1)(x^3+1)}{x-1} + x^8\frac{(x^5-1)(x^4+1)}{(x-1)}$$

$$= \frac{(x^9-1)(x^8+x^4+1)}{(x-1)} = \frac{x^9-1}{x-1}\ \frac{x^{12}-1}{x^4-1}$$

Hence

$$p(E_6) = \frac{(x^2-1)(x^5-1)(x^6-1)(x^8-1)(x^9-1)(x^{12}-1)}{(x-1)(x-1)(x-1)(x-1)(x-1)(x-1)}\ .$$

(E_7):



There is only one conjugacy class $I_1 = I$. Let $J = \{2,3,4,5,6,7\}$. Then W_J is of type (E_6). By proposition

31 w_1 is a ddcr of $W_J\backslash W/W_J$, and the stabilizer of w_1W_J is W_K where $K = \{3,4,5,6,7\}$. W_K is of type (D_5). Proposition 34 applied to the subset $L = \{2,3,4,5,6\}$ shows that $u = 1234564321$ is a ddcr of $W_J\backslash W/W_J$. By the discussion of type (D_ℓ), u induces, by conjugation, the nontrivial graph automorphism of $\{2,3,4,5,6\}$. Thus by theorem 11 the stabilizer of uW_J in W_J is W_L because there is no way to extend this graph automorphism to $\{2,3,4,5,6,7\}$. w_0 is central in W and hence by proposition 32, $w_0 = w_*w_J$ where w_* is a ddcr of $W_J\backslash W/W_J$; w_* is the unique dcr of W/W_J of maximal length, and the stabilizer of w_*W_J in W_J is equal to W_J. Now we have

$$\begin{aligned}
&|(W_J \cup W_Jw_1W_J \cup W_JuW_J \cup W_Jw_*W_J)/W_J| \\
&= 1 + [W_J\colon W_K] + [W_J\colon W_L] + 1 \\
&= 1 + [W(E_6)\colon W(D_5)] + [W(E_6)\colon W(D_5)] + 1 \\
&= 1 + 27 + 27 + 1 \\
&= 56 \\
&= [W(E_7)\colon W(E_6)] = [W\colon W_J].
\end{aligned}$$

Hence $\{1,w_1,u,w_*\}$ is the complete set of ddcr for $W_J\backslash W/W_J$.

Note that $\ell(w_*) = \ell(w_0) - \ell(w_J) = 27$. By proposition 36

$$\frac{p(E_7)}{p(E_6)} = 1 + p(w_1)\frac{p(E_6)}{p(D_5)} + p(u)\frac{p(E_6)}{p(D_5)} + p(w_*)$$

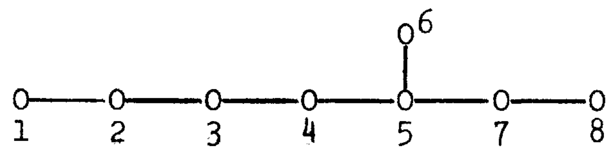
$$= 1 + x\,\frac{(x^9-1)(x^8+x^4+1)}{(x-1)} + x^{10}\frac{(x^9-1)(x^8+x^4+1)}{(x-1)} + x^{27}$$

$$= \frac{(x^{14}-1)(x^5+1)(x^9+1)}{(x-1)} = \frac{x^{14}-1}{x-1} \cdot \frac{x^{10}-1}{x^5-1} \cdot \frac{x^{18}-1}{x^9-1}$$

Hence $p(E_7) = \dfrac{(x^2-1)(x^6-1)(x^8-1)(x^{10}-1)(x^{12}-1)(x^{14}-1)(x^{18}-1)}{(x-1)(x-1)(x-1)(x-1)(x-1)(x-1)(x-1)}$.

REMARK: Note that $w_* = w_0 w_J = w_J w_0$. w_0 is central, but w_J is not central in W_J. Hence w_* induces the unique nontrivial graph automorphism of $\{2,3,4,5,6,7\}$; i.e., the nontrivial graph automorphism of (E_6).

(E_8):



There is only one conjugacy class $I_1 = I$. Let $J = \{2,3,4,5,6,7,8\}$, and $K = \{3,4,5,6,7,8\}$. Then W_J is of type (E_7), W_K is of type (E_6). By proposition 31 w_1 is a ddcr of $W_J \backslash W / W_J$, and the stabilizer of $w_1 W_J$ in W_J is W_K. Proposition 34 applied to the subset $L = \{2,3,4,5,6,7\}$ shows that $u = 123456754321$ is a ddcr of $W_J \backslash W / W_J$. The discussion of type (D_ℓ) shows that under conjugation, u induces the unique nontrivial graph automorphism of $\{2,3,4,5,6,7\}$. By theorem 11 the stabilizer of uW_J in W_J is W_L because this graph automorphism cannot be extended to a graph automorphism of $J = \{2,3,4,5,6,7,8\}$. Now w_0, the unique element of maximal length in W is central as W is of type (E_8); hence by proposition 32 w_* is a ddcr of $W_J \backslash W / W_J$ where $w_* = w_0 w_J = w_J w_0$, and the stabilizer of $w_* W_J$ in W_J is W_J. Note that $\ell(w_*) = \ell(w_0) - \ell(w_J) = 57$. By the discussion of type (E_7), γ_* = the unique dcr of

W_J/W_K is an involution. γ_* is also a ddcr of $W_K \backslash W_J/W_K$. Hence the conditions of proposition 33 are satisfied, and we have that $w_1\gamma_* w_1 = v$ is a ddcr of $W_J\backslash W/W_J$. Under conjugation, γ_* induces the unique nontrivial graph automorphism of $K = \{3,4,5,6,7,8\}$ and w_1 commutes with the elements w_k, $k \in K$. Hence by theorem 11 the stabilizer of vW_J in W_J is W_K as this graph automorphism cannot be extended to $\{2,3,4,5,6,7,8\}$. Note that $\ell(v) = 2 + \ell(\gamma_*) = 2 + 27 = 29$. We present a resumé of what we have found so far in the following table:

ddcr	length	stabilizer
1	0	$W_J=W(E_7)$
w	1	$W_K=W(E_6)$
u	12	$W_L=W(D_6)$
v	29	$W_K=W(E_6)$
w_*	57	$W_J=W(E_7)$

$$|(W_J \cup W_J w_1 W_J \cup W_J u W_J \cup W_J v W_J \cup W_J w_* W_J)/W_J|$$
$$= 1 + [W(E_7)\colon W(E_6)] + [W(E_7)\colon W(D_6)] + [W(E_7)\colon W(E_6)] + 1$$
$$= 1 + 56 + 126 + 56 + 1 = 240 = [W(E_8)\colon W(E_7)].$$

Hence $\{1,w_1,u,v,w_*\}$ is the complete set of ddcr of $W_J\backslash W/W_J$. By proposition 36 we have

$$\frac{p(E_8)}{p(E_7)} = 1 + p(w_1)\frac{p(E_7)}{p(E_6)} + p(u)\frac{p(E_7)}{p(D_6)} + p(v)\frac{p(E_7)}{p(E_6)} + p(w_*)$$

$$= 1 + \frac{x(x^{14}-1)(x^5+1)(x^9+1)}{(x-1)} + x^{12}\frac{(x^{14}-1)(x^{12}+x^6+1)(x^8+x^4+1)}{(x-1)}$$

$$+ x^{29}\frac{(x^{14}-1)(x^5+1)(x^9+1)}{(x-1)} + x^{57}$$

$$= \frac{(x^{30}-1)(x^{18}+x^{12}+x^6+1)(x^{10}+1)}{(x-1)}$$

$$= \frac{(x^{30}-1)(x^{24}-1)(x^{20}-1)}{(x-1)(x^6-1)(x^{10}-1)} .$$

It follows that

$$p(E_8) = \frac{(x^2-1)(x^8-1)(x^{12}-1)(x^{14}-1)(x^{18}-1)(x^{20}-1)(x^{24}-1)(x^{30}-1)}{(x-1)(x-1)(x-1)(x-1)(x-1)(x-1)(x-1)(x-1)}.$$

(F_4): $\quad \underset{1}{\circ}\!-\!\!-\!\underset{2}{\circ}\!\Rightarrow\!\underset{3}{\circ}\!-\!\!-\!\underset{4}{\circ}$

There are two conjugacy classes $I_1 = \{1,2\}$, $I_2 = \{3,4\}$. Let $J = \{2,3,4\}$, $K = \{3,4\}$. By proposition 31 w_1 is a ddcr of $W_J\backslash W/W_J$, and the stabilizer of w_1W_J in W_J is W_K. Note that $p(w_1) = x$. Proposition 34 applied to the subset $L = \{2,3\}$ shows that $u = 12321$ is a ddcr of $W_J\backslash W/W_J$. Note that u is an involution. From the discussion of type (B_ℓ) it follows that u stabilizes the set L under conjugation. It is easily seen that $uw_4u \neq w_4$. Thus by theorem 11, the stabilizer of uW_J in W_J is W_L. Now w_0 is central in W. Hence by proposition 32 w_* is a ddcr of $W_J\backslash W/W_J$, where $w_* = w_0w_J = w_Jw_0$, and the stabilizer of w_*W_J in W_J is W_J. w_* is also the unique dcr of W/W_J of maximal length. Note that $\ell(w_*) = \ell(w_0)-\ell(w_J)$ $= 15$. One has $p(w_0) = x^{12}y^{12}$, $p(w_J) = x^3y^6$, and hence $p(w_*) = x^9y^6$. Now let $\gamma_* = 232432$. It is easy to see that

γ_* is an involution and that $\gamma_* w_3 \gamma_* = w_4$. Hence γ_* is a dcr of W_J/W_K and also a ddcr of $W_K\, W_J/W_K$. It follows from lemma 13 that γ_* is the unique dcr of W_J/W_K of maximal length. Thus the conditions of proposition 33 are satisfied, and $v = w_1 \gamma_* w_1$ is a ddcr of $W_J\, W/W_J$. $\ell(v)=8$, and so again by proposition 33, the stabilizer of vW_J in W_J is W_K. Note that one has $p(v) = x^5y^3$. We list the information we have accumulated so far in the following chart:

w=ddcr	$\ell(w)$	p(w)	W_J-stabilizer of wW_J
1	0	1	W_J
w_1	1	x	W_K
u	5	x^4y	W_L
v	8	x^5y^3	W_K
w_*	15	x^9y^6	W_J

$$|(W_J \cup W_J w_1 W_J \cup W_J u W_J \cup W_J v W_J \cup W_J w_* W_J)/W_J|$$
$$= 1 + [W_J : W_K] + [W_J : W_L] + [W_J : W_K] + 1$$
$$= 1 + [W(B_3): W(A_2)] + [W(B_3): W(B_2)] + [W(B_3): W(A_2)] + 1$$
$$= 1 + 8 + 6 + 8 + 1 = 24 = [W(F_4): W(B_3)] = [W: W_J].$$

Hence $\{1, w_1, u, v, w_*\}$ is the complete list of ddcr for $W_J\, W/W_J$. By proposition 36 we have

$$\frac{p(F_4)}{p(B_3)} = 1 + x\frac{p(W_J)}{p(W_K)} + x^4y\frac{p(W_J)}{p(W_L)} + x^5y^3\frac{p(W_J)}{p(W_K)} + x^9y^6.$$

Now $$p(W_J) = \frac{y^2-1}{y-1}\cdot\frac{y^3-1}{y-1}\,(x+1)(xy+1)(xy+1)$$

$$p(W_K) = \frac{y^2-1}{y-1}\cdot\frac{y^3-1}{y-1}$$

$$p(W_L) = (x+1)(y+1)(xy+1).$$

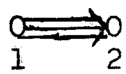
It follows that

$$\frac{p(F_4)}{p(B_3)} = (x^2+x+1)(xy+1)(x^2y+1)(x^2y^2+1)(x^2y^2-xy+1).$$

Hence

$$\begin{aligned} p(F_4) &= (x+1)(y+1)(x^2+x+1)(y^2+y+1)(xy+1)(xy+1)(x^2y+1) \\ &\quad \cdot (xy^2+1)(x^2y^2-xy+1)(x^2y^2+1) \\ &= (x+1)(y+1)(x^2+x+1)(y^2+y+1)(x^2y+1)(xy^2+1)(xy+1) \\ &\quad \cdot (x^2y^2+1)(x^3y^3+1) \end{aligned}$$

(G_2)



There are two classes of involutions $I_1 = \{1\}$, $I_2 = \{2\}$. Let $J = \{2\}$. It is easy to see that the ddcr of W/W_J are $\{1, w_1, w_1w_2w_1, w_1w_2w_1w_2w_1\}$. One has $\frac{p(W)}{p(W_J)} = (x+1)(x^2y^2+xy+1)$. Hence $p(G_2) = (x+1)(y+1)(x^2y^2+xy+1)$.

<u>REMARK</u>: One can show that if W is the dihedral group of order $2m$, then

$$p(W) = \begin{cases} \dfrac{x^2-1}{x-1} \cdot \dfrac{x^m-1}{x-1} & \text{if } m \text{ is odd,} \\[2ex] \dfrac{x^2-1}{x-1} \cdot \dfrac{y^2-1}{y-1} \cdot \dfrac{(xy)^{\frac{m}{2}}-1}{xy-1} & \text{if } m \text{ is even.} \end{cases}$$

§8. THE DEGREES OF THE IRREDUCIBLE CHARACTERS OF G WHOSE RESTRICTIONS TO $H_C(G,B)$ ARE ONE-DIMENSIONAL

Let G be a finite irreducible group with BN pair whose associated Coxeter system is (W,I). I_1 and I_2 represent the two conjugacy classes of the elements $\{w_i | i \in I\}$. If there is only one conjugacy class we put $I_2 = \emptyset$, $I_1 = I$. Adhering to our usual convention we put $\zeta(X_{w_i}) = p$, $i \in I_1$, and $\zeta(X_{w_i}) = q$, $i \in I_2$.

By proposition (17) there are two one-dimensional characters ζ and σ of $H = H_C(G,B)$ if $I_2 = \emptyset$, while if $I_2 \neq \emptyset$ there are two additional one-dimensional characters σ_1 and σ_2. For convenience we repeat the definition of $\zeta,\sigma,\sigma_1,\sigma_2$:

$$\zeta(X_w) = p^{\ell_1(w)}q^{\ell_2(w)} \;,$$
$$\sigma(X_w) = (-1)^{\ell(w)} \;,$$
$$\sigma_1(X_w) = p^{\ell_1(w)}(-1)^{\ell_2(w)} \;,$$
$$\sigma_2(X_w) = (-1)^{\ell_1(w)}q^{\ell_2(w)} \;,$$

where $\ell_1(w)$ and $\ell_2(w)$ are defined as in the corollary to proposition 17.

THEOREM 12: Denote by $\tilde{\zeta}$, $\tilde{\sigma}$, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ the unique irreducible characters of G whose restrictions to $H_C(G,B)$ are ζ, σ, σ_1, σ_2 respectively. Then

(i) $\tilde{\zeta} = 1_G$ is the trivial character of G,

(ii) $\tilde{\sigma}(1) = \zeta(X_{w_0}) = [B: B \cap w_0Bw_0^{-1}] = (p^{\ell_1}q^{\ell_2})^{h/2}$,

(iii) $\tilde{\sigma}_1(1) = f(p,q)/f(p,q^{-1})$,

(iv) $\tilde{\sigma}_2(1) = f(p,q)/f(p^{-1},q)$,

where $\ell_1 = |I_1|$, $\ell_2 = |I_2|$ and $f(x,y)$ is the Poincaré polynomial of (W,I); h is the Coxeter number of W.

PROOF: (i) has been observed in §1.

(ii) Apply the formula (ii) of theorem 1 with $\tilde{\chi} = \tilde{\sigma}$. Then

$$\begin{aligned}\tilde{\sigma}(1) &= [G:\,B][\sum_{w\in W} \zeta(X_w)^{-1}]^{-1}\\ &= [\sum_{w\in W} \zeta(X_w)][\sum_{w\in W} \zeta(X_w)^{-1}]^{-1}\\ &= \zeta(X_{w_0})\cdot[\sum_{w\in W} \zeta(X_w)][\sum_{w\in W} \zeta(X_w)]^{-1}\\ &= \zeta(X_{w_0}).\end{aligned}$$

(iii) Apply the formula (ii) of theorem 1 with $\tilde{\chi} = \tilde{\sigma}_1$. Then $\tilde{\sigma}_1(1) = [G:B].\sum_{w\in W} p^{\ell_1(w)}q^{-\ell_2(w)} = f(p,q)/f(p,q^{-1})$.

(iv) Interchange p and q in (iii).

$\tilde{\sigma}$ is the Steinberg character of G. The fact that $\tilde{\sigma}(1) = [B:\, B \cap w_0Bw_0^{-1}]$ was proved by R. Steinberg [18] when G is a finite Lie group, and by C. Curtis [5] for an arbitrary finite group G with BN pair. Note that if $p = q$ and if the Coxeter system (W,I) is of Lie type $(\mathfrak{g})$, then $\tilde{\sigma}(1) = p^N$ where N is the number of positive roots of $\mathfrak{g}$. The specific formulas for $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ as functions of p and q are easy to calculate using theorem 12 and the Poincaré polynomials of section 7. We give the formulas below for the cases when (W,I) is irreducible of Lie type $(\mathfrak{g})$ such that the Dynkin diagram of $(\mathfrak{g})$ is multiply laced.

$\underline{(B_\ell)}$: $\tilde{\sigma}_1(1) = \prod_{i=1}^{\ell} \frac{q(p^{i-1}q + 1)}{(p^{i-1} + q)} = q^\ell \prod_{i=1}^{\ell} \frac{p^{i-1}q + 1}{p^{i-1} + q}$

$$\tilde{\sigma}_2(1) = \prod_{i=1}^{\ell} \frac{p^{2(i-1)}(p^{i-1}q+1)}{(p^{i-1}+q)} = p^{\ell(\ell-1)} \prod_{i=1}^{\ell} \frac{p^{i-1}q+1}{p^{i-1}+q}$$

If $p = q$, these formulas become:

$$\tilde{\sigma}_1(1) = \frac{p(p^{\ell-1} + 1)(p^\ell + 1)}{2(p + 1)}$$

$$\tilde{\sigma}_2(1) = \frac{p^{(\ell-1)^2}(p^{\ell-1} + 1)(p^\ell + 1)}{2(p + 1)}$$

$\underline{(F_4)}$: $\tilde{\sigma}_1(1) = \frac{q^{12}(p^2q+1)(pq^2+1)(pq+1)(p^2q^2+1)(p^3q^3+1)}{(p^2+q)(p+q^2)(p+q)(p^2+ q^2)(p^3+q^3)}$

$$\tilde{\sigma}_2(1) = \frac{p^{12}(p^2q+1)(pq^2+1)(pq+1)(p^2q^2+1)(p^3q^3+1)}{(p^2+q)(p+q^2)(p+q)(p^2+q^2)(p^3+q^3)}$$

If $p = q$ these formulas become

$$\tilde{\sigma}_1(1) = \tilde{\sigma}_2(1) = \frac{p^4(p^3+1)^2(p^2+1)(p^4+1)(p^6+1)}{8(p+1)^2}$$

$\underline{(G_2)}$: $\tilde{\sigma}_1(1) = \frac{q^3(p^2q^2 + pq + 1)}{(p^2 + pq + q^2)}$

$$\tilde{\sigma}_2(1) = \frac{p^3(p^2q^2 + pq + 1}{(p^2 + pq + q^2)}$$

If $p = q$ these formulas become

$$\tilde{\sigma}_1(1) = \tilde{\sigma}_2(1) = \frac{p(p^4 + p^2 + 1)}{3} .$$

§9. THE DEGREE OF THE REFLECTION CHARACTER AND ITS DUAL

Let G be a finite irreducible group with BN pair whose associated Coxeter system (W,I) is of Lie type $(\mathfrak{g})$ where $\mathfrak{g}$ is a simple complex Lie algebra. Let $H = H_C(G,B)$ and $\pi\colon H \to \operatorname{End} h$ the reflection representation of H. We use the notations and conventions established in section 5. By proposition 2 there exists a unique irreducible complex character $\tilde{\chi}$ of G such that $\tilde{\chi}|H = \chi$, the reflection character of H. We call $\tilde{\chi}$ the reflection character of G. In this section we calculate the degree of $\tilde{\chi}$.

By theorem 9 if J is a maximal proper subset of I, then $(\tilde{\chi},(1_{G_J})^G)_G = 1$. Hence by proposition 2 the restriction of $\tilde{\chi}$ to $H_C(G,G_J)$ is a linear character of $H_C(G,G_J)$. Consider $H_C(G,G_J)$ as a subalgebra of H. There is no confusion if we also denote by χ the restriction of $\tilde{\chi}$ to $H_C(G_J,G)$. Now if $\{u_1,\dots,u_m\}$ is the complete set of distinguished double coset representatives for $W_J\backslash W/W_J$, then this set can also be taken as a complete set of double coset representatives for $G_J\backslash G/G_J$. Thus $\{Y_{u_i} | 1 \le i \le m\}$ is our canonical basis for $H_C(G,G_J)$, where $Y_{u_i} = |G_J|^{-1} \sum_{x\in G_J u_i G_J} x$.

By theorem 1 we have the degree of $\tilde{\chi}$:

$$\tilde{\chi}(1) = \Big(\sum_{i=1}^{m} \chi(Y_{u_i})\chi(Y_{u_i^{-1}})\zeta(Y_{u_i})^{-1}\Big)^{-1}\cdot[G\colon G_J].$$

Thus the degree of $\tilde{\chi}$ will be determined once we know $\chi(Y_{u_i})$, $\zeta(Y_{u_i})$, and $[G\colon G_J]$. We describe a method for finding these quantities using the results of sections 6 and 7. Recall that in section 7 we defined for any subset S of W: $p(S) = \sum_{w \varepsilon S} x^{\ell_1(w)} y^{\ell_2(w)} = p(S;\, x,y)$, where I_1 and I_2 represent the two conjugacy classes of the elements $\{w_i | i \varepsilon I\}$. Let $q_i = p$ for all $i \varepsilon I_1$, $q_i = q$ for all $i \varepsilon I_2$ and put $f(S) = p(S;\, p,q)$ for every subset S of W. It is clear then that one has $f(S) = \sum_{w \varepsilon S} \zeta(X_w)$. Now $[G\colon G_J] = [G\colon B]/[G_J\colon B]$. Hence one has $[G\colon G_J] = f(W)/f(W_J)$.

If W is of type $(\mathcal{G})$ we always choose the maximal subset J of I exactly as we did in section 7. Thus $[G\colon G_J]$ has already been computed for every type $(\mathcal{G})$.

Now $Y_{u_i} = |G_J|^{-1} \cdot \sum_{x \varepsilon G_J u_i G_J} x = [G_J\colon B]^{-1} \cdot \sum_{w \varepsilon W_J u_i W_J} X_w.$

(24)

If we let K_i be the unique subset of J such that W_{K_i} is the stabilizer of $u_i W_J$ in W_J (cf. theorem 11), and Γ_i the set of dcr for W_J/W_{K_i}, then $Y_{u_i} = [G_J\colon B]^{-1} \cdot \sum_{\gamma \varepsilon \Gamma_i} X_\gamma X_{u_i} \sum_{w \varepsilon W_J} X_w = \sum_{\gamma \varepsilon \Gamma_i} X_\gamma \cdot X_{u_i} e_J$, where $e_J = |G_J|^{-1} \sum_{x \varepsilon G_J} x = (\sum_{w \varepsilon W_J} \zeta(X_w))^{-1} \sum_{w \varepsilon W_J} X_w$. Thus $\zeta(Y_{u_i}) = \sum_{\gamma \varepsilon \Gamma} \zeta(X_\gamma) \cdot \zeta(X_{u_i}) = f(u_i) \dfrac{f(W_J)}{f(W_{K_i})}$. Now for our particular choice of the subset J, the complete set of ddcr of $W_J \backslash W/W_J$ have been listed in §7 for each case along with $p(u_i)$ and $p(W_J)/p(W_{K_i})$; thus to obtain $\zeta(Y_{u_i})$ one only needs to replace x by p and y by q.

The linear character χ of $H(G,G_J)$ is afforded by the one-dimensional space $e_J \cdot h$. We have chosen $J = I-\{1\}$ in each case. $e_J \cdot h$ contains the weight λ_1 (cf. §5). Let C be the Cartan matrix of H, $d = \det C$. Put $\mu = d\lambda_1 = m\alpha_1 + \sum_{i=2}^{\ell} a_i \alpha_i$. By Cramer's rule, $m = \det M_{11}$ where M_{11} is the (1-1)-minor of C. d and m have been computed in each case in §5. Now $e_J \cdot \alpha_j = 0$ for $j \neq 1$. Thus $\mu = e_J \cdot \mu = m e_J \cdot \alpha_1$; that is, $e_J \cdot \alpha_1 = m^{-1} \cdot \mu$. Note that by proposition 25 we have $X_{w_1} \cdot \mu = \mu - d\alpha_1$. We summarize these facts in the following

PROPOSITION 37: Let the notation be as above. Then

(i) $e_J \cdot \alpha_j = 0,\ j \neq 1$

(ii) $e_J \cdot \alpha_1 = m^{-1}\mu$

(iii) $X_{w_1} \cdot \mu = q_1 \mu - d\alpha_1$

(iv) $X_{w_j} \cdot \mu = q_j \cdot \mu,\ j \neq 1.$

Now from §1 it follows that

$$\begin{aligned} Y_{u_i} &= \zeta(Y_{u_i}) e_J u_i e_J = \zeta(Y_{u_i}) e_J e(B) u_i e(B) e_J \\ &= \zeta(Y_{u_i}) \zeta(X_{u_i})^{-1} e_J X_{u_i} e_J \\ &= \frac{f(W_J)}{f(W_{K_i})} e_J X_{u_i} e_J \ . \end{aligned}$$

Furthermore, $Y_{u_i} \cdot \mu = \chi(Y_{u_i})\mu$. Thus to obtain $\chi(Y_{u_i})$ one has only to compute $e_J X_{u_i} \cdot \mu$. This in turn can be done using proposition 37.

<u>PROPOSITION 38</u>: One has $\sum_{i=1}^{m} \chi(Y_{u_i}) = 0$.

PROOF: Indeed $\sum_{i=1}^{m} \chi(Y_{u_i}) = |G_J|^{-1} \cdot \sum_{x \varepsilon G} \tilde{\chi}(x)$. This must be equal to zero because $\tilde{\chi}$ is not the trivial character of G.

An inspection of the case by case treatment given in §7 reveals that the unique dcr w_* of W/W_J of maximal length is an involution (and hence the ddcr of $W_J \backslash W/W_J$ of maximal length) if and only if W is not of type (A_ℓ) or (E_6). Moreover in this case, when $w_*^2 = 1$, we have $w_0 \cdot \alpha_1 = -\alpha_1$. This enables us to calculate $\chi(Y_{w_*})$ quite easily as follows:

<u>LEMMA 16</u>: Assume G is not of type (A_ℓ) or (E_6), then one has $X_{w_0} \cdot \alpha_1 = -(p^{\ell_1} q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}} \alpha_1$, where $\ell_1 = |I_1|$, $\ell_2 = |I_2|$, and h is the Coxeter number of (W,I).

PROOF: This is an immediate consequence of proposition 28.

<u>PROPOSITION 39</u>: Assume that the unique dcr w_* of W/W_J of maximal length is an involution (and hence by lemma 15 a ddcr of $W_J \backslash W/W_J$), then one has

$$\chi(Y_{w_*}) = -(p^{\ell_1} q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}} \zeta(X_{w_J})^{-1}$$

PROOF: Recall that in this case the stabilizer of $w_* W_J$ in W_J is W_J. Thus $Y_{w_*} = X_{w_*} e_J$. Let $\xi = e_J \cdot \alpha_1$.

Then $\chi(Y_{w_*})\xi = Y_{w_*}\cdot\xi = X_{w_*}e_J\cdot\xi = X_{w_*}\cdot\xi$. We also have $w_0 = w_* w_J$, $X_{w_0} = X_{w_*}X_{w_J}$, and hence $X_{w_0}\cdot\xi = \chi(Y_{w_*})\zeta(X_{w_J})\cdot\xi$. Now by the lemma 16 we have

$X_{w_0}\cdot\alpha_1 = -(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}}\cdot\alpha_1$, while if $j \neq 1$, then

$X_{w_0}\cdot\alpha_j = -(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}}\alpha_{j'}$, for some $j' \neq 1$. But

$\xi = e_J\cdot\alpha_1 = \alpha_1 + \sum_{i=2}^{\ell} a_i\alpha_i$. It follows that

$X_{w_0}\cdot\xi = -(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}}\cdot\xi$. Hence

$$\chi(Y_{w_*})\zeta(X_{w_J}) = -(p^{\ell_1}q^{\ell_2})^{\frac{h(\ell-1)}{2\ell}} .$$

Using the methods outlined in this section it is possible to obtain the degree of the reflection character in each case. We have carried out this computation; and the results are listed below.

It is of interest to notice that if the Dynkin graph of G is simply laced, then $\tilde{\chi}(1) = \sum_{i=1}^{\ell} p^{m_i}$ where $\{m_1, m_2, \ldots, m_\ell\}$ are the exponents of the Weyl group of G.

In the case of (G_2) it is easy to show from the formula given for $\tilde{\chi}(1)$ that one must have $\sqrt{pq} \in Z$.

<u>(A_ℓ):</u>

$$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{\ell}{\circ}$$

u_i = ddcr	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$\frac{p(p^\ell - 1)}{(p - 1)}$	-1

$$[G:G_J] = \frac{p^{\ell+1}-1}{p-1}, \quad \langle\chi,\chi\rangle_{G_J} = \frac{p^{\ell+1}-1}{p(p^{\ell}-1)}$$

$$\tilde{\chi}(1) = \frac{p(p^{\ell}-1)}{(p-1)} = p + p^2 + \cdots + p^{\ell}.$$

(B_ℓ):

$q_i=p\ (1\le i\le \ell\ 1), q_\ell=q.$

u_i = ddcr	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$\frac{p(p^{\ell-1}-1)(p^{\ell-2}q+1)}{(p-1)}$	$p^{\ell-1}-1$
w_*	$p^{2(\ell-1)}q$	$-p^{\ell-1}$

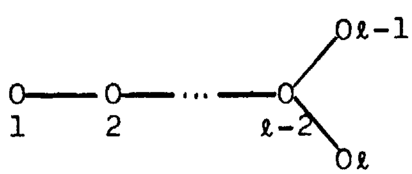
$$[G:G_J] = \frac{(p^{\ell}-1)(p^{\ell-1}q+1)}{p-1}; \qquad \langle\chi,\chi\rangle_{G_J} = \frac{(p+q)(p^{\ell-1}q+1)}{pq(p^{\ell-2}q+1)};$$

$$\tilde{\chi}(1) = \frac{(pq)\cdot(p^{\ell}-1)(p^{\ell-2}q+1)}{(p+q)(p-1)}.$$

If $p=q$, as in the case of the Chevalley groups, this becomes:

$$\tilde{\chi}(1) = \tfrac{1}{2}(q+q^2+\cdots+q^{\ell-1}+2q^{\ell}+q^{\ell+1}+q^{\ell+2}+\cdots+q^{2\ell-1}).$$

(D_ℓ):

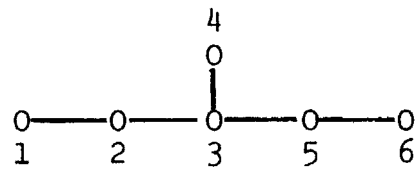


u_i=ddcr	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$\frac{p(p^{\ell-1}-1)(p^{\ell-2}+1)}{(p-1)}$	$p^{\ell-1}-1$
w_*	$p^{2(\ell-1)}$	$-p^{\ell-1}$

$$[G:\, G_J] = \frac{(p^{\ell}-1)(p^{\ell-1}+1)}{(p-1)}\,, \qquad \langle\chi,\chi\rangle_{G_J} = \frac{(p^{\ell-1}+1)(p+1)}{p(p^{\ell-2}+1)}$$

$$\tilde{\chi}(1) = \frac{p(p^{\ell}-1)(p^{\ell-2}+1)}{(p-1)(p+1)} = p+p^3+p^5+\cdots+p^{2\ell-3}+p^{\ell-1}.$$

<u>(E_6)</u>:



u_i=ddcr	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$\dfrac{p(p^8-1)(p^3+1)}{(p-1)}$	$\dfrac{(p^3+p^2-1)(p^4-1)(p^3+1)}{p^2-1}$
u	$\dfrac{p^8(p^5-1)(p^4+1)}{(p-1)}$	$\dfrac{-p^4(p^5-1)}{(p-1)}$

$$d = \det C = \frac{p^6+1}{p^2+1}(p^2+p+1) = p^6+p^5-p^3+p+1$$

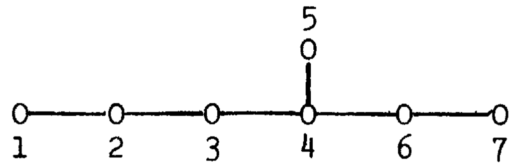
$$m = (p+1)(p^4+1)$$

$$[G:G_J] = \frac{(p^9-1)(p^{12}-1)}{(p-1)(p^4-1)} = (p^2+p+1)^2(p^6+p^3+1)(p^2-p+1)(p^4-p^2+1)$$

$$\langle\chi,\chi\rangle_{G_J} = \frac{(p^2+p+1)^2(p^2-p+1)(p^4-p^2+1)}{p(p^4+1)}$$

$$\tilde{\chi}(1) = p(p^4+1)(p^6+p^3+1) = p + p^4 + p^5 + p^7 + p^8 + p^{11}$$

$\underline{(E_7)}$:



u_i=ddcr	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$\frac{p(p^9-1)(p^{12}-1)}{(p-1)(p^4-1)}$	$\frac{(p^2+1)(p-1)(p^3+p^2-1)}{(p^6+1)(p^3-1)(p-1)(p^4-1)} \cdot (p^9-1)(p^{12}-1)$
u	$\frac{p^{10}(p^9-1)(p^{12}-1)}{(p-1)(p^4-1)}$	$\frac{p^5(p^9-1)(p^3+1)(p^3-p-1)}{p^2-1}$
w_*	p^{27}	$-p^{18}$

$\pi(X_{w_0}) = -p^{54}\cdot I$, $\zeta(X_{w_J}) = p^{36}$

$$d = \det C = \frac{(p+1)(p^9+1)}{(p^3+1)} = p^7+p^6-p^4-p^3+p+1$$

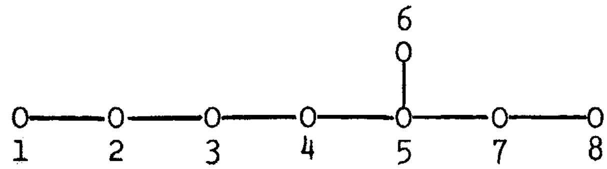
$$m = \frac{(p^6+1)(p^2+p+1)}{p^2+1} = \frac{p^6+1}{p^2+1}\cdot\frac{p^3-1}{p-1} = p^6+p^5-p^3+p+1$$

$$[G:G_J] = \frac{(p^{14}-1)(p^5+1)(p^9+1)}{(p-1)} = \frac{p^{14}-1}{p-1}\cdot\frac{p^{10}-1}{p^5-1}\cdot\frac{p^{18}-1}{p^9-1}$$

$$\langle\chi,\chi\rangle_{G_J} = \frac{(p^9+1)(p^5+1)(p^4-1)}{p(p^6+1)(p-1)}$$

$$\tilde{\chi}(1) = \frac{p(p^6+1)(p^{14}-1)}{(p^2+1)(p^2-1)} = p + p^5 + p^7 + p^9 + p^{11} + p^{13} + p^{17}.$$

$\underline{(E_8)}$:



ddcr $= u_i$	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$\frac{p(p^{14}-1)(p^9+1)(p^5+1)}{(p-1)}$	$\frac{(p^{14}-1)}{(p-1)}(p^3+p^2-1)(p^2-p+1) \cdot (p^5+1)$
u	$\frac{p^{12}(p^{14}-1)(p^{12}+p^6+1)}{(p-1)} \cdot (p^8+p^4+1)$	$p^6\frac{(p^{14}-1)}{(p-1)}(p^6+p^3+1) \cdot (p^8+p^4+1)(p^3-1)$
v	$\frac{p^{29}(p^{14}-1)(p^5+1)(p^9+1)}{(p-1)}$	$p^{19}\frac{(p^{14}-1)}{(p-1)}(p^3-p-1) \cdot (p^2-p+1)(p^5+1)$
w_*	p^{57}	$-p^{42}$

$$d = \frac{(p^{15}+1)(p+1)}{(p^5+1)(p^3+1)} = p^8+p^7-p^5-p^4-p^3+p+1$$

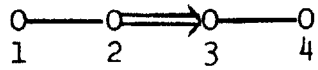
$$m = (p+1)(p^6-p^3+1)$$

$$\pi(X_{w_0}) = -p^{105} \cdot I, \quad \zeta(X_{w_J}) = p^{63}$$

$$[G:\, G_J] = \frac{(p^{30}-1)(p^{24}-1)(p^{20}-1)}{(p-1)(p^6-1)(p^{10}-1)}$$

$$\langle \chi, \chi \rangle_{G_J} = \frac{p^{30}-1}{p(p-1)}$$

$$\tilde{\chi}(1) = \frac{p(p^{24}-1)(p^{10}+1)}{(p^6-1)} = p+p^7+p^{11}+p^{13}+p^{17}+p^{19}+p^{23}+p^{29}$$

$\underline{(F_4)}$:  $q_i=p \ (i=1,2); \ q_i=q \ (i=3,4)$

ddcr $= u_i$	$\zeta(Y_{u_i})$	$\chi(Y_{u_i})$
1	1	1
w_1	$p(p+1)(pq+1(pq^2+1)$	$(p+1)(pq+1)(p+pq-1)$
u	$p^4q(q^2+q+1)(pq^2+1)$	$p^2(p^2q-1)(q^2+q+1)$
v	$p^5q^3(p+1)(pq+1)(pq^2+1)$	$p^3q(pq+1)(p+1(pq-q-1)$
w_*	p^9q^6	$-p^6q^3$

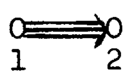
$$d = \det C = p^2q^2 - pq + 1 = \frac{p^3q^3 + 1}{pq + 1}$$

$$m = pq^2 + 1$$

$$[G: G_J] = (p^2+p+1)(pq+1)(p^2q+1)(p^2q^2+1)(p^2q^2-pq+1)$$

$$\langle\chi,\chi\rangle_{G_J} = \frac{(p^2+p+1)(p^3q^3+1)(p+q)}{pq(pq^2 + 1)}$$

$$\tilde{\chi}(1) = \frac{pq(pq^2+1)(p^2q+1)(p^2q^2+1)}{(p+q)}$$

$\underline{(G_2)}$:  $q = p, \ q = q$

u=ddcr	$\zeta(Y_u)$	$\chi(Y_u)$
1	1	1
w_1	$p(q+1)$	$p+\sqrt{pq} -1$
$w_1w_2w_1$	$p^2q(q+1)$	$p\sqrt{pq} - p - \sqrt{pq}$
$w_*=w_1w_2w_1w_2w_1$	p^3q^2	$-p^{3/2}q^{1/2}$

$d = \det C = pq - \sqrt{pq} + 1, \ \pi(X_{w_0}) = -(pq)^{3/2}\cdot I.$

$m = q + 1$

$[G: G_J] = (p + 1)(p^2q^2 + pq + 1)$

$$\langle\chi,\chi\rangle_{G_J} = \frac{2(p + \sqrt{pq} + q)(pq - \sqrt{pq} + 1}{pq(q + 1)}$$

$$\tilde{\chi}(1) = \frac{pq(p+1)(q+1)(pq+\sqrt{pq}+1)}{2(p + \sqrt{pq} + q)}$$

Finally we give a formula for the dual $\tilde{\chi}^{(\ell-1)}$ of the reflection character. Recall that $\tilde{\chi}^{(\ell-1)}$ is the unique irreducible complex character of G whose restriction to $H = H_C(G,B)$ is the character $\hat{\chi}$, where χ is the reflection character of H; and $\hat{\chi}(x) = \chi(\hat{x})$ for all $x \in H$ (cf. section 3). From theorem 1 we have

$$\tilde{\chi}^{(\ell-1)}(1) = \sum_{w\in W}\zeta(X_w)[\sum_{w\in W}\hat{\chi}(X_w)\hat{\chi}(X_{w^{-1}})\zeta(X_w)]^{-1} . \tag{25}$$

Now $\hat{\chi}(X_w) = \chi(\hat{X}_w) = (-1)^{\ell(w)}\zeta(X_w)\chi(X_{w^{-1}}^{-1})$. Moreover,

$$\sum_{w}\zeta(X_w) = \zeta(X_{w_0})\sum_{w\in W}\zeta(X_w^{-1}) = p^{\ell_1}q^{\ell_2}\sum_{w\in W}\zeta(X_w)^{-1}.$$

From the definition of the reflection representation $\pi: H \to \operatorname{End} h$ it is easy to see that A_{w_i} is the matrix representing $\pi(X_{w_i})$ relative to the basis $\{\alpha_1,\ldots,\alpha_\ell\}$, then $A_{w_i}^{-1}$ is obtained by replacing p by p^{-1} and q by q^{-1} in all the entries of A_{w_i}. It follows that for any $w \in W$, the inverse of A_w, the matrix representing $\pi(X_w)$, can be obtained by replacing p by p^{-1}, q by q^{-1} in the entries of $A_{w^{-1}}$. Hence we have the following theorem.

<u>THEOREM 13</u>: Let $\tilde{\chi}$ be the reflection character of G and $\hat{\tilde{\chi}}$ the dual of $\tilde{\chi}$ in the above sense. Then there exists a rational function $r(x,y) \in Q(x,y)$ such that $\tilde{\chi}(1) = r(p,q)$ and $\hat{\tilde{\chi}}(1) = p^{\ell_1}q^{\ell_2}r(p^{-1},q^{-1})$.

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BIOGRAPHICAL NOTE

Robert W. Kilmoyer, Jr. was born July 13, 1939 in Lebanon, Pennsylvania. He entered Lebanon Valley College at Annville, Pennsylvania, in 1957, and received his A.B. from that institution in 1961. From 1961 to 1966 he was a graduate student in mathematics at the Massachusetts Institute of Technology. He has been a member of the mathematics faculty at Clark University since September, 1966.

He is married, has two children, and currently resides at 18 Pocasset Avenue, Worcester, Massachusetts.