OPERATIONS RESEARCH CENTER

Working Paper

Lead Time Distribution of Three-Machine Two-Buffer Lines with Unreliable Machines and Finite Buffers

by

Chuan Shi
Stanley B. Gershwin

OR 398-16 **August 2016**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lead Time Distribution of Three-Machine Two-Buffer Lines with Unreliable Machines and Finite Buffers

Chuan Shi and Stanley B. Gershwin

Department of Mechanical Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139-4307 USA
mitcshi@gmail.com, gershwin@mit.edu

August 18, 2016

Abstract

The lead time of a manufacturing system is the amount of time a part spends in it. This quantity is important because customers demand short and reliable lead times, and because many products lose value in storage. It is random because of some of the events that occur during the production process, including unpredictable machine failures, uncertain processing times, and quality variations. Knowledge of the probability distribution of lead time can be useful in deciding how to design or operate a system, and in making delivery date commitments.

We describe an analytic method for determining the steady-state probability distribution of the lead time of a three-machine, two-buffer production line in which the buffers are finite. The method is an extension of recent work by the authors on the probability distribution of the sojourn time of a two-machine line. We consider the movement of a *reference part* from its arrival until its departure. We first compute the conditional probability that the lead time $T = \tau$, given the state of the line when the part arrives. This is done by solving a set of recurrence equations which are developed from a detailed analysis of the reference part's movement through the first buffer, from the first to the second buffer, and through the second buffer. The conditioning is removed by using the steady-state probability distribution of the three-machine line.

We provide two kinds of numerical evidence for the accuracy of this method. First, we show that it satisfies Little's Law. Then we compare the distribution calculated by the new method with the simulated lead time distribution for several cases and show very close agreement. Several numerical examples then are examined to observe the shapes of the probability distributions and how they are influenced by the parameters of the machines and the sizes of the buffers. Other numerical experiments demonstrate the effect of the existence and location of a bottleneck. Finally, we suggest future research directions.

Keywords: lead time, transfer line, production line, unreliable machines, finite buffers

Contents

1 Introduction

1.1 Problem and Motivation

The lead time of a manufacturing system is the amount of time a part spends in it. This quantity is important because customers demand short and reliable lead times and because many products lose value in storage. It is uncertain because of the random events that occur during the production process, including machine failures, varying processing times, quality problems, and other events. The probability distribution of lead time is important to determine the reliability of meeting proposed delivery dates. Vendors can never predict lead times with certainty, but customers often require them to make firm delivery promises. (They may have a contractual obligation to compensate a customer if they fail to meet the promise. Even without such an obligation, they suffer a loss of reputation and goodwill if they do not deliver on time.) They therefore attempt to design and operate their production system so that no more than a small fraction of their deliveries are late.

Lead time limitation is important for other reasons. The quality of food deteriorates if too much time elapses between the farm and the customer (or between the farm and the freezing or canning process). In the semiconductor industry, lead times are on the order of weeks or months, and technological progress is so rapid that products can lose significant value during such time periods. The same is true for the clothing industry, where value loss is due to the changing of fashions. Again, manufacturers attempt to ensure that no more than a specified fraction of their products spend more than a small period in factories or storage.

1.2 Literature

Some prior literature on lead time analysis is based on classical queueing theory. In such models, machines are reliable with random service times. In most such models, buffers are assumed to be infinite. Chow (1980) determines the lead time distribution of a cyclic queue with two exponential servers and infinite buffers. Leemans (2001) analyzes a Markovian two-class two-server queue with non-preemptive heterogeneous priority structures. The lead time distribution is derived based on the technique of tagging and randomization. Ayhan et al. (2004) consider a multiple-stage cyclic queueing network with N customers, general service times, and infinite buffers. The bounds on the nth departure time from each stage are investigated. Azaron et al. (2006) investigate the optimal design of multi-stage assemblies modeled as an open queueing network. The arrival process of product orders is Poisson and each station has a single server with exponential service times. They obtain the lead time distribution by applying the longest path analysis. In Wu and McGinnis (2012), the authors model manufacturing systems as general queueing networks and analyze their mean queue time. Lagershausen and Tan (2015) focus on closed queueing networks where machines have phase-type service time distributions and buffers are finite. They model such a network as a continuous time Markov chain with finite state space. By conducting first passage time analysis, the authors find the distributions of inter-departure, inter-start and cycle time.

On the other hand, researchers have studied lead time using models with unreliable machines and, in most cases, finite buffers. Tan (2003) proposes a performance evaluation methodology that can be applied to a wide range of discrete production systems with unreliable machines and finite buffers. The methodology first generates the transition matrix of the Markov chain, and then solves the transition equations to find the steady-state probabilities of the system, from which the performance measures are computed. The author then derives the conditional transient lead time distribution given the initial state of the system. Shi and Gershwin (2016) develop an analytical solution for the lead time distribution of two-machine one-buffer production lines with unreliable machines and a finite buffer. Machines are assumed to have geometric failure and repair probabilities. They first find the conditional lead time probability distribution of a part based on the position of the part and the state of the downstream machine. The unconditional probability is then derived by applying the total probability theorem. The research reported below is an extension of this work. Shi (2012) uses this method to study the production line profit maximization problem subject to both a production rate constraint and a part sojourn time constraint in a given buffer. The author extends Shi and Gershwin (2009) to develop an algorithm that solves the optimization problem efficiently and accurately. Colledani et al. (2014) extend Shi and Gershwin (2016) to two-machine one-buffer lines where machines follow a general Markovian model. The lead time distribution is derived. In addition, they conduct integrated analysis of quality and production logistics performance in their study.

Biller et al. (2013) study a model in which machines obey Bernoulli reliability but buffers are infinite. The first machine is a release machine and it controls the availability of raw material. The authors maximize the production rate of the line subject to an average lead time constraint by controlling the parameters of the release machine. Meerkov and Yan (2014) advance Biller et al. (2013) to production lines where machines have exponentially distributed up and down times. They also assume that buffer sizes are infinite.

1.3 Outline

Section 2 defines the material flow model and introduces a new model which focuses on the movement of a single part. The analysis of the probability distribution of the lead time is presented in Section 3. The numerical experiments in Section 4 provide evidence that the method calculates the distribution correctly and they demonstrate some of the effects of the system parameters on the distribution. Section 5 concludes and summarizes the contributions of the paper, and it suggests research directions. Appendices A–E provide the equations that are needed to calculate the distribution and they describe the algorithms that were used for the numerical results in Section 4.

2 Flow Line Model

The technique we present to determine the lead time of a production line requires the analysis of two dynamic systems. The first, which we refer to here as the *Material Flow System* (MFS) (Section 2.1), describes the flow of material in the line. It is used to determine the steady-state probability distribution of inventory and machine repair states. This distribution is then used to derive expressions for the production rate and average in-process inventory.

The second, which we call the *Reference Part Movement System* (RPMS), is developed in Section 2.2. It is based on the MFS, but it focuses on the movement of a single part. We use it to determine the probability distribution of the lead time of that part, conditioned on the state of the system when it arrives. We calculate the unconditional distribution of the lead time from it.

2.1 Material Flow System Model

We need the MSF for two reasons: the RPMS model is built on it; and its steady-state probability distribution is used in Section 3 to derive the lead time probability distribution.

2.1.1 Description

The MFS model considered in this paper is the Gershwin (1994) version of the Buzacott model of a three-machine, two-buffer transfer line. (See Figure 1.) The processing times of all machines are equal, deterministic, and constant. Time is scaled so that operations take one time unit. Transportation time is ignored. Buffer B_i is finite and can hold $N_i < \infty$ parts (for $i = 1, 2$). All machines are unreliable with geometrically distributed times to failure and to repair. The probabilities of failure and repair of machine M_i during one time unit are p_i and r_i, respectively. M_i is blocked when its downstream buffer B_i is full and is idle. Similarly, M_i is starved and consequently idle when its upstream buffer B_{i-1} is empty. Idle machines cannot fail or affect the number of parts in the buffer.

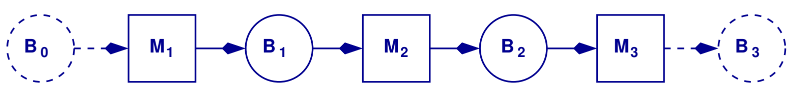


Figure 1: Three-machine line. $M_1 - B_1 - M_2 - B_2 - M_3$ is the system that is modeled in detail; B_0 and B_3 are external fictitious buffers that are introduced to insure that M_1 is never starved and M_3 is never blocked.

The first machine is never starved and the last is never blocked. It is equivalent, and sometimes convenient, to say that there is an infinite buffer (B_0) which is never empty upstream of the first machine. That buffer is called the *raw material buffer.* Similarly, there is an infinite *finished goods buffer* (B_3) downstream of the last machine which is never full. Parts in B_0 and B_3 are not considered to be in the system. The calculation of lead time and inventory only considers parts while they are in B_1 and B_2.

The lead time of a part is the time that the part spends in the three-machine two-buffer line. To provide a precise definition of the lead time of the production line model considered, we must first explain how inventory is defined. The model assumes that

- there is no space for a part at a machine, and therefore the inventory of the line is the total number of parts in the two buffers B_1 and B_2.

- when M_i processes a part, it moves the part from its upstream buffer to its downstream buffer. During the time unit when the part is being processed, it is still considered residing in the upstream buffer, rather than in the machine.

- if M_i attempts to process a part and fails, the part remains, undamaged, in the buffer upstream of M_i.

These assumptions imply that when M_1 performs an operation during some time unit t, it moves a part from the raw material buffer into B_1. During that time unit t, that part is considered outside the line and therefore it does not contribute to the inventory. The part enters the line as soon as M_1 completes its operation and adds it to buffer B_1. As a result, the time that a part spends in the line starts from the instant it enters B_1, after being processed by M_1. Similarly, when M_3 performs an operation during a time unit, it moves a part from B_2 to the finished goods buffer, which is outside of the line. During that time unit, that part is still residing in B_2 and therefore it is still part of the inventory. The part leaves the line as soon as M_3 completes its operation and removes it from B_2. Consequently, the time that a part spends in the line ends at the instant it leaves B_2, after being processed by M_3. As a result, the lead time of a part is computed from the instant it enters B_1 until the instant it leaves B_2. In addition, in this model, all events occur at integer times. At every event, each machine adds one part, removes one part, or does neither. Therefore, lead times are always integers.

2.1.2 Notation and Dynamics

The state of the MFS at time t is a set of five random variables, $\nu_1(t), \nu_2(t), \alpha_1(t), \alpha_2(t), \alpha_3(t)$. $\nu_i(t)$ is the number of parts in B_i at time t and satisfies $0 \leq \nu_i(t) \leq N_i$, $i = 1, 2$, where N_i is the size of B_i. $\alpha_i(t)$ is the repair state of M_i at time t. $\alpha_i(t) = 0, 1, i = 1, 2, 3$. $\alpha_i(t) = 1$ means M_i is operational at time t; $\alpha_i(t) = 0$ means M_i is under repair at time t.

Figure 2 illustrates the sequence of events in this model. The convention is that the states of machines are determined at the beginning of a time unit while the buffer levels are computed at the end of a time unit. Both time instants and time units are plotted on a horizontal line. A *time instant* ($t-1$, t, or $t+1$ in the figure) is the end of one time unit and the beginning of the next. The interval between the time instants t and $t+1$ is *time unit* t.

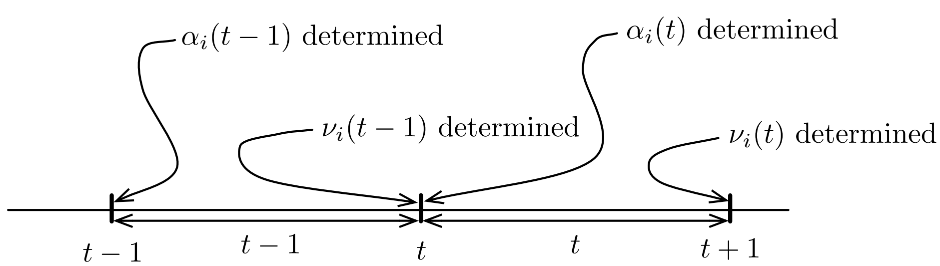


Figure 2: Convention of the Gershwin (1994) version of the Buzacott model

Each machine, if it is operational and not idle, attempts to perform an operation and, if the machine fails, that state change is considered to occur at the beginning of the time unit. Similarly, if a machine is under repair, and that repair is completed during the current time unit, that state change is also considered to occur at the beginning of the time unit. (After a repair, the machine successfully performs an operation during the same time unit.) The dynamics of the machine state $\alpha_i(t)$ is therefore described by a pair of Bernoulli random processes. In particular

$$\text{If } \alpha_i(t) = 0, \qquad \alpha_i(t+1) = \begin{cases} 1 & \text{with probability } r_i \\ 0 & \text{with probability } 1 - r_i \end{cases}$$

The failure process can only occur if a machine is not starved or blocked:

$$\text{If } \alpha_i(t) = 1, \qquad \begin{cases} \alpha_i(t+1) = 1 \quad \text{if } M_i \text{ is starved or blocked,} \\ \\ \alpha_i(t+1) = \left\{ \begin{array}{ll} 1 & \text{with probability } 1 - p_i \\ 0 & \text{with probability } p_i \end{array} \right\} \\ \qquad\qquad\qquad \text{if } M_i \text{ is neither starved nor blocked.} \end{cases}$$

If M_i succeeds in doing an operation on a part, that part will leave buffer B_{i-1} and enter B_i at the *end* of the current time unit. The change of ν_i, the level of B_i is therefore

$$\nu_i(t+1) = \nu_i(t) + \alpha_i(t+1) - \alpha_{i+1}(t+1), \quad i = 1, 2$$

if the buffer is neither empty nor full at time t, that is, if $1 \leq \nu_i(t) \leq N_i - 1$.

The general form of this equation (Gershwin 1994) also accounts for the cases in which B_i is either starved or blocked. It can be written

$$\nu_i(t+1) = \nu_i(t) + I_i(t+1) - I_{i+1}(t+1), \quad i = 1, 2 \tag{1}$$

in which random variable $I_i(t+1)$ is the indicator of whether a part is moved by M_i from B_{i-1} to B_i. That is,

$$I_1(t+1) = \begin{cases} 1 & \text{if } \alpha_1(t+1) = 1 \text{ and } \nu_1(t) < N_1 \\ & \quad \text{(i.e., } M_1 \text{ is up and not blocked)} \\ \\ 0 & \text{if } \alpha_1(t+1) = 0 \text{ or } \{a_1(t+1) = 1 \text{ and } \nu_1(t) = N_1\} \\ & \quad \text{(i.e., } M_1 \text{ is down or blocked)} \end{cases}$$

$$I_2(t+1) = \begin{cases} 1 & \text{if } \alpha_2(t+1) = 1, \nu_1(t) > 0, \text{ and } \nu_2(t) < N_2 \\ & \quad \text{(i.e., } M_2 \text{ is up, not starved, and not blocked)} \\ \\ 0 & \text{if } \alpha_2(t+1) = 0 \text{ or } \Big\{a_2(t+1) = 1 \text{ and } \{\nu_1(t) = 0 \text{ or } \nu_2(t) = N_2\}\Big\} \\ & \quad \text{(i.e., } M_2 \text{ is down or starved or blocked)} \end{cases}$$

$$I_3(t+1) = \begin{cases} 1 & \text{if } \alpha_3(t+1) = 1 \text{ and } \nu_2(t) > 0 \\ & \text{(i.e., } M_3 \text{ is up and not starved)} \\ \\ 0 & \text{if } \alpha_3(t+1) = 0 \text{ or } \{a_3(t+1) = 1 \text{ and } \nu_2(t) = 0\} \\ & \text{(i.e., } M_3 \text{ is down or starved)} \end{cases}$$

2.2 Reference Part Movement System Model

2.2.1 Description

We assume that parts in the buffer follow a first-in first-out (FIFO) discipline. In previous analyses of this and similar systems, the discipline was not specified because it did not affect the production rate or average inventory. However, the discipline does affect the probability distribution of the lead time, so we must specify it here.

Because we have made the FIFO assumption, we can make the following definition. The *position* of a part in a buffer is one more than the number of parts that will leave the buffer before it (Shi and Gershwin 2016). If a part's position is k, we also say that it is the kth part in the buffer. Note that the position of a part is always 1 or greater.

Assume a part enters the system at the end of time unit t'. We call it the *reference part*. Assume there are $\nu_1(t')$ parts in buffer B_1 (including the reference part) and $\nu_2(t')$ parts in buffer B_2 when the reference part arrives. The reference part experiences the following sequence of events:

1. It goes into buffer B_1 at position $\nu_1(t')$ after being processed by M_1. It cannot enter B_2 without going through B_1 first.

2. It stays in B_1 until the $\nu_1(t') - 1$ parts in front of it are processed by M_2. Then it is processed by M_2 at some time $s > t'$.

3. After it is processed by M_2, it is added to B_2 at the end of time unit s. The level of B_2 (which is now the position of the reference part) at the end of s is $\nu_2(s)$.

4. After it enters B_2, it stays in B_2, waiting for M_3 until the $\nu_2(s) - 1$ parts in front of it are processed by M_3.

5. It is processed by M_3 at some time $u > s$, and it leaves B_2 and therefore the line at the end of time unit u.

The reference part enters B_1 at the end of time unit t' and leaves B_2 at the end of time unit u. Consequently, its lead time is $T = u - t'$.

2.2.2 Notation and Dynamics

Define $\chi_i(t)$ to be the position of the reference part in buffer B_i at time t. $\chi_i(t)$ is only meaningful when the reference part is in buffer B_i. We call the dynamic system which describes the movement of the reference part the *Reference Part Movement System* (RPMS). The movement of the part is determined only by events downstream of it. Therefore, when the reference part is in B_1, the state of the RPMS consists of the random variables χ_1, ν_2, α_2 and α_3. When the reference part is in B_2, the state of this system consists of χ_2 and α_3.

We refer to the period from t' to s in which the reference part is in B_1 as phase 1. Phase 2 is the period from s to u during which the reference part is in B_2. Since the part is only in B_i during phase i, $\chi_1(t)$ is only meaningful for $t' \leq t < s$ and $\chi_2(t)$ is only meaningful for $s \leq t < u$.

Upon the arrival of the part at B_1, phase 1 starts and $\chi_1(t') = \nu_1(t')$. The RPMS at time t' is shown in Figure 3. At the end of time unit s, the reference part leaves B_1 and enters B_2. Therefore, phase 1 ends and phase 2 starts. At the end of s, $\chi_2(s) = \nu_2(s)$ and the RPMS is shown in Figure 4.

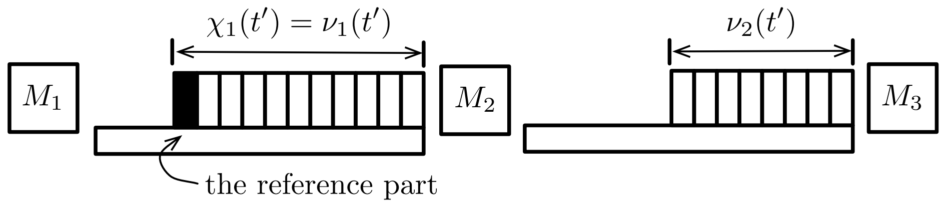


Figure 3: Start of phase 1 (at the end of time unit t')

To find the lead time of the reference part, we need to consider the lengths of time that the reference part spends in the two phases by studying its movements in each.

Phase 1 For $t' \leq t < s$, we model the dynamics of both $\chi_1(t)$ and $\nu_2(t)$ as M_2 moves parts from B_1 to B_2 and M_3 moves parts out of the system from B_2. The dynamics of $\chi_1(t)$ are

$$\chi_1(t+1) = \chi_1(t) - I_2(t+1) \tag{2}$$

Equation (2) indicates that once the reference part enters B_1, its position is unaffected to anything that happens to the upstream machine M_1. In addition, (2) shows that χ_1 cannot increase with t. It decreases whenever M_2 moves parts out of B_1. It stays unchanged if M_2 either fails or is blocked by a full B_2. Therefore, χ_1 is affected by M_2 directly and M_3 indirectly. On the other hand, equation (1) for $i = 1$ shows that ν_1 can increase, decrease, or remain unchanged with t, depending on the part inflow from M_1 and outflow to M_2.

Changes in χ_1 and ν_2 are related because

1. Whenever M_2 performs an operation, it removes a part from B_1 and adds it to B_2 (i.e., $I_2 = 1$). That is, the operation reduces χ_1 by 1 and increases ν_2 by 1.

2. (a) If $\chi_1(t') + \nu_2(t') > N_2$ (i.e., if the total number of parts in the two buffers when the reference part arrives is greater than the size of B_2), and M_3 is down, ν_2 will increase as M_2 keeps

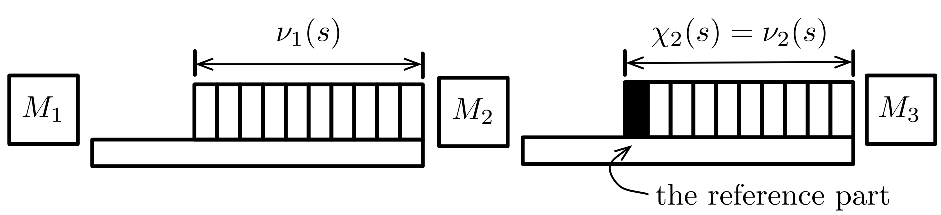


Figure 4: Start of phase 2 (at the end of time unit s)

transferring parts from B_1 to B_2. If M_3 stays down long enough, B_2 will become full ($\nu_2 = N_2$) while the reference part is in B_1. A full B_2 will in turn block M_2. As a result, I_2 will be 0 and χ_1 will remain unchanged until the blockage of M_2 ends.

(b) If $\chi_1(t') + \nu_2(t') \leq N_2$, B_2 will not become full before the reference part is processed by M_2 no matter how long M_3 is down.

The reference part can only leave B_1 at the end of time unit t if $\chi_1(t-1)$ is 1 and M_2 is up and not blocked during t. If both conditions are satisfied, the reference part leaves B_1 at the end of t, and we refer to the value of t as s.

Phase 2 The reference part enters B_2 at the end of time unit s. Upon arrival, the position of the reference part in B_2 is $\chi_2(s) = \nu_2(s)$. Its remaining time in the line depends on how $\chi_2(t)$ changes for $t > s$. The dynamics of χ_2 are

$$\chi_2(t+1) = \chi_2(t) - \alpha_3(t+1) \tag{3}$$

That is, when the reference part is in B_2, its position is unaffected by anything that happens to the upstream machines M_1 and M_2. χ_2 decreases with t as long as M_3 moves parts out of B_2, and it remains constant when M_3 is down. By contrast, ν_2 can increase, decrease, or remain unchanged with t, depending on the part inflow from M_2 and outflow to M_3. Note that M_3 cannot be starved as the reference part is in B_2; it cannot be blocked either due to the assumption that there is an infinite finished goods buffer downstream of M_3. The reference part can only leave B_2 (and therefore the line) at the end of some time unit t if its position $\chi_2(t-1)$ is 1 and M_3 is up during t. If both conditions are satisfied, the reference part leaves the system at the end of t, and we refer to that value of t as u. The time that the reference part spends in B_2 is $u - s$, and u depends on $\chi_2(s)$ and the state of M_3 for $s < t \leq u$.

To summarize, once the reference part enters the three-machine two-buffer line at the end of time unit t', its lead time $T = u - t'$ depends only on $\chi_1(t')$, $\nu_2(t')$, and whether M_2 and M_3 are up, down, blocked, or starved for each $t \geq t'$ until it leaves the line.

3 Derivation of the Lead Time Distribution

3.1 Overview

We derive the steady-state probability distribution of the lead time of the three-machine two-buffer line in this section. In our approach, we assume that the MFS is in steady state. We analyze the movement of a reference part that enters the line at the end of some time unit t' when the RPMS is in state $(\chi_1(t'), \nu_2(t'), \alpha_2(t'), \alpha_3(t'))$ and derive equations for a set of conditional probabilities for the lead time in Sections 3.2 and 3.3. The unconditional probability distribution is determined from them in Section 3.4.

We no longer need to make a distinction between t' and a generic t. We use **prob**() to represent the probability of an event. For example the probability mass function of the lead time is denoted by $\mathbf{prob}(T = \tau)$.

Define $A(t)$ to be the event that the reference part enters the system at the end of time unit t. To calculate $\mathbf{prob}(T = \tau)$:

- We derive, in Section 3.2, a set of recurrence equations for $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$. This is the conditional probability that the reference part has lead time $T = \tau$ given that $\chi_1(t) = x_1$, $\nu_2(t) = n_2$, $\alpha_2(t) = a_2$, $\alpha_3(t) = a_3$, and given that it arrived at time t ($1 \le x_1 \le N_1$, $0 \le n_2 \le N_2$, $a_2 = 0, 1$ and $a_3 = 0, 1$);

- We find, in Section 3.3, the steady-state conditional probability $\mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t))$. This is the probability that $\chi_1(t) = x_1$, $\nu_2(t) = n_2$, $\alpha_2(t) = a_2$, $\alpha_3(t) = a_3$, given that the reference part arrived at time t ($1 \le x_1 \le N_1, 0 \le n_2 \le N_2, a_2 = 0, 1, a_3 = 0, 1$). This derivation requires the steady-state probability distribution of the MFS.

- Finally, in Section 3.4, we find the steady-state probability $\mathbf{prob}(T = \tau | A(t))$ by using the Total Probability Theorem (Bertsekas and Tsitsiklis 2008):

$$\mathbf{prob}(T = \tau | A(t)) =$$

$$\sum_{x_1=1}^{N_1} \sum_{n_2=0}^{N_2} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \left[\begin{array}{r} \mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) \times \\ \mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t)) \end{array} \right]. \tag{4}$$

Note that $\mathbf{prob}(T = \tau | A(t))$ is exactly the lead time distribution we are looking for, i.e.,

$$\mathbf{prob}(T = \tau) = \mathbf{prob}(T = \tau | A(t)) \tag{5}$$

This is because the event $\{T = \tau\}$ is actually the event that a part arrives at the line at some time t *and* it spends τ time units in it. Since $A(t)$ is the event that a part arrives at some time t, $\{T = \tau\}$ is a subset of $A(t)$. This implies equation (5). (Note that since we are considering the MFS in steady state, the value of t does not affect the probability.)

However this reasoning does *not* imply that we can drop $A(t)$ from the conditions in any of the probabilities on the right side of equation (4). This is because all of those probabilities involve $\{\chi_1(t) = x_1\}$. That event occurs at the same time as the arrival of the part in event $A(t)$. That is, even though we are considering the system in steady state, the quantities on the right side of (4) involve events that happen at the *same* time t.

Before we proceed with the derivation of the lead time distribution, it is important to observe that the set of possible values of the lead time T depends on the state of the RPMS when the reference part enters the line.

Assume that the reference part is at position χ_1 in B_1 and that there are ν_2 parts in B_2. Then there are $\chi_1 - 1$ parts in B_1 and ν_2 parts in B_2 ahead of the reference part. χ_1 and ν_2 determine the minimum possible value of T. For example, if $\chi_1 = \nu_2 = 10$, then we know that the lead time T cannot be 1 time unit. To determine the minimum precisely, we must consider two cases.

- First, suppose that $\nu_2 > 0$. If M_2 and M_3 stay up, M_3 will process those ν_2 parts (while M_2 moves parts from B_1 to B_2), as well as the $\chi_1 - 1$ parts that were originally in B_1, before it can work on the reference part. In other words, given no failures of M_2 and M_3, the reference part will wait for $\chi_1 + \nu_2 - 1$ time units before it can be processed by M_3, and therefore, its minimum lead time is $\chi_1 + \nu_2$ time units. If either of M_2 or M_3 fails during the process, the lead time will be longer. Consequently, $T \ge \chi_1 + \nu_2$ when $\nu_2 > 0$.

- Next, suppose that $\nu_2 = 0$. After one time unit, if M_2 is up, it will put one part into B_2; and M_3 will be starved during that time unit. The reference part will then be the $(\chi_1 - 1)$th part in B_1, and there will be one part in B_2. Therefore there will still be $\chi_1 - 1$ parts in front of the reference part in the line. If M_2 and M_3 stay up, M_3 will have to process those $\chi_1 - 1$ parts before it can work on the reference part. As a result, assuming no failures of M_2 and M_3 occur, the reference part must wait for the time unit during which M_3 is starved plus $\chi_1 - 1$ time units during which M_3 processes the preceding $\chi_1 - 1$ parts. That is, the reference part must wait for $1 + (\chi_1 - 1) = \chi_1$ time units before it can be processed by M_3. Consequently, the minimum lead time of the reference part is $\chi_1 + 1$ time units. If any of M_2 or M_3 fails during the process, the lead time will be longer. As a result, $T \geq \chi_1 + 1$ when $\nu_2 = 0$.

Combining the two cases, T must satisfy $T \geq \chi_1 + \max(\nu_2, 1)$ and therefore

$$\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) = 0 \quad \text{if } \tau < x_1 + \max(n_2, 1) \tag{6}$$

Equation (6) indicates that the minimum value of the lead time of a three-machine two-buffer line is 2 time units, when $\chi_1 = 1$ and $\nu_2 = 0$ or 1. On the other hand, there is no upper bound on the lead time.

3.2 Derivation of $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$

In this section, we find $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$, the first set of factors in (4), by tracking the movement of the reference part in the reference part movement system. For example, suppose that the RPMS is in state $(\chi_1(t), \nu_2(t), \alpha_2(t), \alpha_3(t)) = (x_1, n_2, 1, 1)$ where $x_1 \geq 2, n_2 \geq 1$ after the reference part enters B_1. If both M_2 and M_3 stay up and not blocked, then at the end of the next time unit, M_2 will move a part from B_1 to B_2 and M_3 will move a part from B_2 to out of the line and the RPMS will be in state $(x_1 - 1, n_2, 1, 1)$. By tracing how the reference part moves from time step to time step, we are able to determine the relationship between the lead time of the reference part and the state of the RPMS, and therefore to find $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$. This leads to a set of recursive equations for the probabilities.

To present the derivation equations, we define the following notation for convenience. For $a_2 = 0, 1$; $a_3 = 0, 1$; $1 \leq x_1 \leq N_1, 0 \leq n_2 \leq N_2$:

$$\begin{aligned}
\Pi_t^{a_2 a_3}(\tau, x_1, n_2) &= \mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) \\
\Pi_t^{B a_3}(\tau, x_1) &= \mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = N_2, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t)) \\
\Pi_t^{a_2 S}(\tau, x_1) &= \mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = 0, \alpha_2(t) = a_2, \alpha_3(t) = 1, A(t))
\end{aligned}$$

In this notation, B indicates that buffer B_2 is full (i.e., $\nu_2(t) = N_2$) and therefore M_2 is blocked (i.e., $\alpha_2(t) = 1$); and S indicates that buffer B_2 is empty (i.e., $\nu_2(t) = 0$) and therefore M_3 is starved (i.e., $\alpha_3(t) = 1$).

We need to find $\Pi_t^{a_2 a_3}(\tau, x_1, n_2)$, $\Pi_t^{B a_3}(\tau, x_1)$, and $\Pi_t^{a_2 S}(\tau, x_1)$ for all applicable τ, x_1 and n_2. According to (6), $\Pi_t^{a_2 a_3}(\tau, x_1, n_2) = \Pi_t^{B a_3}(\tau, x_1) = \Pi_t^{a_2 S}(\tau, x_1) = 0$ for all $\tau < x_1 + \max(n_2, 1)$. Therefore, we just need to find the probabilities for all other combinations of τ, x_1 and n_2.

It is important to point out that for the RPMS, the states of the form $(1, n_2, a_2, a_3)$ (i.e., where $x_1 = 1$) are special. This is because if M_2 stays up and not blocked, the reference part will leave B_1 and enter B_2 at the end of next time unit. Consequently, the reference part leaves phase 1 and enters phase 2 as described in 2.2.2, and the state of the RPMS is changed from (x_1, n_2, a_2, a_3) to (x_2, a_3).

3.2.1 $2 \le x_1 \le N_1$

When $2 \le x_1 \le N_1$, there are five sets of recurrence equations depending on the value of n_2. They are cases in which $n_2 = 0$, $n_2 = 1$, $n_2 = N_2 - 1$, $n_2 = N_2$, and $2 \le n_2 \le N_2 - 2$.

We use the following example to demonstrate how to construct the recurrence equations. Suppose that $\chi_1(t) = x_1 \ge 2$ and $2 \le \nu_2(t) = n_2 \le N_2 - 2$ and $\alpha_2(t) = \alpha_3(t) = a_2 = a_3 = 1$ when the reference part enters the line at the end of time unit t. We discuss what may happen during time unit $t+1$. There are four possibilities depending on the new states of M_2 and M_3. Note the reference part is in B_1 at the end of time unit t, and it remains in B_1 at the end of $t+1$ in all four cases. As a consequence, $\chi_1(t)$ and $\chi_1(t+1)$ are meaningful and $\chi_1(t+1)$ must be determined. On the other hand, neither $\chi_2(t)$ nor $\chi_2(t+1)$ are meaningful.

1. Both M_2 and M_3 fail, with probability $p_2 p_3$. There is no change of the position of the reference part or the level of B_2. The reference part will still be the x_1th part in B_1 and the level of B_2 is still n_2. Therefore, $\chi_1(t+1) = x_1$, $\nu_2(t+1) = n_2$, $\alpha_2(t+1) = 0$ and $\alpha_3(t+1) = 0$.

2. M_2 fails while M_3 stays up, with probability $p_2(1-p_3)$. M_2 does not change the level of B_1 or B_2 while M_3 removes a part from B_2. The reference part is still the x_1th part in B_1 and the level of B_2 is decreased by 1. Therefore, $\chi_1(t+1) = x_1$, $\nu_2(t+1) = n_2 - 1$, $\alpha_2(t+1) = 0$ and $\alpha_3(t+1) = 1$.

3. M_2 stays up while M_3 fails, with probability $(1-p_2)p_3$. M_2 moves a part from B_1 to B_2 while M_3 does not remove anything from B_2. The reference part will then be the $(x_1 - 1)$st part in B_1 and the level of B_2 is increased by 1. Therefore, $\chi_1(t+1) = x_1 - 1$, $\nu_2(t+1) = n_2 + 1$, $\alpha_2(t+1) = 1$ and $\alpha_3(t+1) = 0$.

4. Both M_2 and M_3 stay up, with probability $(1-p_2)(1-p_3)$. M_2 moves a part from B_1 to B_2 while M_3 removes a part from B_2. The reference part will then be the $(x_1 - 1)$st part in B_1, and the number of parts in B_2 is unchanged. Therefore, $\chi_1(t+1) = x_1 - 1$, $\nu_2(t+1) = n_2$, $\alpha_2(t+1) = 1$ and $\alpha_3(t+1) = 1$.

To develop a set of equations to determine the probabilities, note that no matter what happens to M_2 and M_3 during time unit $t+1$, this time unit has passed. For the reference part to have a lead time of τ time units counted from the end of time unit t (when it arrived), it must have a residual lead time of $\tau - 1$ time units counted from the end of time unit $t+1$. At the end of time unit $t+1$, the four state variables of the RPMS are $\chi_1(t+1)$, $\nu_2(t+1)$, $\alpha_2(t+1)$, and $\alpha_3(t+1)$.

Consider a hypothetical second reference part that enters the line at the end of time unit $t+1$, and assume that the values of the state variables are the same as the first scenario discussed above for the original reference part, i.e., $\chi_1(t+1) = x_1$, $\nu_2(t+1) = n_2$, $\alpha_2(t+1) = 0$ and $\alpha_3(t+1) = 0$. The probability that the second reference part has a lead time of $\tau - 1$ given these variables is $\mathbf{prob}(T = \tau - 1 | \chi_1(t+1) = x_1, \nu_2(t+1) = n_2, \alpha_2(t+1) = 0, \alpha_3(t+1) = 0)$. Both the original and the second reference part have the same conditions in terms of $\chi_1(t+1)$, $\nu_2(t+1)$, $\alpha_2(t+1)$ and $\alpha_3(t+1)$. Consequently, the distribution of the residual lead time of the original reference part is the same as the distribution of the lead time of the second reference part.

Repeating this analysis for the other three scenarios and then applying the Total Probability Theorem lead to the following recurrence equation:

$$\begin{aligned}
&\textbf{prob}\Big(T=\tau|\chi_1(t)=x_1,\nu_2(t)=n_2,\alpha_2(t)=1,\alpha_3(t)=1,A(t)\Big)=\\
&\quad p_2p_3\textbf{prob}\Big(T=\tau-1|\chi_1(t+1)=x_1,\nu_2(t+1)=n_2,\alpha_2(t+1)=0,\alpha_3(t+1)=0,A(t+1)\Big)\\
&+p_2(1-p_3)\textbf{prob}\Big(T=\tau-1|\chi_1(t+1)=x_1,\nu_2(t+1)=n_2-1,\alpha_2(t+1)=0,\alpha_3(t+1)=1,A(t+1)\Big)\\
&+(1-p_2)p_3\textbf{prob}\Big(T=\tau-1|\chi_1(t+1)=x_1-1,\nu_2(t+1)=n_2+1,\alpha_2(t+1)=1,\alpha_3(t+1)=0,A(t+1)\Big)\\
&+(1-p_2)(1-p_3)\textbf{prob}\Big(T=\tau-1|\chi_1(t+1)=x_1-1,\nu_2(t+1)=n_2,\alpha_2(t+1)=1,\alpha_3(t+1)=1,A(t+1)\Big)
\end{aligned} \tag{7}$$

We disregard the t arguments in (7) because the conditional lead time probability distribution is in steady state so it depends only on the current state of the RPMS. In the new notation, equation (7) becomes

$$\begin{aligned}
\Pi^{11}(\tau,x_1,n_2) \;=\; & p_2p_3\,\Pi^{00}(\tau-1,x_1,n_2) \;+\; p_2(1-p_3)\,\Pi^{01}(\tau-1,x_1,n_2-1)\\
+\; & (1-p_2)p_3\,\Pi^{10}(\tau-1,x_1-1,n_2+1) \;+\; (1-p_2)(1-p_3)\,\Pi^{11}(\tau-1,x_1-1,n_2).
\end{aligned} \tag{8}$$

Equation (8) is the recurrence equation for the conditional probability that the lead time of the reference part is τ given that $\chi_1(t)=x_1,\nu_2(t)=n_2,\alpha_2(t)=1,\alpha_3(t)=1$ ($x_1\geq 2$, $2\leq n_2\leq N_2-2$) upon its arrival. With the same analysis, we derive such recurrence equations for all combinations of τ, x_1, n_2, α_2 and α_3 for $2\leq x_1\leq N_1$, $0\leq n_2\leq N_2$, $a_2=0,1$ and $a_3=0,1$. We list them in Appendix A.

3.2.2 $x_1=1$

When the reference part enters the line, it is the first (in fact, the only) part in buffer B_1. If M_2 is up and not blocked during the next time unit, the reference part will move from B_1 to B_2. If M_2 is blocked or down, the reference part will stay in B_1. As an example, suppose that both M_2 and M_3 are up (i.e., $a_2=a_3=1$) and $2\leq n_2\leq N_2-1$ when the reference part arrives. As we did in Section 3.2.1, we discuss what could happen during time unit $t+1$. Since the reference part is in B_1 during time unit t, $\chi_1(t)$ is meaningful and $\chi_2(t)$ is not. In the cases in which the reference part stays in B_1 at the end of time unit $t+1$, $\chi_1(t+1)$ is meaningful and $\chi_2(t+1)$ is not. However, in the cases where the reference part moves to B_2, $\chi_1(t+1)$ is not meaningful and $\chi_2(t+1)$ is.

1. Both M_2 and M_3 fail, with probability p_2p_3. Consequently, the reference part will still be the first part in B_1 and the level of B_2 will remain unchanged. As a result, $\chi_1(t+1)=1$, $\nu_2(t+1)=n_2$, $\alpha_2(t+1)=0$ and $\alpha_3(t+1)=0$.

2. M_2 fails while M_3 stays up, with probability $p_2(1-p_3)$. At the end of time unit $t+1$, M_2 does not move the reference part from B_1 to B_2 while M_3 removes a part from B_2. The reference part will still be the first part in B_1 and level of B_2 is decreased by 1. As a result, $\chi_1(t+1)=1$, $\nu_2(t+1)=n_2-1$, $\alpha_2(t+1)=0$ and $\alpha_3(t+1)=1$.

3. Both M_2 and M_3 stay up, with probability $(1-p_2)(1-p_3)$. At the end of time unit $t+1$, M_2 moves the reference part from B_1 to B_2 while M_3 removes a part from B_2. As a result, $\chi_1(t+1)$ is not meaningful, while and $\chi_2(t+1)$ is. In particular, $\chi_2(t+1)=\nu_2(t+1)=n_2$, $\alpha_2(t+1)=1$ and $\alpha_3(t+1)=1$.

4. M_2 stays up while M_3 fails, with probability $(1-p_2)p_3$. At the end of time unit $t+1$, M_2 moves the reference part from B_1 to B_2 while M_3 cannot remove anything from B_2. Therefore, $\chi_1(t+1)$ is not meaningful, while and $\chi_2(t+1)$ is. In particular, $\chi_2(t+1) = \nu_2(t+1) = n_2+1$, $\alpha_2(t+1) = 1$ and $\alpha_3(t+1) = 0$.

The movement of the reference part from B_1 to B_2 requires new probabilities to be defined. For $w \geq 1$ and $1 \leq x_2 \leq N_2$, let

$$\begin{aligned}
\pi^1(w, x_2) &= \mathbf{prob}(\text{the reference part spends } w \text{ time units at } B_2 | \chi_2(s) = x_2, \alpha_3(s) = 1, B(s)) \\
\pi^0(w, x_2) &= \mathbf{prob}(\text{the reference part spends } w \text{ time units at } B_2 | \chi_2(s) = x_2, \alpha_3(s) = 0, B(s))
\end{aligned}$$

where $B(s)$ is the event that the reference part enters B_2 at the end of some time unit s. Then, from the Total Probability Theorem, we establish the following recurrence equation:

$$\begin{aligned}
\Pi^{11}(\tau, 1, n_2) = \; & p_2 p_3 \, \Pi^{00}(\tau-1, 1, n_2) + p_2(1-p_3)\, \Pi^{01}(\tau-1, 1, n_2-1) \\
+ \; & (1-p_2)(1-p_3)\, \pi^1(\tau-1, n_2) + (1-p_2)p_3\, \pi^0(\tau-1, n_2+1)
\end{aligned} \tag{9}$$

Equation (9) does not have the same form as (8). This is because of $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$, which do not appear in the previous equations. To determine these quantities, we observe that once the reference part enters B_2, the time it spends there is the same as the time that a part spends in the buffer of a two-machine one-buffer line consisting of M_2, B_2, and B_3. This is precisely the quantity calculated by Shi and Gershwin (2016), in which they developed the equations that $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ satisfy for all $w \geq 1$ and $1 \leq x_2 \leq N_2$. The equations for $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ from Shi and Gershwin (2016) are provided in Appendix D.

We can now derive the recurrence equations for all combinations of τ, x_1, n_2, a_2 and a_3 for $x_1 = 1$, $0 \leq n_2 \leq N_2$, $a_2 = 0, 1$ and $a_3 = 0, 1$. There are different sets of equations for $n_2 = 0$ or 1 (which involve the possibility of starvation of M_3), $n_2 = N_2 - 1$ or N_2 (which involve the possibility of blockage of M_2), and $2 \leq n_2 \leq N_2 - 2$. They are listed in Appendix B. Recall that the equations in Appendices A and B are only for the values of τ, x_1, and n_2 that satisfy (6). The probabilities of all other values of τ, x_1, and n_2 are 0.

Finally, in order to determine all these probabilities using recurrence equations (A.1) to (A.5) and (B.1) to (B.4), we must specify a set of initial conditions. The initial conditions should involve only the states (x_1, n_2, a_2, a_3) of the RPMS and the value of τ such that probabilities $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3)$ can be determined from $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ alone. The RHS of the equations for these probabilities should only contain $\pi^1(w, x_2)$ or $\pi^0(w, x_2)$ or both.

The initial conditions are the equations that involve $(1, 0, a_2, a_3)$ and $(1, 1, a_2, a_3)$ for $\tau = 2$. Consequently we have the following initial conditions:

$$
\begin{aligned}
\Pi^{0S}(2,1) &= r_2\pi^1(1,1) \\
\Pi^{1S}(2,1) &= (1-p_2)\pi^1(1,1) \\
\Pi^{00}(2,1,0) &= r_2r_3\pi^1(1,1) + r_2(1-r_3)\pi^0(1,1) \\
\Pi^{10}(2,1,0) &= (1-p_2)r_3\pi^1(1,1) + (1-p_2)(1-r_3)\pi^0(1,1) \\
\Pi^{00}(2,1,1) &= r_2r_3\pi^1(1,1) \\
\Pi^{01}(2,1,1) &= r_2(1-p_3)\pi^1(1,1) \\
\Pi^{10}(2,1,1) &= (1-p_2)r_3\pi^1(1,1) \\
\Pi^{11}(2,1,1) &= (1-p_2)(1-p_3)\pi^1(1,1)
\end{aligned}
\tag{10}
$$

According to Shi and Gershwin (2016), $\pi^1(1,1) = 1-p_3$ and $\pi^0(1,1) = r_3$. See Appendix D.

The set (10) of initial conditions, together with $\pi^1(w,x_2)$ and $\pi^0(w,x_2)$ for $w \geq 1$, $1 \leq x_2 \leq N_2$, and recurrence equations (A.1) to (A.5) and (B.1) to (B.4) determine the non-zero $\Pi^{\alpha_2\alpha_3}(\tau,x_1,n_2)$, $\Pi^{B\alpha_3}(\tau,x_1)$, and $\Pi^{\alpha_2 S}(\tau,x_1)$ for all $\tau \geq x_1 + \max(1,n_2)$, $1 \leq x_1 \leq N_1$ and $0 \leq n_2 \leq N_2$. The procedure is summarized in Algorithm 1 which is provided in Appendix E.

3.3 Derivation of prob$(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$

3.3.1 Reformulation

In this section, we find $\textbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$, the second set of factors in (4), from the steady-state probabilities of the MFS. We start the analysis by noting that when the reference part enters B_1 at the end of time unit t, its position $\chi_1(t)$ is equal to $\nu_1(t)$, the level of B_1. In other words, $\chi_1(t) = x_1 = \nu_1(t) = n_1$. Therefore,

$$
\begin{aligned}
& \textbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t)) \\
= \; & \textbf{prob}(\chi_1(t) = n_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t)) \\
= \; & \textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))
\end{aligned}
$$

The last quantity, $\textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$, is the same as $\textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$ because a part can only enter the line (at time t) if M_1 is up (i.e., $\alpha_1(t) = 1$).

Therefore,

$$
\begin{aligned}
& \textbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t)) \\
= \; & \textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t)) \\
= \; & \frac{\textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))}{\textbf{prob}(A(t))}
\end{aligned}
\tag{11}
$$

This can be further simplified. $A(t)$ is the event that a part enters the three-machine two-buffer line at the end of time unit t. Therefore $\textbf{prob}(A(t))$ is the probability that a part enters the line. (We no longer need to speak of the reference part.) Conservation of flow requires this to be the same as the probability that a part leaves the line. That is precisely the production rate P of the MFS model of the three-machine two-buffer line. As a consequence, (11) becomes

$$\begin{aligned} &\mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t)) \\ =\ &\frac{1}{P}\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) \end{aligned} \tag{12}$$

In the following, we find $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ and P in terms of the steady-state probabilities of the MFS. We introduce the following notation: in steady state,

$$\mathbf{p}(n_1, n_2, a_1, a_2, a_3) = \mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = a_1, \alpha_2(t) = a_2, \alpha_3(t) = a_3)$$

These probabilities are found by solving the steady-state Markov transition equation of the three-machine line. (A general description of the transition equations appears in Gershwin 1994.) Gershwin and Schick (1983) derived an analytical solution, but it is difficult to implement. In this study, we use the exact numerical solution of Tan (2003) to calculate the steady-state probabilities. In the following, we treat all $\mathbf{p}(n_1, n_2, a_1, a_2, a_3)$ as known quantities.

The steady-state production rate P of the line is computed from those probabilities according to either of the following expressions.

$$P = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \mathbf{p}(n_1, n_2, 1, a_2, a_3) = \sum_{n_1=0}^{N_1} \sum_{n_2=1}^{N_2} \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \mathbf{p}(n_1, n_2, a_1, a_2, 1)$$

In the first expression, the sum is taken over all states in which M_1 is operational and not blocked. It is the probability that a part enters the system. In the second, which is the probability that a part exits from the system, the sum is taken over all states in which M_3 is operational and not starved.

3.3.2 $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$

In order to evaluate (12), we show next how to express $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ in terms of $\mathbf{p}(n_1, n_2, a_1, a_2, a_3)$. To do so, we divide the set of non-transient states $(\nu_1(t), \nu_2(t), \alpha_1(t), \alpha_2(t), \alpha_3(t)) = (n_1, n_2, 1, a_2, a_3)$ into two subsets: $\mathcal{A}$, those that can be reached *only* when $A(t)$ occurs (i.e., when a new part arrives); and $\bar{\mathcal{A}}$, those that can be reached whether or not $A(t)$ occurs.

Subset $\mathcal{A}$: $\mathcal{A}$ is the set of $(\nu_1(t), \nu_2(t), \alpha_1(t), \alpha_2(t), \alpha_3(t))=(n_1, n_2, 1, a_2, a_3)$ that satisfy one of the following conditions:

$$a_1 = 1; a_2 = 0, 1; a_3 = 0, 1; 1 \leq n_1 \leq N_1 - 2; 0 \leq n_2 \leq N_2, \tag{13}$$

or

$$a_1 = 1; a_2 = 0; a_3 = 0, 1; n_1 = N_1 - 1; 0 \leq n_2 \leq N_2 \tag{14}$$

- Consider a state that satisfies (13). According to (1), it can be reached only from states of the form $(\nu_1(t-1), \nu_2(t-1), \alpha_1(t-1), \alpha_2(t-1),\ \alpha_3(t-1)) = (n_1', n_2', a_1', a_2', a_3')$ where $a_i' = 0$ or 1 and n_i' satisfy

$$\begin{array}{lcl} n_1 = n_1' + 1 - a_2 & \text{or} & n_1' = n_1 - 1 + a_2 \\ n_2 = n_2' + a_2 - a_3 & \text{or} & n_2' = n_2 - 1 + a_3 \end{array}$$

  Since $n_1 \leq N_1 - 2$, the equation for n_1 implies that $n_1' \leq N_1 - 1$.

  That equation means that one part entered B_1 (and one part may have left it). Consequently, if the system is in any state in $\mathcal{A}$ that satisfies (13), $A(t)$ must have occurred. Therefore,

$$\begin{aligned} & \mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) \\ = \ & \mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3) \qquad (15) \\ = \ & \mathbf{p}(n_1, n_2, 1, a_2, a_3) \end{aligned}$$

- Now consider a state that satisfies (14). It can be reached only from states of the form $(\nu_1(t-1), \nu_2(t-1), \alpha_1(t-1), \alpha_2(t-1),\ \alpha_3(t-1)) = (n_1', n_2', a_1', a_2', a_3')$ where $a_i' = 0$ or 1 and n_i' satisfy

$$\begin{array}{rcl} N_1 - 1 = n_1' + 1 & \text{or} & n_1' = N_1 - 2 \\ n_2 = n_2' & \text{or} & n_2' = n_2 \end{array}$$

  In this case, one part entered B_1 and no part left it. Again, $A(t)$ must have occurred. Therefore,

$$\begin{aligned} & \mathbf{prob}(\nu_1(t) = N_1 - 1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 0, A(t)) \\ = \ & \mathbf{prob}(\nu_1(t) = N_1 - 1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 0) \qquad (16) \\ = \ & \mathbf{p}(N_1 - 1, n_2, 1, 0, 0) \end{aligned}$$

Subset $\bar{\mathcal{A}}$: $\bar{\mathcal{A}}$ is the set of $(\nu_1(t), \nu_2(t), \alpha_1(t), \alpha_2(t),\ \alpha_3(t))$=$(n_1, n_2, 1, a_2, a_3)$ that satisfy one of the following conditions:

$$a_1 = 1; a_2 = 1; a_3 = 0, 1; n_1 = N_1 - 1; 0 \leq n_2 \leq N_2 \qquad (17)$$

or

$$a_1 = 1; a_2 = 0, 1; a_3 = 0, 1; n_1 = N_1; 0 \leq n_2 \leq N_2 \qquad (18)$$

In order to develop an expression for $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ under condition (18) we first state $A(t)$ (the event that a part arrives in B_1 at time t) explicitly. $A(t)$ is the event that M_1 is operational at time t and B_1 is not full at time $t-1$. That is,

$$A(t) = \{\alpha_1(t) = 1 \text{ and } \nu_1(t-1) \le N_1 - 1\}$$

Consequently,

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) =$$

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, \nu_1(t-1) \le N_1 - 1)$$

Our objective is to express this as a function of probabilities $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = a_1, \alpha_2(t) = a_2, \alpha_3(t) = a_3) = \mathbf{p}(n_1, n_2, a_1, a_2, a_3)$, the steady-state probability distribution of the MFS. Note that if $\nu_1(t) = N_1$, $\nu_1(t-1)$ cannot be less than $N_1 - 1$. Therefore, $\nu_1(t-1) = N_1 - 1$ and this expression is

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, \nu_1(t-1) = N_1 - 1)$$

Furthermore, for the level of B_1 to increase from $\nu_1(t-1) = N_1 - 1$ to $\nu_1(t) = N_1$, we must have $\alpha_1(t) = 1$ and either $\{\alpha_2(t) = 0 \text{ and } \nu_2(t-1) < N_2\}$ or $\{\alpha_2(t) = 1 \text{ and } \nu_2(t-1) = N_2\}$. Therefore, the expression becomes

$$\begin{aligned} &\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3, A(t)) \\ =&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3, \nu_1(t-1) = N_1 - 1, \nu_2(t-1) < N_2) \end{aligned} \tag{19}$$

if $\alpha_2(t) = 0$ and

$$\begin{aligned} &\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t)) \\ =&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, \nu_1(t-1) = N_1 - 1, \nu_2(t-1) = N_2) \end{aligned} \tag{20}$$

if $\alpha_2(t) = 1$.

The expressions in (19) and (20) can now ne written as the sum of some of the steady-state probabilities of the MFS at time $t-1$. For example, (19) can be expressed as

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3, A(t)) =$$

$$\begin{aligned} \sum &\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3, \\ &\nu_1(t-1) = N_1 - 1, \nu_2(t-1) = n_2', \alpha_1(t-1) = a_1', \alpha_2(t-1) = a_2', \alpha_3(t-1) = a_3') \end{aligned}$$

where the sum is taken over all $n_2' < N_2, a_1', a_2', a_3'$.

This can be written

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3, A(t)) =$$

$$\sum \mathbf{prob}\Big(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3 \;\Big|$$
$$\nu_1(t-1) = N_1 - 1, \nu_2(t-1) = n_2', \alpha_1(t-1) = a_1', \alpha_2(t-1) = a_2', \alpha_3(t-1) = a_3'\Big) \times$$
$$\mathbf{prob}(\nu_1(t-1) = N_1 - 1, \nu_2(t-1) = n_2', \alpha_1(t-1) = a_1', \alpha_2(t-1) = a_2', \alpha_3(t-1) = a_3')$$

The conditional probability on the right side of the equation is the transition probability from a state of the system at time $t-1$ to a state at time t. It is a function of the repair and failure probabilities r_i and p_i. We can abbreviate the last equation as

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = a_3, A(t)) =$$

$$\sum \mathbf{prob}(N_1, n_2, 1, 0, a_3 \,\Big|\, N_1 - 1, n_2', a_1', a_2', a_3') \; \mathbf{p}(N_1 - 1, n_2', a_1', a_2', a_3')$$

To analyze this expression, we must deal with cases separately. For example, let $a_3 = 1$. Then, from (1) and the definitions of I_2 and I_3,

$$n_2 = n_2' + I_2(t) - I_3(t)$$

where

$$I_3(t) = 1$$

Consequently,

$$n_2 = n_2' - 1$$

Therefore.

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 1, A(t)) =$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 0, 0, 0)\ \mathbf{p}(N_1 - 1, n_2 + 1, 0, 0, 0) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 0, 0, 1)\ \mathbf{p}(N_1 - 1, n_2 + 1, 0, 0, 1) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 0, 1, 0)\ \mathbf{p}(N_1 - 1, n_2 + 1, 0, 1, 0) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 0, 1, 1)\ \mathbf{p}(N_1 - 1, n_2 + 1, 0, 1, 1) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 1, 0, 0)\ \mathbf{p}(N_1 - 1, n_2 + 1, 1, 0, 0) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 1, 0, 1)\ \mathbf{p}(N_1 - 1, n_2 + 1, 1, 0, 1) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 1, 1, 0)\ \mathbf{p}(N_1 - 1, n_2 + 1, 1, 1, 0) +$$

$$\mathbf{prob}(N_1, n_2, 1, 0, 1 \,\big|\, N_1 - 1, n_2 + 1, 1, 1, 1)\ \mathbf{p}(N_1 - 1, n_2 + 1, 1, 1, 1)$$

The first factor of the first term is the conditional probability that the machine states go from (0,0,0) to (1,0,1) and the first buffer gains a part and the second buffer loses a part. But since the change of the buffer levels is a consequence of the new machine states (1,0,1), the conditional probability is simply the probability of the changes of all the machine states, which is $r_1(1-r_2)r_3$. The rest of the conditional probabilities can be evaluated similarly, and the last expression reduces to the last equation of (C.8). All the other cases of equation (19) and all the cases of equation (20) can be treated similarly.

Expressions for $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ for all states that are in $\bar{\mathcal{A}}$ are provided in Appendix C.

Equations (C.1) to (C.10) are used to find $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$, $1 \le n_1 \le N_1$, $0 \le n_2 \le N_2, a_2 = 0, 1, a_3 = 0, 1$. (12) is then applied to find $\mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t))$. The process is summarized in Algorithm 2, which appears in Appendix E.

3.4 Calculating $\mathbf{prob}(T = \tau)$

The procedures to find the quantities discussed in Sections 3.2 and 3.3 are provided in Algorithms 1 and 2 in Appendix E. Once $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ and $\mathbf{prob}(\chi_1(t) =$

$x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$ are found, applying (4) and (5) gives $\mathbf{prob}(T = \tau)$.

The procedure to compute the lead time distribution $\mathbf{prob}(T = \tau)$ is provided in Algorithm 3 which also appears in Appendix E.

4 Numerical Studies

In this section, we provide numerical evidence for the correctness and accuracy of the analytical solution of the lead time distribution for three-machine two-buffer lines derived here. We also indicate how the distribution can provide insight to help line design.

Evidence for the correctness and accuracy of the distribution is obtained by applying Little's law as well as from comparisons with simulation. For insight, we describe the lead time distributions in a line and its reverse. We also show an example of the relationship between the size of a buffer and the variance and 95th percentile of the lead time distribution. Finally, we show examples of how the mean time between failures (MTBF) affects the variance and 95th percentile of the lead time distribution.

4.1 Test with Little's Law

In this section, we verify the calculation of $\mathbf{prob}(T = \tau)$ by applying Little's Law (Little 1961) to a few three-machine two-buffer systems. In our notation, Little's law is written as $(\bar{n}_1 + \bar{n}_2)/P = \mathbf{E}[T]$ in which $\bar{n}_1$, $\bar{n}_2$, and P are found from the three-machine two-buffer MFS model, and $\mathbf{E}[T]$ is computed from the PMF of T:

$$\mathbf{E}[T] = \sum_{\tau=2}^{\infty} \tau \mathbf{prob}(T = \tau). \tag{21}$$

Table 1 shows that $\mathbf{E}[T] = (\bar{n}_1 + \bar{n}_2)/P$ in all these experiments.

Table 1: Test with Little's Law

case	1	2	3	4	5
r_1, p_1	.1, .01	.8, .096	.07, .01	.2, .02	.12, .009
r_2, p_2	.1, .01	.1, .01	.12, .008	.2, .02	.15, .009
r_3, p_3	.1, .01	.1, .01	.12, .008	.4, .048	.07, .01
N_1	10	30	16	18	19
N_2	10	22	23	35	17
P	.819137	.861210	.847203	.874546	.848478
$\bar{n}_1$	5.983370	14.496551	6.606718	9.860994	12.468434
$\bar{n}_2$	4.016630	9.393820	6.665852	16.534348	9.779282
$(\bar{n}_1 + \bar{n}_2)/P$	12.207969	27.740476	15.666334	30.181748	26.220720
$\mathbf{E}[T]$	12.207969	27.740476	15.666334	30.181748	26.220720

4.2 Comparison with Simulation

In this section, we verify the accuracy of the lead time distribution by comparing it with a discrete time simulation. In each example, the length of each simulation is 21,000,000 time units with the first

1,000,000 time units being the warm up period, and we run the simulation 30 times and use the average as the simulation result.

4.2.1 Example 1

Consider a balanced line first, where machines are identical with parameters $p_i = .01$ and $r_i = .1$ for $i = 1, 2, 3$. The buffers are the same size, with $N_1 = N_2 = 10$. The analytical and simulation results for the distribution are shown in Figure 5. They demonstrate the accuracy of the analytical lead time distribution for this case.

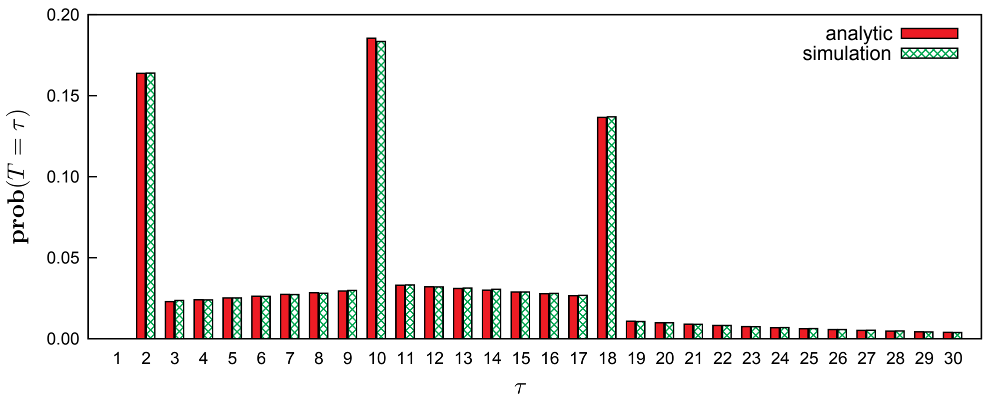


Figure 5: Example 1, **prob**$(T = \tau)$, analytical solution vs. simulation

The shape of **prob**$(T = \tau)$ indicates that most of the parts entering this line have a lead time of 2, 10, or 18 time units. In addition, there is a small fraction of parts whose lead times are longer than 18 time units. This is because the size of each buffer is equal to the mean time to repair (MTTR) of each machine. Consequently, when one machine fails, both of the buffers tend to be full or empty frequently and the other two machines are forced to be idle. In particular,

- if M_1 fails for a long time, both B_1 and B_2 will become empty. After M_1 is repaired, the system will run with an inventory level of 1 in both buffers before the next machine failure occurs[1]. During this period, each part spends 1 time unit in each buffer, which leads to a total lead time of 2 time units.

- if M_2 fails for a long time, B_1 becomes full and B_2 becomes empty. After M_2 is repaired, the system will run with an inventory level of 9 in B_1 and an inventory level of 1 in B_2[2]. In other words, during this period, each part spends 9 time units in B_1 and 1 time unit in B_2, adding to a total of 10 time units.

[1]Immediately after the repair, M_1 adds one part to B_1. Since M_2 was starved, it cannot remove a part, so $n_1 = 1$ and B_2 remains empty. At the next time step, M_1 adds a part to B_1 and M_2 can now remove one, so n_1 remains equal to 1. M_2 adds that part to B_2. Since M_3 was starved, it cannot remove a part so n_2 is 1. After that, the buffer levels stay constant at $n_1 = n_2 = 1$ until the next machine failure.

[2]The reason for these buffer levels is that the system behavior in this case is similar to that of the previous case. Immediately after the repair, M_2 removes one part from B_1 and adds it to B_2. M_1 does not add a part to B_1 because it was blocked, while M_3 cannot remove a part from B_2 because it was starved. After that, all machines are up and not idle, so the buffer levels stay constant until the next failure.

- finally, if M_3 fails for a long time, both buffers become full. After M_3 is repaired, the system will run with an inventory level of 9 in both buffers before the next failure takes place. During this period, each part spends 9 time units in both B_1 and B_2 and therefore has a lead time of 18 time units. If M_3 fails again when the system is running with an inventory level of 9 in both buffers, the lead times for those parts in the buffer will be longer than 18 time units.

4.2.2 Example 2

Table 2 has the parameters of the line of Example 2. It also contains $e_i = r_i/(r_i + p_i)$, the isolated efficiencies of the machines. The machine with the smallest e_i is the bottleneck. Figure 6 shows that the analytical and simulation results are very close.

i	p_i	r_i	e_i	N_i
1	.01	.07	.875	16
2	.008	.12	.938	23
3	.008	.12	.938	

Table 2: Example 2 parameters

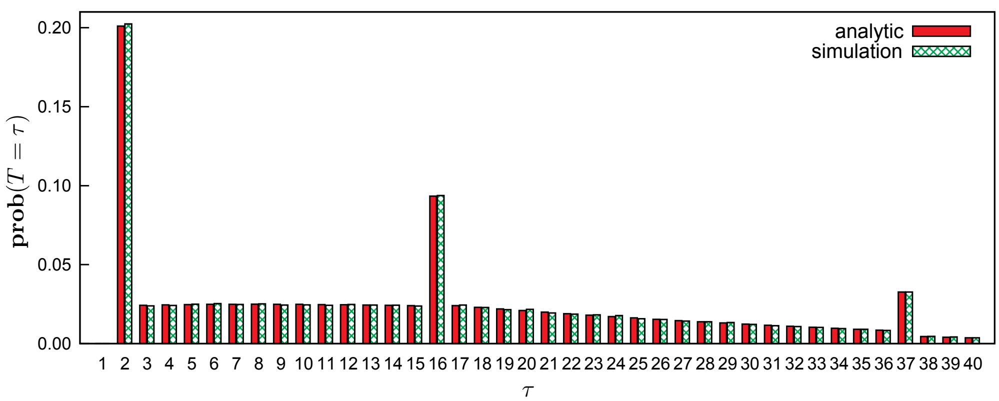


Figure 6: Example 2, **prob**$(T = \tau)$, analytical solution vs. simulation

In this line, the first machine is the bottleneck. Whenever it fails for a long time, the two buffers become empty. After M_1 is repaired, the system will run with an inventory level of 1 in both buffers until the next machine failure. During this period, each part spends two time units in the system and therefore has a lead time of 2. This explains why **prob**$(T = 2)$ is large. The values of **prob**$(T = 16)$ and **prob**$(T = 37)$ can be explained similarly. Moreover, since M_1 is the bottleneck, most of the parts have a small lead time.

4.2.3 Example 3

The parameters of this line are shown in Table 3. Figure 7 compares the analytical and simulation results and illustrates the accuracy of the analytical solution.

M_3 is the bottleneck of the line. Whenever it fails for a long time, the two buffers become full. After M_3 is repaired, the system will run with inventory levels of 18 in B_1 and 16 in B_2 until the next machine

i	p_i	r_i	e_i	N_i
1	.009	.12	.930	19
2	.009	.15	.943	17
3	.01	.07	.875	

Table 3: Example 3 parameters

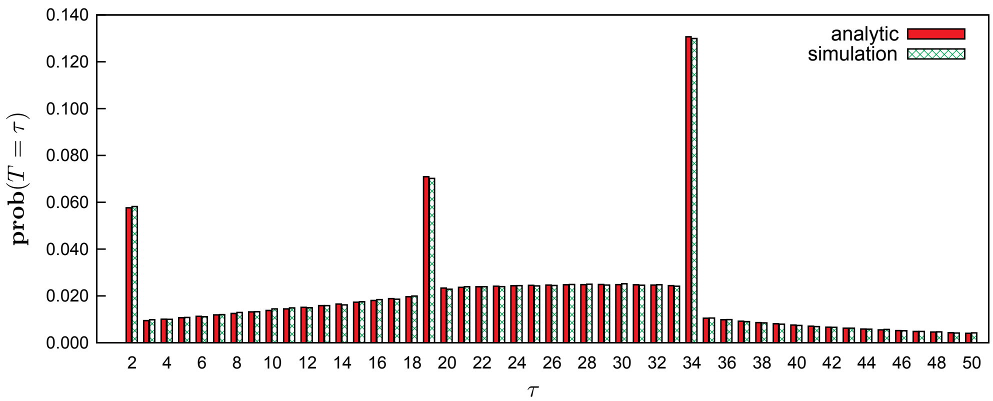


Figure 7: Example 3, $\mathbf{prob}(T = \tau)$, analytical solution vs. simulation

failure. During this period, each part spends 34 time units in the system and therefore $\mathbf{prob}(T = 34)$ is large. The values of $\mathbf{prob}(T = 2)$ and $\mathbf{prob}(T = 19)$ can be explained similarly. Most of the parts have a long lead time because M_3 is the bottleneck.

4.3 Example 4: A Line and Its Reverse

Consider a three-machine line with parameters $(p_1^o, r_1^o, p_2^o, r_2^o, p_3^o, r_3^o, N_1^o, N_2^o)$. The reverse of that line has parameters $(p_1^r, r_1^r, p_2^r, r_2^r, p_3^r, r_3^r, N_1^r, N_2^r) = (p_3^o, r_3^o, p_2^o, r_2^o, p_1^o, r_1^o, N_2^o, N_1^o)$. Then the production rates of the original line and its reverse are the same and the average buffer levels satisfy (Gershwin 1994)

$$\bar{n}_i^o + \bar{n}_{3-i}^r = N_i^o = N_{3-i}^r, \quad i = 1, 2 \tag{22}$$

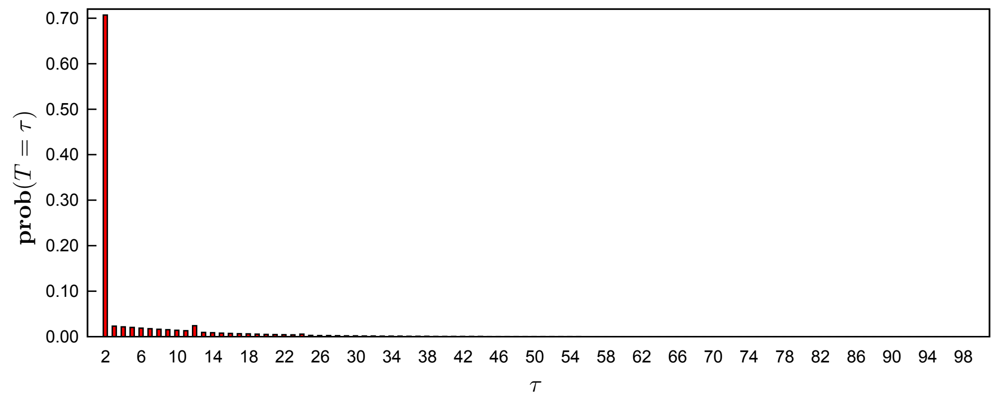
where $\bar{n}_i^o$ and $\bar{n}_{3-i}^r$ are the average inventory levels of buffers B_i^o and B_{3-i}^r in the original and the reversed lines, respectively. In other words, by reversing a transfer line, we achieve the same production rate with completely different average inventories. A line with bottleneck machine at its beginning is better than its reverse which has the bottleneck at its end, because the former will achieve lower average inventory levels and therefore smaller lead times. We illustrate with an example.

Consider the original line and its reverse in Table 4. M_1^o is the bottleneck machine of the original line. When the line is reversed, it becomes M_3^r and it becomes the bottleneck of the reversed line. The two lines have the same production rate, but the original line has much lower average inventory levels in both buffers than the reversed line. The lead time distributions of the two lines are shown in Figure 8.

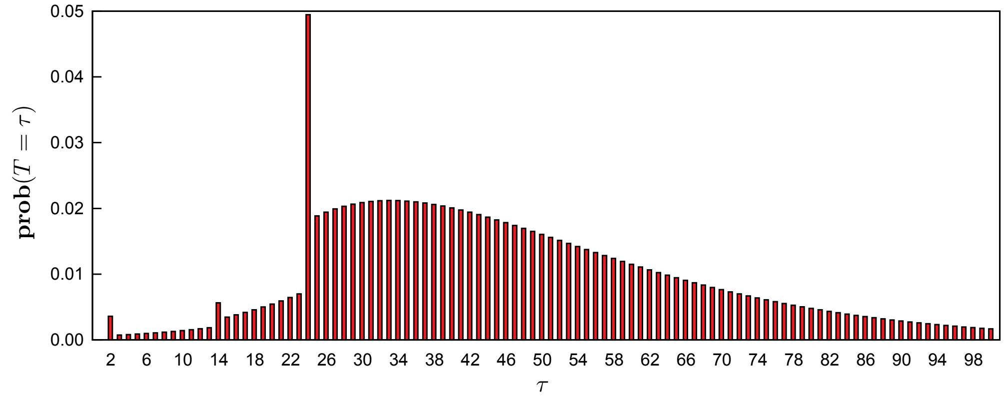
Figure 8 reveals that the two lines exhibit completely different lead time distributions. In the original line, since the bottleneck machine M_1^o is at the beginning of the line, its frequent breakdowns cause the system to run with an inventory level of 1 at both buffers most of the time. As a result, most of the parts have a lead time of 2 (Figure 8(a)). On the other hand, since the bottleneck machine M_3^r is the last machine of the reversed line, its breakdowns cause the system to run with inventory levels of 13 and

Table 4: A line and its reverse

	r_1	p_1	r_2	p_2	r_3	p_3	N_1	N_2	P	$\bar{n}_1$	$\bar{n}_2$
original line	.1	.1	.1	.01	.1	.01	12	14	.493214	1.284711	1.293355
reversed line	.1	.01	.1	.01	.1	.1	14	12	.493214	12.706645	10.715289



(a) Lead time distribution of the original line



(b) Lead time distribution of the reversed line

Figure 8: Lead time distributions of a line and its reverse

11. Consequently, a large number of parts have lead times of 24. Moreover, when M_3^r fails while there are 24 parts in the system, the parts in the system will have even longer lead times. This is why there is a large tail in Figure 8(b) for $\tau > 24$.

The average lead time of the original line is 5.23 time units, while that of the reversed line is 47.49 time units. From the lead time perspective, the original line is much better than the reversed line, as it produces parts at the same rate of its reversed line, but with a much lower lead time on average. The example emphasizes the key production line design principle that, when it is possible, it is better to put the bottleneck machine at the beginning, rather than at the end, of a line.

This principle is also illustrated by comparing the variance of the lead time and the probability distribution of the lead time in the original and the reversed line. The variance of the lead time of the original line is 54.86. The variance of the lead time of the reversed line is 500.23.

We compute $\mathbf{prob}(T \leq \tau)$ of the original line and the reversed line in Table 5 and Figure 9.

Table 5: $\mathbf{prob}(T \leq \tau)$ of the original and reversed lines

τ	$\mathbf{prob}(T \leq \tau)$ of the original line	$\mathbf{prob}(T \leq \tau)$ of the reversed line
10	.8546	.0120
20	.9489	.0492
30	.9816	.2381
40	.9929	.4467
50	.9972	.6266
60	.9989	.7619
70	.9996	.8548
80	.9998	.9147
90	.9999	.9514
100	1.0000	.9730

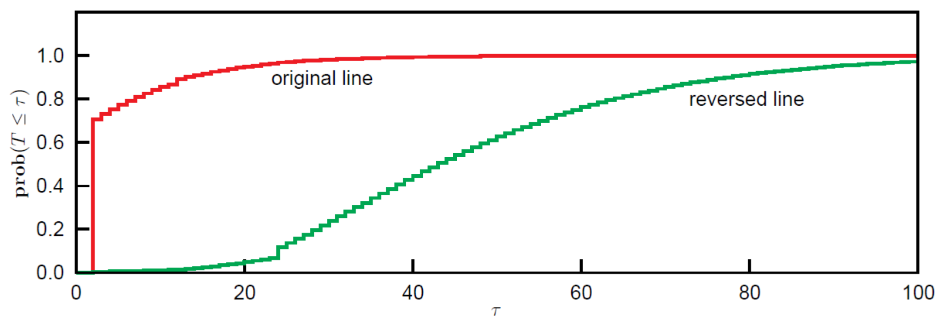


Figure 9: $\mathbf{prob}(T \leq \tau)$ of the original and reversed lines

4.4 Example 5: Variance and $\mathbf{prob}(T \leq \tau)$ vs. N_2

Consider a three-machine line with parameters: $r_1 = .07$, $p_1 = .01$, $r_2 = .12$, $p_2 = .008$, $r_3 = .12$, $p_3 = .008$, and $N_1 = 100$. We vary N_2. The variance of the lead time and $\tau_{.95}$, the minimum value of τ such that $\mathbf{prob}(T \leq \tau) \geq .95$, are illustrated in Figures 10 and 11.

These graphs show that there is a size of Buffer B_2 that minimizes the variance of the lead time and an optimal size of B_2 to minimize $\tau_{.95}$. However, a similar experiment in which N_1 was varied and N_2 was held constant did not show the same behavior.

This experiment illustrates the importance of modeling buffers as finite. Such an observation could not be made if buffers were modeled as infinite. This phenomenon should be studied systematically.

4.5 Example 6: Variance and $\mathbf{prob}(T \leq \tau)$ vs. MTBF

In this set of experiments, we observe the effect of the mean time between failures (MTBF) of a machine on the variance and 95th percentile of the lead time of the line. Because of our assumption of geometric up- And down-times of machines, the mean time to fail (MTTF$_i$) and the mean time to repair (MTTR$_i$) of machine M_i are $1/p_i$ and $1/r_i$ respectively, and the mean time between failures is given by

$$\text{MTBF}_i = \frac{1}{p_i} + \frac{1}{r_i}$$

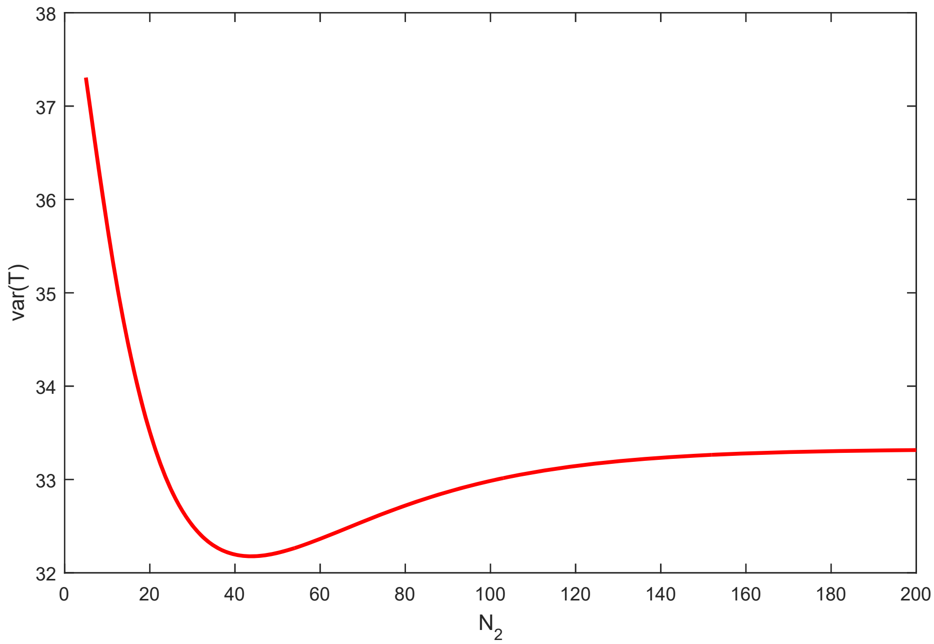


Figure 10: The variance of lead time vs. N_2

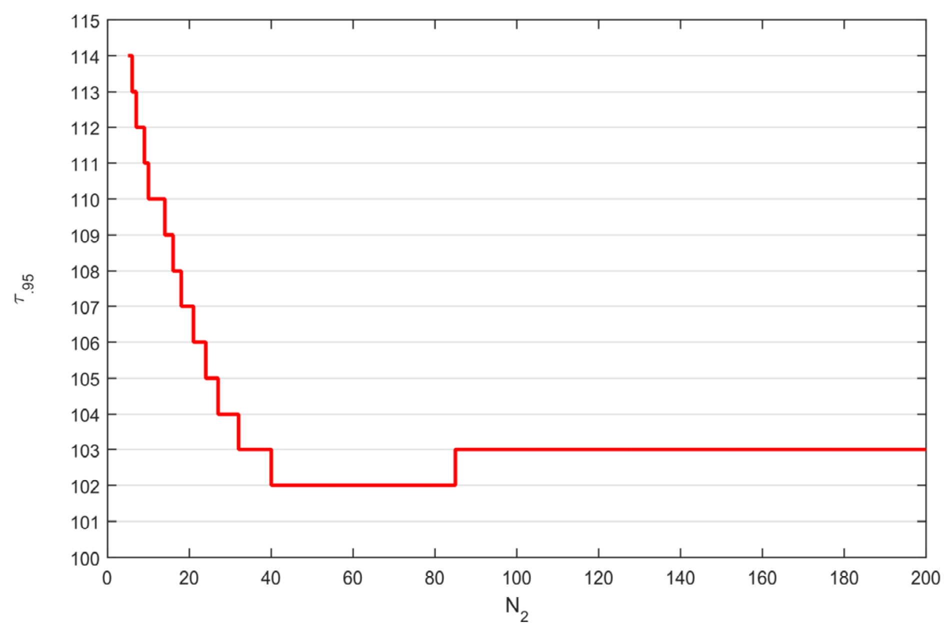


Figure 11: The minimum value of τ such that $\mathbf{prob}(T \leq \tau) \geq .95$

We vary MTBF_i by varying p_i and r_i together to keep

$$e_i = \frac{r_i}{r_i + p_i}$$

constant. This allows us to focus on the sensitivity of the results to the duration of the average up-down cycle of a machine, and to eliminate the effect of changing the isolated production rates of machines.

4.5.1 Varying MTBF$_1$

In this experiment, we vary p_1 and r_1 together so that $e_1 = .8$. The other parameters of the line are $r_2 = .1$, $p_2 = .01$, $r_3 = .1$, $p_3 = .01$, and $N_1 = N_2 = 20$. Since $e_2 = e_3 = .9091$, M_1 is the bottleneck of the line. Figure 12 shows the effect of MTBF$_1$ on the variance of the lead time and Figure 13 shows the effect of MTBF$_1$ on the 95th percentile of the lead time.

Figure 12 shows that the variance has a maximum at MTBF$_i \approx 250$. The 95th percentile of the lead time increases with MBTF$_1$, as shown in Figure 13.

4.5.2 Varying MTBF$_2$

Here we vary r_2 and p_2 such that $e_2 = .8$ and we choose $r_1 = .1$, $p_1 = .01$, $r_3 = .1$, $p_3 = .01$, and $N_1 = N_2 = 20$. Now M_2 is the bottleneck. The qualitative properties of these graphs differ from those in Section 4.5.1. In Figure 14, the variance of the lead time appears almost linear in MTBF$_2$. In Figure 15, τ_{95} has a maximum.

4.5.3 Varying MTBF$_3$

Now we vary r_3 and p_3 such that $e_3 = .8$ and we choose $r_1 = .1$, $p_1 = .01$, $r_2 = .1$, $p_2 = .01$, and $N_1 = N_2 = 20$. M_3 is the bottleneck. The qualitative properties of the graphs of Figures 16 and 17 are similar to those of Figures 14 and 15.

5 Conclusion and Future Work

5.1 Summary

In this paper, we describe an analytical approach to find the lead time distribution of a Buzacott (discrete-state, discrete-time) model of a three-machine two-buffer line with unreliable machines and finite buffers. Using this distribution, we can find the mean and standard deviation, as well as any given percentile, of the lead time.

This is of practical importance because make-to-order manufacturers must make early and reliable delivery promises to maintain good customer relations. In the absence of a practical way to determine the probability that delivery will take place on or before a certain time, either the manufacturer will risk losing customers, or it will have to design its production system very conservatively or hold large inventories, both of which will increase costs. With the method described here, effective factories can be built with less excess productive capacity or with less need to hold inventory.

It is also important, for similar reasons, for producers of goods whose value deteriorates rapidly, either because of spoilage (such as food) or obsolescence (such as fashion or technological advances.)

The approach is based on tracking the movement of a reference part in the MFS model of the three-machine two-buffer line. Numerical experiments, including verification by Little's law and comparison with simulation, are provided to show the correctness of the distribution. In other numerical experiments, we compare the lead time distributions of a line and its reverse and we show how the mean time between failures (MTBF) of each of the three machines in a an example of a line affect the variance of the lead

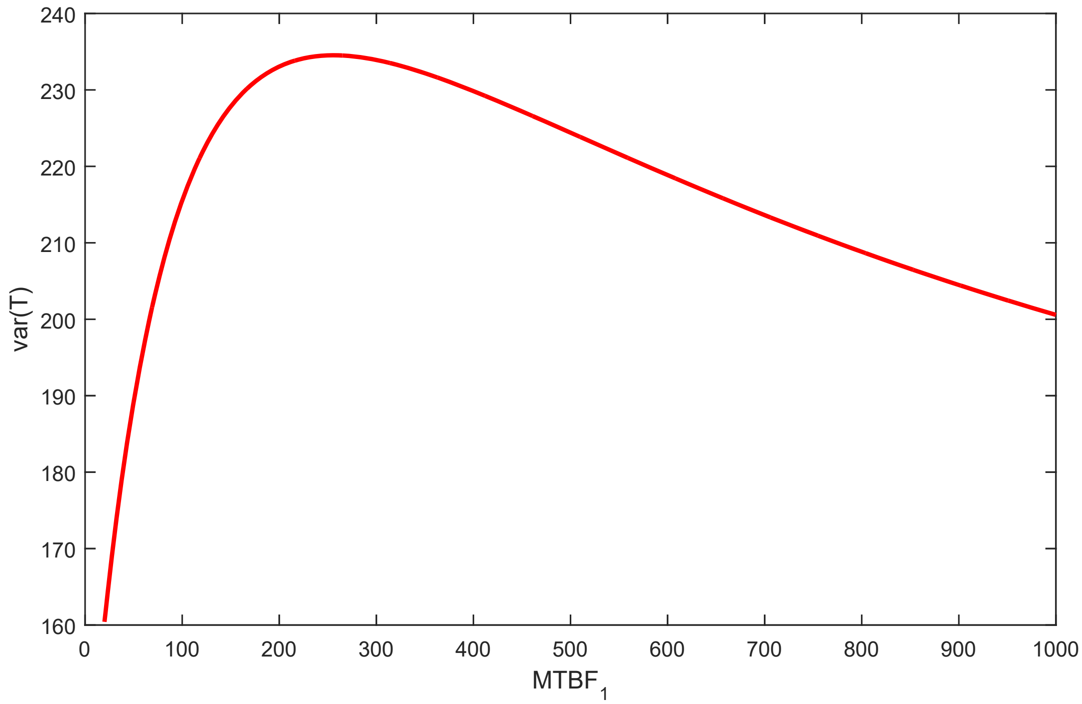


Figure 12: Variance of lead time vs. MTBF_1

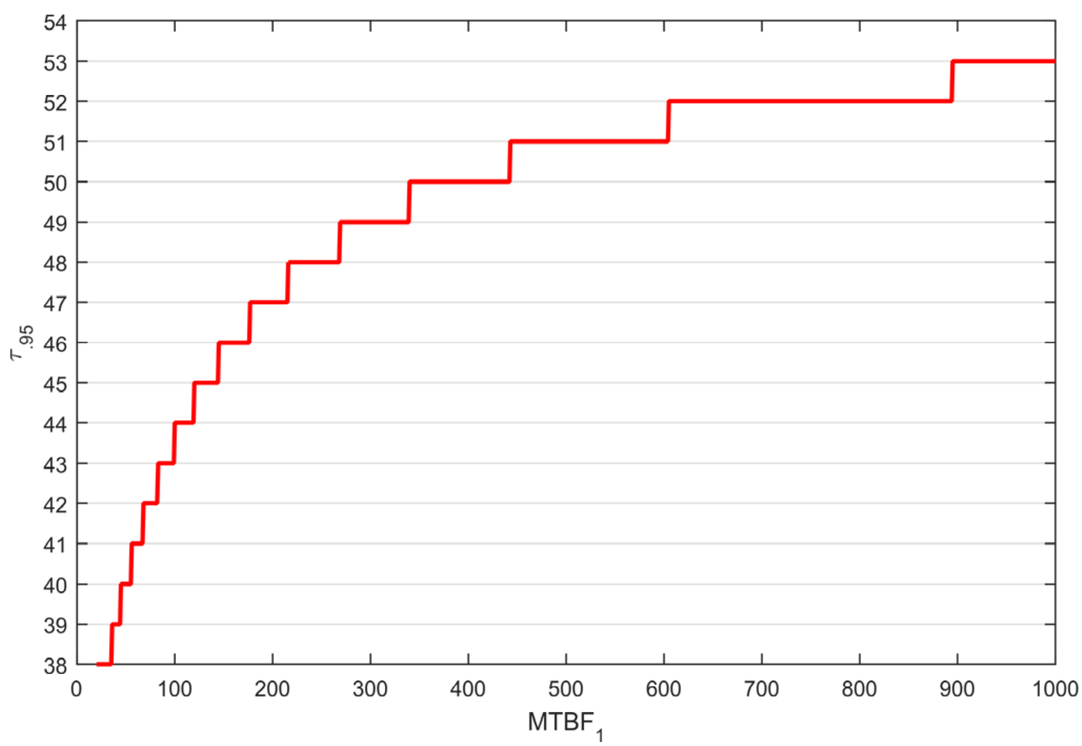


Figure 13: τ_{95} vs. MTBF_1

time and $\tau_{.95}$, the minimum value of the time such that the probability that the lead time is greater than that time is at least .95.

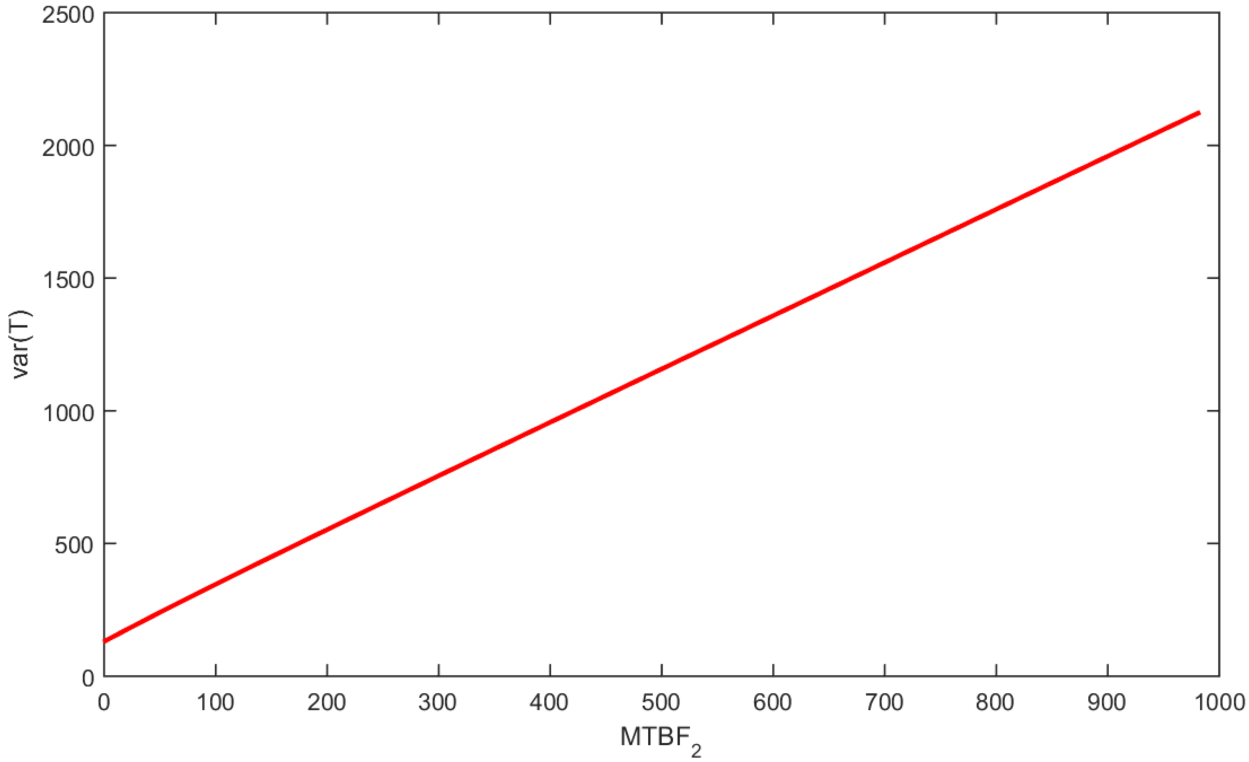


Figure 14: Variance of lead time vs. $MTBF_2$

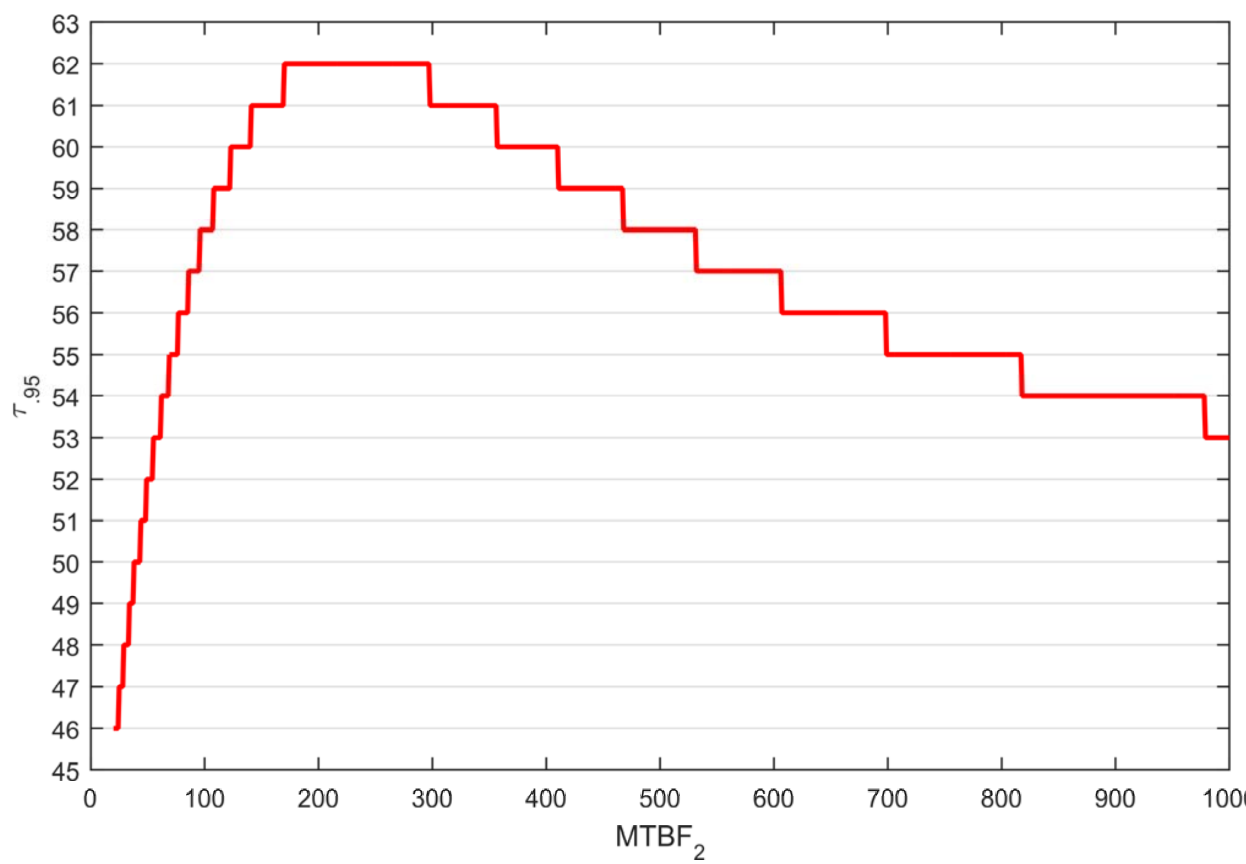


Figure 15: τ_{95} vs. $MTBF_2$

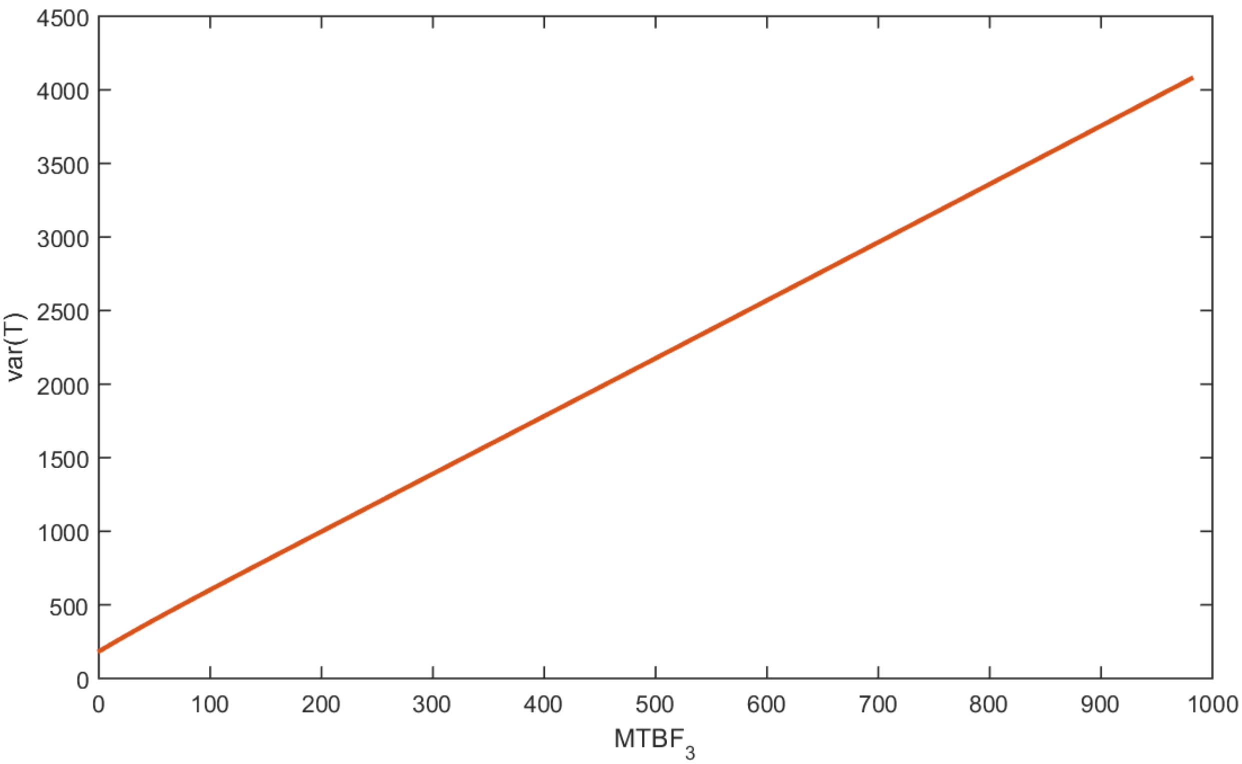


Figure 16: Variance of lead time vs. MTBF_3

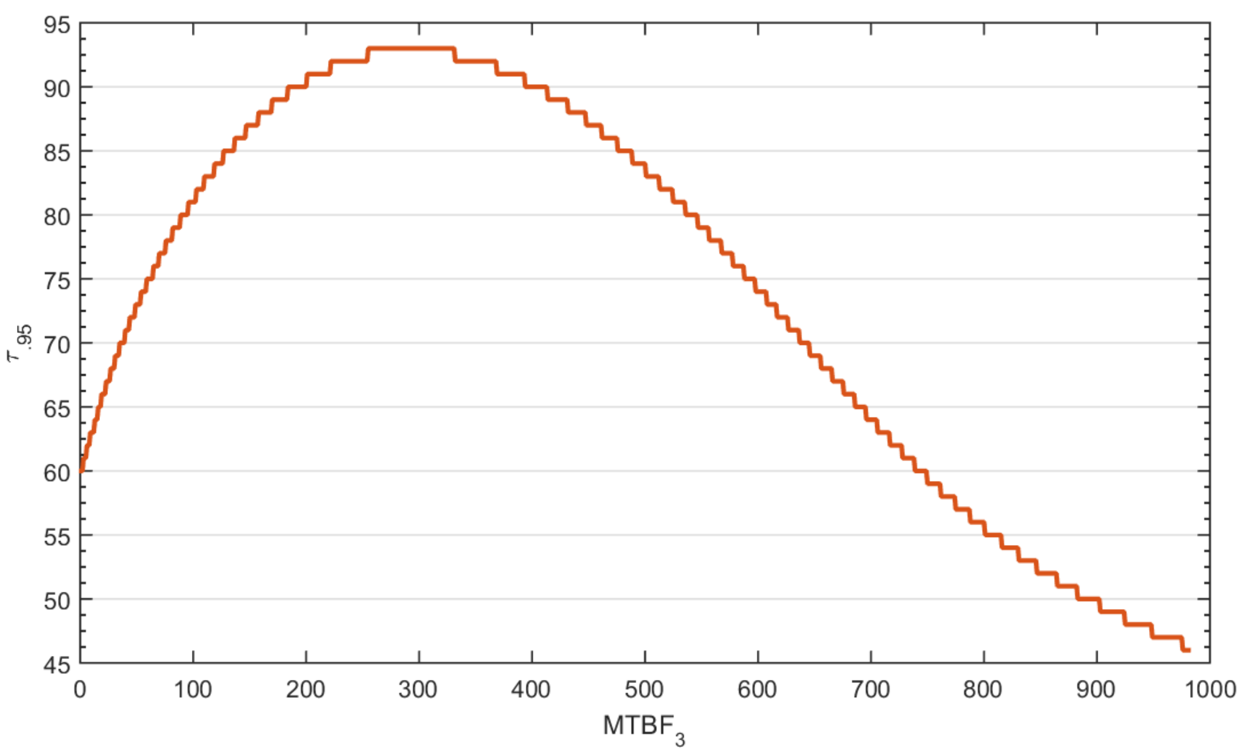


Figure 17: τ_{95} vs. MTBF_3

5.2 Future Research

There are many future research directions that can follow from this work.

1. The methodology can be extended to lines with machines that have multiple failure modes. The same approach can be applied but we need to consider different failure modes of M_2 and M_3 in the recurrence equations. A further extension would be to consider machines whose repair/failure behavior is described by general Markov chains. See Colledani et al. (2015). More general still would be to consider continuous-time systems with discrete or continuous material.

2. It will be of interest to study the shapes of the lead time distribution of three-machine two-buffer lines qualitatively and systematically. The study of three-machine two-buffer lines of Shi and Gershwin (2013) classifies such production systems into five different types according to the machine repair and failure parameters. It demonstrates that the qualitative behavior of average inventory levels as a function of buffer sizes is very distinct in different types. Each type demonstrates a different sensitivity of average inventories to buffer sizes. Since the average lead time of such a line is closely related to the average inventory levels, the study of the lead time distribution for lines of each type may provide deeper insights into the relationship between system lead time, machine parameters, and buffer sizes. Shi and Gershwin (2013) show how their qualitative observations are also relevant to longer lines.

3. This method can be extended to longer lines. For example, the movement of a reference part in a four-machine system can be tracked. When it leaves B_1 and enters B_2 at position x_2, the conditional lead time probabilities of three-machine lines can be used to construct the initial conditions for the four-machine line recurrence equations. This will allow us to determine the conditional lead time probabilities of four-machine lines. Using the steady-state probability that a four-machine line is in a given state, the lead time distribution of a four-machine line can be found. The lead time distribution of four-machine lines can in turn be used to find that of five-machine lines. Repeating this approach, the lead time distribution of a k-machine, $k-1$-buffer lines can be found for any k. Extensions to assembly/disassembly systems and to networks with loops are also useful. See Colledani et al. (2015). An extension to re-entrant flow systems would be of great value to semiconductor manufacturers.

4. Other buffer disciplines are sometimes observed in factories, such as Last In, First Out (LIFO) and random selection of parts. The lead time distributions resulting from such disciplines will be very different from those discussed here, although the mean lead times will be the same.

5. As noted above, Shi and Gershwin (2016) reported that the waiting time in a single buffer of a long line can be approximately determined by applying their two-machine, one-buffer sojourn time analysis to the buffer in the two-machine, one-buffer line corresponding to that buffer in the decomposition method for calculating production rate and average buffer levels. That research can now be extended to a three-machine, two-buffer segment of a long line or other network topologies.

6. An important problem of practical interest is the performance optimization of manufacturing systems with a constraint on $\tau_{.95}$ for a single buffer, for a segment of the system, or for the entire line.

7. The exact numerical steady-state probability distribution of a line is needed in order to determine its exact lead time distribution in the method described here. This is impractical for lines that are too long. It would be useful to determine an approximate lead time distribution for long lines.

A Recurrence Equations for prob$(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ where $2 \leq x_1 \leq N_1$

The recurrence equations to find **prob**$(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ where $2 \leq x_1 \leq N_1$ are listed in this section.

- $n_2 = 0$ (and $2 \leq x_1 \leq N_1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, x_1, 0) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, x_1, 0) & + & r_2(1-r_3)\,\Pi^{10}(\tau-1, x_1-1, 1) \\
& + & r_2 r_3\,\Pi^{11}(\tau-1, x_1-1, 1) & + & (1-r_2)r_3\,\Pi^{0S}(\tau-1, x_1) \\
\Pi^{10}(\tau, x_1, 0) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1, x_1, 0) & + & (1-p_2)(1-r_3)\,\Pi^{10}(\tau-1, x_1-1, 1) \\
& + & (1-p_2)r_3\,\Pi^{11}(\tau-1, x_1-1, 1) & + & p_2 r_3\,\Pi^{0S}(\tau-1, x_1) \\
\Pi^{0S}(\tau, x_1) & = & (1-p_2)\,\Pi^{11}(\tau-1, x_1-1, 1) & + & p_2\,\Pi^{0S}(\tau-1, x_1) \\
\Pi^{1S}(\tau, x_1) & = & r_2\,\Pi^{11}(\tau-1, x_1-1, 1) & + & (1-r_2)\,\Pi^{0S}(\tau-1, x_1)
\end{array} \tag{A.1}$$

- $n_2 = 1$ (and $2 \leq x_1 \leq N_1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, x_1, n_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & r_2(1-r_3)\,\Pi^{10}(\tau-1, x_1-1, n_2+1) \\
& + & r_2 r_3\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & (1-r_2)r_3\,\Pi^{0S}(\tau-1, x_1, 0) \\
\Pi^{01}(\tau, x_1, n_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & r_2 p_3\,\Pi^{10}(\tau-1, x_1-1, n_2+1) \\
& + & r_2(1-p_3)\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & (1-r_2)(1-p_3)\,\Pi^{0S}(\tau-1, x_1, 0) \\
\Pi^{10}(\tau, x_1, n_2) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-p_2)(1-r_3)\,\Pi^{10}(\tau-1, x_1-1, n_2+1) \\
& + & (1-p_2)r_3\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & p_2 r_3\,\Pi^{0S}(\tau-1, x_1) \\
\Pi^{11}(\tau, x_1, n_2) & = & p_2 p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-p_2)p_3\,\Pi^{10}(\tau-1, x_1-1, n_2+1) \\
& + & (1-p_2)(1-p_3)\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & p_2(1-p_3)\,\Pi^{0S}(\tau-1, x_1)
\end{array} \tag{A.2}$$

- $2 \leq n_2 \leq N_2 - 2$ (and $2 \leq x_1 \leq N_1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, x_1, n_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-r_2)r_3\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & r_2(1-r_3)\,\Pi^{10}(\tau-1, x_1-1, n_2+1) & + & r_2 r_3\,\Pi^{11}(\tau-1, x_1-1, n_2) \\
\Pi^{01}(\tau, x_1, n_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-r_2)(1-p_3)\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & r_2 p_3\,\Pi^{10}(\tau-1, x_1-1, n_2+1) & + & r_2(1-p_3)\,\Pi^{11}(\tau-1, x_1-1, n_2) \\
\Pi^{10}(\tau, x_1, n_2) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & p_2 r_3\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & (1-p_2)(1-r_3)\,\Pi^{10}(\tau-1, x_1-1, n_2+1) & + & (1-p_2)r_3\,\Pi^{11}(\tau-1, x_1-1, n_2) \\
\Pi^{11}(\tau, x_1, n_2) & = & p_2 p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & p_2(1-p_3)\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & (1-p_2)p_3\,\Pi^{10}(\tau-1, x_1-1, n_2+1) & + & (1-p_2)(1-p_3)\,\Pi^{11}(\tau-1, x_1-1, n_2)
\end{array} \tag{A.3}$$

- $n_2 = N_2 - 1$ (and $2 \le x_1 \le N_1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, x_1, n_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-r_2)r_3\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & r_2 r_3\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & r_2(1-r_3)\,\Pi^{B0}(\tau-1, x_1-1) \\
\Pi^{01}(\tau, x_1, n_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-r_2)(1-p_3)\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & r_2(1-p_3)\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & r_2 p_3\,\Pi^{B0}(\tau-1, x_1-1) \\
\Pi^{10}(\tau, x_1, n_2) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & p_2 r_3\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & (1-p_2)r_3\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & (1-p_2)(1-r_3)\,\Pi^{B0}(\tau-1, x_1-1) \\
\Pi^{11}(\tau, x_1, n_2) & = & p_2 p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & p_2(1-p_3)\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & (1-p_2)(1-p_3)\,\Pi^{11}(\tau-1, x_1-1, n_2) & + & (1-p_2)p_3\,\Pi^{B0}(\tau-1, x_1-1)
\end{array} \tag{A.4}$$

- $n_2 = N_2$ (and $2 \le x_1 \le N_1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, x_1, N_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-r_2)r_3\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & r_2 r_3\,\Pi^{11}(\tau-1, x_1, n_2-1) & + & r_2(1-r_3)\,\Pi^{B0}(\tau-1, x_1) \\
\Pi^{01}(\tau, x_1, N_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1, x_1, n_2) & + & (1-r_2)(1-p_3)\,\Pi^{01}(\tau-1, x_1, n_2-1) \\
& + & r_2(1-p_3)\,\Pi^{11}(\tau-1, x_1, n_2-1) & + & r_2 p_3\,\Pi^{B0}(\tau-1, x_1) \\
\Pi^{B0}(\tau, x_1) & = & r_3\,\Pi^{11}(\tau-1, x_1, n_2-1) & + & (1-r_3)\,\Pi^{B0}(\tau-1, x_1) \\
\Pi^{B1}(\tau, x_1) & = & (1-p_3)\,\Pi^{11}(\tau-1, x_1, n_2-1) & + & p_3\,\Pi^{B0}(\tau-1, x_1)
\end{array} \tag{A.5}$$

B Recurrence Equations for prob$(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ where $x_1 = 1$

The recurrence equations to find **prob**$(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ where $x_1 = 1$ are listed in this section.

- $n_2 = 0$ (and $x_1 = 1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, 1, 0) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, 1, 0) & + & (1-r_2)r_3\,\Pi^{0S}(\tau-1, 1) \\
& + & r_2 r_3\,p(\tau-1, 1) & + & r_2(1-r_3)\,q(\tau-1, 1) \\
\Pi^{10}(\tau, 1, 0) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1, 1, 0) & + & p_2 r_3\,\Pi^{0S}(\tau-1, 1) \\
& + & (1-p_2)r_3\,p(\tau-1, 1) & + & (1-p_2)(1-r_3)\,q(\tau-1, 1) \\
\Pi^{0S}(\tau, 1) & = & (1-r_2)\,\Pi^{0S}(\tau-1, 1) & + & r_2\,p(\tau-1, 1) \\
\Pi^{1S}(\tau, 1) & = & p_2\,\Pi^{0S}(\tau-1, 1) & + & (1-p_2)\,p(\tau-1, 1)
\end{array} \tag{B.1}$$

- $n_2 = 1$ (and $x_1 = 1$)

$$\begin{array}{rcrcr}
\Pi^{00}(\tau, 1, n_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1, 1, n_2) & + & (1-r_2)r_3\,\Pi^{0S}(\tau-1, 1) \\
& + & r_2 r_3\,p(\tau-1, n_2) & + & r_2(1-r_3)\,q(\tau-1, n_2+1) \\
\Pi^{01}(\tau, 1, n_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1, 1, n_2) & + & (1-r_2)(1-p_3)\,\Pi^{0S}(\tau-1, 1) \\
& + & r_2(1-p_3)\,p(\tau-1, n_2) & + & r_2 p_3\,q(\tau-1, n_2+1) \\
\Pi^{10}(\tau, 1, n_2) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1, 1, n_2) & + & p_2 r_3\,\Pi^{0S}(\tau-1, 1) \\
& + & (1-p_2)r_3\,p(\tau-1, n_2) & + & (1-p_2)(1-r_3)\,q(\tau-1, n_2+1) \\
\Pi^{11}(\tau, 1, n_2) & = & p_2 p_3\,\Pi^{00}(\tau-1, 1, n_2) & + & p_2(1-p_3)\,\Pi^{0S}(\tau-1, 1) \\
& + & (1-p_2)(1-p_3)\,p(\tau-1, n_2) & + & (1-p_2)p_3\,q(\tau-1, n_2+1)
\end{array} \tag{B.2}$$

- $2 \leq n_2 \leq N_2 - 1$ (and $x_1 = 1$)

$$
\begin{array}{rcrcr}
\Pi^{00}(\tau,1,n_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1,1,n_2) & + & (1-r_2)r_3\,\Pi^{01}(\tau-1,1,n_2-1) \\
 & + & r_2r_3\,p(\tau-1,n_2) & + & r_2(1-r_3)\,q(\tau-1,n_2+1) \\
\Pi^{01}(\tau,1,n_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1,1,n_2) & + & (1-r_2)(1-p_3)\,\Pi^{01}(\tau-1,1,n_2-1) \\
 & + & r_2(1-p_3)\,p(\tau-1,n_2) & + & r_2p_3\,q(\tau-1,n_2+1) \\
\Pi^{10}(\tau,1,n_2) & = & p_2(1-r_3)\,\Pi^{00}(\tau-1,1,n_2) & + & p_2r_3\,\Pi^{01}(\tau-1,1,n_2-1) \\
 & + & (1-p_2)r_3\,p(\tau-1,n_2) & + & (1-p_2)(1-r_3)\,q(\tau-1,n_2+1) \\
\Pi^{11}(\tau,1,n_2) & = & p_2p_3\,\Pi^{00}(\tau-1,1,n_2) & + & p_2(1-p_3)\,\Pi^{01}(\tau-1,1,n_2-1) \\
 & + & (1-p_2)(1-p_3)\,p(\tau-1,n_2) & + & (1-p_2)p_3\,q(\tau-1,n_2+1)
\end{array}
\tag{B.3}
$$

- $n_2 = N_2$ (and $x_1 = 1$)

$$
\begin{array}{rcrcr}
\Pi^{00}(\tau,1,n_2) & = & (1-r_2)(1-r_3)\,\Pi^{00}(\tau-1,1,n_2) & + & (1-r_2)r_3\,\Pi^{01}(\tau-1,1,n_2-1) \\
 & + & r_2r_3\,\Pi^{11}(\tau-1,1,n_2-1) & + & r_2(1-r_3)\,\Pi^{B0}(\tau-1,1) \\
\Pi^{01}(\tau,1,n_2) & = & (1-r_2)p_3\,\Pi^{00}(\tau-1,1,n_2) & + & (1-r_2)(1-p_3)\,\Pi^{01}(\tau-1,1,n_2-1) \\
 & + & r_2(1-p_3)\,\Pi^{11}(\tau-1,1,n_2-1) & + & r_2p_3\,\Pi^{B0}(\tau-1,1) \\
\Pi^{B0}(\tau,1) & = & r_3\,\Pi^{11}(\tau-1,1,n_2-1) & + & (1-r_3)\,\Pi^{B0}(\tau-1,1) \\
\Pi^{B1}(\tau,1) & = & (1-p_3)\,\Pi^{11}(\tau-1,1,n_2-1) & + & p_3\,\Pi^{B0}(\tau-1,1)
\end{array}
\tag{B.4}
$$

C Equations for $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$

The equations for all $\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$, $a_2 = 0,1, a_3 = 0,1, 1 \leq n_1 \leq N_1, 0 \leq n_2 \leq N_2$, are listed in this section. Recall that, in the equations below, $\mathbf{p}(n_1, n_2, a_1, a_2, a_3)$ is the steady-state probability that the MFS is in state $(n_1, n_2, a_1, a_2, a_3)$.

First, recall that, for $a_1 = 1; a_2 = 0,1, a_3 = 0,1, 1 \leq n_1 \leq N_1 - 2, 0 \leq n_2 \leq N_2$ and $a_1 = 1, a_2 = 0, a_3 = 0,1, n_1 = N_1 - 1, 0 \leq n_2 \leq N_2$, from (15)

$$
\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) = \mathbf{p}(n_1, n_2, 1, a_2, a_3) \tag{C.1}
$$

For $a_1 = 1, a_2 = 1, a_3 = 0,1, n_1 = N_1 - 1, 0 \leq n_2 \leq N_2$:

1. $n_2 = 0$

$$
\begin{array}{rcl}
\mathbf{prob}(\nu_1(t) = N_1 - 1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t)) & = & 0 \\
\mathbf{prob}(\nu_1(t) = N_1 - 1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t)) & = & 0
\end{array}
\tag{C.2}
$$

2. $n_2 = 1$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1-1, \nu_2(t) = 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t))} \\
= & r_1 r_2 (1-r_3)\,\mathbf{p}(N_1-1,0,0,0,0) & + & r_1(1-p_2)(1-r_3)\,\mathbf{p}(N_1-1,0,0,1,0) \\
+ & (1-p_1) r_2 (1-r_3)\,\mathbf{p}(N_1-1,0,1,0,0) & + & (1-p_1)(1-p_2)(1-r_3)\,\mathbf{p}(N_1-1,0,1,1,0)
\end{array}$$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1-1, \nu_2(t) = 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t))} \\
= & r_1 r_2 r_3\,\mathbf{p}(N_1-1,1,0,0,0) & + & r_1 r_2 (1-p_3)\,\mathbf{p}(N_1-1,1,0,0,1) \\
+ & r_1(1-p_2) r_3\,\mathbf{p}(N_1-1,1,0,1,0) & + & r_1(1-p_2)(1-p_3)\,\mathbf{p}(N_1-1,1,0,1,1) \\
+ & (1-p_1) r_2 r_3\,\mathbf{p}(N_1-1,1,1,0,0) & + & (1-p_1) r_2 (1-p_3)\,\mathbf{p}(N_1-1,1,1,0,1) \\
+ & (1-p_1)(1-p_2) r_3\,\mathbf{p}(N_1-1,1,1,1,0) & + & (1-p_1)(1-p_2)(1-p_3)\,\mathbf{p}(N_1-1,1,1,1,1) \\
+ & r_1 r_2 r_3\,\mathbf{p}(N_1-1,0,0,0,0) & + & r_1 r_2\,\mathbf{p}(N_1-1,0,0,0,1) \\
+ & r_1(1-p_2) r_3\,\mathbf{p}(N_1-1,0,0,1,0) & + & r_1(1-p_2)\,\mathbf{p}(N_1-1,0,0,1,1) \\
+ & (1-p_1) r_2 r_3\,\mathbf{p}(N_1-1,0,1,0,0) & + & (1-p_1) r_2\,\mathbf{p}(N_1-1,0,1,0,1) \\
+ & (1-p_1)(1-p_2) r_3\,\mathbf{p}(N_1-1,0,1,1,0) & + & (1-p_1)(1-p_2)\,\mathbf{p}(N_1-1,0,1,1,1)
\end{array} \qquad (C.3)$$

3. $2 \le n_2 \le N_2 - 2$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1-1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t))} \\
= & r_1 r_2 (1-r_3)\,\mathbf{p}(N_1-1,n_2-1,0,0,0) & + & r_1 r_2 p_3\,\mathbf{p}(N_1-1,n_2-1,0,0,1) \\
+ & r_1(1-p_2)(1-r_3)\,\mathbf{p}(N_1-1,n_2-1,0,1,0) & + & r_1(1-p_2) p_3\,\mathbf{p}(N_1-1,n_2-1,0,1,1) \\
+ & (1-p_1) r_2 (1-r_3)\,\mathbf{p}(N_1-1,n_2-1,1,0,0) & + & (1-p_1) r_2 p_3\,\mathbf{p}(N_1-1,n_2-1,1,0,1) \\
+ & (1-p_1)(1-p_2)(1-r_3)\mathbf{p}(N_1-1,n_2-1,1,1,0) & + & (1-p_1)(1-p_2) p_3\,\mathbf{p}(N_1-1,n_2-1,1,1,1)
\end{array}$$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1-1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t))} \\
= & r_1 r_2 r_3\,\mathbf{p}(N_1-1,n_2,0,0,0) & + & r_1 r_2 (1-p_3)\,\mathbf{p}(N_1-1,n_2,0,0,1) \\
+ & r_1(1-p_2) r_3\,\mathbf{p}(N_1-1,n_2,0,1,0) & + & r_1(1-p_2)(1-p_3)\,\mathbf{p}(N_1-1,n_2,0,1,1) \\
+ & (1-p_1) r_2 r_3\,\mathbf{p}(N_1-1,n_2,1,0,0) & + & (1-p_1) r_2 (1-p_3)\,\mathbf{p}(N_1-1,n_2,1,0,1) \\
+ & (1-p_1)(1-p_2) r_3\,\mathbf{p}(N_1-1,n_2,1,1,0) & + & (1-p_1)(1-p_2)(1-p_3)\,\mathbf{p}(N_1-1,n_2,1,1,1)
\end{array} \qquad (C.4)$$

4. $n_2 = N_2 - 1$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1-1, \nu_2(t) = N_2-1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t))} \\
= & r_1 r_2 (1-r_3)\,\mathbf{p}(N_1-1,N_2-2,0,0,0) & + & r_1 r_2 p_3\,\mathbf{p}(N_1-1,N_2-2,0,0,1) \\
+ & r_1(1-p_2)(1-r_3)\,\mathbf{p}(N_1-1,N_2-2,0,1,0) & + & r_1(1-p_2) p_3\,\mathbf{p}(N_1-1,N_2-2,0,1,1) \\
+ & (1-p_1) r_2 (1-r_3)\,\mathbf{p}(N_1-1,N_2-2,1,0,0) & + & (1-p_1) r_2 p_3\,\mathbf{p}(N_1-1,N_2-2,1,0,1) \\
+ & (1-p_1)(1-p_2)(1-r_3)\,\mathbf{p}(N_1-1,N_2-2,1,1,0) & + & (1-p_1)(1-p_2) p_3\,\mathbf{p}(N_1-1,N_2-2,1,1,1)
\end{array}$$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1-1, \nu_2(t) = N_2-1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t))} \\
= & r_1 r_2 r_3\,\mathbf{p}(N_1-1,N_2-1,0,0,0) & + & r_1 r_2 (1-p_3)\,\mathbf{p}(N_1-1,N_2-1,0,0,1) \\
+ & r_1(1-p_2) r_3\,\mathbf{p}(N_1-1,N_2-1,0,1,0) & + & r_1(1-p_2)(1-p_3)\,\mathbf{p}(N_1-1,N_2-1,0,1,1) \\
+ & (1-p_1) r_2 r_3\,\mathbf{p}(N_1-1,N_2-1,1,0,0) & + & (1-p_1) r_2 (1-p_3)\,\mathbf{p}(N_1-1,N_2-1,1,0,1) \\
+ & (1-p_1)(1-p_2) r_3\,\mathbf{p}(N_1-1,N_2-1,1,1,0) & + & (1-p_1)(1-p_2)(1-p_3)\,\mathbf{p}(N_1-1,N_2-1,1,1,1) \\
+ & r_1 r_2 r_3\,\mathbf{p}(N_1-2,N_2,0,0,0) & + & r_1 r_2 (1-p_3)\,\mathbf{p}(N_1-2,N_2,0,0,1) \\
+ & r_1 r_3\,\mathbf{p}(N_1-2,N_2,0,1,0) & + & r_1 (1-p_3)\,\mathbf{p}(N_1-2,N_2,0,1,1) \\
+ & (1-p_1) r_2 r_3\,\mathbf{p}(N_1-2,N_2,1,0,0) & + & (1-p_1) r_2 (1-p_3)\,\mathbf{p}(N_1-2,N_2,1,0,1) \\
+ & (1-p_1) r_3\,\mathbf{p}(N_1-2,N_2,1,1,0) & + & (1-p_1)(1-p_3)\,\mathbf{p}(N_1-2,N_2,1,1,1)
\end{array} \qquad (C.5)$$

5. $n_2 = N_2$

$$\mathbf{prob}(\nu_1(t) = N_1 - 1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t)) = 0$$

$$\begin{array}{rrcr}
\multicolumn{4}{l}{\mathbf{prob}(\nu_1(t) = N_1 - 1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t))} \\
= & r_1 r_2 (1-r_3)\, \mathbf{p}(N_1 - 1, N_2 - 1, 0, 0, 0) & + & r_1 r_2 p_3\, \mathbf{p}(N_1 - 1, N_2 - 1, 0, 0, 1) \\
+ & r_1 (1-p_2)(1-r_3)\, \mathbf{p}(N_1 - 1, N_2 - 1, 0, 1, 0) & + & r_1 (1-p_2) p_3\, \mathbf{p}(N_1 - 1, N_2 - 1, 0, 1, 1) \\
+ & (1-p_1) r_2 (1-r_3)\, \mathbf{p}(N_1 - 1, N_2 - 1, 1, 0, 0) & + & (1-p_1) r_2 p_3\, \mathbf{p}(N_1 - 1, N_2 - 1, 1, 0, 1) \\
+ & (1-p_1)(1-p_2)(1-r_3)\, \mathbf{p}(N_1 - 1, N_2 - 1, 1, 1, 0) & + & (1-p_1)(1-p_2) p_3\, \mathbf{p}(N_1 - 1, N_2 - 1, 1, 1, 1) \\
+ & r_1 r_2 (1-r_3)\, \mathbf{p}(N_1 - 2, N_2, 0, 0, 0) & + & r_1 r_2 p_3\, \mathbf{p}(N_1 - 2, N_2, 0, 0, 1) \\
+ & r_1 (1-r_3)\, \mathbf{p}(N_1 - 2, N_2, 0, 1, 0) & + & r_1 p_3\, \mathbf{p}(N_1 - 2, N_2, 0, 1, 1) \\
+ & (1-p_1) r_2 (1-r_3)\, \mathbf{p}(N_1 - 2, N_2, 1, 0, 0) & + & (1-p_1) r_2 p_3\, \mathbf{p}(N_1 - 2, N_2, 1, 0, 1) \\
+ & (1-p_1)(1-r_3)\, \mathbf{p}(N_1 - 2, N_2, 1, 1, 0) & + & (1-p_1) p_3\, \mathbf{p}(N_1 - 2, N_2, 1, 1, 1)
\end{array} \tag{C.6}$$

For $a_1 = 1, a_2 = 0, 1, a_3 = 0, 1, n_1 = N_1, 0 \le n_2 \le N_2$:

1. $n_2 = 0$

$$
\begin{aligned}
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t)) = 0 \\
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t)) = 0 \\
\\
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 0, A(t)) \\
&\quad = \quad r_1(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1,0,0,0,0) \quad + \quad r_1p_2(1-r_3)\,\mathbf{p}(N_1-1,0,0,1,0) \\
&\quad + \quad (1-p_1)(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1,0,1,0,0) \quad + \quad (1-p_1)p_2(1-r_3)\,\mathbf{p}(N_1-1,0,1,1,0) \\
\\
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 1, A(t)) \\
&\quad = \quad r_1(1-r_2)r_3\,\mathbf{p}(N_1-1,1,0,0,0) \quad + \quad r_1(1-r_2)(1-p_3)\,\mathbf{p}(N_1-1,1,0,0,1) \\
&\quad + \quad r_1p_2r_3\,\mathbf{p}(N_1-1,1,0,1,0) \quad + \quad r_1p_2(1-p_3)\,\mathbf{p}(N_1-1,1,0,1,1) \\
&\quad + \quad (1-p_1)(1-r_2)r_3\,\mathbf{p}(N_1-1,1,1,0,1) \quad + \quad (1-p_1)(1-r_2)(1-p_3)\,\mathbf{p}(N_1-1,1,1,0,1) \\
&\quad + \quad (1-p_1)p_2r_3\,\mathbf{p}(N_1-1,1,1,1,0) \quad + \quad (1-p_1)p_2(1-p_3)\,\mathbf{p}(N_1-1,1,1,1,1) \\
&\quad + \quad r_1(1-r_2)r_3\,\mathbf{p}(N_1-1,0,0,0,0) \quad + \quad r_1(1-r_2)\,\mathbf{p}(N_1-1,0,0,0,1) \\
&\quad + \quad r_1p_2r_3\,\mathbf{p}(N_1-1,0,0,1,0) \quad + \quad r_1p_2\,\mathbf{p}(N_1-1,0,0,1,1) \\
&\quad + \quad (1-p_1)(1-r_2)r_3\,\mathbf{p}(N_1-1,0,1,0,0) \quad + \quad (1-p_1)(1-r_2)\,\mathbf{p}(N_1-1,0,1,0,1) \\
&\quad + \quad (1-p_1)p_2r_3\,\mathbf{p}(N_1-1,0,1,1,0) \quad + \quad (1-p_1)p_2\,\mathbf{p}(N_1-1,0,1,1,1)
\end{aligned}
\tag{C.7}
$$

2. $1 \le n_2 \le N_2 - 2$

$$
\begin{aligned}
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t)) = 0 \\
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t)) = 0 \\
\\
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 0, A(t)) \\
&\quad = \quad r_1(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1,n_2,0,0,0) \quad + \quad r_1(1-r_2)p_3\,\mathbf{p}(N_1-1,n_2,0,0,1) \\
&\quad + \quad r_1p_2(1-r_3)\,\mathbf{p}(N_1-1,n_2,0,1,0) \quad + \quad r_1p_2p_3\,\mathbf{p}(N_1-1,n_2,0,1,1) \\
&\quad + \quad (1-p_1)(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1,n_2,1,0,0) \quad + \quad (1-p_1)(1-r_2)p_3\,\mathbf{p}(N_1-1,n_2,1,0,1) \\
&\quad + \quad (1-p_1)p_2(1-r_3)\,\mathbf{p}(N_1-1,n_2,1,1,0) \quad + \quad (1-p_1)p_2p_3\,\mathbf{p}(N_1-1,n_2,1,1,1) \\
\\
&\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 1, A(t)) \\
&\quad = \quad r_1(1-r_2)r_3\,\mathbf{p}(N_1-1,n_2+1,0,0,0) \quad + \quad r_1(1-r_2)(1-p_3)\,\mathbf{p}(N_1-1,n_2+1,0,0,1) \\
&\quad + \quad r_1p_2r_3\,\mathbf{p}(N_1-1,n_2+1,0,1,0) \quad + \quad r_1p_2(1-p_3)\,\mathbf{p}(N_1-1,n_2+1,0,1,1) \\
&\quad + \quad (1-p_1)(1-r_2)r_3\,\mathbf{p}(N_1-1,n_2+1,1,0,0) \quad + \quad (1-p_1)(1-r_2)(1-p_3)\,\mathbf{p}(N_1-1,n_2+1,1,0,1) \\
&\quad + \quad (1-p_1)p_2r_3\,\mathbf{p}(N_1-1,n_2+1,1,1,0) \quad + \quad (1-p_1)p_2(1-p_3)\,\mathbf{p}(N_1-1,n_2+1,1,1,1)
\end{aligned}
\tag{C.8}
$$

3. $n_2 = N_2 - 1$

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2 - 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t)) = 0$$

$$\begin{array}{lrlrl}
\multicolumn{5}{l}{\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2 - 1, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 0, A(t))} \\
= & r_1(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1, N_2-1, 0, 0, 0) & + & r_1(1-r_2)p_3\,\mathbf{p}(N_1-1, N_2-1, 0, 0, 1) \\
+ & r_1p_2(1-r_3)\,\mathbf{p}(N_1-1, N_2-1, 0, 1, 0) & + & r_1p_2p_3\,\mathbf{p}(N_1-1, N_2-1, 0, 1, 1) \\
+ & (1-p_1)(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1, N_2-1, 1, 0, 0) & + & (1-p_1)(1-r_2)p_3\,\mathbf{p}(N_1-1, N_2-1, 1, 0, 1) \\
+ & (1-p_1)p_2(1-r_3)\,\mathbf{p}(N_1-1, N_2-1, 1, 1, 0) & + & (1-p_1)p_2p_3\,\mathbf{p}(N_1-1, N_2-1, 1, 1, 1)
\end{array}$$

$$\begin{array}{lrlrl}
\multicolumn{5}{l}{\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2 - 1, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 1, A(t))} \\
= & r_1(1-r_2)r_3\,\mathbf{p}(N_1-1, N_2, 0, 0, 0) & + & r_1(1-r_2)(1-p_3)\,\mathbf{p}(N_1-1, N_2, 0, 0, 1) \\
+ & (1-p_1)(1-r_2)r_3\,\mathbf{p}(N_1-1, N_2, 1, 0, 0) & + & (1-p_1)(1-r_2)(1-p_3)\,\mathbf{p}(N_1-1, N_2, 1, 0, 1)
\end{array}$$

$$\begin{array}{lrlrl}
\multicolumn{5}{l}{\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2 - 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t))} \\
= & r_1r_2r_3\,\mathbf{p}(N_1-1, N_2, 0, 0, 0) & + & r_1r_2(1-p_3)\,\mathbf{p}(N_1-1, N_2, 0, 0, 1) \\
+ & r_1r_3\,\mathbf{p}(N_1-1, N_2, 0, 1, 0) & + & r_1(1-p_3)\,\mathbf{p}(N_1-1, N_2, 0, 1, 1) \\
+ & (1-p_1)r_2r_3\,\mathbf{p}(N_1-1, N_2, 1, 0, 0) & + & (1-p_1)r_2(1-p_3)\,\mathbf{p}(N_1-1, N_2, 1, 0, 1) \\
+ & (1-p_1)r_3\,\mathbf{p}(N_1-1, N_2, 1, 1, 0) & + & (1-p_1)(1-p_3)\,\mathbf{p}(N_1-1, N_2, 1, 1, 1)
\end{array} \tag{C.9}$$

4. $n_2 = N_2$

$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 1, A(t)) = 0$$
$$\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, A(t)) = 0$$

$$\begin{array}{lrlrl}
\multicolumn{5}{l}{\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 0, \alpha_3(t) = 0, A(t))} \\
= & r_1(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1, N_2, 0, 0, 0) & + & r_1(1-r_2)p_3\,\mathbf{p}(N_1-1, N_2, 0, 0, 1) \\
+ & (1-p_1)(1-r_2)(1-r_3)\,\mathbf{p}(N_1-1, N_2, 1, 0, 0) & + & (1-p_1)(1-r_2)p_3\,\mathbf{p}(N_1-1, N_2, 1, 0, 1)
\end{array}$$

$$\begin{array}{lrlrl}
\multicolumn{5}{l}{\mathbf{prob}(\nu_1(t) = N_1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 0, A(t))} \\
= & r_1r_2(1-r_3)\,\mathbf{p}(N_1-1, N_2, 0, 0, 0) & + & r_1r_2p_3\,\mathbf{p}(N_1-1, N_2, 0, 0, 1) \\
+ & r_1(1-r_3)\,\mathbf{p}(N_1-1, N_2, 0, 1, 0) & + & r_1p_3\,\mathbf{p}(N_1-1, N_2, 0, 1, 1) \\
+ & (1-p_1)r_2(1-r_3)\,\mathbf{p}(N_1-1, N_2, 1, 0, 0) & + & (1-p_1)r_2p_3\,\mathbf{p}(N_1-1, N_2, 1, 0, 1) \\
+ & (1-p_1)(1-r_3)\,\mathbf{p}(N_1-1, N_2, 1, 1, 0) & + & (1-p_1)p_3\,\mathbf{p}(N_1-1, N_2, 1, 1, 1)
\end{array} \tag{C.10}$$

D Recurrence Equations for $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ from Shi and Gershwin (2016)

Shi and Gershwin (2016) derived the lead time distribution for two-machine one-buffer lines. They showed that for $w < x_2$, $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ are 0; and for $w \geq x_2$, $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ are found using the following recurrence equations:

- $w = 1$

$$\pi^1(1, 1) = 1 - p_3, \tag{D.1}$$
$$\pi^0(1, 1) = r_3, \tag{D.2}$$

- $2 \le w \le N_2$

$$\begin{aligned}
\pi^1(w,1) &= p_3\pi^0(w-1,1), && (D.3)\\
\pi^0(w,1) &= (1-r_3)\pi^1(w-1,1), && (D.4)\\
\pi^1(w,x_2) &= p_3\pi^0(w-1,x_2)+(1-p_3)\pi^1(w-1,x_2-1), \quad 2 \le x_2 \le w-1, && (D.5)\\
\pi^0(w,x_2) &= r_3\pi^1(w-1,x_2-1)+(1-r_3)\pi^0(w-1,x_2), \quad 2 \le x_2 \le w-1, && (D.6)\\
\pi^1(w,x_2) &= (1-p_3)\pi^1(w-1,x_2-1), \quad x_2 = w, && (D.7)\\
\pi^0(w,x_2) &= r_3\pi^1(w-1,x_2-1), \quad x_2 = w, && (D.8)
\end{aligned}$$

- $w > N_2$

$$\begin{aligned}
\pi^1(w,1) &= p_3\pi^0(w-1,1), && (D.9)\\
\pi^0(w,1) &= (1-r_3)\pi^0(w-1,1), && (D.10)\\
\pi^1(w,x_2) &= p_3\pi^0(w-1,x_2)+(1-p_3)\pi^1(w-1,x_2-1), \quad 2 \le x_2 \le N_2, && (D.11)\\
\pi^0(w,x_2) &= r_3\pi^1(w-1,x_2-1)+(1-r_3)\pi^0(w-1,x_2), \quad 2 \le x_2 \le N_2. && (D.12)
\end{aligned}$$

E Algorithms to Find the Lead Time Distribution

Algorithm 1: Find prob$(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)), a_2 = 0, 1, a_3 = 0, 1, 1 \le x_1 \le N_1, 0 \le n_2 \le N_2$

Step 1: Find $\pi^1(w, x_2)$ and $\pi^0(w, x_2)$ according to Appendix D
Step 2: Compute initial conditions according to (10), and set $x_1 = 1$
Step 3: **for** $\tau \ge 3$ **do**
    for $n_2 = 0$ *to* $\min(N_2, \tau - 1)$ **do**
        if $n_2 = 0$ **then**
            evaluate (B.1)
        else if $n_2 = 1$ **then**
            evaluate (B.2)
        else if $n_2 = N_2$ **then**
            evaluate (B.4)
        else
            evaluate (B.3)
        end
    end
end
Step 4: **for** $x_1 = 2$ *to* N_1 **do**
    for $\tau \ge x_1 + 1$ **do**
        for $n_2 = 0$ *to* $\min(N_2, \tau - x_1)$ **do**
            if $n_2 = 0$ **then**
                evaluate (A.1)
            else if $n_2 = 1$ **then**
                evaluate (A.2)
            else if $n_2 = N_2 - 1$ **then**
                evaluate (A.4)
            else if $n_2 = N_2$ **then**
                evaluate (A.5)
            else
                evaluate (A.3)
            end
        end
    end
end
Step 5: End

Algorithm 2: Find prob$(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t)), a_2 = 0,1; a_3 = 0,1; 1 \leq x_1 \leq N_1; 0 \leq n_2 \leq N_2$

Step 1: Find the steady-state probabilities $\mathbf{p}(n_1, n_2, a_1, a_2, a_3)$, $0 \leq n_1 \leq N_1$; $0 \leq n_2 \leq N_2$; $a_1 = 0,1$; $a_2 = 0,1$; $a_3 = 0,1$, and the production rate P of the three-machine two-buffer line using the exact numeric solution of Tan (2003)

Step 2: **for** $n_1 = 1$ *to* N_1 **do**
  for $n_2 = 0$ *to* N_2 **do**
    if $n_1 = N_1 - 1$, $n_2 = 0$ *and* $a_2 = 1$ **then**
      determine **prob**$(\nu_1(t) = N_1 - 1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t))$ from (C.2)
    else if $n_1 = N_1 - 1$, $n_2 = 1$ *and* $a_2 = 1$ **then**
      determine **prob**$(\nu_1(t) = N_1 - 1, \nu_2(t) = 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t))$ from (C.3)
    else if $n_1 = N_1 - 1$, $2 \leq n_2 \leq N_2 - 2$ *and* $a_2 = 1$ **then**
      determine **prob**$(\nu_1(t) = N_1 - 1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t))$ from (C.4)
    else if $n_1 = N_1 - 1$, $n_2 = N_2 - 1$ *and* $a_2 = 1$ **then**
      determine **prob**$(\nu_1(t) = N_1 - 1, \nu_2(t) = N_2 - 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t))$ from (C.5)
    else if $n_1 = N_1 - 1$, $n_2 = N_2$ *and* $a_2 = 1$ **then**
      determine **prob**$(\nu_1(t) = N_1 - 1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = a_3, A(t))$ from (C.6)
    else if $n_1 = N_1$ *and* $n_2 = 0$ **then**
      determine **prob**$(\nu_1(t) = N_1, \nu_2(t) = 0, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ from (C.7)
    else if $n_1 = N_1$ *and* $1 \leq n_2 \leq N_2 - 2$ **then**
      determine **prob**$(\nu_1(t) = N_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ from (C.8)
    else if $n_1 = N_1$ *and* $n_2 = N_2 - 1$ **then**
      determine **prob**$(\nu_1(t) = N_1, \nu_2(t) = N_2 - 1, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ from (C.9)
    else if $n_1 = N_1$ *and* $n_2 = N_2$ **then**
      determine **prob**$(\nu_1(t) = N_1, \nu_2(t) = N_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ from (C.10)
    else
      determine **prob**$(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ from (C.1)
    end
  end
end

Step 3: Find **prob**$(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t))$ from (12)

Step 4: End

Algorithm 3: Find $\mathbf{prob}(T = \tau)$

Step 1: Determine $\mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t))$,
$a_2 = 0, 1; a_3 = 0, 1; 1 \leq x_1 \leq N_1; 0 \leq n_2 \leq N_2$ according to Algorithm 2
Step 2: Set $\tau = 2$
Step 3:
Step 3a: Determine $\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$,
$a_2 = 0, 1; a_3 = 0, 1; 1 \leq x_1 \leq N_1; 0 \leq n_2 \leq N_2$, by recurrence according to Algorithm 1
Step 3b: Determine $\mathbf{prob}(T = \tau)$ from (4)
Step 3c: Evaluate the stopping criterion
if *the stopping criterion is satisfied* **then**
| go to Step 4
else
| Set $\tau = \tau + 1$ and go back to the beginning of Step 3
end

Step 4: End

References

Ayhan, H., Z. Palmowski, and S. Schlegel (2004). Cyclic queueing networks with subexponential service times. *Journal of Applied Probability 41*(3), 791–801.

Azaron, A., H. Katagiri, K. Kato, and M. Sakawa (2006). Modelling complex assemblies as a queueing network for lead time control. *European Journal of Operational Research 174*(1), 150–168.

Bertsekas, D. P. and J. N. Tsitsiklis (2008). *Introduction to Probability, 2nd Edition.* Athena Scientific.

Biller, S., S. M. Meerkov, and C.-B. Yan (2013). Raw material release rates to ensure desired production lead time in bernoulli serial lines. *International Journal of Production Research 51*(14), 4349–4364.

Chow, W.-M. (1980). The cycle time distribution of exponential cyclic queues. *Journal of the Association for Computing Machinery 27*(2), 281–286.

Colledani, M., A. Angius, and A. Horvath (2014). Lead time distribution in unreliable production lines processing perishable products. In *IEEE Conference on Emerging Technology and Factory Automation*, Barcelona, Spain, pp. 1–8.

Colledani, M., A. Angius, A. Horvath, and S. B. Gershwin (2015). Lead time dependent product deterioration in manufacturing systems with serial, assembly and closed-loop layout. In *10th Conference on Stochastic Models of Manufacturing and Service Operations (SMMSO 2015)*, Volos, Greece.

Gershwin, S. B. (1994). *Manufacturing Systems Engineering.* Prentice-Hall. Currently available at `http://home.comcast.net/~hierarchy/MSE/mse.html`. For corrections, see `http://web.mit.edu/manuf-sys/www/gershwin.errata.html`.

Gershwin, S. B. and I. C. Schick (1983). Modeling and analysis of three stage transfer lines with unreliable machines and finite buffers. *Operations Research 31*(2), 354–377.

Lagershausen, S. and B. Tan (2015). On the exact inter-departure, inter-start, and cycle time distribution of closed queueing networks subject to blocking. *IIE Transactions 47*(7), 673–692.

Leemans, H. (2001). Waiting time distribution in a two-class two-server heterogeneous priority queue. *Performance Evaluation 43*(2-3), 133 – 150.

Little, J. D. C. (1961). A proof of the queueing formula $L = \lambda W$. *Operations Research 9*, 383–387.

Meerkov, S. M. and C.-B. Yan (2014). Production lead time in serial lines: Evaluation, analysis, and control. *IEEE Transactions on Automation Science and Engineering PP*(99), 1–13.

Shi, C. (2012). *Efficient buffer design algorithms for production line profit maximization.* Ph. D. thesis, Massachusetts Institute of Technology Department of Mechanical Engineering.

Shi, C. and S. B. Gershwin (2009). An efficient buffer design algorithm for production line profit maximization. *International Journal of Production Economics 122*(2), 725–740.

Shi, C. and S. B. Gershwin (2013). Qualitative behavior of average buffer levels. In *IX Conference on Stochastic Models of Manufacturing and Service Operations*, Kloster Seeon, Germany, pp. 197–204.

Shi, C. and S. B. Gershwin (2016). Part sojourn time distribution in a two-machine line. *European Journal of Operational Research 248*(1), 146 – 158.

Tan, B. (2003). State-space modeling and analysis of pull-controlled production systems. In S. B. Gershwin, Y. Dallery, C. T. Papadopoulos, and J. M. Smith (Eds.), *Analysis and Modeling of Manufacturing Systems*, Volume 60 of *International Series in Operations Research & Management Science*, pp. 363–398. Springer US.

Wu, K. and L. McGinnis (2012). Performance evaluation for general queueing networks in manufacturing systems: Characterizing the trade-off between queue time and utilization. *European Journal of Operational Research 221*(2), 328 – 339.