Design of a lane departure driver-assist system under safety specifications: Theorems and Proofs

Daniel Hoehener* Geng Huang† Domitilla Del Vecchio‡

Version 1.0
March 1, 2016

Abstract

This note contains full proofs of all the results provided in the submission "On a semi-autonomous lane departure assist system" to the 2016 Conference on Decision and Control by the same authors, [2].

Introduction

This note is a complement to [2] and its main objective is to provide full proofs of the results presented in the mentioned paper. Unless stated otherwise all the notations are as in [2] and also the numbers of theorems refer to that paper. Additional material is provided in the Appendix.

Main results

Theorem 3.1. *Let* $x^0 = (U^0, V^0, r^0, \psi^0, d_l^0, d_r^0) \in \mathcal{W}_+ \cap \mathcal{W}_-$ *be such that*

(i) $(U^0, V^0, r^0) \in [U_{min}, U_{max}] \times [-\tilde{V}, \tilde{V}] \times [-\tilde{r}, \tilde{r}]$;

(ii) $\psi^0 \in]-\tilde{\psi}, \tilde{\psi}[$.

Define $\pi^s \colon X \rightsquigarrow [-\bar{\tau}_w, \bar{\tau}_w] \times [-\bar{\delta}_f, \bar{\delta}_f]$ *by*

$$\pi^s(x) := \begin{cases} (\tau_w^s, \bar{\delta}_f) & \text{if } x \in \overline{\mathcal{W}_+^c}, \\ (\tau_w^s, -\bar{\delta}_f) & \text{if } x \in \overline{\mathcal{W}_-^c} \setminus \overline{\mathcal{W}_+^c}, \\ [-\bar{\tau}_w, \bar{\tau}_w] \times [-\bar{\delta}_f, \bar{\delta}_f] & \text{otherwise.} \end{cases}$$

Then the corresponding flow satisfies for every $t \in \mathbb{R}_+$ *such that* $\mathbf{x}_1(s; \pi^s, x^0) \in [U_{min}, U_{max}]$ *for all* $s \in [0, t]$,

$$\mathbf{x}(t; \pi^s, x^0) \in \mathcal{S} \qquad \text{and} \qquad \mathbf{x}_4(t; \pi^s, x^0) \in]-\tilde{\psi}, \tilde{\psi}[.$$

*Department of Mechanical Engineering, MIT, 77 Massachusetts Avenue, Cambridge, MA
†Department of Mechanical Engineering, MIT, 77 Massachusetts Avenue, Cambridge, MA
‡Department of Mechanical Engineering, MIT, 77 Massachusetts Avenue, Cambridge, MA

Proof. The proof requires some technical results that are provided in Appendix A. Fix an arbitrary x^0 satisfying the assumptions of Theorem 3.1. It suffices to show that $\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_+ \cap \mathcal{W}_-$ and $\hat{\mathbf{x}}_4(t;\pi^s,x^0) \in]-\tilde{\psi},\tilde{\psi}[$ for all $t \in \mathbb{R}_+$ such that

$$\hat{\mathbf{x}}_1(s;\pi^s,x^0) \in [U_{min},U_{max}] \qquad \forall s \in [0,t]. \tag{1}$$

It will be convenient to denote the set of times $t \in \mathbb{R}_+$ for which (1) is satisfied by T. We start by showing that

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_+ \qquad \forall t \in T.$$

Assume to the contrary that there exists $t_1 \in T$ such that

$$x_1 \coloneqq \hat{\mathbf{x}}(t_1;\pi^s,x^0) \in \mathcal{W}_+^c. \tag{2}$$

Lemmas A.1 and A.2 state that in this case there exists $t_2 \in [0,t_1[\subset T$ satisfying

$$x_2 \coloneqq \hat{\mathbf{x}}(t_2;\pi^s,x^0) \in \partial\mathcal{W}_+ \subset \mathcal{W}_+, \tag{3}$$

and for all $t \in [t_2,t_1]$,

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \overline{\mathcal{W}_+^c}.$$

Thus from the very definition of π^s and (3) it follows that

$$x_1 = \hat{\mathbf{x}}(t_1 - t_2;\bar{\mathbf{d}}_f,x_2) \in \mathcal{W}_+,$$

contradicting (2). Next, let us define

$$t^* \coloneqq \inf\left\{t \in T \;\middle|\; \hat{\mathbf{x}}_4(t;\pi^s,x^0) \in \{-\tilde{\psi},\tilde{\psi}\}\right\},$$

where we set $t^* = \infty$ if this set is empty. We show that for all $t \in [0,t^*[\cap T$,

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_-, \tag{4}$$

and that this in turn implies that $t^* = \infty$ which establishes the statement of the theorem.

Let us start by showing that (4) holds for all $t \in [0,t^*[\cap T$. We argue again by contradiction and assume that there exists $t_3 \in [0,t^*[$ such that $\hat{\mathbf{x}}(t_3;\pi^s,x^0) \in \mathcal{W}_-^c$. Applying as before Lemmas A.1 and A.2 there exists $t_4 \in [0,t_3[$ such that

$$x_4 \coloneqq \hat{\mathbf{x}}(t_4;\pi^s,x^0) \in \partial\mathcal{W}_- \subset \mathcal{W}_-,$$

and for all $t \in]t_4,t_3]$,

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \mathcal{W}_-^c. \tag{5}$$

Next we define

$$t_5 \coloneqq \min\left\{t \in [t_4,t_3] \;\middle|\; \hat{\mathbf{x}}(t;\pi^s,x^0) \in \partial W_+\right\},$$

where we use the convention that $t_5 = t_3$ if the set $\left\{t \in [t_4,t_3] \;\middle|\; \hat{\mathbf{x}}(t;\pi^s,x^0) \in \partial W_+\right\}$ is empty. Notice also that this set is compact, thus if it is not empty, then this minimum is well-defined. If $t_5 > t_4$, then for all $t \in [t_4,(t_5+t_4)/2]$

$$\hat{\mathbf{x}}(t;\pi^s,x^0) \in \overline{\mathcal{W}_-^c} \setminus \overline{\mathcal{W}_+^c}.$$

A similar argument as above allows then to conclude that $\hat{\mathbf{x}}((t_5+t_4)/2;\pi^s,x^0)\in\mathcal{W}_-$, contradicting (5). It remains to consider the case $t_4=t_5$, that is $x_4\in\partial\mathcal{W}_+\cap\partial\mathcal{W}_-$. We will show that in fact $x_4\in\mathcal{W}_{0+}\cap\mathcal{W}_{0-}$ which then by Condition 2 establishes the contradiction.

Since $t_4<t^*$, it is clear that

$$\hat{\mathbf{x}}_4(t_4;\pi^s,x^0)\in\,]-\tilde{\psi},\tilde{\psi}[. \tag{6}$$

Thus, by the characterization of the boundaries of $\mathcal{W}_+$ and $\mathcal{W}_-$ given in Proposition A.6 and Lemma A.4 there exist $t_+\in\mathcal{T}_+(x_4)$ and $t_-\in\mathcal{T}_-(x_4)$ satisfying

$$\hat{\mathbf{x}}_6(t_+;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)=\hat{\mathbf{x}}_5(t_-;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)=0.$$

Hence, if $t_+>0$ or $t_->0$ then the maps $t\mapsto\hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)$ and $t\mapsto\hat{\mathbf{x}}_5(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)$ have a local minimum in t_+, respectively t_-. This establishes that in this case $\dot{\hat{\mathbf{x}}}_6(t_+;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)=0$, respectively $\dot{\hat{\mathbf{x}}}_5(t_-;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)=0$. It remains to consider the cases when either t_+ or t_- equals zero. Let us start with the case when $t_+=0$. Since $x_4\in\mathcal{W}_+$, $\hat{\mathbf{x}}(t;\pi^s,x^0)\in\mathcal{W}_+$ for all $t\in\mathbb{R}_+$ and (6) holds true, there exists $\epsilon>0$ such that

$$\begin{aligned}\hat{\mathbf{x}}_6(t;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)&\geq 0\quad\forall t\in[0,\epsilon],\\ \hat{\mathbf{x}}_6(t;\pi^s,x^0)&\geq 0\quad\forall t\in[t_4-\epsilon,t_4].\end{aligned}$$

This yields

$$0\leq\dot{\hat{\mathbf{x}}}_6(0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)=\dot{\hat{\mathbf{x}}}_6(t_4;\pi^s,x^0)\leq 0.$$

Consequently, $\dot{\hat{\mathbf{x}}}_6(0;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x_4)=0$ as desired.

Consider now the case when $t_-=0$, that is $\hat{\mathbf{x}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)=0$. By (5), for every $n\in\mathbb{N}$ large enough,

$$x^n:=\hat{\mathbf{x}}(1/n;\pi^s,x_4)\in\mathcal{W}_-^c.$$

That is, there exists $t^n\in\mathcal{T}_-(x^n)$ such that

$$\hat{\mathbf{x}}_5(t^n,\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n)<0. \tag{7}$$

By continuity and Lemma A.3 it is clear that the sequence $(t^n,x^n)_{n\in\mathbb{N}}$ is contained in a compact set. Consequently there exists a convergent subsequence, for convenience still denoted by $(t^n,x^n)_n$. Its limit is denoted by $(\bar{t},\bar{x})$. Clearly $\bar{x}=x_4$ and by continuity of the flows

$$\hat{\mathbf{x}}_4(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)\leq\tilde{\psi}\quad\text{and}\quad\hat{\mathbf{x}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)\leq 0.$$

In fact, since $x_4\in\mathcal{W}_-$ and $\hat{\mathbf{x}}_4(t_4;\pi^s,x^0)>-\tilde{\psi}$, it follows from Lemma A.4 and Corollary A.5 that in fact

$$\hat{\mathbf{x}}_4(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)<\tilde{\psi}\quad\text{and}\quad\hat{\mathbf{x}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)=0. \tag{8}$$

We show that $\dot{\hat{\mathbf{x}}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)=0$. This is clear if $\bar{t}>0$ because $\hat{\mathbf{x}}_5(\bar{t};\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)$ has a local minimum in $\bar{t}$ in this case. We can therefore assume that $\bar{t}=0$. Moreover, since it follows readily from $x_4\in\mathcal{W}_-$ and (8) that $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)\geq 0$, it suffices to exclude the case $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)>0$. Indeed, if $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4)>0$ then there exist $\eta>0$ and $\epsilon>0$ such that

$$\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x)\geq\eta\qquad\forall x\in B(x_4,\epsilon).$$

By continuity of $t \mapsto \hat{\mathbf{x}}(t;\pi^s, x_4)$ there exists $\epsilon' > 0$ such that for all $t \in [0,\epsilon']$, $\hat{\mathbf{x}}(t;\pi^s,x^4) \in B(x_4,\epsilon/2)$. Similarly, the continuous function $(t,x) \mapsto \hat{\mathbf{x}}(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x)$ is uniformly continuous on the compact set $[0,1] \times \overline{B(x_4,1)}$. This implies in particular that for some $\epsilon'' > 0$ we have that for all $x \in \overline{B(x_4,1)}$ and all $t_1,t_2 \in [0,1]$ satisfying $|t_1 - t_2| \leq \epsilon''$,

$$\left\|\hat{\mathbf{x}}(t_1;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x) - \hat{\mathbf{x}}(t_2;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x)\right\| \leq \frac{\epsilon}{2}.$$

Using convergence of the sequence $t^n \to 0$ when $n \to \infty$, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} \leq \epsilon' \quad \text{and} \quad t^n \leq \epsilon''.$$

It is then easy to check that for all $s \in [0,1/n]$ and all $t \in [0,t^n]$,

$$\hat{\mathbf{x}}(s;\pi^s,x^4) \in B(x_4,\epsilon/2) \quad \text{and} \quad \hat{\mathbf{x}}(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n) \in B(x_4,\epsilon).$$

Finally, noticing that $x^n = \hat{\mathbf{x}}(1/n;\pi^s,x^4)$ we find

$$\begin{aligned}
\hat{\mathbf{x}}_5(t^n;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n) &= \hat{\mathbf{x}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n) + \int_0^{t^n} \dot{\hat{\mathbf{x}}}_5(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n)dt \\
&= \hat{\mathbf{x}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) + \int_0^{1/n} \hat{\mathbf{x}}(t;\pi^s,x^4)dt + \int_0^{t^n} \dot{\hat{\mathbf{x}}}_5(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x^n)dt \\
&\geq \left(\frac{1}{n} + t^n\right)\eta > 0.
\end{aligned}$$

This however contradicts with (7). Hence $\dot{\hat{\mathbf{x}}}_5(0;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x_4) = 0$ in the case when $\bar{t} = 0$. The proof of (4) is complete.

It remains to deduce that $t^* = \infty$. Indeed, if $t^* < \infty$, then by (4)

$$\min\left\{\hat{\mathbf{x}}_6(t^*;\pi^s,x^0), \hat{\mathbf{x}}_5(t^*;\pi^s,x^0)\right\} \geq 0.$$

Furthermore, since by Lemma A.2 the sets $\mathcal{W}_+$ and $\mathcal{W}_-$ are closed, it is also clear that

$$\hat{\mathbf{x}}(t^*;\pi^s,x^0) \in \mathcal{W}_+ \cap \mathcal{W}_-,$$

which however by [2, Condition 3] contradicts with the fact that $\hat{\mathbf{x}}_4(t^*;\pi^s,x^0) \in \{-\tilde{\psi},\tilde{\psi}\}$. The proof is complete. □

Proposition 3.2. *If $\mathcal{V}_1 > 0$ then $\mathcal{W}_{0+} \cap \mathcal{W}_{0-} = \emptyset$.*

Proof. Clearly it is equivalent to show that if $\mathcal{W}_{0+} \cap \mathcal{W}_{0-} \neq \emptyset$ then $\mathcal{V}_1 = 0$. For this let $x \in \mathcal{W}_{0+} \cap \mathcal{W}_{0-}$. By the very definition of these sets, see [2, Eq. (3)], it follows that there exist $s \in \overline{\mathcal{T}_+(x)}$ and $t \in \overline{\mathcal{T}_-(x)}$ such that

$$\begin{aligned}
\hat{\mathbf{x}}_6(s;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) &= \hat{\mathbf{x}}_5(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x)) = 0 \\
\dot{\hat{\mathbf{x}}}_6(s;\bar{\mathbf{t}}_w,\bar{\mathbf{d}}_f,x) &= \dot{\hat{\mathbf{x}}}_5(t;\bar{\mathbf{t}}_w,-\bar{\mathbf{d}}_f,x) = 0.
\end{aligned}$$

Clearly, (s,t,x) is a feasible point of the optimization problem [2, Eq. (5)] achieving the lower bound 0. This allows to conclude that $\mathcal{V}_1 = 0$ and completes the proof. □

Proposition 3.3. *If $\mathcal{V}_2 > 0$ then for all $x \in X$ such that $\psi = -\tilde{\psi}$ and $d_r \leq W_\ell / \cos(\tilde{\psi})$ we have that $x \in \mathcal{W}_+^c$.*

Proof. It is equivalent to show that if there exits $x = (U, V, r, \psi, d_l, d_r) \in X$ such that $\psi = -\tilde{\psi}$ and $d_r \leq W_\ell \cos(\tilde{\psi})$ satisfying $x \in \mathcal{W}_+$ then $\mathcal{V}_2 \leq 0$. Let x be as above. As we discuss in more detail in Appendix A, see in particular the proof of Proposition A.6, the map $x \mapsto \hat{\mathbf{x}}_6(\cdot; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ is such that if $x_1 = (U, V, r, \psi)$ and $d_r^1 \leq d_r^2$ then for all $t \in \mathbb{R}_+$, we have

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, (x_1, d_l^1, d_r^1)) \leq \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, (x_1, d_l^2, d_r^2)) \qquad \forall t \in \mathbb{R}_+. \tag{9}$$

Since $x = (x_1, d_l, d_r) \in \mathcal{W}_+$, $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0$ for all $t \in \mathcal{T}_+(x)$. Furthermore, as $d_r \leq W_\ell / \cos(\tilde{\psi})$, using (9) and defining $\tilde{x} = (x_1, 0, W_\ell / \cos(\tilde{\psi}))$ it follows that

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \tilde{x})) \geq 0 \qquad \forall t \in \mathcal{T}_+(\tilde{x}) = \mathcal{T}_+(x).$$

Finally, using that by Lemma A.3 the set $\mathcal{T}_+(\tilde{x})$ is bounded, it follow from the intermediate value theorem that there exists $\bar{t} \in \mathbb{R}_+$ for which $\hat{\mathbf{x}}_4(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \tilde{x}) = 0$. Consequently $\mathcal{V}_2 \leq 0$. ☐

Appendix

A Some properties of the sets $\mathcal{W}_+$ and $\mathcal{W}_-$

In this section we provide some technical results used in the proof of the safety result Theorem 3.1.

Lemma A.1. *Let $A \subset \mathbb{R}^n$ be a closed set and $\phi\colon \mathbb{R}_+ \to \mathbb{R}^n$ be continuous. Let $\phi(0) \in A$ and assume there exists $\bar{t} \in \mathbb{R}_+$ such that $\phi(\bar{t}) \in A^c$. Then there exists $t^* \in [0, \bar{t}[$ such that $\phi(t^*) \in \partial A$ and $\phi(t) \in A^c$ for all $t \in]t^*, \bar{t}]$.*

Proof. The set $\mathcal{T} \coloneqq \phi^{-1}(A) \cap [0, \bar{t}]$ is compact by continuity of ϕ and does not contain $\bar{t}$. Defining $t^* \coloneqq \max \mathcal{T}$, we observe that $\phi(t^*) \in A$, $t^* < \bar{t}$ and $\phi(t) \in A^c$ for all $t \in]t^*, \bar{t}]$. It remains to show that $\phi(t^*) \in \partial A$.

For this we show that for every $\epsilon > 0$ there exists $y \in B(\phi(t^*), \epsilon) \cap A^c$. Let $\epsilon > 0$ be arbitrary. By continuity of ϕ there exists $\delta > 0$ such that for all $t \in]t^* - \delta, t^* + \delta[$ we have $|\phi(t^*) - \phi(t)| \leq \epsilon$. Thus it follows that $y \coloneqq \phi(t^* + \delta/2) \in B(\phi(t^*), \epsilon) \cap A^c$. The proof is complete. ☐

We use the statement of Lemma A.1 to prove properties of the sets $\mathcal{W}_+$ and $\mathcal{W}_-$. This, in particular can be done since these sets are closed by the following result.

Lemma A.2. *The sets $\mathcal{W}_+$ and $\mathcal{W}_-$ are closed.*

Proof. We show that $\mathcal{W}_+$ is closed, the proof for $\mathcal{W}_-$ is analogous. For all $t \in \mathbb{R}_+$ let us define

$$\mathcal{W}_+(t) = \left\{ x \;\middle|\; \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0 \wedge \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < \tilde{\psi} \right\} \cup \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi} \right\}. \tag{10}$$

It follows from the very definition that $\mathcal{W}_+ = \bigcap_{t \in \mathbb{R}_+} \mathcal{W}_+(t)$. Thus it suffices to show that for all $t \in \mathbb{R}_+$, $\mathcal{W}_+(t)$ is closed. This is however clear since (10) can be written as

$$\begin{aligned} \mathcal{W}_+(t) &= \left(\left\{ x \mid \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0 \right\} \cap \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < \tilde{\psi} \right\} \right) \cup \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi} \right\} \\ &= \left\{ x \mid \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)) \geq 0 \right\} \cup \left\{ x \;\middle|\; \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi} \right\}. \end{aligned}$$

$\square$

The rest of the results are used to characterize the boundary of the sets $\mathcal{W}_+$ and $\mathcal{W}_-$. We start with some preliminary results.

Lemma A.3. *There exists a constant $K > 0$ such that for all $x = (U, V, r, \psi, d_l, d_r) \in X$ with $\psi \in [-\tilde{\psi}, \tilde{\psi}]$ we have*

$$\mathcal{T}_+(x) \subset B(0, K) \qquad \textit{and} \qquad \mathcal{T}_-(x) \subset B(0, K).$$

Proof. Let $x = (U, V, r, \psi, d_l, d_r) \in X$ be as in the statement of the lemma. By the system dynamics [2, Eq. 1] it is clear that for all $t \in \mathbb{R}_+$ we have

$$\dot{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \mathbf{d}_f, x) = \mathbf{x}_3(t; \bar{\mathbf{t}}_w, \mathbf{d}_f, x) \qquad \forall \mathbf{d}_f \in S([-\bar{\delta}_f, \bar{\delta}_f]).$$

The first step consists of showing that there exist $\bar{t} \in \mathbb{R}_+$ and $\eta > 0$ such that for all $t \geq \bar{t}$,

$$\mathbf{x}_3(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \eta \qquad \text{and} \qquad \mathbf{x}_3(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) \leq -\eta. \tag{11}$$

We show only the former, the later is analogous. For this, let us introduce the notations

$$a_{11} = \frac{c_f + c_r}{m}, \qquad a_{12} = \frac{c_r l_r - c_f l_f}{m}, \qquad a_{21} = \frac{c_r l_r - c_f l_f}{J_z},$$
$$a_{22} = \frac{c_f l_f^2 + c_r l_r^2}{J_z}, \qquad b_1 = \frac{c_f}{m}, \qquad b_2 = \frac{c_f l_f}{J_z}.$$

By Assumption 2 all these constants are positive and for all $U \in [U_{min}, U_{max}]$ defining

$$A(U) = \begin{pmatrix} -\frac{a_{11}}{U} & \frac{a_{12}}{U} - U \\ \frac{a_{21}}{U} & -\frac{a_{22}}{U} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

the lateral dynamics can be written as

$$\begin{pmatrix} \dot{V} \\ \dot{r} \end{pmatrix} = A(U) \begin{pmatrix} V \\ r \end{pmatrix} + B\delta_f,$$

see [2, Eq. 1]. In the following we fix $U \in [U_{min}, U_{max}]$. Using Condition 1, a simple computation shows that the matrix $A(U)$ has conjugate complex eigenvalues with real part $\lambda_{Re}(U)$ and imaginary part $\lambda_{Im}(U)$ given by

$$\lambda_{Re}(U) = -\frac{a_{11} + a_{22}}{2U} < 0 \qquad \text{and} \qquad \lambda_{Im}(U) = \frac{\sqrt{4a_{21}U^2 - 4a_{12}a_{21} - (a_{11} - a_{22})^2}}{2U} > 0.$$

Notice that both $\lambda_{Re}(U)$ and $\lambda_{Im}(U)$ are increasing functions of U. Using Fulmer's method, see for instance [1, Section 9.4], we know that by setting

$$M(U) = (1/\lambda_{Im}(U))(A(U) - \lambda_{Re}(U)\mathrm{Id}),$$

we get that

$$e^{A(U)t} = e^{\lambda_{Re}(U)t} \left(\mathrm{Id} \cos(\lambda_{Im}(U)t) + M(U) \sin(\lambda_{Im}(U)t) \right).$$

Then, using that for all $t \in \mathbb{R}_+$,

$$\begin{pmatrix} \mathbf{x}_2(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \\ \mathbf{x}_3(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \end{pmatrix} = e^{A(U)t} \begin{pmatrix} V \\ r \end{pmatrix} + \bar{\delta}_f \int_0^t e^{A(U)(t-s)} B ds,$$

a simple computation shows that there exists a constant $C > 0$ such that

$$\mathbf{x}_3(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \\ \geq -e^{\lambda_{Re}(U_{max})t} C + \bar{\delta}_f \left(\frac{2a_{21}b_1 + (a_{11} - a_{22})b_2}{2U_{max} \left(\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2\right)} - \frac{\lambda_{Re}(U_{max})}{\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2} \right).$$

Using Assumption 2, it is not difficult to check that $2a_{21}b_1 + (a_{11} - a_{22})b_2 > 0$. Thus setting,

$$\eta = \frac{\bar{\delta}_f}{2} \left(\frac{2a_{21}b_1 + (a_{11} - a_{22})b_2}{2U_{max} \left(\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2\right)} - \frac{\lambda_{Re}(U_{max})}{\lambda_{Re}(U_{min})^2 + \lambda_{Im}(U_{max})^2} \right),$$

it follows from the fact that $e^{\lambda_{Re}(U_{max} t)} \to 0$ when $r \to \infty$ that there exists $\bar{t} \in \mathbb{R}_+$ such that $e^{\lambda_{Re}(U_{max})t} C \leq \eta$ for all $t \geq \bar{t}$. The first statement of (11) follows.

Next, we use (11) to deduce that there exists $K > 0$ such that for all $t \geq K$ and all $x \in X$ as in the Lemma, $\mathbf{x}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi}$. First notice that by Condition 1 there exists a bounded over approximation $\mathcal{R}$ of the reachable set $\mathcal{R}(\tilde{V}, \tilde{r})$ and hence $\bar{V}$ and $\bar{r}$ such that $\mathcal{R} \in [-\bar{V}, \bar{V}] \times [-\bar{r}, \bar{r}]$. Therefore $\mathbf{x}_3(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \in [-\bar{r}, \bar{r}]$ for all $t \in \mathbb{R}_+$ and by assumption $\psi \in [-\tilde{\psi}, \tilde{\psi}]$. This implies that

$$\mathbf{x}_4(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \psi - \int_0^{\bar{t}} \bar{r} ds \geq -\tilde{\psi} - \bar{t}\bar{r}.$$

Thus for all $t \geq \bar{t}$ we have

$$\mathbf{x}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = \mathbf{x}_4(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) + \int_{\bar{t}}^t \mathbf{x}_3(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq -\tilde{\psi} - \bar{t}\bar{r} + (t - \bar{t})\eta. \tag{12}$$

Setting $K = 2(\tilde{\psi} + \bar{t}\bar{r} + \eta\bar{t})/\eta$, it follows from (12) that for all $t \geq K$, $\mathbf{x}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \tilde{\psi}$. Notice that a similar argument allows to prove that for all $t \geq K$, and all $x \in X$ as in the Lemma, $\mathbf{x}_4(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) \leq -\tilde{\psi}$.

Finally, since by definition of $\hat{\mathbf{x}}$ this implies that $\hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = \tilde{\psi}$ and $\hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, -\bar{\mathbf{d}}_f, x) = -\tilde{\psi}$ for all $t \geq K$ the assertion of the lemma follows. □

Lemma A.4. *Let $x = (U, V, r, \psi, d_l, d_r) \in \mathcal{W}_+$ be such that $\psi < \tilde{\psi}$ and there exists $\bar{t} \in \mathbb{R}_+$ such that $\hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0$. Then $\bar{t} \in \mathcal{T}_+(x)$. Moreover, an analogous statement holds for $x \in \mathcal{W}_-$.*

Proof. We argue by contradiction and assume that $\hat{\mathbf{x}}_4(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = \tilde{\psi}$. From the very definition of $\hat{\mathbf{x}}_6$ it follows then that

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) &= \left(\hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)\hat{\mathbf{x}}_3(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) + U\right) \tan(\hat{\mathbf{x}}_4(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)) + \hat{\mathbf{x}}_2(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \\ &= U \tan(\tilde{\psi}) + \hat{\mathbf{x}}_2(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \\ &\geq U_{min} \tan(\tilde{\psi}) - \bar{V}, \end{aligned}$$

where $\bar{V}$ is as in the proof of Lemma A.3. From Condition 3 it follows then that $\dot{\hat{\mathbf{x}}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) > 0$. By continuity of the map $t \mapsto \dot{\hat{\mathbf{x}}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$, there exists $\epsilon > 0$ such that

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < 0 \quad \forall t \in [t - \epsilon, \bar{t}[.$$

Moreover, since $x \in \mathcal{W}_+$ and $\psi < \tilde{\psi}$, it is clear that $d_r \geq 0$. It follows therefore from Lemma A.1 that there exists $t_0 \in [0, t - \epsilon[$ for which

$$\hat{\mathbf{x}}_6(t_0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \quad \text{and} \quad \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < 0 \; \forall t \in]t_0, t - \epsilon]. \tag{13}$$

The fact that $x \in \mathcal{W}_+$ implies then also that $\hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = \tilde{\psi}$ for all $t \in]t_0, t - \epsilon]$ and hence by continuity, $\hat{\mathbf{x}}_4(t_0; \bar{\mathbf{d}}_f, x) = \tilde{\psi}$. The same arguments as above allow therefore to conclude that $\dot{\hat{\mathbf{x}}}_6(t_0; x) > 0$. However, by (13) we have also that

$$\dot{\hat{\mathbf{x}}}_6(t_0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = \lim_{s \to t_0+} \frac{\hat{\mathbf{x}}_6(s; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) - \hat{\mathbf{x}}_6(t_0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)}{s - t_0} \leq 0,$$

which is impossible. □

The lemma has the following useful Corollary.

Corollary A.5. *Let $x = (U, V, r, \psi, d_l, d_r) \in \mathcal{W}_+$ such that $\psi < \tilde{\psi}$. Then*

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0 \qquad \forall t \in \mathbb{R}_+.$$

A similar statement holds true for $x \in \mathcal{W}_-$.

Proof. Assume to the contrary that there exists $\bar{t} \in \mathbb{R}_+$ such that $\hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < 0$. Since $x \in \mathcal{W}_+$ and $\psi < \tilde{\psi}$, $\hat{\mathbf{x}}_6(0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq 0$. It follows then from Lemma A.1 that there exists $t_0 \in [0, \bar{t}[$ for which (13) holds true. As above in the proof of Lemma A.4 it then also follows that $\hat{\mathbf{x}}_4(t_0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = \tilde{\psi}$. This however contradicts with the fact that by Lemma A.4 $t_0 \in \mathcal{T}_+(x)$. □

Finally we are ready to provide a description of the boundary of $\mathcal{W}_+$ and $\mathcal{W}_-$ respectively.

Proposition A.6. *The following inclusions hold true:*

$$\begin{aligned} &\left\{ x \in \mathcal{W}_+ \;\middle|\; \exists t \in \overline{\mathcal{T}_+(x)} \; s.t. \; \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \right\} \subset \partial \mathcal{W}_+, \\ &\left\{ x \in \mathcal{W}_- \;\middle|\; \exists t \in \overline{\mathcal{T}_-(x)} \; s.t. \; \hat{\mathbf{x}}_5(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \right\} \subset \partial \mathcal{W}_-. \end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned} \left\{ x \in \partial \mathcal{W}_+ \;\middle|\; \psi < \tilde{\psi} \right\} &\subset \left\{ x \in \mathcal{W}_+ \mid \exists t \in \mathcal{T}_+(x) \; s.t. \; \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \right\}, \\ \left\{ x \in \partial \mathcal{W}_- \;\middle|\; \psi > -\tilde{\psi} \right\} &\subset \left\{ x \in \mathcal{W}_- \mid \exists t \in \mathcal{T}_-(x) \; s.t. \; \hat{\mathbf{x}}_5(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \right\}. \end{aligned} \tag{15}$$

Remark A.7. Notice that the inclusion in (14) is strict. Indeed, consider $x \in X$ with $U \in [U_{min}, U_{max}]$ arbitrary and V and r be the steady state lateral velocity and yaw rate corresponding to U and the constant control input $\bar{\mathbf{d}}_f$. It is easy to check that in this case $r > 0$ and hence $t \mapsto \hat{\mathbf{x}}_4(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ is an increasing function of time. Let $\psi = \tilde{\psi}$ and $d_r = 1$, i.e. $\hat{\mathbf{x}}_6(0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < -1$. Then it is clear that $x \in \mathcal{W}_+$ and $\overline{\mathcal{T}_+(x)} = \emptyset$. However, for all $\epsilon > 0$ small enough, $(U, V, r, \tilde{\psi} - \epsilon, d_l, d_r) \in \mathcal{W}_+^c$. Hence $x \in \partial\mathcal{W}_+$.

Proof of Proposition A.6. We show the result only in the case of $\mathcal{W}_+$. Similar arguments allow to prove the other case.

We start by showing (14). Let $\hat{x} \in \left\{ x \in \mathcal{W}_+ \,\middle|\, \exists t \in \overline{\mathcal{T}_+(x)} \text{ s.t. } \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) = 0 \right\}$. Since $\hat{x} \in \mathcal{W}_+$ it suffices to show that for all $\epsilon > 0$ there exists $x \in B(\bar{x}, \epsilon)$ such that $x \notin \mathcal{W}_+$. Fix $\epsilon > 0$, recall that $\hat{x} = (\hat{U}, \hat{V}, \hat{r}, \hat{\psi}, d_l, \bar{d}_r)$ and set $x := (\hat{U}, \hat{V}, \hat{r}, \hat{\psi}, d_l, \bar{d}_r - \epsilon/2) \in B(\hat{x}, \epsilon)$. Consider next the linear time varying system with dynamics

$$f\colon (t, d_r) \mapsto \hat{\mathbf{x}}_2(t; \bar{\mathbf{d}}_f, \hat{U}, \hat{V}, \hat{r}) + \left(\hat{U} + d_r \hat{\mathbf{x}}_3(t; \bar{\mathbf{d}}_f, \hat{U}, \hat{V}, \hat{r})\right) \tan(\hat{\mathbf{x}}_4(t; \bar{\mathbf{d}}_f, \hat{U}, \hat{V}, \hat{r}, \hat{\psi})), \tag{16}$$

where we omitted the arguments the flows do not depend on. By the definition of $\hat{\mathbf{x}}_6$ it is clear that for all $t \in \mathbb{R}_+$

$$\dot{\hat{\mathbf{x}}}_6(t; x) = f(t, \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)) \quad \text{and} \quad \dot{\hat{\mathbf{x}}}_6(t; \hat{x}) = f(t, \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x})).$$

Thus $t \mapsto \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ and $t \mapsto \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x})$ are the flows of the linear time varying system with dynamics given by (16). Since this is a 1-dimensional system, by uniqueness of solutions, it follows from $\bar{d}_r - \epsilon/2 < \bar{d}_r$ that for all $t \in \overline{\mathcal{T}_+(x)} = \overline{\mathcal{T}_+(\hat{x})}$,

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) < \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x}).$$

Finally, since there exists $\bar{t} \in \overline{\mathcal{T}_+(\hat{x})}$ such that

$$0 = \hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \hat{x}) > \hat{\mathbf{x}}_6(\bar{t}; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x).$$

This shows that $x \notin \mathcal{W}_+$ and completes the proof of (14).

It remains to show (15). Let $\bar{x} = (U, V, r, \psi, d_l, d_r) \in \partial\mathcal{W}_+$ be such that $\psi < \tilde{\psi}$. Since by Lemma A.2 the set $\mathcal{W}_+$ is closed, it is clear that $\bar{x} \in \mathcal{W}_+$. We show that there exists $t \in \overline{\mathcal{T}_+(\bar{x})}$ such that $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = 0$. We argue by contradiction. Thus, assume that

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) > 0 \quad \forall t \in \overline{\mathcal{T}_+(\bar{x})}.$$

By Lemma A.3 there exists $K > 0$ such that

$$\mathcal{T}_+(x) \subset B(0, K) \qquad \forall x \in X_1 \times \mathbb{R}. \tag{17}$$

This in particular implies that $\overline{\mathcal{T}_+(\bar{x})}$ is compact. Consequently there exists $\eta_1 > 0$ such that

$$\min_{t \in \overline{\mathcal{T}_+(\bar{x})}} \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = \eta_1. \tag{18}$$

Next notice that by [2, Condition 3] there exists $\eta_2 > 0$ such that

$$\bar{V} / \tan(\tilde{\psi}) + \eta_2 = U_{min},$$

and therefore setting $\eta := \min\{\eta_1/2, \eta_2/(2\bar{r})\}$ we obtain

$$\bar{V}/\tan(\tilde{\psi}) + \eta\bar{r} < U_{min}. \tag{19}$$

We claim that for all $t \in \mathbb{R}_+$,

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) \geq \eta. \tag{20}$$

Assume to the contrary that there exists $\bar{t} \in \mathbb{R}_+$ such that $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) < \eta$. Since $\psi < \tilde{\psi}$ it follows from (18) that $\hat{\mathbf{x}}_6(0; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) \geq \eta$. From Lemma A.1 we obtain then that there exists $t^* \in [0, \bar{t}[$ such that $\hat{\mathbf{x}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = \eta$ and

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) < \eta \qquad \forall t \in]t^*, \bar{t}]. \tag{21}$$

It follows readily from (21) that

$$\dot{\hat{\mathbf{x}}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) \leq 0.$$

On the other hand, since $\eta < \eta_1$ it follows from (18) that $t^* \in \mathcal{T}_+(\bar{x})^c$, that is, $\hat{\mathbf{x}}_4(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = \tilde{\psi}$. Furthermore by [2, Condition 3],

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) &= U\tan(\tilde{\psi}) + \hat{\mathbf{x}}_2(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) + \hat{\mathbf{x}}_6(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x})\hat{\mathbf{x}}_3(t^*; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x})\tan(\tilde{\psi}) \\ &\geq U_{min}\tan(\tilde{\psi}) - \bar{V} - \eta\bar{r}\tan(\tilde{\psi}) > 0,\end{aligned}$$

where the last inequality follows from (19). This establishes the desired contradiction and thus proves (20).

Next, the continuous function $(t, x) \mapsto \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x)$ is uniformly continuous on the compact set $\overline{B(0,K)} \times \overline{B(\bar{x}, 1)}$. One can therefore find an $\epsilon > 0$ such that for all $x_1, x_2 \in B(\bar{x}, 1)$ with $\|x_1 - x_2\| \leq \epsilon$,

$$\left|\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x_1) - \hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x_2)\right| \leq \frac{\eta}{2} \qquad \forall t \in \overline{B(0,K)}. \tag{22}$$

Take $x \in B(\bar{x}, \epsilon)$ arbitrary. By (17), (20) and (22),

$$\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, x) \geq \frac{\eta}{2} \qquad \forall t \in \mathcal{T}_+(x).$$

We conclude that $B(\bar{x}, \epsilon) \subset \mathcal{W}_+$ which contradicts with the fact that $\bar{x} \in \partial\mathcal{W}_+$. This proves that there exists $t \in \overline{\mathcal{T}_+(\bar{x})}$ such that $\hat{\mathbf{x}}_6(t; \bar{\mathbf{t}}_w, \bar{\mathbf{d}}_f, \bar{x}) = 0$. Applying Lemma A.4 it is clear that actually $t \in \mathcal{T}_+(x)$ and the proof is complete. □

References

[1] W. A. Adkins and M. G. Davidson, *Ordinary Differential Equations*, Springer, 2012.

[2] D. Hoehener, G. Huang, and D. Del Vecchio, *Design of a lane departure driver-assist system under safety specifications*, in Conference on Decision and Control, 2016. submitted.