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DYNAMIC RESPONSE OF SYSTEMS WITH STRUCTURAL DAMPING

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ABSTRACT

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The response of simple vibratory systems containing structural damping is studied analytically when the excitation is an impulse and when the excitation is stationary random vibration. It is found that in a strict sense the assumption of ideal structural damping represents a physically unrealizable model because a small precursor response occurs before the application of an impulsive load. For stationary random excitation exact solutions for the mean square response are compared with approximate solutions obtained from two increasingly accurate "equivalent viscous" substitutes for structural damping.

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Linear structural damping has been widely used in analytical studies of vibration and flutter as a simple model for damping mechanisms in which the dissipation increases with the stress level of the vibration. The linear structural damping assumption is easily described in terms of complex frequency response but difficulties arise in translating this back into the time domain. In the present note the impulse response of a simple oscillatory system with structural damping is represented as a Fourier integral and an approximation solution is plotted. It is found that in a strict sense the linear structural damping model is physically unrealizable because it implies a (small) response <u>prior</u> to the application of the excitation. When the excitation is a stationary random process with an ideally white spectrum the exact solution for the mean square response is obtained for a system with structural damping and for a system with both structural damping <u>and</u> viscous damping. References are made throughout to two increasingly accurate "equivalent viscous" approximations of structural damping.

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1. Equivalent viscous approximations for structural damping. In Fig. 1(a) a linear spring with linear structural damping is indicated. The specification for linear structural damping is that under simple harmonic motion of the form $e^{i\omega t}$ the complex ratio of spring force to elongation is $k(1+ig)$ where g is the damping coefficient; i.e., the magnitude of the damping force is proportional to the elastic force but its phase is advanced by 90°. The damping force is independent of the frequency ω, at least if only positive frequencies are considered. If, however, negative frequencies are to be considered the correct phase relation for the damping force requires that the complex force displacement ratio be written

$$k(1+ ig \quad \operatorname{sgn} \omega) \tag{1.1}$$

where $\operatorname{sgn} \omega$ is +1 if ω is positive, zero if $\omega = 0$ and -1 if ω is negative.

In Fig. 1(b) the same elastic spring is in parallel with a linear viscous dashpot. Under simple harmonic motion of the form $e^{i\omega t}$ the complex ratio of force to displacement across the pair is

$$k + i\omega c_e \tag{1.2}$$

In Fig. 2 the imaginary parts of (1.1) and (1.2) are sketched as functions of frequency. Because of the awkward singularity at the origin for structural damping it is sometimes convenient to substitute for the structural damping an "equivalent viscous" damping. An equivalence is obtained by taking the cross-over frequency ω_o in Fig. 2 equal to the (undamped) natural frequency $\omega_n = \sqrt{k/m}$. At this frequency the viscous force amplitude is $c_e\sqrt{k/m}$

while the structural damping force amplitude is gk (for unit displacement amplitude in both cases). These forces will be equal if

$$c_e = g\sqrt{mk} \tag{1.3}$$

The rationalization for choosing $\omega_o = \omega_n$ can be based on the fact that for oscillatory excitation of fixed amplitude the viscous damping force magnitude is greatest when the exciting frequency is equal to ω_n and on the fact that at this frequency the system response is most sensitive to changes in the dashpot parameter. At frequencies far removed from resonance the system response is not greatly affected by the damping and hence an approximation with incorrect damping away from resonance might be acceptable if it gave the correct damping at resonance.

This is the basis for the first equivalent viscous substitution. The original system with structural damping is in Fig. 1(a). The "equivalent viscous" system in 1(b) is taken to have the same mass m and the same spring constant k, but instead of the structural damping a dashpot with constant c_e given by (1.3) is inserted. The equivalent dashpot constant depends on all three parameters of the original system but only one element (the damping element) is altered.

In the second equivalent viscous substitution of Fig. 1(c) we permit the magnitude of the elastic spring constant to differ from that in Fig. 1(a). The equivalence here is based on the requirement that the complex frequency responses in the two cases should have the same poles.

Let the external exciting force in Fig. 1 be of the form $f = \mathrm{Re}\left\{e^{i\omega t}\right\}$. The steady state response will then have the form $x = \mathrm{Re}\left\{H(\omega)\, e^{i\omega t}\right\}$

where $H(\omega)$ is the complex frequency response. For the system in Fig. 1(a) with structural damping we have

$$H_a(\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + i g \omega_n^2 \operatorname{sgn} \omega} \tag{1.4}$$

for real ω. This may be extended to complex ω if we replace sgn ω by sgn $(R_e \{\omega\})$. The poles of $H_a(\omega)$ are then at

$$\omega = (\pm \mu + i\lambda)\, \omega_n \tag{1.5}$$

where

$$\mu^2 = \frac{\sqrt{1+g^2} + 1}{2} \tag{1.6}$$

$$\lambda^2 = \frac{\sqrt{1+g^2} - 1}{2}$$

Turning next to the system with viscous damping in Fig. 1(c) let the spring constant and dashpot constant be k_e and c_e respectively. The complex frequency response is

$$H_c(\omega) = \frac{1/m}{k_e/m - \omega^2 + i\omega c_e/m} \tag{1.7}$$

which has poles at

$$\omega = \pm \sqrt{\frac{k_e}{m} - \frac{c_e^2}{4m^2}} + i\,\frac{c_e}{2m} \tag{1.8}$$

These poles will coincide with those of (1.5) if

$$k_e = k\sqrt{1+g^2}$$

$$c_e = g\sqrt{km}\left[\frac{2\left(\sqrt{1+g^2}-1\right)}{g^2}\right]^{1/2} \tag{1.9}$$

This result is due to Soroka [1].

For comparison with (1.4) we list the complex frequency responses for the equivalent viscous systems of Fig. 1(b) and 1(c) under the conditions (1.3) and (1.9) respectively.

$$H_b(\omega) = \frac{1}{m}\,\frac{1}{\omega_n^2-\omega^2+ig\omega_n\omega}$$

$$H_c(\omega) = \frac{1}{m}\,\frac{1}{(\mu^2+\lambda^2)\omega_n^2-\omega^2+i2\lambda\omega_n\omega} \tag{1.10}$$

2. Impulse response. If in Fig. 1 the excitation f(t) is the unit impulse function $\delta(t)$ the response h(t) is called the impulse response function. In principle, the impulse response function is implied as soon as the complex frequency response $H(\omega)$ is known since h(t) and $H(\omega)$ satisfy the Fourier transform relation [2]

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega \tag{2.1}$$

For the systems with viscous damping in Fig. 1(b) and 1(c) the impulse response functions corresponding to (1.10) are well known [3]. For $t > 0$

$$h_b(t) = \frac{e^{-\frac{g\omega_n t}{2}}}{m\omega_n\sqrt{1-g^2/4}} \sin\sqrt{1-g^2/4}\,\omega_n t \qquad (2.2)$$

$$h_c(t) = \frac{e^{-\lambda\omega_n t}}{m\omega_n\mu} \sin\mu\omega_n t$$

where λ and μ are given in (1.6) while $h_b(t) = h_c(t) = 0$ for $t < 0$.

In the case of structural damping the singularity at $\omega = 0$ in (1.4) rules out the usual evaluation of (2.1) by contour integration. A real integral from zero to infinity can be obtained in this case by noting that the imaginary part of the integrand is an odd function and that the real part is an even function.

$$h_a(t) = \frac{1}{m\pi}\int_0^\infty \frac{(\omega_n^2-\omega^2)\cos\omega t + g\omega_n^2\sin\omega t}{(\omega_n^2-\omega^2)^2+g^2\omega_n^4}\,d\omega \qquad (2.3)$$

We have been unable to evaluate (2.3) in closed form but an asymptotic procedure which permits the attainment of any desired accuracy will be outlined. Some characteristics of $h_a(t)$ can however be ascertained directly. The most important is that $h_a(0)$ does not vanish. Setting $t=0$ yields a standard integral [4] from which we obtain

$$h_a(0) = -\frac{\lambda}{2m\omega_n(\lambda^2+\mu^2)} \qquad (2.4)$$

Furthermore $h_a(t)$ is continuous at t=0 so that a non-zero response occurs for $t < 0$. For light damping we will see that this precursor response is small but strictly speaking it signifies that the linear structural damping model is physically unrealizable [5]. It is possible that this may vitiate the usefulness of the structural damping model for some purposes.

We now develop an asymptotic approximation to (2.3) and evaluate the leading term of the discrepancy between $h_a(t)$ and $h_c(t)$. We begin by subtracting the second of (2.2) from (2.3) using (2.1) and the second of (1.10)....

$$h_a(t) - h_c(t) = \frac{2\lambda\omega_n}{\pi m}\int_0^\infty \frac{-\lambda\omega_n \cos\omega t + (\mu\omega_n - \omega)\sin\omega t}{(\omega_n^2-\omega^2)^2 + 4\mu^2\lambda^2\omega_n^4}\,d\omega \quad (2.5)$$

Let

$$C = \frac{2\omega_n^2}{\pi}\int_0^\infty \frac{-\lambda\omega_n \cos\omega t}{(\omega_n^2-\omega^2)^2 + 4\mu^2\lambda^2\omega_n^4}\,d\omega \quad (2.6)$$

$$S = \frac{2\omega_n^2}{\pi}\int_0^\infty \frac{(\mu\omega_n - \omega)\sin\omega t}{(\omega_n^2-\omega^2)^2 + 4\mu^2\lambda^2\omega_n^4}\,d\omega$$

be integrals to be evaluated for $t > 0$. Then we have

$$h_a(t) - h_c(t) = \begin{cases} \frac{\lambda}{m\omega_n}(C+S) & \text{for } t>0 \\ \frac{\lambda}{m\omega_n}(C-S) & \text{for } t<0 \end{cases} \quad (2.7)$$

The cosine integral C is a standard case [6]

$$C = -\frac{e^{-\lambda\omega_n t}}{2(\mu^2+\lambda^2)}\left(\cos\theta + \frac{\lambda}{\mu}\sin\theta\right) \qquad (2.8)$$

where θ is an abbreviation for $\mu\omega_n t$.

We turn now to the sine integral S which is apparently non-elementary. In the lightly damped case (λ small, $\mu \approx 1$) the integrand changes rapidly in the neighborhood of $\omega = \mu\omega_n$. Accordingly we make a partial fraction expansion and split the range of integration to isolate the term responsible for this behavior.

$$S = \frac{2\omega_n^2}{\pi}\left[\int_0^{2\mu\omega_n} + \int_{2\mu\omega_n}^{\infty}\right]\frac{(\mu\omega_n-\omega)\sin\omega t}{4\mu\omega_n^3(\lambda^2+\mu^2)}\left[\frac{\omega+2\mu\omega_n}{(\omega+\mu\omega_n)^2+\lambda^2\omega_n^2} - \frac{\omega-2\mu\omega_n}{(\omega-\mu\omega_n)^2+\lambda^2\omega_n^2}\right]d\omega \qquad (2.9)$$

The second term in the first range of integration is very sensitive to the damping. If λ were to vanish the other contributions would not change much but the contribution of this term would become infinite. We therefore proceed further with this term alone (S_2, say) before expanding in powers of λ. Setting $\Omega = \omega - \mu\omega_n$ we have

$$S_2 = \frac{1}{2\pi\mu\omega_n(\lambda^2+\mu^2)}\int_{-\mu\omega_n}^{\mu\omega_n}\left(1 - \frac{\mu\omega_n\Omega + \lambda^2\omega_n^2}{\Omega^2+\lambda^2\omega_n^2}\right)\left(\sin\mu\omega_n t\cos\Omega t + \cos\mu\omega_n t\sin\Omega t\right)d\Omega$$

$$= \frac{1}{2\pi\mu\omega_n(\lambda^2+\mu^2)}\left(\frac{2\sin^2\mu\omega_n t}{t} - \lambda^2\omega_n^2 \sin\mu\omega_n t \int_{-\mu\omega_n}^{\mu\omega_n} \frac{\cos\Omega t}{\Omega^2+\lambda^2\omega_n^2}\,d\Omega \right. \tag{2.10}$$

$$\left. - \mu\omega_n \cos\mu\omega_n t \int_{-\mu\omega_n}^{\mu\omega_n} \frac{\Omega \sin\Omega t}{\Omega^2+\lambda^2\omega_n^2}\,d\Omega \right)$$

Finally the two integrals remaining in (2.10) are known for infinite limits [7] so that we have

$$S_2 = \frac{1}{2\pi\mu\omega_n(\lambda^2+\mu^2)}\left[\frac{2\sin^2\mu\omega_n t}{t} - \pi\omega_n e^{-\lambda\omega_n t}(\lambda\sin\mu\omega_n t + \mu\cos\mu\omega_n t) \right. \tag{2.11}$$

$$\left. + 2\lambda^2\omega_n^2 \sin\mu\omega_n t \int_{\mu\omega_n}^{\infty} \frac{\cos\Omega t\,d\Omega}{\Omega^2+\lambda^2\omega_n^2} + 2\mu\omega_n\cos\mu\omega_n t \int_{\mu\omega_n}^{\infty} \frac{\Omega\sin\Omega t\,d\Omega}{\Omega^2+\lambda^2\omega_n^2} \right]$$

The two integrals in (2.11) are no longer sensitive to variations in λ when λ is small.

To complete the evaluation of (2.9) it remains to evaluate these two integrals and the three other integrals in (2.9). In all five of these integrals it is possible to introduce Ω as either $\omega \pm \mu\omega_n$ so as to reduce all denominators to the form $\Omega^2 + \lambda^2\omega_n^2$ with the origin, $\Omega = 0$, excluded from the range of integration. A systematic method of approximation can then be based on the following expansion in powers of λ.

$$\frac{1}{\Omega^2+\lambda^2\omega_n^2} = \frac{1}{\Omega^2}\left(1 - \lambda^2\frac{\omega_n^2}{\Omega^2} + \lambda^4\frac{\omega_n^4}{\Omega^4} - \dots\right) \tag{2.12}$$

All integrals can then be evaluated in terms of the Ci and Si functions [8] which have been tabulated [9]. Here we give the results of using only the first term of (2.12). This produces a discrepancy between $h_a(t)$ and $h_c(t)$ which is $O(\lambda)$. Utilizing one more term of (2.12) would give further terms of order λ^3, etc. Note that for small λ the relation (1.6) yields $\lambda \approx g/2$.

Carrying out the above program leads to the following result for S.

$$S = \frac{1}{\lambda^2+\mu^2}\Big[-\frac{e^{-\lambda\omega_n t}}{2}\left(\cos\theta + \frac{\lambda}{\mu}\sin\theta\right) + \frac{1}{\pi}\left(\sin\theta\, Ci\,\theta - \cos\theta\, si\,\theta\right) - \frac{\theta}{\pi}\left(\sin\theta\, si\,\theta + \cos\theta\, Ci\,\theta\right)\Big] + O(\lambda^2) \qquad (2.13)$$

where we have used the notations $\theta = \mu\omega_n t$ and $-si\,\theta = Si\,\infty - Si\,\theta$. Neglecting the $O(\lambda^2)$ term in (2.13) and combining with (2.8) according to (2.7) gives our final result for the (approximate) discrepancy between $h_a(t)$ and $h_c(t)$. This result is plotted in Fig. 3 for the particular case g=0.05(Q=20). Note the precursor response for $t < 0$. Since $h_c(t) = 0$ for $t < 0$ this precursor response is just $h_a(t)$. It should be pointed out that the magnitude of the precursor response is <u>small</u> compared to $h_a(t)$ for $t > 0$. The plot in Fig. 3 shows only the <u>discrepancy</u> between h_a and h_c. The amplitude of $m\omega_n h_a(t)$ itself runs from approximately unity at t=0 to about 0.6 at t=20. For the damping chosen in Fig. 3 the magnitude of the precursor response is thus only about one percent of the post impulse response.

For additional interest the discrepancy between $h_b(t)$ and $h_c(t)$ as given by (2.2) is also plotted in Fig. 3 for the same amount of damping. The magnitude of the difference between the two equivalent viscous responses is about 40 times smaller than the difference between either one of them and our asymptotic approximation to the structural damping response.

3. Random vibration of system with structural damping. Let the excitation $f(t)$ in Fig. 1 be a stationary random process with an ideally white spectrum; i.e., the mean square spectral density $S_f(\omega)$ of $f(t)$ is taken to be S_o a constant over all frequencies from $\omega = -\infty$ to $\omega = \infty$. The dimensions of S_o are those of force squared per unit of circular frequency. The mean square spectral density $S_\chi(\omega)$ of the response $\chi(t)$ in case (a) is [10.]

$$S_\chi(\omega) = H(\omega)\ H(-\omega)\ S_f(\omega)$$

$$= \frac{S_o\ /m^2}{(\omega_n^2 - \omega^2 + ig\omega_n^2)(\omega_n^2 - \omega^2 - ig\omega_n^2)} \tag{3.1}$$

using (1.4). Note that although $H(\omega)$ has a singularity at the origin, $S_\chi(\omega)$ does not. The response is thus a stationary random process with a well-behaved spectrum, analytic except for poles at the roots of the denominator. When g is small $S_\chi(\omega)$ has a typical narrow band appearance. The expected mean square of the response [11] .

$$E\ [\chi^2] = \int_{-\infty}^{\infty} S_\chi\ (\omega)\, d\omega \tag{3.2}$$

can be obtained by contour integration and the method of residues. Alternatively, since (3.1) is an even real function (3.2) can be reduced to the following real integral

$$E\left[x^2\right] = \frac{2S_o}{m^2}\int_0^{\infty}\frac{d\omega}{(\omega_n^2-\omega^2)^2+g^2\omega_n^4} \tag{3.3}$$

which can be found in tables [12]. Either way the result of the integration for case (a) is

$$E\left[x^2\right] = \frac{\pi S_o}{g\, m^2\omega_n^3}\cdot\left[\frac{1+\sqrt{1+g^2}}{2\,(1+g^2)}\right]^{1/2} \tag{3.4}$$

For the equivalent viscous approximations of Fig. 1(b) and 1(c) the calculation of the mean square response is well known [12]. In case (b) we find

$$E\left[x^2\right] = \frac{\pi S_o}{k\, c_e} = \frac{\pi S_o}{g\, m^2\omega_n^3} \tag{3.5}$$

while for case (c) the result turns out to be identical to (3.4); i.e., the requirement that the complex frequency response in case (c) should have the same poles as in case (a) leads to the result that the mean square response to white stationary random excitation is the same in the two cases.

In Fig. 4 the mean square responses according to (3.4) and (3.5) are plotted as functions of the damping parameter g. For small damping the approximation (3.5) furnished by the simpler equivalent viscous system of Fig. 1(b) is quite satisfactory. For large damping the relative error of the approximation is large, but for $g=0.4$ the relative error is already down to about five percent and for smaller damping the relative error is $O(g^2)$.

4. Random vibration of system with both structural and viscous damping. In more complicated systems containing structural damping it can be expected that neither of the equivalent viscous substitutions will produce the correct mean square response under random excitation. As an illustration consider the system in Fig. 5(a) in which a spring with structural damping is in parallel with a viscous damping element. The complex frequency response for x when the excitation is the force f is

$$H_a(\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + i\,(g\omega_n^2 \operatorname{sgn}\omega + 2\zeta\omega_n\omega)} \tag{4.1}$$

where

$$\omega_n^2 = \frac{k}{m} \qquad\qquad 2\zeta\omega_n = \frac{c}{m} \tag{4.2}$$

If the excitation $f(t)$ is a stationary random process with an ideally white spectrum of density S_o the expected mean square response is

$$E[x^2] = \frac{S_o}{m^2}\int_{-\infty}^{\infty} \frac{d\omega}{\left[(\omega_n^2-\omega^2)+i(g\omega_n^2\operatorname{sgn}\omega+2\zeta\omega_n\omega)\right]\left[(\omega_n^2-\omega^2)-i(g\omega_n^2\operatorname{sgn}\omega+2\zeta\omega_n\omega)\right]} \tag{4.3}$$

In this case the singularity at the origin remains and it is necessary to evaluate (4.3) as a real integral over the range 0 to ∞.

$$E[x^2] = \frac{2S_0}{m^2}\int_0^\infty \frac{d\omega}{(\omega_n^2-\omega^2)^2 + (g\omega_n^2 + 2\zeta\omega_n\omega)^2} \tag{4.4}$$

The singularity in (4.3) is reflected by the fact that whereas the poles of the integrand of (4.3) are $(\pm a \pm ib)\omega_n$ where

$$a = \left(\frac{\sqrt{(1-\zeta^2)^2+g^2} + 1-\zeta^2}{2}\right)^{1/2} \tag{4.5}$$

$$b = \zeta + \left(\frac{\sqrt{(1-\zeta^2)^2+g^2} - 1+\zeta^2}{2}\right)^{1/2}$$

the poles of the integrand of (4.4) are $(a \pm ib)\omega_n$ and $[-a \pm i(2\zeta - b)]\omega_n$. The integration of (4.4) is accomplished by a partial fraction expansion into four logarithmic integrals of complex quantities. Careful attention to the changes in arguments during integration leads finally to the following result

$$E[x^2] = \frac{2S_0}{m^2\omega_n^3}\,\frac{\left\{\begin{array}{l}\zeta(\pi-\tan^{-1} b/a)\left[a(a-g/2\zeta)+(b-\zeta)(b-2\zeta)\right]\\ +\zeta\left(\tan^{-1}\frac{2\zeta-b}{a}\right)\left[a(a+g/2\zeta)+(b-\zeta)b\right]\\ +\left[a\zeta^2-(b-\zeta)g/2\right]\log\left[\frac{a^2+b^2}{a^2+(2\zeta-b)^2}\right]^{1/2}\end{array}\right\}}{\left[a^2+(b-\zeta)^2\right](2\zeta+g)(2\zeta-g)} \tag{4.6}$$

which is valid for $0 < 2\zeta/g < \infty$ although it is indeterminate when $g = 2\zeta$. The expression (4.6) is continuous however and may be evaluated at $g = 2\zeta$ by l'Hospital's rule. When $g = 2\zeta$ we have $a = 1$ and $b = 2\zeta = g$, the integrand of (4.4) has a double pole at $\omega = -\omega_n$ and (4.6) reduces to

$$E\left[x^2\right] = \frac{S_o}{m^2\omega_n^3}\frac{1}{4\zeta(1+\zeta^2)}\left[\frac{1-\zeta^2}{1+\zeta^2}(\pi - \tan^{-1}2\zeta) + 2\zeta + \frac{\zeta\log(1+4\zeta^2)}{1+\zeta^2}\right] \qquad (4.7)$$

in terms of ζ. This is plotted as curve (a) in Fig. 6.

The two equivalent viscous substitutions for this system are shown in Fig. 5(b) and (c). In Fig. 5(b) we have the first equivalence in which only the damping element is changed to a single viscous element which gives the same magnitude of damping force in simple harmonic motion at $\omega = \omega_n$ as the original system of Fig. 5(a). The expected mean square response under the same random excitation is

$$E\left[x^2\right] = \frac{\pi S_o}{k(c + kg/\omega_n)} = \frac{\pi S_o}{m^2\omega_n^3} \cdot \frac{1}{2\zeta + g} \qquad (4.8)$$

For the particular case $2\zeta = g$ this is plotted as curve (b) in Fig. 6.

The second equivalence is based on the requirement that the equivalent viscous system of Fig. 5(c) should have the same poles as the original system of Fig. 5(a). Here both the elastic element and the damping element are adjusted to meet this requirement. We find

$$k_e = k(a^2+b^2)$$

$$c_e = 2\sqrt{km}\ b \tag{4.9}$$

where a and b are given by (4.5). The expected mean square response under the same random excitation is

$$E\left[x^2\right] = \frac{\pi S_o}{k_e c_e} = \frac{\pi S_o}{m^2 \omega_n^3} \cdot \frac{1}{2b\,(a^2+b^2)} \tag{4.10}$$

This is plotted as curve (c) in Fig. 6 for the particular case where $2\zeta = g$. Studying Fig. 6 we note that the exact solution (a) lies between the approximations (b) and (c). When $2\zeta = g = 0.4$ the relative error between (a) and (c) is about 5% while the relative error between (a) and (b) is about 10%. For smaller values of damping the relative error is $O(g^2)$. Some notion of the behavior of the error for other combinations of structural and viscous damping can be had by remembering that Fig. 4 represents the case where $2\zeta/g = 0$, that Fig. 6 represents the case $2\zeta/g = 1$ and that for $2\zeta/g = \infty$ (i.e., all damping is viscous) all three models become identical.

Acknowledgment

It is a pleasure to acknowledge the help of G.R. Khabbaz who carried out the computations and plotted the curves in Fig. 3.

Captions for Figures

Fig. 1 (a) Vibratory system with linear structural damping. (b) First equivalent viscous system with same elastic element and equal damping force at $\omega = \omega_n$. (c) Second equivalent viscous system in which both elastic and damping elements are adjusted to produce some poles as in (a).

Fig. 2 Comparison of the behavior of structural and viscous damping forces as functions of frequency for fixed amplitude of displacement.

Fig. 3 Discrepancies between impulse responses for the systems of Fig. 1 when $g = 0.05$.

Fig. 4 Mean square response to white random excitation for the systems of Fig. 1.

Fig. 5 (a) Vibratory system with structural damping and viscous damping. (b) First equivalent viscous system with same elastic element and equal damping force at $\omega = \omega_n$. (c) Second equivalent viscous system in which both elastic and damping elements are adjusted to produce same poles as in (a).

Fig. 6 Mean square response to white random excitation for the systems of Fig. 5, for the case where $g = 2\zeta$

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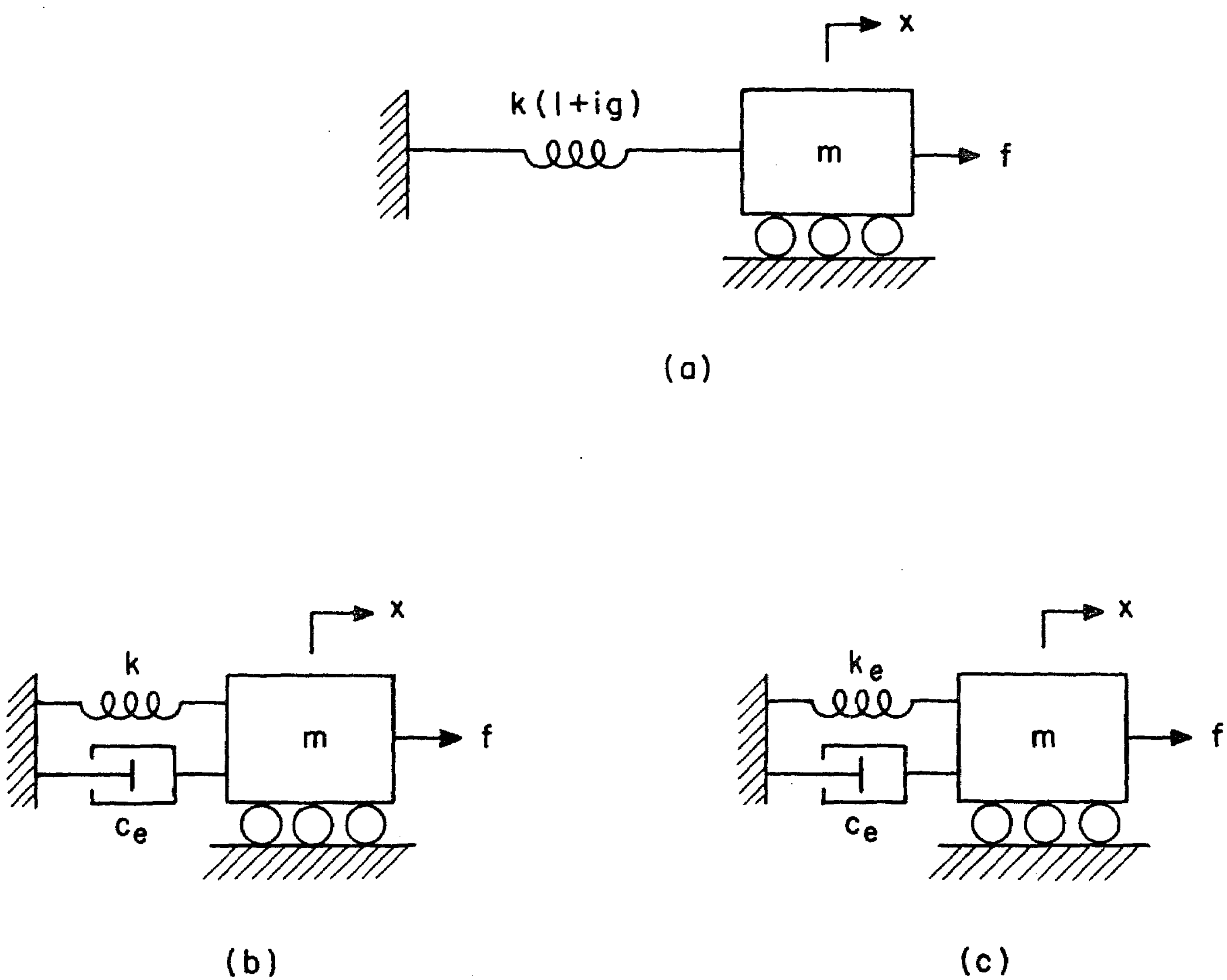
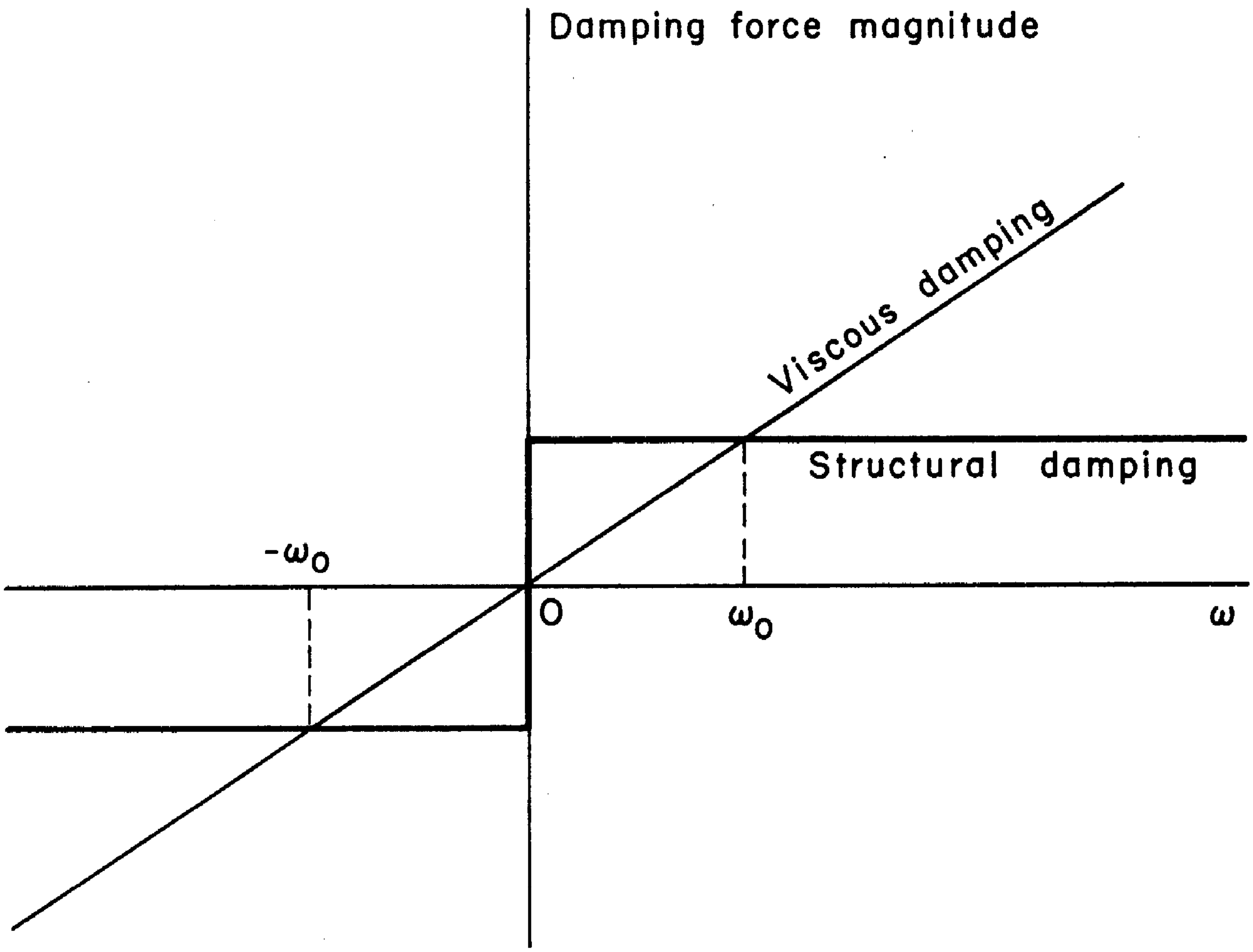


FIGURE 1



Damping force magnitude
Viscous damping
Structural damping
$-\omega_0$
0
ω_0
ω

FIGURE 2

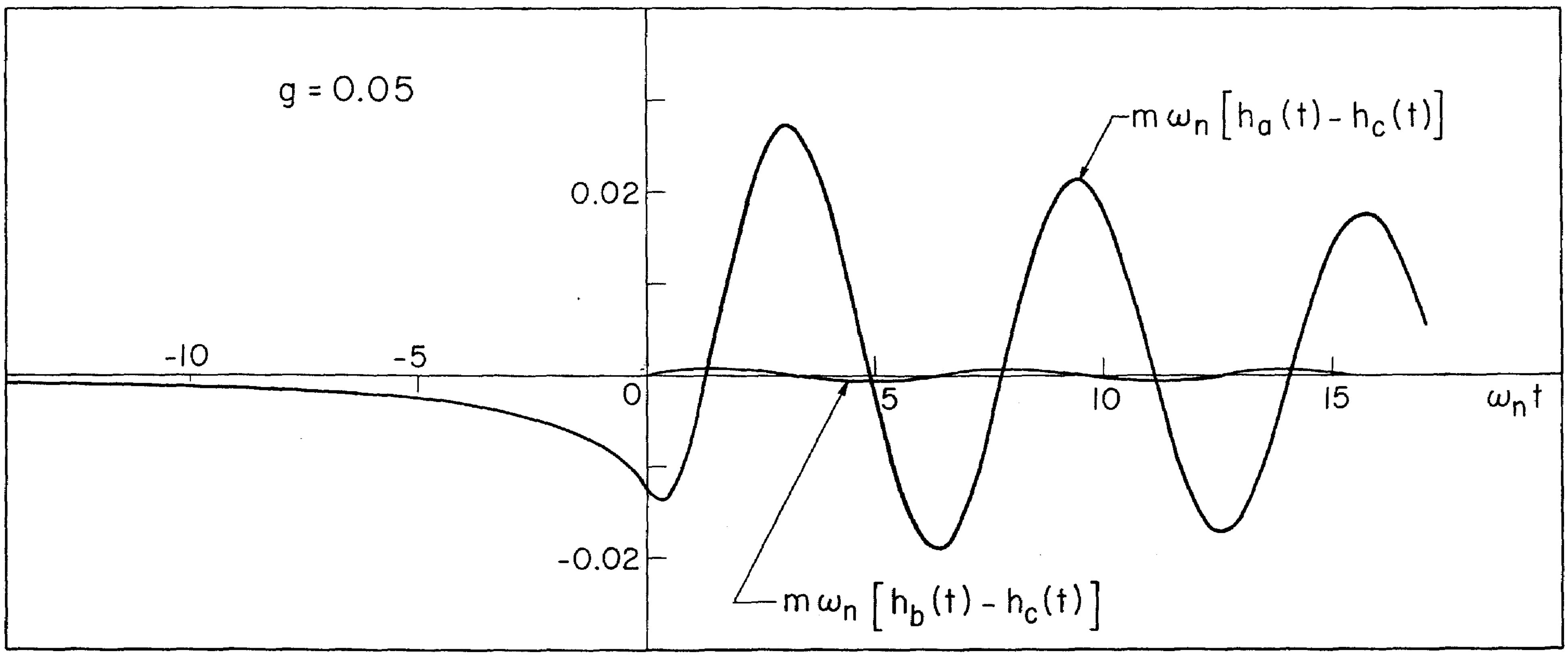

g = 0.05
0.02
-0.02
-10
-5
0
5
10
15
$\omega_n t$
$m\omega_n [h_a(t) - h_c(t)]$
$m\omega_n [h_b(t) - h_c(t)]$

FIGURE 3

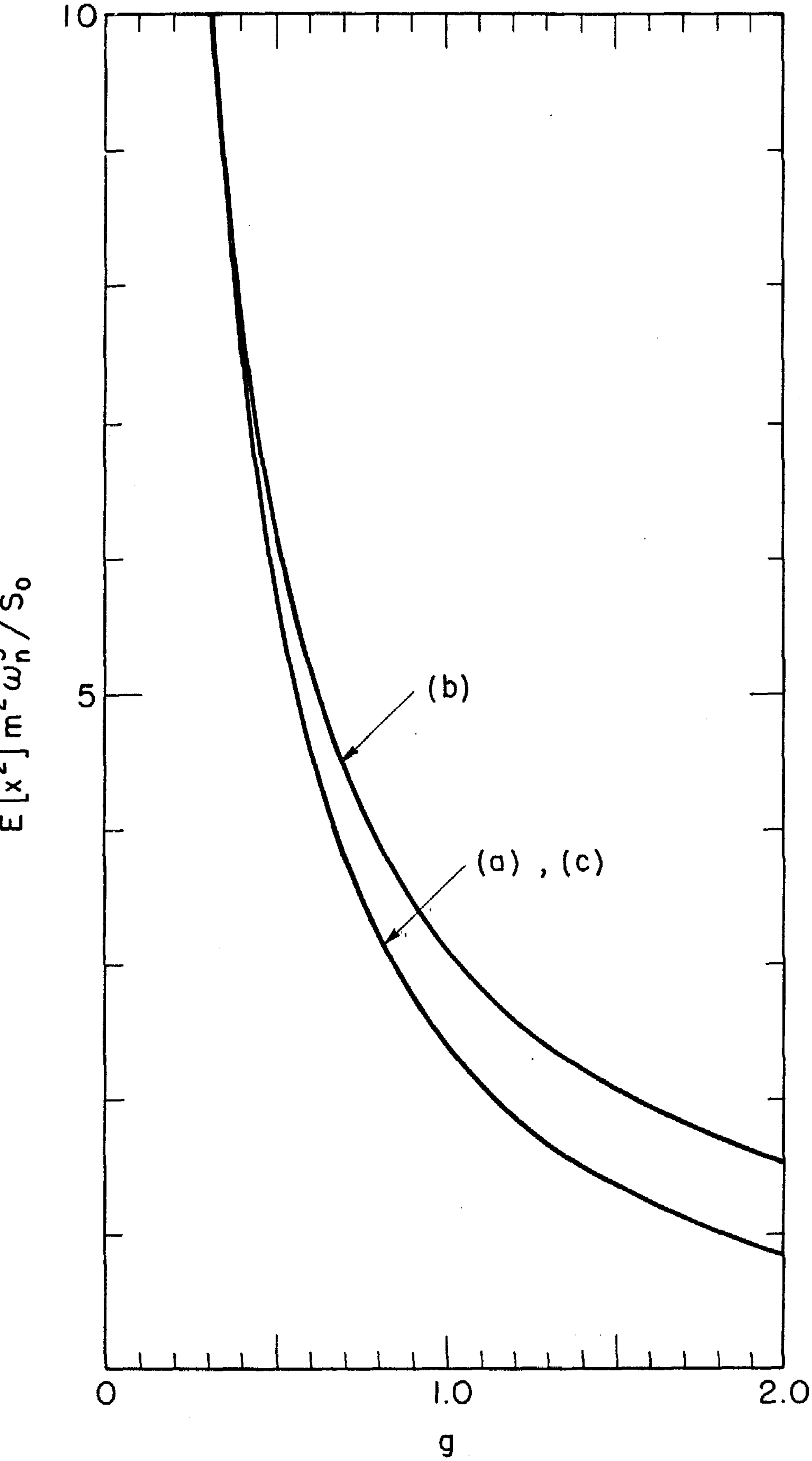


FIGURE 4

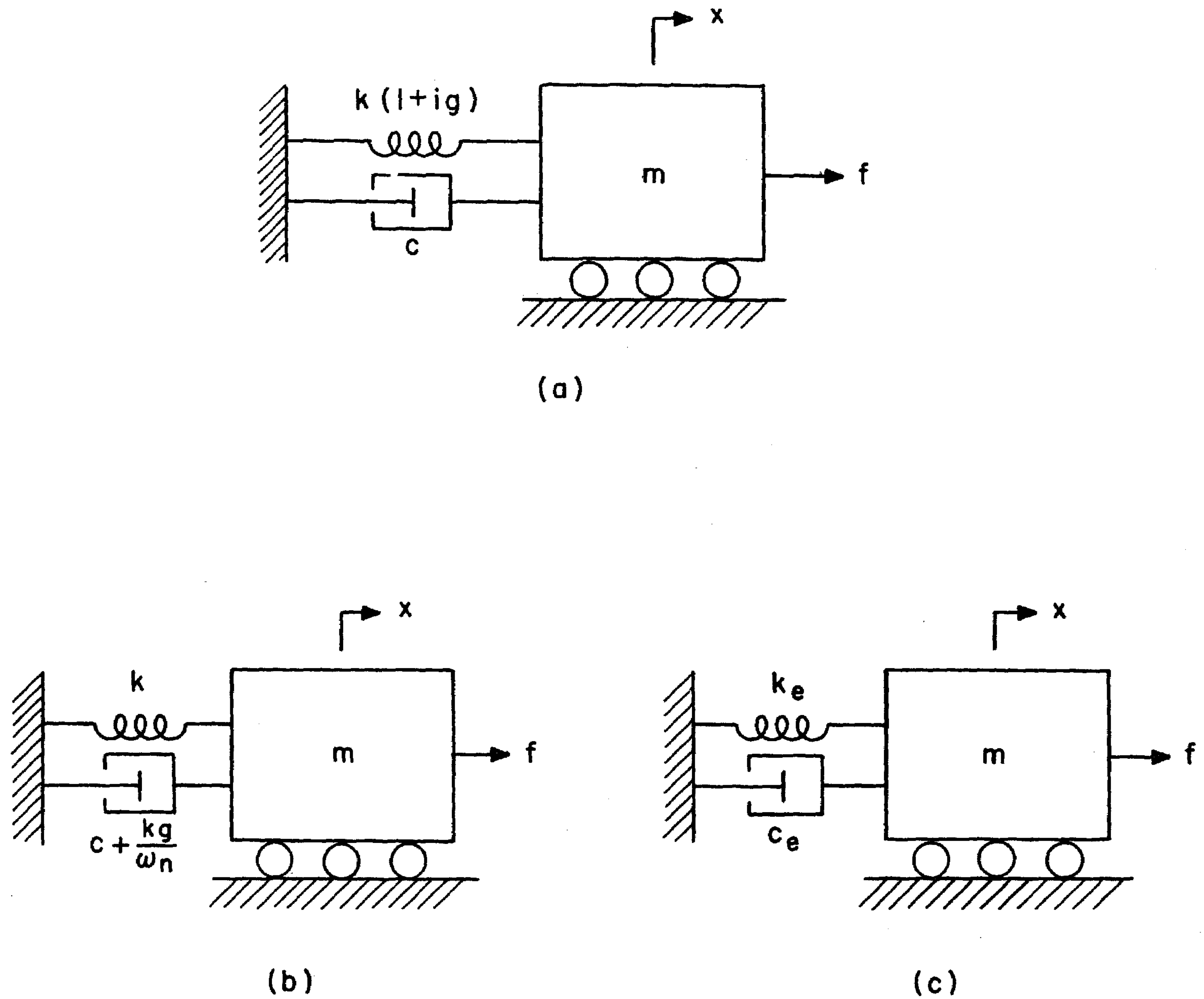


FIGURE 5

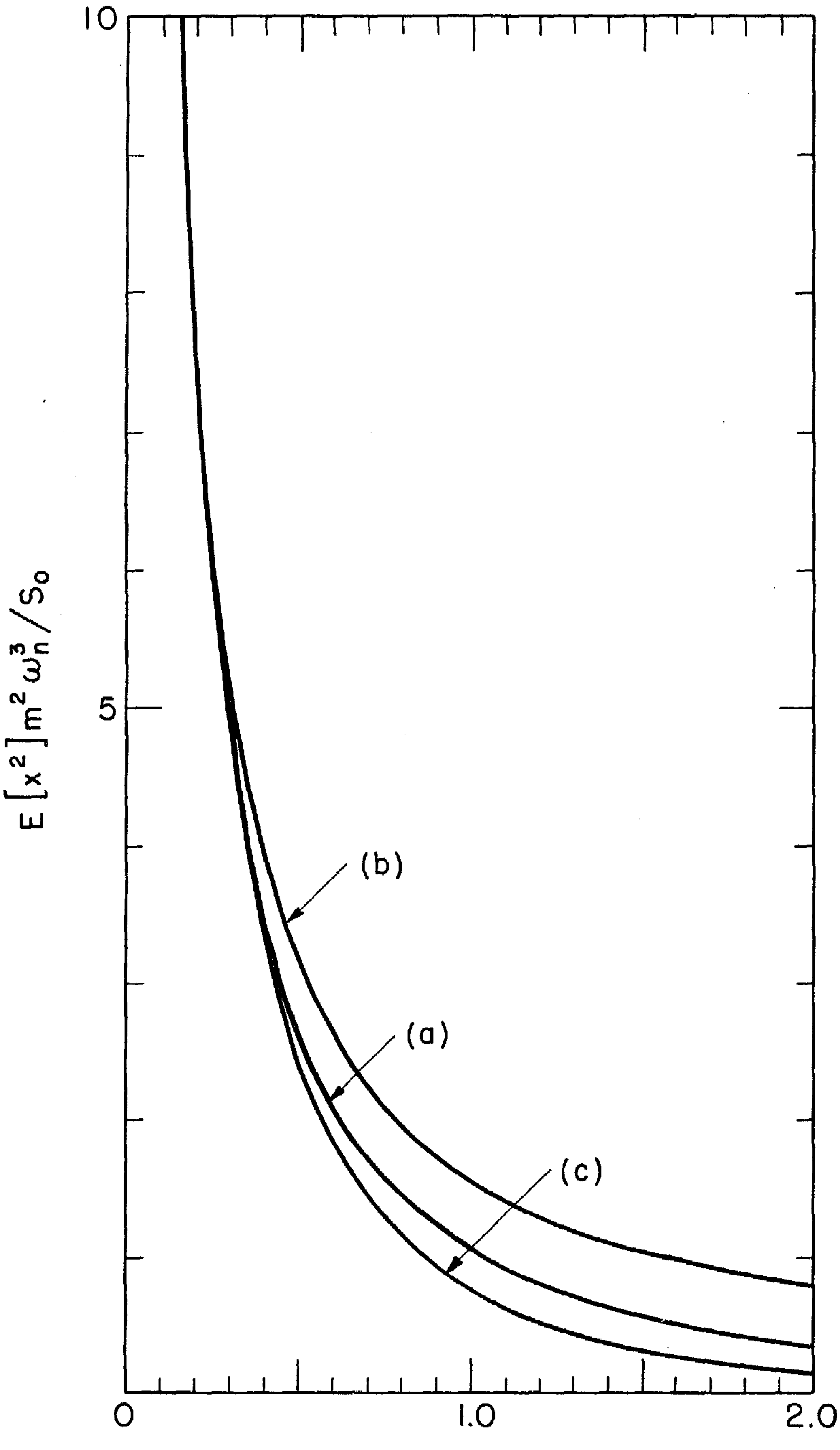


FIGURE 6